Decay of correlations and absence of superfluidity in the disordered Tonks-Girardeau gas

Robert Seiringer
IST Austria

IST AUSTRIA
Institute of Science and Technology

Joint work with Simone Warzel

New J. Phys. **18**, *035002 (2016)*

QMath13: Mathematical Results in Quantum Physics

Georgia Institute of Technology, October 8–11, 2016

Preface: Many-Body Localization

Many-body localization (MBL) concerns the effect of **disorder** on interacting many-body quantum systems or, equivalently, the effect of interparticle **interactions** on disordered quantum systems.

While one-body localization is reasonably well understood, MBL represents a big challenge.

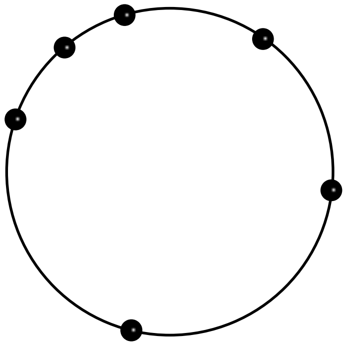
Expected effects:

- **Non-thermalization.** E.g., particle density does not become uniform at large times
- **Exponential decay** of (static) correlation functions even at zero temperature
- Absence of **superfluidity** even at zero temperature
- ...

We shall prove the occurrence of these effects in a simple toy model, the Tonks-Girardeau gas.

The Disordered Girardeau-Tonks Gas

Consider N bosons on a ring of length L with point interactions:



$$H_g = \sum_{j=1}^{N} h_j + g \sum_{1\le j<k\le N} \delta(x_j - x_k)$$

on the symmetric tensor product $\bigotimes_{\rm sym}^N L^2([0,L])$, with **random** one-particle Hamiltonian

$$h = -\frac{d^2}{dx^2} + V_\omega(x)$$

Tonks-Girardeau limit: $g \to \infty$, i.e., Dirichlet conditions on $\{x_j = x_k\}$. Equivalent to non-interacting fermions. Eigenfunctions of H_∞ are given by

$$\psi(x_1,\dots,x_N) = \frac{1}{\sqrt{N!}} \det\left(\varphi_{j_\alpha}(x_\beta)\right)_{\alpha,\beta=1}^N \prod_{1\le j<k\le N} \mathrm{sign}(x_j - x_k)$$

where the φ_{j_α} are eigenfunctions of h, either with **periodic** (N odd) or **anti-periodic** (N even) boundary conditions.

Quantities of Interest 1

The **one-particle density matrix** of a many-body wave functions ψ, defined as

$$\gamma_\psi(x,y) = N \int \psi(x,x_2,\ldots,x_N)\overline{\psi(y,x_2,\ldots,x_N)}dx_2\cdots dx_N$$

satisfying $0 \leq \gamma \leq N$ and $\mathrm{Tr}\,\gamma = N$. For an **eigenfunction** of H_∞, it takes the form

$$\gamma_\psi(x,y) = \det \begin{pmatrix} 0 & \varphi_{j_1}(x)\cdots\varphi_{j_N}(x) \\ \hline \varphi_{j_1}(y) & \\ \vdots & K_N(x,y) \\ \hline \varphi_{j_N}(y) & \end{pmatrix}$$

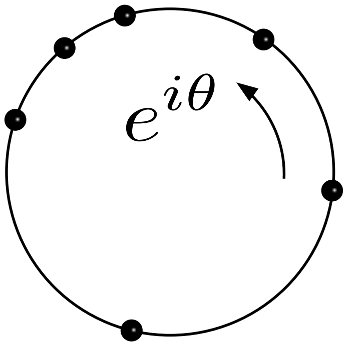
where

$$[K_N(x,y)]_{\alpha,\beta} = \delta_{\alpha,\beta} - 2\int_{[x,y]} \varphi_{j_\alpha}(z)\overline{\varphi_{j_\beta}(z)}dz \qquad \text{for all } x \leq y$$

Note: $\gamma_\psi(x,y) \neq \sum_\alpha \varphi_{j_\alpha}(x)\overline{\varphi_{j_\alpha}(y)}$, but $\varrho(x) = \gamma_\psi(x,x) = \sum_\alpha |\varphi_{j_\alpha}(x)|^2$.

Quantities of Interest 2

The **superfluid density** is defined via the increase of the **ground state energy** as one twists the boundary conditions:



$$\psi(\ldots, x_j + L, \ldots) = e^{i\theta}\psi(\ldots, x_j, \ldots)$$

For any $\mu \in \mathbb{R}$, we define $N_\mu = \mathrm{Tr}\, \mathbb{1}_{(-\infty,\mu)}(h)$ and

$$\varrho_{\mathrm{s}}(\mu) = \limsup_{\theta \to 0} \frac{1}{\theta^2} \limsup_{L \to \infty} L \left(E_0(N_\mu, \theta) - E_0(N_\mu, 0) \right)$$

Note: $E_0(N, \theta)$ is simply the sum of the first N eigenvalues of h with θ boundary conditions. In the absence of disorder, one easily checks that

$$\varrho_{\mathrm{s}}(\mu) = \varrho = \lim_{L \to \infty} \frac{N_\mu}{L} = \frac{\sqrt{[\mu]_+}}{2\pi^2}$$

One-Body Localization Assumption

We shall **assume** that $\exists C, \ell \in (0,\infty)$ such that for all $1 \leq n, m \leq L$ and all $L \in \mathbb{N}$

$$\mathbb{E}\left[Q_L(n,m;J)\right] \leq C \exp\left(-\frac{\mathrm{dist}(n,m)}{\ell}\right)$$

for suitable $J \subset \mathbb{R}$, where $\mathrm{dist}(\cdot,\cdot)$ denotes the Euclidean distance on the torus and

$$Q_L(n,m;J) := \sum_{j,\, E_j \in J} \Phi_j(n)\Phi_j(m)\,,$$

with $\Phi_j(n) := \left(\int_{I_n} |\varphi_j(x)|^2\, dx\right)^{\frac{1}{2}}$ and $I_n := [n, n+1)$.

This assumption is known for many models of the form $-d^2/dx^2 + V_\omega(x)$, e.g., **Anderson type models** with $V_\omega(x) = \sum_{j\in\mathbb{Z}} \omega_j u(x-j)$, for *any* $J \subset \mathbb{R}$ bounded above.

It implies exponential decay of eigenfunctions, and also **dynamical localization**.

Consequences of Localization Assumption

Consider an N-body wave function which is a superposition of eigenfunctions made up of φ_js with $E_j \subset J$ for some $J \subset \mathbb{R}$ where **ECL** holds. Let ϱ_t by the one-particle density at some later time $t > 0$. Then

THEOREM 1. *For any* $I \subset [0, L]$,

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}}\left|\int_I \varrho_t(x)dx - \int_I \varrho_0(x)dx\right|\right] \le A$$

and for any pair of subsets $I \subset K \subset [0, L]$

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}}\int_I \varrho_t(x)dx\right] \le \mathbb{E}\left[\int_K \varrho_0(x)dx\right] + A\exp\left(-\frac{\mathrm{dist}(I, K^c)}{\ell}\right)$$

for some $A \in (0, \infty)$ *independent of* N, L.

This facet of **many-body localization** is an easy consequence of one-body localization, and is equivalent to the corresponding statement for free fermions.

MAIN RESULTS. 1. EXPONENTIAL DECAY OF CORRELATIONS

Assume again **ECL** for some $J \subset \mathbb{R}$, and let ψ be an eigenfunction of H_∞ corresponding to φ_js with $E_j \in J$.

THEOREM 2. *There exists and* $A \in (0,\infty)$ *independent of* L *and* N *such that*

$$\mathbb{E}\left[\|1_{I_n}\gamma_\psi 1_{I_m}\|_2^\sigma\right] \leq A \exp\left(-\frac{1}{3}(1-\sigma)\frac{\operatorname{dist}(n,m)}{\ell}\right)$$

for all $1 \leq n, m \leq L$ *and all* $2/5 \leq \sigma < 1$, $I_n := [n, n+1)$.

This result implies **absence of BEC** provided the local density fluctuations are reasonably bounded: If

$$\sup_{n,L} \mathbb{E}\left[(\operatorname{Tr} 1_{I_n} P_J(h))^p\right] < \infty$$

for some $p > 2$ then

$$\|\gamma_\psi\|_\infty \leq o(L^r) \qquad \text{for any } \frac{2}{p} < r \leq 1$$

Note: In the absence of disorder, $\|\gamma_\psi\| = O(L^{1/2})$ in the ground state.

Main Results. 2. Absence of Superfluidity

Recall the definition of the **superfluid density**:

$$\varrho_{\rm s}(\mu) = \limsup_{\theta\to 0} \frac{1}{\theta^2} \limsup_{L\to\infty} L\left(E_0(N_\mu,\theta) - E_0(N_\mu,0)\right)$$

for $\mu \in \mathbb{R}$ and $N_\mu = \operatorname{Tr} \mathbb{1}_{(-\infty,\mu)}(h)$.

THEOREM 3. *Assume that* **ECL** *holds on* $J = (-\infty, \mu]$. *Then*

$$\limsup_{L\to\infty} L\left(E_0(N_\mu,\theta) - E_0(N_\mu,0)\right) = 0$$

almost surely for any $\theta \in [0, 2\pi)$, *and hence* $\varrho_{\rm s} = 0$ *almost surely.*

Conclusions

- We showed **absence of BEC and superfluidity** for the Tonks-Girardeau model with generic disorder
- In particular, this model exhibits **many-body localization**
- The model is the continuum analog for the XY spin chain: previous work by [Hamza/Sims/Stolz 2012, Sims/Warzel 2015, Abdul-Rahman/Stolz 2015]
- Nothing is known for finite interaction strength $g < \infty$, except:
- In the **mean field limit** $g \sim N^{-2}$, there is always BEC, and superfluidity may or may not occur depending on the strength of the disorder [Seiringer/Yngvason/Zagrebnov 2012, Könenberg/Moser/Seiringer/Yngvason 2015]