Discriminating quantum states: the \textit{multiple Chernoff distance}

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Outline

1. The problem
2. The answer
3. History review
4. Proof sketch
5. One-shot case
6. Open questions
Accessing quantum systems: quantum measurement

- Quantum measurement: formulated as positive operator-valued measure (POVM)
  \[ \mathcal{M} = \{M_i\}_i, \text{ with } 0 \leq M_i \leq \mathbb{1} \text{ and } \sum_i M_i = \mathbb{1}; \]
  when performing the POVM on a system in the state \( \omega \), we obtain outcome \( "i" \) with probability \( \text{Tr}(\omega M_i) \).

- von Neumann measurement: special case of POVM, with the POVM elements being orthogonal projectors:
  \[ M_i M_j = \delta_{ij} M_i, \text{ where } \delta_{ij} \text{ is the Kronecker delta.} \]
Quantum state discrimination (quantum hypothesis testing)

- Suppose a quantum system is in one of a set of states \( \{ \omega_1, \ldots, \omega_r \} \), with a given prior \( \{ p_1, \ldots, p_r \} \). The task is to detect the true state with a minimal error probability.

- Method: making quantum measurement \( \{ M_i \}_{i=1}^r \).

- Error probability (let \( A_i := p_i \omega_i \))

\[
P_e (\{A_1, \ldots, A_r\}; \{M_1, \ldots, M_r\}) := \sum_{i=1}^r \text{Tr} A_i (\mathbb{1} - M_i).
\]

- Optimal error probability

\[
P_e^* (\{A_1, \ldots, A_r\}) := \min \left\{ P_e (\{A_1, \ldots, A_r\}; \{M_1, \ldots, M_r\}) : \text{POVM } \{M_1, \ldots, M_r\} \right\}.
\]
Asymptotics in quantum hypothesis testing

- What's the asymptotic behavior of
  \[ P_e^* \left( \left\{ p_1 \rho_1^\otimes n, \ldots, p_r \rho_r^\otimes n \right\} \right), \quad \text{as } n \to \infty ? \]

- Exponentially decay! (Parthasarathy '2001)
  \[ P_e^* \sim \exp(-\xi n) \]

- But, what's the error exponent
  \[ \xi = \liminf_{n \to \infty} \frac{-1}{n} \log P_e^* \left( \left\{ p_1 \rho_1^\otimes n, \ldots, p_r \rho_r^\otimes n \right\} \right) \]

  It has been an open problem (except for \( r=2 \))?
Our result:
error exponent = multiple Chernoff distance

We prove that

**Theorem**  Let $\{\rho_1, \ldots, \rho_r\}$ be a finite set of quantum states on a finite-dimensional Hilbert space $\mathcal{H}$. Then the asymptotic error exponent for testing $\{\rho_1^\otimes n, \ldots, \rho_r^\otimes n\}$, for an arbitrary prior $\{p_1, \ldots, p_r\}$, is given by the multiple quantum Chernoff distance:

$$
\lim_{n \to \infty} \frac{-1}{n} \log P_e^* (\{p_1 \rho_1^\otimes n, \ldots, p_r \rho_r^\otimes n\}) = \min_{(i,j):i \neq j} \max_{0 \leq s \leq 1} \left\{ -\log \text{Tr} \rho_i^s \rho_j^{1-s} \right\}. \quad (1)
$$
Remarks

Remark 1: Our result is a multiple-hypothesis generalization of the $r=2$ case. Denote the **multiple quantum Chernoff distance** (r.h.s. of eq. (1)) as $C(\rho_1, \ldots, \rho_r)$, then

$$C(\rho_1, \ldots, \rho_r) = \min_{(i,j): i \neq j} C(\rho_i, \rho_j),$$

with the **binary quantum Chernoff distance** is defined as

$$C(\rho_1, \rho_2) := \max_{0 \leq s \leq 1} \{-\log \text{Tr} \rho_1^s \rho_2^{1-s}\}.$$

Remark 2: when $\rho_1, \ldots, \rho_r$ commute, the problem reduces to classical statistical hypothesis testing. Compared to the classical case, the difficulty of quantum statistics comes from **noncommutativity** & *entanglement*. 
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Some history review

- The classical Chernoff distance as the optimal error exponent for testing two probability distributions was given in H. Chernoff, Ann. Math. Statist. 23, 493 (1952).

- The multiple generalizations were subsequently made in
  
  N. P. Salihov, Dokl. Akad. Nauk SSSR 209, 54 (1973);
  E. N. Torgersen, Ann. Statist. 9, 638 (1981);
Quantum hypothesis testing (state discrimination) was the main topic in the early days of quantum information theory in 1970s.

Maximum likelihood estimation
- for two states: Holevo-Helstrom tests
  \[ (\{\rho_1 - \rho_2 > 0\}, \mathbb{I} - \{\rho_1 - \rho_2 > 0\}) \]


- for more than two states: only formulated in a complex and implicit way. Competitions between pairs make the problem complicated!

Some history review

In 2001, Parthasarathy showed exponential decay.

In 2006, two groups [Audenaert et al] and [Nussbaum & Szkola] together solved the r=2 case.

In 2010/2011, Nussbaum & Szkola conjectured the solution (our theorem), and proved that \( \frac{C}{3} \leq \xi \leq C \).

In 2014, Audenaert & Mosonyi proved that \( \frac{C}{2} \leq \xi \leq C \).
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Sketch of proof

- We only need to prove the achievability part " $\xi \geq C$ ".
  For this purpose, we construct an asymptotically optimal quantum measurement, and show that it achieves the quantum multiple Chernoff distance as the error exponent.

- Motivation: consider detecting two weighted pure states.
  
  Big overlap: give up the light one;

  Small overlap: make a projective measurement, using orthonormalized version of the two states.
Sketch of proof

Spectral decomposition:

\[ \rho_i \otimes n = \bigoplus_{k=1}^{T_i} \lambda_{ik}^{(n)} Q_{ik}^{(n)} , \]

\[ T := \max \{ T_i \}_i \leq (n + 1)^d \]

Overlap between eigenspaces:

\[ \text{Olap} \left( \text{supp} \left( Q_{ik}^{(n)} \right), \text{supp} \left( Q_{j\ell}^{(n)} \right) \right) \]

\[ := \max \left\{ |\langle \varphi | \phi \rangle| : \varphi \in \text{supp} \left( Q_{ik}^{(n)} \right), \phi \in \text{supp} \left( Q_{j\ell}^{(n)} \right) \right\} \]
Sketch of proof

"Dig holes" in every eigenspaces to reduce overlaps

\[ \tilde{\rho}_1^\otimes n \quad \tilde{\rho}_2^\otimes n \quad \tilde{\rho}_r^\otimes n \]

\[ \text{Sketch of proof} \]

\[ \text{\"Dig holes\" in every eigenspaces to reduce overlaps} \]

\[ \tilde{\rho}_i^\otimes n = \bigoplus_{k=1}^{T_i} \lambda_i^{(n)} Q_i^{(n)}_{ik}, \quad \text{Olap} \left( \text{supp} \left( Q_i^{(n)}_{ik} \right), \text{supp} \left( Q_j^{(n)}_{j\ell} \right) \right) \leq \epsilon \]

\[ \epsilon\text{-subtraction:} \]

Let \[ P_1 P_2 P_1 = \bigoplus_{x} \lambda_x Q_x \]

Define \[ P_1 \ominus \epsilon P_2 := P_1 - \sum_{x: \lambda_x \geq \epsilon^2} Q_x \]
The next step is to orthogonalize these eigenspaces

1. Order the eigenspaces according to the their eigenvalues, in the decreasing order.
2. Orthogonalization using the Gram-Schmidt process.

Now the supporting space of the hypothetic states have small overlaps. For \( i \neq j \),

\[
\text{Olap}\left(\text{supp}\left(\tilde{\rho}_{i}^{\otimes n}\right), \text{supp}\left(\tilde{\rho}_{j}^{\otimes n}\right)\right) \leq T\varepsilon
\]
Sketch of proof

- Now the eigenspaces are all orthogonal.

\[ \rho_i \otimes n = \bigoplus_{k=1}^{T_i} \lambda_{ik}^{(n)} Q_{ik}^{(n)} \]

- We construct a projective measurement

\[ \Pi_i = \bigoplus_{k} Q_{ik}^{(n)} \]

- Use this to discriminate the original states:

\[ P_{\text{succ}} = \sum_{i=1}^{r} p_i \text{Tr} \rho_i \otimes n \Pi_i \]
Sketch of proof

- Loss in "digging holes":
  \[
  \text{Tr} \left( Q_{ik}^{(n)} - \tilde{Q}_{ik}^{(n)} \right) \leq \frac{1}{\epsilon^2} \sum_{(j,\ell):\lambda_{j\ell}^{(n)} > \lambda_{ik}^{(n)}} \text{Tr} Q_{ik}^{(n)} Q_{j\ell}^{(n)}
  \]

- Mismatch due to orthogonalization:
  \[
  \text{Tr} \left[ \tilde{Q}_{ik}^{(n)} \left( \mathbb{1} - \tilde{Q}_{ik}^{(n)} \right) \right] \leq \frac{1 - (r - 1)T\epsilon}{1 - 2(r - 1)T\epsilon} \sum_{(j,\ell):\lambda_{j\ell}^{(n)} > \lambda_{ik}^{(n)}} \text{Tr} Q_{ik}^{(n)} Q_{j\ell}^{(n)}
  \]

- Estimation of the total error:
  \[
  P_e \leq \sum_{i,k} \lambda_{ik}^{(n)} \text{Tr} \left[ Q_{ik}^{(n)} \left( \mathbb{1} - \tilde{Q}_{ik}^{(n)} \right) \right] \leq \sum_{i,k} \lambda_{ik}^{(n)} \left\{ \text{Tr} \left( Q_{ik}^{(n)} - \tilde{Q}_{ik}^{(n)} \right) + \text{Tr} \left[ \tilde{Q}_{ik}^{(n)} \left( \mathbb{1} - \tilde{Q}_{ik}^{(n)} \right) \right] \right\}
  \]
Sketch of proof

\[
P_e \leq \left( \frac{1}{\epsilon^2} + \frac{1 - (r - 1)T\epsilon}{1 - 2(r - 1)T\epsilon} \right) \sum_{(i,j): i \neq j} \sum_{k, \ell} \min\{\lambda_{ik}^{(n)}, \lambda_{j\ell}^{(n)}\} \text{Tr} Q_{ik}^{(n)} Q_{j\ell}^{(n)} \leq \left( \lambda_{ik}^{(n)} \right)^s \left( \lambda_{j\ell}^{(n)} \right)^{(1-s)} \]

\[
\leq p(n) \sum_{(i,j): i \neq j} \min_{0 \leq s \leq 1} \left( \text{Tr} \rho_i^s \rho_j^{(1-s)} \right)^n
\]

\[
\sim \exp \left\{ -n \left( \min_{(i,j): i \neq j} \max_{0 \leq s \leq 1} \left\{ -\log \text{Tr} \rho_i^s \rho_j^{1-s} \right\} \right) \right\}
\]
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Result for the one-shot case

**Theorem** Let \(A_1, \ldots, A_r \in \mathcal{P}(\mathcal{H})\) be nonnegative matrices on a finite-dimensional Hilbert space \(\mathcal{H}\). For all \(1 \leq i \leq r\), let \(A_i = \bigoplus_{k=1}^{T_i} \lambda_{ik} Q_{ik}\) be the spectral decomposition of \(A_i\), and write \(T := \max\{T_1, \ldots, T_r\}\). Then

\[
P_e^* (\{A_1, \ldots, A_r\}) \leq 10(r-1)^2 T^2 \sum_{(i,j):i<j} \sum_{k,\ell} \min\{\lambda_{ik}, \lambda_{j\ell}\} \operatorname{Tr} Q_{ik} Q_{j\ell}.
\]

**Remark 1:** It matches a lower bound up to some states-dependent factors:

\[
P_e^* (\{A_1, \ldots, A_r\}) \geq \frac{1}{2(r-1)} \sum_{(i,j):i<j} \sum_{k,\ell} \min\{\lambda_{ik}, \lambda_{j\ell}\} \operatorname{Tr} Q_{ik} Q_{j\ell}.
\]

Result for the one-shot case

- Remark 2: for the case $r=2$, we have

$$P_e^* \left( \{A_1, A_2\} \right) \leq 10T^2 \sum_{k,\ell} \min\{\lambda_{1k}, \lambda_{2\ell}\} \Tr Q_{1k}Q_{2\ell}.$$ 

On the other hand, it is proved in [K. Audenaert et al, PRL, 2007] that

$$P_e^* \left( \{A_1, A_2\} \right) \leq \min_{0 \leq s \leq 1} \Tr A_1^s A_2^{1-s}.$$ 

(note that it is always true that

$$\sum_{k,\ell} \min\{\lambda_{1k}, \lambda_{2\ell}\} \Tr Q_{1k}Q_{2\ell} \leq \min_{0 \leq s \leq 1} \Tr A_1^s A_2^{1-s}. \)$$
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Open questions

1. Applications of the bounds:

\[ P_e^*(\{A_1, \ldots, A_r\}) \begin{cases} \leq 10(r-1)^2 T^2 \sum_{(i,j):i<j} \sum_{k,\ell} \min\{\lambda_{ik}, \lambda_{j\ell}\} \operatorname{Tr} Q_{ik} Q_{j\ell} \\ \geq \frac{1}{2(r-1)} \sum_{(i,j):i<j} \sum_{k,\ell} \min\{\lambda_{ik}, \lambda_{j\ell}\} \operatorname{Tr} Q_{ik} Q_{j\ell} \end{cases} \]

2. Strengthening the states-dependent factors

3. Testing composite hypotheses:

\[ \rho^{\otimes n} \quad \text{Vs} \quad \sum_i q_i \sigma_i^{\otimes n} \quad \text{(or, } \int \sigma^{\otimes n} \, d\mu(\sigma)) \]

Thank you!