SPECIAL TK_5 IN GRAPHS CONTAINING K_4^-

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Dawei He

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SPECIAL TK_5 IN GRAPHS CONTAINING K_4^-

Approved by:

Professor Xingxing Yu,
Committee Chair
School of Mathematics
Georgia Institute of Technology

Professor Xingxing Yu, Advisor
School of Mathematics
Georgia Institute of Technology

Professor Robin Thomas
School of Mathematics
Georgia Institute of Technology

Professor William T. Trotter
School of Mathematics
Georgia Institute of Technology

Professor Greg Blekherman
School of Mathematics
Georgia Institute of Technology

Professor Geoffrey Ye Li
School of Electrical and Computer
Engineering
Georgia Institute of Technology

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To my parents, Renhui He and Dianfeng Huang

PREFACE

One important task in structural graph theory is to obtain good characterizations of various classes of graphs. A well-known example is the Kuratowski's theorem [17], which states that a graph is planar if and only if it contains no $TK_{3,3}$ and TK_5. Given a graph K, TK is used to denote a subdivision of K, which is a graph obtained from K by substituting some edges for paths.

It is natural to ask for structural characterizations of graphs containing no TK_5 and of graphs containing no $TK_{3,3}$. It can easily be derived from Kuratowski's theorem that every 3-connected nonplanar graph has a subgraph isomorphic to a $TK_{3,3}$ unless it is isomorphic to K_5.

Kelmans [15], and independently, Seymour [23] conjectured that every 5-connected nonplanar graph contains a TK_5. $K_{4,4}$ indicates that 4-connectedness is not sufficient.

In [19], J. Ma and X. Yu proved Kelmans-Seymour conjecture for graphs containing K_4^-. A strategy to prove this conjecture for graphs containing no K_4^- is to strengthen this result of Ma and Yu. In this dissertation, we show that if G is a 5-connected nonplanar graph containing K_4^-, then it contains TK_5 which avoids certain edges or vertices.

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SUMMARY

Given a graph K, TK is used to denote a subdivision of K, which is a graph obtained from K by substituting some edges for paths. The well-known Kelmans-Seymour conjecture states that every nonplanar 5-connected graph contains TK_5. Ma and Yu proved the conjecture for graphs containing K_4^-. In this dissertation, we strengthen their result in two ways. The results will be useful for completely resolving the Kelmans-Seymour conjecture.

Let G be a 5-connected nonplanar graph and let $x_1, x_2, y_1, y_2 \in V(G)$ be distinct, such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1 y_2 \notin E(G)$.

We show that one of the following holds: $G - y_2$ contains K_4^-, or G contains a TK_5 in which y_2 is not a branch vertex, or G has a special 5-separation, or for any distinct $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$, $G - \{y_2 v : v \notin \{x_1, x_2, w_1, w_2, w_3\}\}$ contains TK_5.

We show that one of the following holds: $G - x_1$ contains K_4^-, or G contains a TK_5 in which x_1 is not a branch vertex, or G contains a K_4^- in which x_1 is of degree 2, or $\{x_2, y_1, y_2\}$ may be chosen so that for any distinct $z_0, z_1 \in N(x_1) - \{x_2, y_1, y_2\}$, $G - \{x_1 v : v \notin \{z_0, z_1, x_2, y_1, y_2\}\}$ contains TK_5.

CHAPTER I

INTRODUCTION TO GRAPH THEORY

1.1 Basics

We use notation and terminology from [1, 5].

A *graph* is an ordered pair $G = (V, E)$ comprising a finite set V of *vertices*, together with a set E of *edges*, which are 2-element subsets of V.

Let $G = (V, E)$ be a graph. For an edge $\{x, y\} \subseteq V$, graph theorists usually use the shorter notation xy. The vertices x, y are said to be *adjacent* to each other. The edge xy is said to be *incident* to the vertices x and y.

Let U be a subset of V. The *neighbors* of U are the vertices in $V \setminus U$ adjacent to some vertex in U, and their set is denoted by $N_G(U)$, or briefly $N(U)$. We write $N_G(v)$ for $N_G(\{v\})$.

Let $v \in V$ be a vertex in G. The *degree* of v is the number of neighbors of v, which is also equal to the number of edges incident to v, denoted by $d_G(v)$.

A *walk* W in G of *length* k is an alternating sequence of vertices and edges $v_0, e_0, v_1, e_1, v_2, \ldots, v_{k-1}, e_{k-1}, v_k$, such that $v_0, v_1, \ldots, v_k \in V$, $e_0, \ldots, e_{k-1} \in E$, and $e_i = v_i v_{i+1}$ for $0 \leq i \leq k-1$. W is said to be a *path* if $v_0, v_1, \ldots, v_k$ are all distinct. If W is a path, we write $W = v_0 v_1 \ldots v_k$ by the natural sequence of its vertices and call W a path from v_0 to v_k and $v_1, \ldots, v_{k-1}$ the *internal vertices*. W is said to be a *cycle* if $v_0, v_1, \ldots, v_k$ are all distinct except that $v_0 = v_k$.

Let S, T be two subsets of V and P be a path from v_0 to v_k. We call P an $S-T$ *path* if $V(P) \cap S = \{v_0\}$ and $V(P) \cap T = \{v_k\}$.

Let $G = (V, E)$ be a graph. G is said to be a *bipartite graph* if V can be divided into two disjoint *parts* A and B such that every edge in E connects a vertex in A to one in B, and we also write $G = (A, B, E)$. A bipartite graph $G = (A, B, E)$ is said to be a *complete bipartite graph* if every vertex in A is connected to every vertex in B, and we also denote G by $K_{m,n}$ if $|A| = m$ and $|B| = n$.

Let $G = (V, E)$ be a graph. G is said to be a *complete graph* if every pair of vertices is connected by an edge, and we also denote G by K_n if $|V| = n$. In this dissertation we use K_4^- to denote the graph obtained from K_4 by deleting a single edge.

Let $G = (V, E)$ be a graph. A graph $G' = (V', E')$ is said to be a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$, written as $G' \subseteq G$. In this dissertation when we call a graph *minimal* or *maximal* with some property but have not specified any particular ordering, we are referring to the subgraph relation.

Let $G = (V, E)$ be a graph and U be a subset of V. We denote by $G[U]$ the graph on U whose edges are precisely those in E with both ends in U. A subgraph G' is said to be an *induced subgraph* of G if $G' = G[U]$ for some $U \subseteq V$. An *induced path* (or *induced cycle*) of G is a path (or cycle) that is an induced subgraph of G. Let U be a subset of V. We write $G - U$ for $G[V \setminus U]$. Let v be a vertex in V. We write $G - v$ for $G - \{v\}$. Let G' be a subgraph of G. We write $G - G'$ for $G - V(G')$. For a set F of 2-element subsets of V, we write $G - F := (V, E \setminus F)$ and $G + F := (V, E \cup F)$. As above, $G - \{e\}$ and $G + \{e\}$ are abbreviated to $G - e$ and $G + e$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. $G_1 \cup G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, and $G_1 \cap G_2$ is the graph with vertex set $V_1 \cap V_2$ and edge set $E_1 \cap E_2$.

Let $G = (V, E)$ be a graph and $e = xy$ be an edge in E. By G/e we denote the graph obtained from G by *contracting* the edge e into a new vertex v_e, which becomes adjacent to all the former neighbors of x and of y. For a connected subgraph M of G, we use G/M to denote the graph obtained from G by contracting M into a new vertex v_M, which becomes adjacent to all the former neighbors of vertices in M. A graph K is called a *minor* of G if K can be formed from G by deleting edges and vertices and by contracting edges.

Let $G = (V, E)$ and $uv \in E$. We may form an *elementary subdivision* of G by adding a new vertex w and replacing the edge uv by edges uw and vw. A graph H is said to be a *subdivision* of G if H can be obtained from G by a sequence of elementary subdivisions. We use TG to denote a subdivision of G. The vertices of TG corresponding to those in V are its *branch vertices*.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. An *isomorphism* of graphs G_1 and G_2 is a bijection between V_1 and V_2

$$f : V_1 \longrightarrow V_2$$

such that any two vertices x and y of G_1 are adjacent if and only if $f(x)$ and $f(y)$ are adjacent in G_2, and G_1, G_2 are called isomorphic and denoted as $G_1 \cong G_2$.

1.2 Connectivity

Let $G = (V, E)$ be a graph. If $S, T \subseteq V$, $X \subseteq V \cup E$ and every S-T path in G contains a vertex or an edge from X, we say that X *separates* S from T in G or X *separates* G, and call X a *separating set* in G. Furthermore, we call X a *vertex cut* of G if $X \subseteq V$. A vertex $v \in V$ is said to be a *cutvertex* if $\{v\}$ is a vertex cut of G. We call X an *edge cut* of G if $X \subseteq E$. An edge $e \in E$ is said to be a *bridge* if $\{e\}$ is an edge cut of G.

A *k-separation* of a graph G is a pair (G_1, G_2) of subgraphs of G such that $E(G) = E(G_1) \cup E(G_2)$, $E(G_1) \cap E(G_2) = \emptyset$, neither G_1 nor G_2 is a subgraph of the other, and $|V(G_1 \cap G_2)| = k$.

Let $G = (V, E)$ be a graph. We say that G is *connected* if there is a path from any vertex to any other vertex in G. A maximal connected subgraph is called a *component* of G. A maximal connected subgraph without a cutvertex is called a *block* of G.

Let $G = (V, E)$ be a graph and k be a positive integer. G is *k-connected* if $|G| > k$ and $G - X$ is connected for any subset $X \subseteq V$ with $|X| < k$. G is *(k, A)-connected* if every component of $G - X$ contains a vertex from A for any vertex cut $X \subseteq V$ with $|X| < k$.

Every graph is connected if and only if it is 1-connected. Every block of a graph is either a maximal 2-connected subgraph, or a bridge (with its ends), or an isolated vertex. We call a block *nontrivial* if it is 2-connected.

1.3 Planarity

Let $G = (V, E)$ be a graph. We say that G is *plane* if G is drawn in the plane with no crossing edges. Let $A \subseteq V$. We say that (G, A) is *plane* if G is drawn in a closed disc in the plane with no crossing edges such that the vertices in A are incident with the boundary of the closed disc. Moreover, for vertices $a_1, \ldots, a_k \in V(G)$, we say $(G, a_1, \ldots, a_k)$ is *plane* if G is drawn in a closed disc in the plane with no crossing edges such that $a_1, \ldots, a_k$ occur on the boundary of the disc in this cyclic order.

We say that G is *planar* if G has a plane drawing. Otherwise, G is said to be *nonplanar*. We say that (G, A) is *planar* if (G, A) has a plane representation such that (G, A) is plane. Similarly, we say that $(G, a_1, \ldots, a_k)$ is *planar* if $(G, a_1, \ldots, a_k)$ has a plane representation such that $(G, a_1, \ldots, a_k)$ is plane.

A 3-*planar graph* $(G, \mathcal{A})$ consists of a graph G and a collection $\mathcal{A} = \{A_1, \ldots, A_k\}$ of pairwise disjoint subsets of $V(G)$ (possibly $\mathcal{A} = \emptyset$) such that

- for distinct $i, j \in [k]$, $N(A_i) \cap A_j = \emptyset$,
- for $i \in [k]$, $|N(A_i)| \leq 3$, and
- if $p(G, \mathcal{A})$ denotes the graph obtained from G by (for each $i \in [k]$) deleting A_i and adding new edges joining every pair of distinct vertices in $N(A_i)$, then $p(G, \mathcal{A})$ can be drawn in a closed disc with no crossing edges.

If, in addition, $b_1, \ldots, b_n$ are vertices in G such that $b_j \notin A_i$ for all $i \in [k]$ and $j \in [n]$, $p(G, \mathcal{A})$ can be drawn in a closed disc in the plane with no crossing edges, and $b_1, \ldots, b_n$ occur on the boundary of the disc in this cyclic order, then we say that $(G, \mathcal{A}, b_1, \ldots, b_n)$ is 3-*planar*. If there is no need to specify $\mathcal{A}$, we will simply say that $(G, b_1, \ldots, b_n)$ is 3-*planar*.

1.4 Other notions

A collection of paths in a graph are said to be *independent* if no internal vertex of any path in the collection belongs to another path in the collection.

Let $G = (V, E)$ be a graph and u, v be two vertices in V. We say that a sequence of blocks $B_1, \ldots, B_k$ in G is a *chain of blocks* from u to v if $|V(B_i) \cap V(B_{i+1})| = 1$ for $i \in [k-1]$, $V(B_i) \cap V(B_j) = \emptyset$ for any $1 \leq i < i+1 < j \leq k$, $u, v \in V(B_1)$ are distinct when $k = 1$, and $u \in V(B_1) - V(B_2)$ and $v \in V(B_k) - V(B_{k-1})$ when $k \geq 2$. For convenience, we also view this chain of blocks as $\bigcup_{i=1}^{k} B_i$, a subgraph of G.

For a graph G and a subgraph L of G, an L-*bridge* of G is a subgraph of G that is induced by an edge in $E(G) - E(L)$ with both incident vertices in $V(L)$, or is induced by the edges in a component of $G - L$ as well as edges from that component to L.

CHAPTER II

BACKGROUND AND PREVIOUS LEMMAS

2.1 Background of Kelmans-Seymour conjecture

The well-known Kuratowski's theorem [17] can be stated as follows: A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$. It is known that any 3-connected nonplanar graph other than K_5 contains a subdivision of $K_{3,3}$ (see [27] for more results). Seymour [23] conjectured in 1977 that every 5-connected nonplanar graph contains a subdivision of K_5. This was also posed by Kelmans [15] in 1979.

K. Kawarabayashi, J. Ma and X. Yu proved Kelmans-Seymour conjecture for graphs containing $K_{2,3}$ in [14]. J. Ma and X. Yu also proved Kelmans-Seymour conjecture for graphs containing K_4^- in [19]. In this dissertation, we will generalize the second result in two different ways.

Now we mention several results and problems related to the Kelmans-Seymour conjecture. G. A. Dirac in 1964 [6] conjectured that every graph on n vertices with at least $3n-5$ edges contains a subdivision of the complete graph K_5 on five vertices, which was also mentioned by P. Erdős and A. Hajnal in [7]. Maximal planar graphs show that this is best possible for every $n \geq 5$.

K. Wagner in [32] characterized all edge-maximal graphs not contractible to K_5. It follows easily from this result that every graph G on n vertices with at least $3n-5$ edges is contractible to K_5.

Z. Skupień [26] proved that Dirac's conjecture is true for locally Hamiltonian graphs, i.e. graphs where every vertex has a Hamiltonian neighborhood. It was proved by C. Thomassen in [28] that every graph on n vertices with at least $4n-10$ contains a subdivision of K_5. Then he improved the bound to $\frac{7}{2}n-7$ in [30], and proved in [31] that a subdivision of K_5 can be selected such that a prescribed vertex is no branch vertex, and with this condition the result is best possible. W. Mader finally proved Dirac's conjecture in [20]. Kézdy and McGuiness [16] showed that Kelmans-Seymour conjecture if true would imply Mader's result.

A conjecture of Hajós states that every graph containing no subdivision of K_{k+1} is k-colorable. A graph G is said to be *k-colorable* if there is a map $c : V \to S$ such that $c(u) \neq c(v)$ whenever u and v are adjacent. The smallest number of colors needed to color a graph G is called its *chromatic number*. A graph that can be assigned a k-coloring is *k-colorable*. P. Catlin [2] showed that Kelmans-Seymour conjecture is related to Hajós' conjecture, and Hajós' conjecture is false for $k \geq 6$ and true for $k = 1, 2, 3$, and remains open for the case $k = 4$ and $k = 5$.

2.2 Motivation for our work

As mentioned in the previous section, the motivation of this dissertation is to generalize J. Ma and X. Yu's result on Kelmans-Seymour conjecture for graphs containing K_4^-. In this section, we state a strategy to prove the Kelmans-Seymour conjecture, which is systematically outlined in [8].

Let H be a 5-connected nonplanar graph not containing K_4^-. Then by a result of Kawarabayashi [12], H contains an edge e such that H/e is 5-connected. If H/e is planar, we can apply a discharging argument (see [8] for more details). So assume that H/e is not planar. Let M be a maximal connected subgraph of H such that H/M is 5-connected and nonplanar. Let z denote the vertex representing the contraction of M, and let $G = H/M$. Then one of the following holds.

(a) G contains a K_4^- in which z is of degree 2.

(b) G contains a K_4^- in which z is of degree 3.

(c) G does not contain K_4^-, and there exists $T \subseteq G$ such that $z \in V(T)$, $T \cong K_2$

or $T \cong K_3$, G/T is 5-connected and planar.

(d) G does not contain K_4^-, and for any $T \subseteq G$ with $z \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, G/T is not 5-connected.

In [8] certain special separations are studied and the results can be used to take care of (c). In this dissertation, we prove generalizations of J. Ma and X. Yu's result on graphs containing K_4^-, which can be used for taking care of (a) and (b). The results are collected in [9] and [10], which are prepared to publish.

2.3 Previous lemmas

In this section, we list a number of known results that will be used in the proof of the main results.

First, we state the following result of Seymour [24]; equivalent versions can be found in [3, 25, 29].

Lemma 2.3.1 *Let G be a graph and s_1, s_2, t_1, t_2 be distinct vertices of G. Then exactly one of the following holds:*

(i) G contains disjoint paths from s_1 to t_1 and from s_2 to t_2, respectively.

(ii) (G, s_1, s_2, t_1, t_2) is 3-planar.

We also state a generalization of Lemma 2.3.1, which is a consequence of Theorems 2.3 and 2.4 in [22].

Lemma 2.3.2 *Let G be a graph, $v_1, \ldots, v_n \in V(G)$ be distinct, and $n \geq 4$. Then exactly one of the following holds:*

(i) There exist $1 \leq i < j < k < l \leq n$ such that G contains disjoint paths from v_i, v_j to v_k, v_l, respectively.

(ii) $(G, v_1, v_2, \ldots, v_n)$ is 3-planar.

We will make use of the following result of Menger [11].

Lemma 2.3.3 *Let G be a finite undirected graph and x and y two distinct vertices. Then the size of the minimum vertex cut separating x from y is equal to the maximum number of independent paths from x to y.*

We also need the following result of Perfect [21].

Lemma 2.3.4 *Let G be a graph, $u \in V(G)$, and $A \subseteq V(G-u)$. Suppose there exist k independent paths from u to distinct $a_1, \dots, a_k \in A$, respectively, and otherwise disjoint from A. Then for any $n \geq k$, if there exist n independent paths $P_1, \dots, P_n$ in G from u to n distinct vertices in A and otherwise disjoint from A then $P_1, \dots, P_n$ may be chosen so that $a_i \in V(P_i)$ for $i \in [k]$.*

We will also use a result of Watkins and Mesner [33] on cycles through three vertices.

Lemma 2.3.5 *Let G be a 2-connected graph and let y_1, y_2, y_3 be three distinct vertices of G. There is no cycle in G through y_1, y_2, y_3 if, and only if, one of the following holds:*

(*i*) *There exists a 2-cut S in G and there exist pairwise disjoint subgraphs D_{y_i} of $G-S$, $i \in [3]$, such that $y_i \in V(D_{y_i})$ and each D_{y_i} is a union of components of $G-S$.*

(*ii*) *There exist 2-cuts S_{y_i} of G, $i \in [3]$, and pairwise disjoint subgraphs D_{y_i} of G, such that $y_i \in V(D_{y_i})$, each D_{y_i} is a union of components of $G-S_{y_i}$, there exists $z \in S_{y_1} \cap S_{y_2} \cap S_{y_3}$, and $S_{y_1} - \{z\}, S_{y_2} - \{z\}, S_{y_3} - \{z\}$ are pairwise disjoint.*

(*iii*) *There exist pairwise disjoint 2-cuts S_{y_i} in G, $i \in [3]$, and pairwise disjoint subgraphs D_{y_i} of $G - S_{y_i}$ such that $y_i \in V(D_{y_i})$, D_{y_i} is a union of components of $G - S_{y_i}$, and $G - V(D_{y_1} \cup D_{y_2} \cup D_{y_3})$ has precisely two components, each containing exactly one vertex from S_{y_i} for $i \in [3]$.*

The next result is Theorem 3.2 from [18].

Lemma 2.3.6 *Let G be a 5-connected nonplanar graph and let $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. Suppose $G - x_1x_2$ contains a path X between x_1 and x_2 such that $G - X$ is 2-connected, $X - x_2$ is induced in G, and $y_1, y_2 \notin V(X)$. Let $v \in V(X)$ such that $x_2v \in E(X)$. Then G contains a TK_5 in which x_2v is an edge and x_1, x_2, y_1, y_2 are branch vertices.*

It is easy to see that under the conditions of Lemma 2.3.6, $G - \{x_2u : u \notin \{v, x_1, y_1, y_2\}\}$ contains TK_5. The next result is Corollary 2.11 in [14].

Lemma 2.3.7 *Let G be a connected graph with $|V(G)| \geq 7$, $A \subseteq V(G)$ with $|A| = 5$, and $a \in A$, such that G is $(5, A)$-connected, $(G - a, A - \{a\})$ is plane, and G has no 5-separation (G_1, G_2) with $A \subseteq G_1$ and $|V(G_2)| \geq 7$. Suppose there exists $w \in N(a)$ such that w is not incident with the outer face of $G - a$. Then*

(i) the vertices of $G - a$ cofacial with w induce a cycle C_w in $G - a$, and

(ii) $G - a$ contains paths P_1, P_2, P_3 from w to $A - \{a\}$ such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 3$, and $|V(P_i \cap C_w)| = |V(P_i) \cap A| = 1$ for $i \in [3]$.

The next three results are Theorem 1.1, Theorem 1.2, and Proposition 4.2, respectively, in [8].

Lemma 2.3.8 *Let G be a 5-connected nonplanar graph and let (G_1, G_2) be a 5-separation in G. Suppose $|V(G_i)| \geq 7$ for $i \in [2]$, $a \in V(G_1 \cap G_2)$, and $(G_2 - a, V(G_1 \cap G_2) - \{a\})$ is planar. Then one of the following holds:*

(i) G contains a TK_5 in which a is not a branch vertex.

(ii) $G - a$ contains K_4^-.

(*iii*) *G has a 5-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{a, a_1, a_2, a_3, a_4\}$, $G_1 \subseteq G_1'$, and G_2' is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding a and the edges ab_i for $i \in [4]$.*

Lemma 2.3.9 *Let G be a 5-connected graph and (G_1, G_2) be a 5-separation in G. Suppose that $|V(G_i)| \geq 7$ for $i \in [2]$ and $G[V(G_1 \cap G_2)]$ contains a triangle aa_1a_2a. Then one of the following holds:*

(*i*) *G contains a TK_5 in which a is not a branch vertex.*

(*ii*) *$G - a$ contains K_4^-.*

(*iii*) *G has a 5-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{a, a_1, a_2, a_3, a_4\}$ and G_2' is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4$ a_1 and the 4-cycle $b_1b_2b_3b_4b_1$ by adding a and the edges ab_i for $i \in [4]$.*

(*iv*) *For any distinct $u_1, u_2, u_3 \in N(a) - \{a_1, a_2\}$, $G - \{av : v \notin \{a_1, a_2, u_1, u_2, u_3\}\}$ contains TK_5.*

Lemma 2.3.10 *Let G be a 5-connected nonplanar graph and $a \in V(G)$ such that $G - a$ is planar. Then one of the following holds:*

(*i*) *G contains a TK_5 in which a is not a branch vertex.*

(*ii*) *$G - a$ contains K_4^-.*

(*iii*) *G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a, a_1, a_2, a_3, a_4\}$ and G_2 is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding a and the edges ab_i for $i \in [4]$.*

We also need the following results, which are Porposition 1.3 and Proposition 2.3 in [8], respectively.

Lemma 2.3.11 *Let G be a 5-connected nonplanar graph, (G_1, G_2) a 5-separation in G, $V(G_1 \cap G_2) = \{a, a_1, a_2, a_3, a_4\}$ such that G_2 is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding a and the edges ab_i for $i \in [4]$. Suppose $|V(G_1)| \geq 7$. Then, for any $u_1, u_2 \in N(a) - \{b_1, b_2, b_3\}, G - \{av : v \notin \{b_1, b_2, b_3, u_1, u_2\}\}$ contains TK_5.*

Lemma 2.3.12 *Let G be a graph, $A \subseteq V(G)$, and $a \in A$ such that $|A| = 6$, $|V(G)| \geq 8$, $(G - a, A - \{a\})$ is planar, and G is $(5, A)$-connected. Then one of the following holds:*

(i) *$G - a$ contains K_4^-, or G contains a K_4^- in which the degree of a is 2.*

(ii) *G has a 5-separation (G_1, G_2) such that $a \in V(G_1 \cap G_2)$, $A \subseteq V(G_1)$, $|V(G_2)| \geq 7$, and $(G_2 - a, V(G_1 \cap G_2) - \{a\})$ is planar.*

CHAPTER III

2-VERTICES IN K_4^-

3.1 Main result

In this section, we prove the following theorem.

Theorem 3.1.1 *Let G be a 5-connected nonplanar graph and $\{x_1, x_2, y_1, y_2\} \subseteq V(G)$ such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ with $y_1y_2 \notin E(G)$. Then one of the following holds:*

(*i*) *G contains a TK_5 in which y_2 is not a branch vertex.*

(*ii*) *$G - y_2$ contains K_4^-.*

(*iii*) *G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{y_2, a_1, a_2, a_3, a_4\}$, and G_2 is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4$ a_1 and the 4-cycle $b_1b_2b_3b_4b_1$ by adding y_2 and the edges y_2b_i for $i \in [4]$.*

(*iv*) *For $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$, $G - \{y_2v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$ contains TK_5.*

Before proving Theorem 3.1.1, we show its relation with case (a) in Section 2.2.

Let H be a 5-connected nonplanar graph not containing K_4^-. If case (a) in Section 2.2 occurs, then there is a connected subgraph M of H such that $G := H/M$ is 5-connected and nonplanar. Furthermore, there exists $\{x_1, x_2, y_1, y_2\} \subseteq V(G)$ such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ with $y_1y_2 \notin E(G)$ and y_2 is the vertex representing the contraction of M.

Let P be a path in $H[V(M) \cup \{x_1, x_2\}]$ from x_1 to x_2 and w_1 be a neighbor of y_2 in G other than x_1, x_2. Since M is a connected subgraph, there is a path Q in $H[V(M) \cup \{w_1\}]$ from w_1 to some vertex $v \in V(P) - \{x_1, x_2\}$ independent from P.

It is easy to see that P and Q gives three independent paths from v to x_1, x_2, w_1, respectively. By Lemma 2.3.4, there are five independent paths S_1, S_2, S_3, S_4, S_5 in $H[V(M) \cup \{x_1, x_2, w_1, w_2, w_3\}]$ from v to x_1, x_2, w_1, w_2, w_3, respectively, where $w_1, w_2, w_3 \in N_G(y_2) - \{x_1, x_2\}$.

Now we may assume that one of the four results in Theorem 3.1.1 holds. If (i) holds, i.e. G contains a TK_5 in which y_2 is not a branch vertex, then a TK_5 in H can be easily derived from the one in G.

If (ii) holds, i.e. $G - y_2$ contains a K_4^-, then it implies that H itself contains a K_4^-. By J. Ma and X. Yu's result on Kelmans-Seymour conjecture, H contains a TK_5.

If (iii) holds, by similar discussion as above, we can find five independent paths T_1, T_2, T_3, T_4, T_5 in $H[V(M) \cup \{b_1, b_2, b_3, u_1, u_2\}]$ from some vertex $w \in V(M)$ to b_1, b_2, b_3, u_1, u_2, respectively, where $u_1, u_2 \in N(y_2) - \{b_1, b_2, b_3\}$. By Lemma 2.3.11, there exists a TK_5 in $G - \{av : v \notin \{b_1, b_2, b_3, u_1, u_2\}\}$. Hence, H contains a TK_5.

If (iv) holds, by the existence of the five independent paths S_1, S_2, S_3, S_4, S_5 in $H[V(M) \cup \{x_1, x_2, w_1, w_2, w_3\}]$ from v to x_1, x_2, w_1, w_2, w_3, respectively, then H contains a TK_5.

3.2 Non-separating paths

Our first step for proving Theorem 3.1.1 is to find the path X in G (see Figure 1) whose removal does not affect connectivity too much.

The following result was implicit in [4, 13]. Since it has not been stated and proved explicitly before, we include a proof.

Lemma 3.2.1 *Let G be a graph and let $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that G is $(4, \{x_1, x_2, y_1, y_2\})$-connected. Suppose there exists a path X in $G - x_1x_2$ from x_1 to x_2 such that $G - X$ contains a chain of blocks B from y_1 to y_2. Then one of the following holds:*

(i) *There is a 4-separation* (G_1, G_2) *in* G *such that* $B + \{x_1, x_2\} \subseteq G_1$, $|V(G_2)| \geq 6$, *and* $(G_2, V(G_1 \cap G_2))$ *is planar.*

(ii) *There exists an induced path* X' *in* $G - x_1x_2$ *from* x_1 *to* x_2 *such that* $G - X'$ *is a chain of blocks from* y_1 *to* y_2 *and contains* B.

Proof. Without loss of generality, we may assume that X is induced in $G - x_1x_2$. We choose such X that

(1) B is maximal,

(2) the smallest size of a component of $G - X$ disjoint from B (if exists) is minimal, and

(3) the number of components of $G - X$ is minimal.

We claim that $G - X$ is connected. For, suppose $G - X$ is not connected and let D be a component of $G - X$ other than B such that $|V(D)|$ is minimal. Let $u, v \in N(D) \cap V(X)$ such that uXv is maximal. Since G is $(4, \{x_1, x_2, y_1, y_2\})$-connected, $uXv - \{u, v\}$ contains a neighbor of some component of $G - X$ other than D. Let Q be an induced path in $G[D + \{u, v\}]$ from u to v, and let X' be obtained from X by replacing uXv with Q. Then B is contained in B', the chain of blocks in $G - X'$ from y_1 to y_2. Moreover, either the smallest size of a component of $G - X'$ disjoint from B' is smaller than the smallest size of a component of $G - X$ disjoint from B, or the number of components of $G - X'$ is smaller than the number of components of $G - X$. This gives a contradiction to (1) or (2) or (3). Hence, $G - X$ is connected.

If $G - X = B$, we are done with $X' := X$. So assume $G - X \neq B$. By (1), each B-bridge of $G - X$ has exactly one vertex in B. Thus, for each B-bridge D of $G - X$, let $b_D \in V(D) \cap V(B)$ and $u_D, v_D \in N(D - b_D) \cap V(X)$ such that u_DXv_D is maximal.

We now define a new graph $\mathcal{B}$ such that $V(\mathcal{B})$ is the set of all B-bridges of $G-X$, and two B-bridges in $G-X$, C and D, are adjacent if $u_C X v_C - \{u_C, v_C\}$ contains a neighbor of $D - b_D$ or $u_D X v_D - \{u_D, v_D\}$ contains a neighbor of $C - b_C$. Let $\mathcal{D}$ be a component of $\mathcal{B}$. Then $\bigcup_{D \in V(\mathcal{D})} u_D X v_D$ is a subpath of X. Let $S_{\mathcal{D}}$ be the union of $\{b_D : D \in V(\mathcal{D})\}$ and the set of neighbors in B of the internal vertices of $\bigcup_{D \in V(\mathcal{D})} u_D X v_D$.

Suppose $\mathcal{B}$ has a component $\mathcal{D}$ such that $|S_{\mathcal{D}}| \leq 2$. Let $u, v \in V(X)$ such that $uXv = \bigcup_{D \in V(\mathcal{D})} u_D X v_D$. Then $\{u,v\} \cup S_{\mathcal{D}}$ is a cut in G. Since G is $(4, \{x_1, x_2, y_1, y_2\})$-connected, $|S_{\mathcal{D}}| = 2$. So there is a 4-separation (G_1, G_2) in G such that $V(G_1 \cap G_2) = \{u, v\} \cup S_{\mathcal{D}}$, $B + \{x_1, x_2\} \subseteq G_1$, and $D \subseteq G_2$ for $D \in V(\mathcal{D})$. Hence $|V(G_2)| \geq 6$. If G_2 has disjoint paths S_1, S_2, with S_1 from u to v and S_2 between the vertices in $S_{\mathcal{D}}$, then choose S_1 to be induced and let $X' = x_1 X u \cup S_1 \cup v X x_2$; now $B \cup S_2$ is contained in the chain of blocks in $G - X'$ from y_1 to y_2, contradicting (1). So no such two paths exist. Hence, by Lemma 2.3.1, $(G_2, V(G_1 \cap G_2))$ is planar and thus (i) holds.

Therefore, we may assume that $|S_{\mathcal{D}}| \geq 3$ for any component $\mathcal{D}$ of $\mathcal{B}$. Hence, there exist a component $\mathcal{D}$ of $\mathcal{B}$ and $D \in V(\mathcal{D})$ with the following property: $u_D X v_D - \{u_D, v_D\}$ contains vertices w_1, w_2 and $S_{\mathcal{D}}$ contains distinct vertices b_1, b_2 such that for each $i \in [2]$, $\{b_i, w_i\}$ is contained in a $(B \cup X)$-bridge of G disjoint from $D - b_D$. Let P denote an induced path in $G[D + \{u_D, v_D\}]$ between u_D and v_D, and let X' be obtained from X by replacing $u_D X v_D$ with P. Clearly, the chain of blocks in $G - X'$ from y_1 to y_2 contains B as well as a path from b_1 to b_2 and internally disjoint from $D \cup B$. This is a contradiction to (1). ∎

We now show that the conclusion of Theorem 3.1.1 holds or we can find a path X in G such that $y_1, y_2 \notin V(X)$ and $(G - y_2) - X$ is 2-connected.

Lemma 3.2.2 *Let G be a 5-connected nonplanar graph and let $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ with $y_1 y_2 \notin E(G)$. Then one of the following holds:*

(*i*) *G contains a TK_5 in which y_2 is not a branch vertex.*

(*ii*) *$G - y_2$ contains K_4^-.*

(*iii*) *G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{y_2, a_1, a_2, a_3, a_4\}$ and G_2 is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding y_2 and the edges y_2b_i for $i \in [4]$.*

(*iv*) *For $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$, $G - \{y_2v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$ contains TK_5, or $G - x_1x_2$ has an induced path X from x_1 to x_2 such that $y_1, y_2 \notin V(X)$, $w_1, w_2, w_3 \in V(X)$, and $(G - y_2) - X$ is 2-connected.*

Proof. First, we may assume that

(1) $G - x_1x_2$ has an induced path X from x_1 to x_2 such that $y_1, y_2 \notin V(X)$ and $(G - y_2) - X$ is 2-connected.

To see this, let $z \in N(y_1) - \{x_1, x_2\}$. Since G is 5-connected, $(G - x_1x_2) - \{y_1, y_2, z\}$ has a path X from x_1 to x_2. Thus, we may apply Lemma 3.2.1 to $G - y_2$, X and $B = y_1z$.

Suppose (*i*) of Lemma 3.2.1 holds. Then G has a 5-separation (G_1, G_2) such that $y_2 \in V(G_1 \cap G_2)$, $\{x_1, x_2, y_1, z\} \subseteq V(G_1)$ and $y_1z \in E(G_1)$, $|V(G_2)| \geq 7$, and $(G_2 - y_2, V(G_1 \cap G_2) - \{y_2\})$ is planar. If $|V(G_1)| \geq 7$ then, by Lemma 2.3.8, (*i*) or (*ii*) or (*iii*) holds. If $|V(G_1)| = 5$ then $G_1 - y_2$ has a K_4^- or $G - y_2$ is planar; hence, (*ii*) holds in the former case, and (*i*) or (*ii*) or (*iii*) holds in the latter case by Lemma 2.3.10. Thus we may assume that $|V(G_1)| = 6$. Let $v \in V(G_1 - G_2)$. Then $v \neq y_2$. Since G is 5-connected, v must be adjacent to all vertices in $V(G_1 \cap G_2)$. Thus, $v \neq y_1$ as $y_1y_2 \notin E(G)$. Now $|V(G_1 \cap G_2) \cap \{x_1, x_2, z\}| \geq 2$. Therefore, $G[\{v, y_1\} \cup (V(G_1 \cap G_2) \cap \{x_1, x_2, z\})]$ contains K_4^-; so (*ii*) holds.

So we may assume that (*ii*) of Lemma 3.2.1 holds. Then $(G - y_2) - x_1x_2$ has an induced path, also denoted by X, from x_1 to x_2 such that $(G - y_2) - X$ is a chain of blocks from y_1 to z. Since $zy_1 \in E(G)$, $(G - y_2) - X$ is in fact a block. If $V((G - y_2) - X) = \{y_1, z\}$ then, since G is 5-connected and X is induced in $(G - y_2) - x_1x_2$, $G[\{x_1, x_2, z, y_1\}] \cong K_4$; so (ii) holds. This completes the proof of (1).

We wish to prove (iv). So let $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$ and assume that

$$G' := G - \{y_2v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$$

does not contain TK_5. We may assume that

(2) $w_1, w_2, w_3 \notin V(X)$.

For, suppose not. If $w_1, w_2, w_3 \in V(X)$ then (iv) holds. So, without loss of generality, we may assume $w_1 \in V(X) - \{x_1, x_2\}$ and $w_2 \in V(G - X)$. Since X is induced in $G - x_1x_2$ and G is 5-connected, $(G - y_2) - (X - w_1)$ is 2-connected and, hence, contains independent paths P_1, P_2 from y_1 to w_1, w_2, respectively. Then $w_1Xx_1 \cup w_1Xx_2 \cup w_1y_2 \cup P_1 \cup (y_2w_2 \cup P_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_1, y_2, a contradiction.

(3) For any $u \in V(x_1Xx_2) - \{x_1, x_2\}$, $\{u, y_1, y_2\}$ is not contained in any cycle in $G' - (X - u)$.

For, suppose there exists $u \in V(x_1Xx_2) - \{x_1, x_2\}$ such that $\{u, y_1, y_2\}$ is contained in a cycle C in $G' - (X - u)$. Then $uXx_1 \cup uXx_2 \cup C \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices u, x_1, x_2, y_1, y_2, a contradiction. So we have (3).

Let $y_3 \in V(X)$ such that $y_3x_2 \in E(X)$, and let $H := G' - (X - y_3)$. Note that H is 2-connected. By (3), no cycle in H contains $\{y_1, y_2, y_3\}$. Thus, we apply Lemma 2.3.5 to H. In order to treat simultaneously the three cases in the conclusion of Lemma 2.3.5, we introduce some notation. Let $S_{y_i} = \{a_i, b_i\}$ for $i \in [3]$, such that if Lemma 2.3.5(i) occurs we let $a_1 = a_2 = a_3$, $b_1 = b_2 = b_3$, and $S_{y_i} = S$ for

$i \in [3]$; if Lemma 2.3.5(ii) occurs then $a_1 = a_2 = a_3$; and if Lemma 2.3.5(iii) then $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ belong to different components of $H - V(D_{y_1} \cup D_{y_2} \cup D_{y_3})$. If Lemma 2.3.5($ii$) or Lemma 2.3.5($iii$) occurs then let B_a, B_b denote the components of $H - V(D_{y_1} \cup D_{y_2} \cup D_{y_3})$ such that for $i \in [3]$ $a_i \in V(B_a)$ and $b_i \in V(B_b)$. Note that $B_a = B_b$ is possible, but only if Lemma 2.3.5(ii) occurs.

For convenience, let $D'_i := G'[D_{y_i} + \{a_i, b_i\}]$ for $i \in [3]$. We choose the cuts S_{y_i} so that

(4) $D'_1 \cup D'_2 \cup D'_3$ is maximal.

Since H is 2-connected, D'_i, for each $i \in [3]$, contains a path Y_i from a_i to b_i and through y_i. In addition, since $(G - y_2) - X$ is 2-connected, for any $v \in V(D'_3) - \{a_3, b_3, y_3\}$, $D'_3 - y_3$ contains a path from a_3 to b_3 through v.

(5) If $B_a \cap B_b = \emptyset$ then $|V(B_a)| = 1$ or B_a is 2-connected, and $|V(B_b)| = 1$ or B_b is 2-connected. If $B_a \cap B_b \neq \emptyset$ then $B_a = B_b$ and $B_a - a_3$ is 2-connected.

First, suppose $B_a \cap B_b = \emptyset$. By symmetry, we only prove the claim for B_a. Suppose $|V(B_a)| > 1$ and B_a is not 2-connected. Then B_a has a separation (B_1, B_2) such that $|V(B_1 \cap B_2)| \leq 1$. Since H is 2-connected, $|V(B_1 \cap B_2)| = 1$ and, for some permutation ijk of [3], $a_i \in V(B_1) - V(B_2)$ and $a_j, a_k \in V(B_2)$. Replacing S_{y_i}, D'_i by $V(B_1 \cap B_2) \cup \{b_i\}, D'_i \cup B_1$, respectively, while keeping $S_{y_j}, D'_j, S_{y_k}, D'_k$ unchanged, we derive a contradiction to (4).

Now assume $B_a \cap B_b \neq \emptyset$. Then $B_a = B_b$ by definition, and $a_1 = a_2 = a_3$ by our assumption above. Suppose $B_a - a_3$ is not 2-connected. Then B_a has a 2-separation (B_1, B_2) with $a_3 \in V(B_1 \cap B_2)$. First, suppose for some permutation ijk of [3], $b_i \in V(B_1) - V(B_2)$ and $b_j, b_k \in V(B_2)$. Then replacing S_{y_i}, D'_i by $V(B_1 \cap B_2), D'_i \cup B_1$, respectively, while keeping $S_{y_j}, D'_j, S_{y_k}, D'_k$ unchanged, we derive a contradiction to (4). Therefore, we may assume $\{b_1, b_2, b_3\} \subseteq V(B_1)$. Since G is 5-connected, there exists $rr' \in E(G)$ such that $r \in V(X) - \{y_3, x_2\}$ and $r' \in V(B_2 - B_1)$. Let R be a path

$B_2-(B_1-a_3)$ from a_3 to r', and R' a path in B_1-B_2 from b_1 to b_2. Then $(R\cup r'r\cup rXx_1)\cup(a_3Y_3y_3\cup y_3x_2)\cup a_3Y_1y_1\cup a_3Y_2y_2\cup(y_1Y_1b_1\cup R'\cup b_2Y_2y_2)\cup G[\{x_1,x_2,y_1,y_2\}]$ is a TK_5 in G' with branch vertices a_3,x_1,x_2,y_1,y_2, a contradiction.

(6) D_{y_i} is connected for $i\in[3]$.

Suppose D_{y_i} is not connected for some $i\in[3]$, and let D be a component of D_{y_i} not containing y_i. Since G is 5-connected, there exists $rr'\in E(G)$ such that $r\in V(X)-\{x_2,y_3\}$ and $r'\in V(D)$.

Let R be a path in $G[D+a_i]$ from a_i to r', and R' a path from b_1 to b_2 in B_b-a_3. By (5), let A_1,A_2,A_3 be independent paths in B_a from a_i to a_1,a_2,a_3, respectively. Then $(R\cup r'r\cup rXx_1)\cup(A_1\cup a_1Y_1y_1)\cup(A_2\cup a_2Y_2y_2)\cup(A_3\cup a_3Y_3y_3\cup y_3x_2)\cup(y_1Y_1b_1\cup R'\cup b_2Y_2y_2)\cup G[\{x_1,x_2,y_1,y_2\}]$ is a TK_5 in G' with branch vertices a_i,x_1,x_2,y_1,y_2, a contradiction.

(7) If $a_1=a_2=a_3$ then $N(a_3)\cap V(X-\{x_2,y_3\})=\emptyset$.

For, suppose $a_1=a_2=a_3$ and there exists $u\in N(a_3)\cap V(X-\{x_2,y_3\})$. Let Q be a path in B_b-a_3 between b_1 and b_2, and let P be a path in $D_3'-b_3$ from a_3 to y_3. Then $(a_3u\cup uXx_1)\cup(P\cup y_3x_2)\cup a_3Y_1y_1\cup a_3Y_2y_2\cup(y_1Y_1b_1\cup Q\cup b_2Y_2y_2)\cup G[\{x_1,x_2,y_1,y_2\}]$ is a TK_5 in G' with branch vertices a_3,x_1,x_2,y_1,y_2, a contradiction.

We may assume that

(8) there exists $u\in V(X)-\{x_1,x_2,y_3\}$ such that $N(u)-\{y_2\}\not\subseteq V(X\cup D_3')$.

For, suppose no such vertex exists. Then G has a 5-separation (G_1,G_2) such that $V(G_1\cap G_2)=\{a_3,b_3,x_1,x_2,y_2\}$, $X\cup D_3'\subseteq G_1$, and $D_1'\cup D_2'\cup B_a\cup B_b\subseteq G_2$. Clearly, $|V(G_2)|\geq 7$ since $|N(y_1)|\geq 5$ and $y_1y_2\notin E(G)$. If $|V(G_1)|\geq 7$ then, by Lemma 2.3.9, (i) or (ii) or (iii) or (iv) holds. So we may assume $|V(G_1)|=6$. Then $X=x_1y_3x_2$ and $V(D_{y_3})=\{y_3\}$. Hence, $G[\{x_1,x_2,y_1,y_3\}]\cong K_4^-$; so (ii) holds.

(9) For all $u \in V(X) - \{x_1, x_2, y_3\}$ with $N(u) - \{y_2\} \not\subseteq V(X \cup D_3')$, $N(u) \cap V(D_3' - y_3) = \emptyset$.

For, suppose there exist $u \in V(X) - \{x_1, x_2, y_3\}$, $u_1 \in (N(u) - \{y_2\}) - V(X \cup D_3')$, and $u_2 \in N(u) \cap V(D_3' - y_3)$. Recall (see before (5)) that there is a path Y_3' in $D_3' - y_3$ from a_3 to b_3 through u_2.

Suppose $u_1 \in V(D_{y_i})$ for some $i \in [2]$. Then $D_i' - b_i$ (or $D_i' - a_i$) has a path Y_i' from u_1 to a_i (or b_i) through y_i. If Y_i' ends at a_i then let P_a, P_b be disjoint paths in $B_a \cup B_b$ from a_1, b_3 to a_2, b_{3-i}, respectively; now $Y_i' \cup P_a \cup Y_{3-i} \cup P_b \cup b_3 Y_3' u_2 \cup u_2 u u_1$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). So Y_i' ends at b_i. Let P_b, P_a be disjoint paths in $B_a \cup B_b$ from b_1, a_{3-i} to b_2, a_3, respectively. Then $Y_i' \cup P_b \cup Y_{3-i} \cup P_a \cup a_3 Y_3' u_2 \cup u_2 u u_1$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3).

Thus, $u_1 \in V(B_a \cup B_b)$. By symmetry and (7), assume $u_1 \in V(B_b)$. Note that $u_1 \notin \{a_3, b_3\}$ (by the choice of u_1) and $B_b - a_3$ is 2-connected (by (5)). Hence, $B_b - a_3$ has disjoint paths Q_1, Q_2 from $\{u_1, b_3\}$ to $\{b_1, b_2\}$. By symmetry between b_1 and b_2, we may assume Q_1 is between u_1 and b_1 and Q_2 is between b_3 and b_2. Let P be a path in B_a from a_1 to a_2 (which is trivial if $|V(B_a)| = 1$). Then $Q_1 \cup u_1 u u_2 \cup u_2 Y_3' b_3 \cup Q_2 \cup Y_2 \cup P \cup Y_1$ is a cycle in $G' - (X - u)$ containing $\{y_1, y_2, u\}$, contradicting (3).

(10) For any $u \in V(X) - \{x_1, x_2, y_3\}$ with $N(u) - \{y_2\} \not\subseteq V(X \cup D_3')$, there exists $i \in [2]$ such that $N(u) - \{y_2\} \subseteq V(D_i')$ and $\{a_i, b_i\} \not\subseteq N(u)$.

To see this, let $u_1, u_2 \in (N(u) - \{y_2\}) - V(X \cup D_3')$ be distinct, which exist by (9) (and since X is induced in $G' - x_1x_2$). Suppose we may choose such u_1, u_2 so that $\{u_1, u_2\} \not\subseteq V(D_i')$ for $i \in [2]$.

We claim that $\{u_1, u_2\} \not\subseteq V(B_a)$ and $\{u_1, u_2\} \not\subseteq V(B_b)$. Recall that if $B_a \cap B_b \neq \emptyset$ then $B_a = B_b$ and if $B_a \cap B_b = \emptyset$ then there is symmetry between B_a and B_b. So if the claim fails we may assume that $u_1, u_2 \in V(B_b)$. Then by (5), $B_b - a_3$ is 2-connected; so $B_b - a_3$ contains disjoint paths Q_1, Q_2 from $\{u_1, u_2\}$ to $\{b_1, b_2\}$. If $B_a = B_b$, let $P = a_3$. If $B_a \cap B_b = \emptyset$, then let P be a path in B_a from a_1 to a_2. Now $Q_1 \cup u_1uu_2 \cup Q_2 \cup Y_1 \cup P \cup Y_2$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3).

Next, we show that $\{a_i, b_i\} \not\subseteq N(u)$ for $i \in [2]$. For, suppose $u_1 = a_i$ and $u_2 = b_i$ for some $i \in [2]$. Then, since $\{u_1, u_2\} \cap \{a_3, b_3\} = \emptyset$, $|V(B_a)| \geq 2$ and $|V(B_b)| \geq 2$. By (5), let P_1, P_2 be independent paths in B_a from a_i to a_{3-i}, a_3, respectively, and Q_1, Q_2 be independent paths in B_b from b_i to b_{3-i}, b_3, respectively. Now $ua_i \cup ub_i \cup a_iY_iy_i \cup b_iY_iy_i \cup (y_ix_1 \cup x_1Xu) \cup (P_1 \cup Y_{3-i} \cup Q_1) \cup (P_2 \cup a_3Y_3y_3) \cup (Q_2 \cup b_3Y_3y_3) \cup uXy_3 \cup y_ix_2y_3$ is a TK_5 in G' with branch vertices a_i, b_i, u, y_i, y_3, a contradiction.

Suppose $u_1 \in V(B_a - a_3)$ and $u_2 \in V(B_b - b_3)$. Then $|V(B_a)| \geq 2$ and $|V(B_b)| \geq 2$. Let Y_3' be a path in $D_3' - y_3$ from a_3 to b_3. First, assume that $u_1 \in \{a_1, a_2\}$ or $u_2 \in \{b_1, b_2\}$. By symmetry, we may assume $u_1 = a_1$. So $u_2 \neq b_1$. By (5), $B_a - a_1$ contains a path P from a_2 to a_3, and B_b contains disjoint paths Q_1, Q_2 from $\{b_2, b_3\}$ to b_1, u_2, respectively. Then $Y_1 \cup Q_1 \cup Y_2 \cup P \cup Y_3' \cup Q_2 \cup u_1uu_2$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). So $u_1 \notin \{a_1, a_2\}$ and $u_2 \notin \{b_1, b_2\}$. Then by (5) and symmetry, we may assume that B_a contains disjoint paths P_1, P_2 from u_1, a_3 to a_1, a_2, respectively. By (5) again, B_b contains disjoint paths Q_1, Q_2 from b_1, u_2, respectively to $\{b_2, b_3\}$. Now $P_1 \cup Y_1 \cup Q_1 \cup Y_2 \cup P_2 \cup Y_3' \cup Q_2 \cup u_2uu_1$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3).

Therefore, we may assume $u_1 \in V(D_{y_i})$ for some $i \in [2]$. By symmetry, we may assume that $u_1 \in V(D_{y_1})$ and $D_1' - a_1$ contains a path R_1 from u_1 to b_1 and through y_1. Then $u_2 \notin V(D_1')$ as we assumed $\{u_1, u_2\} \not\subseteq V(D_1')$.

Suppose $u_2 \in V(D_{y_2})$. If $D_2' - a_2$ contains a path R_2 from u_2 to b_2 through y_2 then let Q be a path in B_b from b_1 to b_2; now $R_1 \cup Q \cup R_2 \cup u_2uu_1$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). So $D_2' - b_2$ contains a path R_2 from u_2 to a_2

and through y_2. Now let P be a path in B_a from a_2 to a_3, Q be a path in $B_b - a_3$ from b_1 to b_3. Let Y_3' be a path in $D_3' - y_3$ from a_3 to b_3. Then $R_1 \cup Q \cup Y_3' \cup P \cup R_2 \cup u_2uu_1$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3).

Finally, assume $u_2 \in V(B_a \cup B_b)$. If $u_2 \in V(B_b)$ then, by (5), let Q_1, Q_2 be disjoint paths in $B_b - a_3$ from b_1, u_2, respectively, to $\{b_2, b_3\}$, and let P be a path in B_a from a_2 to a_3; now $u_2uu_1 \cup R_1 \cup Q_1 \cup Q_2 \cup Y_2 \cup P \cup Y_3'$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). So $u_2 \notin V(B_b)$ and $u_2 \in V(B_a - a_1)$; hence $B_a \cap B_b = \emptyset$. Let P be a path in B_a from u_2 to a_2 and Q be a path in B_b from b_1 to b_2. Then $u_2uu_1 \cup R_1 \cup Q \cup Y_2 \cup P$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). This completes the proof of (10).

By (10) and by symmetry, let $u \in V(X) - \{x_1, x_2, y_3\}$ and $u_1, u_2 \in N(u)$ such that $u_1 \in V(D_{y_1})$ and $u_2 \in V(D_1')$. If $G[D_1' + u]$ contains independent paths R_1, R_2 from u to a_1, b_1, respectively, such that $y_1 \in V(R_1 \cup R_2)$, then let P be a path in B_a between a_1 and a_2 and Q be a path in $B_b - a_3$ between b_1 and b_2; now $R_1 \cup P \cup Y_2 \cup Q \cup R_2$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). So such paths do not exist. Then in the 2-connected graph $D_1^* := G[D_1' + u] + \{c, ca_1, cb_1\}$ (by adding a new vertex c), there is no cycle containing $\{c, u, y_1\}$. Hence, by Lemma 2.3.5, D_1^* has a 2-cut T separating y_1 from $\{u, c\}$, and $T \cap \{u, c\} = \emptyset$.

We choose u, u_1, u_2 and T so that the T-bridge of D_1^* containing y_1, denoted B, is minimal. Then $B - T$ contains no neighbor of $X - \{x_1, x_2\}$. Hence, G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{x_1, x_2, y_2\} \cup V(T)$, $B \subseteq G_1$, and $X \cup D_2' \cup D_3' \subseteq G_2$. Clearly, $|V(G_2)| \geq 7$. Since $y_1y_2 \notin E(G)$ and G is 5-connected, $|V(G_1)| \geq 7$. So *(i)* or *(ii)* or *(iii)* or *(iv)* holds by Lemma 2.3.9. ∎

3.3 An intermediate substructure

By Lemma 3.2.2, to prove Theorem 3.1.1 it suffices to deal with the second part of *(iv)* of Lemma 3.2.2. Thus, let G be a 5-connected nonplanar graph and $x_1, x_2, y_1, y_2 \in$

$V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ with $y_1y_2 \notin E(G)$, let $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$ be distinct, and let P be an induced path in $G - x_1x_2$ from x_1 to x_2 such that $y_1, y_2 \notin V(P)$, $w_1, w_2, w_3 \in V(P)$, and $(G - y_2) - P$ is 2-connected.

Without loss of generality, assume x_1, w_1, w_2, w_3, x_2 occur on P in order. Let

$$X := x_1Pw_1 \cup w_1y_2w_3 \cup w_3Px_2,$$

and let

$$G' := G - \{y_2v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}.$$

Then X is an induced path in $G' - x_1x_2$, $y_1 \notin V(X)$, and $G' - X$ is 2-connected. For convenience, we record this situation by calling $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3)$ a 9-*tuple*.

In this section, we obtain a substructure of G' in terms of X and seven additional paths A, B, C, P, Q, Y, Z in G'. See Figure 1, where X is the path in boldface and Y, Z are not shown. First, we find two special paths Y, Z in G' with Lemma 3.3.1 below. We will then use Lemma 3.3.2 to find the paths A, B, C, and use Lemma 3.3.3 to find the paths P and Q. In the next section, we will use this substructure to find the desired TK_5 in G or G'.

Lemma 3.3.1 *Let $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3)$ be a 9-tuple. Then one of the following holds:*

(i) G contains TK_5 in which y_2 is not a branch vertex, or G' contains TK_5.

(ii) $G - y_2$ contains K_4^-.

(iii) G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{y_2, a_1, a_2, a_3, a_4\}$, G_2 is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding y_2 and the edges y_2b_i for $i \in [4]$.

(iv) *There exist* $z_1 \in V(x_1Xy_2) - \{x_1, y_2\}$, $z_2 \in V(x_2Xy_2) - \{x_2, y_2\}$ *such that* $H := G' - (V(X - \{y_2, z_1, z_2\}) \cup E(X))$ *has disjoint paths* Y, Z *from* y_1, z_1 *to* y_2, z_2, *respectively.*

Proof. Let K be the graph obtained from $G - \{x_1, x_2, y_2\}$ by contracting $x_iXy_2 - \{x_i, y_2\}$ to the new vertex u_i, for $i \in [2]$. Note that K is 2-connected; since G is 5-connected, X is induced in $G' - x_1x_2$, and $G - X$ is 2-connected. We may assume that

(1) there exists a collection $\mathcal{A}$ of subsets of $V(K) - \{u_1, u_2, w_2, y_1\}$ such that (K, $\mathcal{A}$, u_1, y_1, u_2, w_2) is 3-planar.

For, suppose this is not the case. Then by Lemma 2.3.1, K contains disjoint paths, say Y, U, from y_1, u_1 to w_2, u_2, respectively. Let v_i denote the neighbor of u_i in the path U, and let $z_i \in V(x_iXy_2) - \{x_i, y_2\}$ be a neighbor of v_i in G. Then $Z := (U - \{u_1, u_2\}) + \{z_1, z_2, z_1v_1, z_2v_2\}$ is a path between z_1 and z_2. Now $Y + \{y_2, y_2w_2\}, Z$ are the desired paths for (iv). So we may assume (1).

Since $G - X$ is 2-connected, $|N_K(A) \cap \{u_1, u_2, w_2\}| \leq 1$ for all $A \in \mathcal{A}$. Let $p(K, \mathcal{A})$ be the graph obtained from K by (for each $A \in \mathcal{A}$) deleting A and adding new edges joining every pair of distinct vertices in $N_K(A)$. Since G is 5-connected and $G - X$ is 2-connected, we may assume that $p(K, \mathcal{A}) - \{u_1, u_2\}$ is a 2-connected plane graph, and for each $A \in \mathcal{A}$ with $N_K(A) \cap \{u_1, u_2\} \neq \emptyset$ the edge joining vertices of $N_K(A) - \{u_1, u_2\}$ occur on the outer cycle D of $p(K, \mathcal{A}) - \{u_1, u_2\}$. Note that $y_1, w_2 \in V(D)$.

Let $t_1 \in V(D)$ with t_1Dy_1 minimal such that $u_1t_1 \in E(p(K, \mathcal{A}))$; and let $t_2 \in V(D)$ with y_1Dt_2 minimal such that $u_2t_2 \in E(p(K, \mathcal{A}))$. (So t_1, y_1, t_2, w_2 occur on D in clockwise order.) Since K is 2-connected and X is induced in $G' - x_1x_2$, there exist $z_1 \in V(x_1Xy_2) - \{x_1, y_2\}$ and independent paths R_1, R_1' in G from z_1 to D and internally disjoint from $V(p(K, \mathcal{A})) \cup V(X)$, such that R_1 ends at t_1 and R_1' ends at some vertex $t'_1 \neq t_1$, and w_2, t'_1, t_1, y_1 occur on D in clockwise order. Similarly, there exist $z_2 \in V(x_2 X y_2) - \{x_2, y_2\}$ and independent paths R_2, R'_2 in G from z_2 to D and internally disjoint from $V(p(K, \mathcal{A})) \cup V(X)$, such that R_2 ends at t_2, R'_2 ends at some vertex $t'_2 \neq t_2$, and y_1, t_2, t'_2, w_2 occur on D in clockwise order.

We may assume that

(2) $K - \{u_1, u_2\}$ has no 2-separation (K', K'') such that $V(K' \cap K'') \subseteq V(t_1 D t_2)$, $|V(K')| \geq 3$, and $V(t_2 D t_1) \subseteq V(K'')$.

For, suppose such a separation (K', K'') does exist in $K - \{u_1, u_2\}$. Then by the definition of u_1, u_2, we see that G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = V(K' \cap K'') \cup \{x_1, x_2, y_2\}$, $K' \subseteq V(G_1)$ and $K'' \cup X \subseteq G_2$. Note that $G[\{x_1, x_2, y_2\}]$ is a triangle in G, $|V(G_2)| \geq 7$, and $|V(G_1)| \geq 6$ (as $|V(K')| \geq 3$). If $|V(G_1)| \geq 7$ then by Lemma 2.3.9, (i) or (ii) or (iii) holds. (Note that if (iv) of Lemma 2.3.9 holds then G' has a TK_5; so (i) holds.) So assume $|V(G_1)| = 6$, and let $v \in V(G_1 - G_2)$. Since G is 5-connected, $N(v) = V(G_1 \cap G_2)$. In particular, $v \neq y_1$ as $y_1 y_2 \notin E(G)$. Then $G[\{v, x_1, x_2, y_1\}]$ contains K_4^-, and (ii) holds. So we may assume (2).

Next we may assume that

(3) each neighbor of x_1 is contained in $V(X)$, or $V(t_1 D y_1)$, or some $A \in \mathcal{A}$ with $u_1 \in N_K(A)$, and each neighbor of x_2 is contained $V(X)$, or $V(y_1 D t_2)$, or some $A \in \mathcal{A}$ with $u_2 \in N_K(A)$.

For, otherwise, we may assume by symmetry that there exists $a \in N(x_1) - V(X)$ such that $a \notin V(t_1 D y_1)$ and $a \notin A$ for $A \in \mathcal{A}$ with $u_1 \in N_K(A)$. Let $a' = a$ and $S = a$ if $a \notin A$ for all $A \in \mathcal{A}$. When $a \in A$ for some $A \in \mathcal{A}$ then by (2), there exists $a' \in N_K(A) - V(t_1 D t_2)$ and let S be a path in $G[A + a']$ from a to a'. By (2) again, there is a path T from a' to some $u \in V(t_2 D t_1) - \{t_1, t_2\}$ in $p(K, \mathcal{A}) - \{u_1, u_2, y_2\} - t_1 D t_2$. Then $t_1 D t_2 \cup R_1 \cup R_2$ and $R'_2 \cup t'_2 D u \cup T$ give independent paths T_1, T_2, T_3 in $G - (X - \{z_1, z_2\})$ with T_1, T_2 from y_1 to z_1, z_2, respectively, and T_3

from a' to z_2. Hence, $z_2Xx_2 \cup z_2Xy_2 \cup T_2 \cup (T_3 \cup S \cup ax_1) \cup (T_1 \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2; so (i) holds.

Label the vertices of w_2Dy_1 and x_1Xy_2 such that $w_2Dy_1 = v_1 \dots v_k$ and $x_1Xy_2 = v_{k+1} \dots v_n$, with $v_1 = w_2$, $v_k = y_1$, $v_{k+1} = x_1$ and $v_n = y_2$. Let G_1 denote the union of x_1Xy_2, $\{v_1, \dots, v_k\}$, $G[A \cup (N_K(A) - u_1)]$ for $A \in \mathcal{A}$ with $u_1 \in N_K(A)$, all edges of G' from x_1Xy_2 to $\{v_1, \dots, v_k\}$, and all edges of G' from x_1Xy_2 to A for $A \in \mathcal{A}$ with $u_1 \in N_K(A)$. Note that G_1 is $(4, \{v_1, \dots, v_n\})$-connected. Similarly, let $y_1Dw_2 = z_1 \dots z_l$ and $x_2Xy_2 = z_{l+1} \dots z_m$, with $z_1 = w_2$, $z_l = y_1$, $z_{l+1} = x_2$ and $z_m = y_2$. Let G_2 denote the union of y_2Xx_2, $\{z_1, \dots, z_l\}$, $G[A \cup (N_K(A) - u_2)]$ for $A \in \mathcal{A}$ with $u_2 \in N_K(A)$, all edges of G' from y_2Xx_2 to $\{z_1, \dots, z_l\}$, and all edges of G' from y_2Xx_2 to A for $A \in \mathcal{A}$ with $u_2 \in N_K(A)$. Note that G_2 is $(4, \{z_1, \dots, z_m\})$-connected.

If both $(G_1, v_1, \dots, v_n)$ and $(G_2, z_1, \dots, z_m)$ are planar then $G - y_2$ is planar; so (i) or (ii) or (iii) holds by Lemma 2.3.10. Hence, we may assume by symmetry that $(G_1, v_1, \dots, v_n)$ is not planar. Then by Lemma 2.3.2, there exist $1 \le q < r < s < t \le n$ such that G_1 has disjoint paths Q_1, Q_2 from v_q, v_r to v_s, v_t, respectively, and internally disjoint from $\{v_1, \dots, v_n\}$.

Since (K, u_1, y_1, u_2, w_2) is 3-planar, it follows from the definition of G_1 that $q, r \le k$ and $s, t \ge k + 1$. Note that the paths y_1Dt_2, $t_2'Dv_q$, v_rDy_1 give rise to independent paths P_1, P_2, P_3 in $K - \{u_1, u_2\}$, with P_1 from y_1 to t_2, P_2 from t_2' to v_q, and P_3 from v_r to y_1. Therefore, $z_2Xx_2 \cup z_2Xy_2 \cup (R_2 \cup P_1) \cup (R_2' \cup P_2 \cup Q_1 \cup v_sXx_1) \cup (P_3 \cup Q_2 \cup v_tXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. So (i) holds. ∎

Conclusion (iv) of Lemma 3.3.1 motivates the concept of 11-tuple. We say that $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3, z_1, z_2)$ is an *11-tuple* if

- $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3)$ is a 9-tuple, and $z_i \in V(x_iXy_2) - \{x_i, y_2\}$ for $i \in [2]$,

- $H := G' - (V(X - \{y_2, z_1, z_2\}) \cup E(X))$ contains disjoint paths Y, Z from y_1, z_1 to y_2, z_2, respectively, and

- subject to the above conditions, $z_1 X z_2$ is maximal.

Since G is 5-connected and X is induced in $G' - x_1x_2$, each z_i ($i \in [2]$) has at least two neighbors in $H - \{y_2, z_1, z_2\}$ (which is 2-connected). Note that y_2 has exactly one neighbor $H - \{y_2, z_1, z_2\}$, namely, w_2. So $H - y_2$ is 2-connected.

Lemma 3.3.2 *Let $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3, z_1, z_2)$ be an 11-tuple and Y, Z be disjoint paths in $H := G' - (V(X - \{y_2, z_1, z_2\}) \cup E(X))$ from y_1, z_1 to y_2, z_2, respectively. Then G contains a TK_5 in which y_2 is not a branch vertex, or G' contains TK_5, or*

(i) for $i \in [2]$, H has no path through z_i, z_{3-i}, y_1, y_2 in order (so $y_1 z_i \notin E(G)$), and

(ii) there exists $i \in [2]$ such that H contains independent paths A, B, C, with A and C from z_i to y_1, and B from y_2 to z_{3-i}.

Proof. First, suppose, for some $i \in [2]$, there is a path P in H from z_i to y_2 such that z_i, z_{3-i}, y_1, y_2 occur on P in order. Then $z_{3-i} X x_{3-i} \cup z_{3-i} X y_2 \cup (z_{3-i} P z_i \cup z_i X x_i) \cup z_{3-i} P y_1 \cup y_1 P y_2 \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. So we may assume that such P does not exist. Hence by the existence of Y, Z in H, we have $y_1 z_1, y_1 z_2 \notin E(G)$, and (i) holds.

So from now on we may assume that (i) holds. For each $i \in [2]$, let H_i denote the graph obtained from H by duplicating z_i and y_1, and let z_i' and y_1' denote the duplicates of z_i and y_1, respectively. So in H_i, y_1 and y_1' are not adjacent, and have the same set of neighbors, namely $N_H(y_1)$; and the same holds for z_i and z_i'.

First, suppose for some $i \in [2]$, H_i contains pairwise disjoint paths A', B', C' from $\{z_i, z_i', y_2\}$ to $\{y_1, y_1', z_{3-i}\}$, with $z_i \in V(A'), z_i' \in V(C')$ and $y_2 \in V(B')$. If $z_{3-i} \notin V(B')$, then after identifying y_1 with y_1' and z_i with z_i', we obtain from $A' \cup B' \cup C'$ a path in H from z_{3-i} to y_2 through z_i, y_1 in order, contradicting our assumption that (i) holds. Hence $z_{3-i} \in V(B')$. Then we get the desired paths for (ii) from $A' \cup B' \cup C'$ by identifying y_1 with y_1' and z_i with z_i'.

So we may assume that for each $i \in [2]$, H_i does not contain three pairwise disjoint paths from $\{y_2, z_i, z_i'\}$ to $\{y_1, y_1', z_{3-i}\}$. Then H_i has a separation (H_i', H_i'') such that $|V(H_i' \cap H_i'')| = 2$, $\{y_2, z_i, z_i'\} \subseteq V(H_i')$ and $\{y_1, y_1', z_{3-i}\} \subseteq V(H_i'')$.

We claim that $y_1, y_2, y_1', z_i', z_1, z_2 \notin V(H_i' \cap H_i'')$ for $i \in [2]$. Note that $\{y_1, y_1'\} \neq V(H_i' \cap H_i'')$, since otherwise y_1 would be a cut vertex in H separating z_{3-i} from $\{y_2, z_i\}$. Now suppose one of y_1, y_1' is in $V(H_i' \cap H_i'')$; then since y_1, y_1' are duplicates, the vertex in $V(H_i' \cap H_i'') - \{y_1, y_1'\}$ is a cut vertex in H separating $\{y_1, z_{3-i}\}$ from $\{y_2, z_i\}$, a contradiction. So $y_1, y_1' \notin V(H_i' \cap H_i'')$. Similar argument shows that $z_i, z_i' \notin V(H_i' \cap H_i'')$. Since $H - y_2$ is 2-connected, $y_2 \notin V(H_i' \cap H_i'')$. Since $H - \{z_{3-i}, y_2\}$ is 2-connected, $z_{3-i} \notin V(H_i' \cap H_i'')$.

For $i \in [2]$, let $V(H_i' \cap H_i'') = \{s_i, t_i\}$, and let F_i' (respectively, F_i'') be obtained from H_i' (respectively, H_i'') by identifying z_i' with z_i (respectively, y_1' with y_1). Then (F_i', F_i'') is a 2-separation in H such that $V(F_i' \cap F_i'') = \{s_i, t_i\}$, $\{y_2, z_i\} \subseteq V(F_i') - \{s_i, t_i\}$, and $\{y_1, z_{3-i}\} \subseteq V(F_i'') - \{s_i, t_i\}$. Let Z_1, Y_2 denote the $\{s_1, t_1\}$-bridges of F_1' containing z_1, y_2, respectively; and let Z_2, Y_1 denote the $\{s_1, t_1\}$-bridges of F_1'' containing z_2, y_1, respectively.

We may assume $Y_1 = Z_2$ or $Y_2 = Z_1$. For, suppose $Y_1 \neq Z_2$ and $Y_2 \neq Z_1$. Since $H - y_2$ is 2-connected, there exist independent P_1, Q_1 in Z_1 from z_1 to s_1, t_1, respectively, independent paths P_2, Q_2 in Z_2 from z_2 to s_1, t_1, respectively, independent paths P_3, Q_3 in Y_1 from y_1 to s_1, t_1, respectively, and a path S in Y_2 from y_2 to one of $\{s_1, t_1\}$ and avoiding the other, say avoiding t_1. Then $z_1 X x_1 \cup z_1 X y_2 \cup y_2 x_1 \cup P_1 \cup S \cup (P_3 \cup y_1 x_1) \cup (Q_2 \cup Q_1) \cup P_2 \cup z_2 X y_2 \cup (z_2 X x_2 \cup x_2 x_1)$ is a TK_5 in G' with branch vertices s_1, x_1, y_2, z_1, z_2.

Indeed, $Y_1 = Z_2$. For, if $Y_1 \neq Z_2$ then $Y_2 = Z_1$, $Y_2 - \{s_1, t_1\}$ has a path from y_2 to

z_1, and $Y_1 \cup Z_2$ has two independent paths from y_1 to z_2 (since $H - y_2$ is 2-connected). Now these three paths contradict the existence of the cut $\{s_2, t_2\}$ in H.

Then $\{s_2, t_2\} \cap V(Y_1 - \{s_1, t_1\}) \neq \emptyset$. Without loss of generality, we may assume that $t_2 \in V(Y_1) - \{s_1, t_1\}$. Suppose $Y_2 = Z_1$. Then $s_2 \in V(Y_2) - \{s_1, t_1\}$ and we may assume that in H, $\{s_2, t_2\}$ separates $\{s_1, y_1, z_1\}$ from $\{t_1, y_2, z_2\}$. Hence, in Y_1, t_2 separates $\{y_1, s_1\}$ from $\{z_2, t_1\}$, and in Y_2, s_2 separates $\{z_1, s_1\}$ from $\{y_2, t_1\}$. But this contradicts the existence of the paths Y and Z in H. So $Y_2 \neq Z_1$. Since $H - y_2$ is 2-connected and $N_{G'}(y_2) = \{w_1, w_2, w_3, x_1, x_2\}$, we must have $s_2 = w_2 \in \{s_1, t_1\}$. By symmetry, we may assume that $s_2 = w_2 = s_1$.

Let Y_1', Z_2' be the $\{s_2, t_2\}$-bridge of Y_1 containing y_1, z_2, respectively. Then $t_1 \notin V(Z_2')$; for, otherwise, $H - \{s_2, t_2\}$ would contain a path from z_2 to z_1, a contradiction. Therefore, because of the paths Y and Z, $t_1 \in V(Y_1')$ and Y_1' contains disjoint paths R_1, R_2 from $s_2 = s_1, t_1$ to y_1, t_2, respectively. Since $H - y_2$ is 2-connected, Z_1 has independent P_1, Q_1 from z_1 to $s_2 = s_1, t_1$, respectively, and Z_2' has independent paths P_2, Q_2 from z_2 to $s_2 = s_1, t_2$, respectively. Now $z_1Xx_1 \cup z_1Xy_2 \cup y_2x_1 \cup P_1 \cup s_1y_2 \cup (R_1 \cup y_1x_1) \cup P_2 \cup (Q_2 \cup R_2 \cup Q_1) \cup z_2Xy_2 \cup (z_2Xx_2 \cup x_2x_1)$ is a TK_5 in G' with branch vertices s_1, x_1, y_2, z_1, z_2. ∎

Lemma 3.3.3 *Let $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3, z_1, z_2)$ be an 11-tuple and Y, Z be disjoint paths in $H := G' - V(X - \{y_2, z_1, z_2\} \cup E(X))$ from y_1, z_1 to y_2, z_2, respectively. Then G contains a TK_5 in which y_2 is not a branch vertex or G' contains TK_5, or*

(i) there exist $i \in [2]$ and independent paths A, B, C in H, with A and C from z_i to y_1, and B from y_2 to z_{3-i},

(ii) for each $i \in [2]$ satisfying (i), $z_{3-i}x_{3-i} \in E(X)$, and

(iii) H contains two disjoint paths from $V(B - y_2)$ to $V(A \cup C) - \{y_1, z_i\}$ and internally disjoint from $A \cup B \cup C$, with one ending in A and the other ending in C.

Proof. By Lemma 3.3.2, we may assume that

(1) for each $i \in [2]$, H has no path through z_i, z_{3-i}, y_1, y_2 in order (so $y_1 z_i \notin E(G)$), and

(2) there exist $i \in [2]$ and independent paths A, B, C in H, with A and C from z_i to y_1, and B from y_2 to z_{3-i}.

Let $J(A, C)$ denote the $(A \cup C)$-bridge of H containing B, and $L(A, C)$ denote the union of $(A \cup C)$-bridges of H each of which intersects both $A - \{y_1, z_i\}$ and $C - \{y_1, z_i\}$. We choose A, B, C such that the following are satisfied in the order listed:

(a) A, B, C are induced paths in H,

(b) whenever possible, $J(A, C) \subseteq L(A, C)$,

(c) $J(A, C)$ is maximal, and

(d) $L(A, C)$ is maximal.

We now show that (ii) and (iii) hold even with the restrictions (a), (b), (c) and (d) above. Let B' denote the union of B and the B-bridges of H not containing $A \cup C$.

(3) If (iii) holds then (ii) holds.

Suppose (iii) holds. Let $V(P \cap B) = \{p\}$, $V(Q \cap B) = \{q\}$, $V(P \cap C) = \{c\}$ and $V(Q \cap A) = \{a\}$. By the symmetry between A and C, we may assume that y_2, p, q, z_{3-i} occur on B in order. We may further choose P, Q so that pBz_{3-i} is maximal.

To prove (ii), suppose there exists $x \in V(z_{3-i}Xx_{3-i}) - \{x_{3-i}, z_{3-i}\}$. If $N(x) \cap V(H) - \{y_1\} \not\subseteq V(B')$ then G' has a path T from x to $(A - y_1) \cup (C - y_1) \cup (P - p) \cup (Q - a)$ and internally disjoint from $A \cup B' \cup C \cup P \cup Q$; so $A \cup B \cup C \cup P \cup Q \cup T$ contain disjoint paths from y_1, z_i to y_2, x, respectively, contradicting the choice of Y and Z

in the 11-tuple (that $z_1 X z_2$ is maximal). So $N(x) \cap V(H) - \{y_1\} \subseteq V(B')$. Consider $B'' := G[(B' - z_{3-i}) + x]$.

If B'' contains disjoint paths P', Q' from y_2, x to p, q, respectively, then $Q' \cup Q \cup aAz_i$ and $P' \cup P \cup cCy_1$ contradict the choice of Y, Z. If B'' contains disjoint paths P'', Q'' from x, y_2 to p, q, respectively, then $Q'' \cup Q \cup aAy_1$ and $P'' \cup P \cup cCz_i$ contradict the choice of Y, Z.

So we may assume that there is a cut vertex z in B'' separating $\{x, y_2\}$ from $\{p, q\}$. Note that $z \in V(y_2 B p)$.

Since x has at least two neighbors in $B'' - y_2$ (because G is 5-connected and X is induced in $G' - x_1 x_2$), the z-bridge of B'' containing $\{x, y_2\}$ has at least three vertices. Therefore, from the maximality of pBz_{3-i} and 2-connectedness of $H - \{y_2, z_1, z_2\}$, there is a path in H from y_1 to $y_2 B z - \{y_2, z\}$ and internally disjoint from $P \cup Q \cup A \cup C \cup B'$. So there is a path Y' in H from y_1 to y_2 and disjoint from $P \cup Q \cup A \cup C \cup pBz_{3-i}$. Now $z_{3-i}Bp \cup P \cup cCz_i \cup A \cup Y'$ is a path in H through z_{3-i}, z_i, y_1, y_2 in order, contradicting (1).

By (2) and (3), it suffices to prove (*iii*). Since $H - \{y_2, z_i\}$ is 2-connected, it contains disjoint paths P, Q from $B - y_2$ to some distinct vertices $s, t \in V(A \cup C) - \{z_i\}$, respectively, and internally disjoint from $A \cup B \cup C$.

(4) We may choose P, Q so that $s \neq y_1$ and $t \neq y_1$.

For, otherwise, $H - \{y_2, z_i\}$ has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{v, y_1\}$ for some $v \in V(H)$, $(A \cup C) - z_i \subseteq H_1$ and $B - y_2 \subseteq H_2$. Recall the disjoint paths Y, Z in H from z_1, y_1 to z_2, y_2, respectively. Suppose $v \notin V(Z)$. Then $Z - z_i \subseteq H_2 - \{y_1, v\}$. Hence we may choose Y (by modifying $Y \cap H_1$) so that $V(Y \cap A) = \{y_1\}$ or $V(Y \cap C) = \{y_1\}$. Now $Z \cup A \cup Y$ or $Z \cup C \cup Y$ is a path in H from z_{3-i} to y_2 through z_i, y_1 in order, contradicting (1). So $v \in V(Z)$. Hence $Y \subseteq H_2 - v$, and we may choose Z (by modifying $Z \cap H_1$) so that $V(Z \cap A) = \{z_i\}$ or $V(Z \cap C) = \{z_i\}$.

Now $Z \cup A \cup Y$ or $Z \cup C \cup Y$ is a path in H from z_{3-i} to y_2 through z_i, y_1 in order, contradicting (1) and completing the proof of (4).

If $s \in V(A - y_1)$ and $t \in V(C - y_1)$ or $s \in V(C - y_1)$ and $t \in V(A - y_1)$, then P, Q are the desired paths for (iii). So we may assume by symmetry that $s, t \in V(C)$. Let $V(P \cap B) = \{p\}$ and $V(Q \cap B) = \{q\}$ such that y_2, p, q, z_{3-i} occur on B in this order. By (1) z_i, s, t, y_1 must occur on C in order. We choose P, Q so that

($*$) sCt is maximal, then pBz_{3-i} is maximal, and then qBz_{3-i} is minimal.

Now consider B', the union of B and the B-bridges of H not containing $A \cup C$. Note that $(P - p) \cup (Q - q)$ is disjoint from B', and every path in H from $A \cup C$ to B' and internally disjoint from $A \cup B' \cup C$ must end in B. For convenience, let $K = P \cup Q \cup A \cup B' \cup C$.

(5) $B' - y_2$ contains independent paths P', Q' from z_{3-i} to p, q, respectively.

Otherwise, $B' - y_2$ has a cut vertex z separating z_{3-i} from $\{p, q\}$. Clearly, $z \in V(qBz_{3-i} - z_{3-i})$, and we choose z so that zBz_{3-i} is minimal.

Let B'' denote the z-bridge of $B' - y_2$ containing z_{3-i}; then $zBz_{3-i} \subseteq B''$. Since $H - \{y_2, z_i\}$ is 2-connected, it contains a path W from some $w' \in V(B'' - z)$ to some $w \in V(P \cup Q \cup A \cup C) - \{z_i\}$ and internally disjoint from K. By the definition of B', $w' \in V(z_iBz_{3-i})$. By (1), $w \notin V(P) \cup V(z_iCt - t)$. By ($*$), $w \notin V(Q) \cup V(tCy_1 - y_1)$.

If $w \in V(A) - \{z_i, y_1\}$ then P, W give the desired paths for (iii). So we may assume $w = y_1$ for any choice of W; hence, $z \in V(Z)$ and $Y \cap (B'' \cup (W - y_1)) = \emptyset$. By the minimality of zBz_{3-i}, B'' has independent paths P'', Q'' from z_{3-i} to z, w', respectively. Note that $z_iZz \cap (B'' - z) = \emptyset$. Now $z_iZz \cup P'' \cup Q'' \cup W \cup Y$ is a path in H through z_i, z_{3-i}, y_1, y_2 in order, contradicting (1).

(6) We may assume that $J(A, C) \nsubseteq L(A, C)$.

For, otherwise, there is a path R from B to some $r \in V(A) - \{y_1, z_i\}$ and internally disjoint from $A \cup B' \cup C$. If $R \cap (P \cup Q) \neq \emptyset$, then it is easy to check that $P \cup Q \cup R$ contains the desired paths for (iii). So we may assume $R \cap (P \cup Q) = \emptyset$. If $y_2 \notin V(R)$, then P, R are the desired paths for (iii). So assume $y_2 \in V(R)$. Recall the paths P', Q' from (5). Then $z_i C s \cup P \cup P' \cup Q' \cup Q \cup t C y_1 \cup y_1 A r \cup R$ is a path in H through z_i, z_{3-i}, y_1, y_2 in order, contradicting (1) and completing the proof of (6).

Let $J = J(A, C) \cup C$. Then by (1), J does not contain disjoint paths from y_2, z_i to y_1, z_{3-i}, respectively. So by Lemma 2.3.1, there exists a collection $\mathcal{A}$ of subsets of $V(J) - \{y_1, y_2, z_1, z_2\}$ such that $(J, \mathcal{A}, z_i, y_1, z_{3-i}, y_2)$ is 3-planar. We choose $\mathcal{A}$ so that every member of $\mathcal{A}$ is minimal and, subject to this, $|\mathcal{A}|$ is minimum. Then

(7) for any $D \in \mathcal{A}$ and any $v \in V(D)$, $(J[D + N_J(D)], N_J(D) \cup \{v\})$ is not 3-planar.

Suppose for some $D \in \mathcal{A}$ and some $v \in D$, there is a collection of subsets $\mathcal{A}'$ of $D - \{v\}$ such that $(J[D + N_J(D)], \mathcal{A}', N_J(D) \cup \{v\})$ is 3-planar. Then, with $\mathcal{A}'' = (\mathcal{A} - \{D\}) \cup \mathcal{A}'$, $(J, \mathcal{A}'', z_i, y_1, z_{3-i}, y_2)$ is 3-planar. So $\mathcal{A}''$ contradicts the choice of $\mathcal{A}$. Hence, we have (7).

Let $v_1, \ldots, v_k$ be the vertices of $L(A, C) \cap (C - \{y_1, z_i\})$ such that $z_i, v_1, \ldots, v_k, y_1$ occur on C in the order listed. We claim that

(8) $(J, z_i, v_1, \ldots, v_k, y_1, z_{3-i}, y_2)$ is 3-planar.

For, suppose otherwise. Since there is only one C-bridge in J and $(J, \mathcal{A}, z_i, y_1, z_{3-i}, y_2)$ is 3-planar, there exist $j \in [k]$ and $D \in \mathcal{A}$ such that $v_j \in D$. Since H is 2-connected, let $c_1, c_2 \in V(C) \cap N_J(D)$ with $c_1 C c_2$ maximal.

Suppose $N_J(D) \subseteq V(C)$. Then, since there is only one C-bridge in J and $(J, \mathcal{A}, z_i, y_1, z_{3-i}, y_2)$ is 3-planar, J has a separation (J_1, J_2) such that $V(J_1 \cap J_2) = \{c_1, c_2\}$, $D \cup V(c_1 C c_2) \subseteq V(J_1)$, and $B \subseteq J_2$. Since J has only one C-bridge and

C is induced in H, we have $J_1 = c_1Cc_2$. Now let $\mathcal{A}'$ be obtained from $\mathcal{A}$ by removing all members of $\mathcal{A}$ contained in $V(J_1)$. Then $(J, \mathcal{A}', z_i, y_1, z_{3-i}, y_2)$ is 3-planar, contradicting the choice of $\mathcal{A}$.

Thus, let $c \in N_J(D) - V(C)$. So $c \in V(J(A,C))$. Let $D' = J[D + \{c_1, c_2, c\}]$. By (7) and Lemma 2.3.1, D' contains disjoint paths R from v_j to c and T from c_1 to c_2. We may assume T is induced. Let C' be obtained from C by replacing c_1Cc_2 with T. We now see that A, B, C' satisfy (a), but $J(A, C')$ intersects both $A - \{y_1, z_i\}$ (by definition of v_j and because $c \in V(J(A,C)) - V(C)$) and $C' - \{y_1, z_i\}$ (because of P, Q), contradicting (b) (via (6)) and completing the proof of (8).

(9) There exist disjoint paths R_1, R_2 in $L(A,C)$ from some $r_1, r_2 \in V(C)$ to some $r_1', r_2' \in V(A)$, respectively, and internally disjoint from $A \cup C$, such that z_i, r_1, r_2, y_1 occur on C in this order and z_i, r_2', r_1', y_1 occur on A in this order.

We prove (9) by studying the $(A \cup C)$-bridges of H other than $J(A,C)$. For any $(A \cup C)$-bridge T of H with $T \neq J(A,C)$, if T intersects A let $a_1(T), a_2(T) \in V(T \cap A)$ with $a_1(T)Aa_2(T)$ maximal, and if T intersects C let $c_1(T), c_2(T) \in V(T \cap C)$ with $c_1(T)Cc_2(T)$ maximal. We choose the notation so that $z_i, a_1(T), a_2(T), y_1$ occur on A in order, and $z_i, c_1(T), c_2(T), y_1$ occur on C in order.

If T_1, T_2 are $(A \cup C)$-bridges of H such that $T_2 \subseteq L(A,C)$, $T_1 \neq J(A,C)$, and T_1 intersects C (or A) only, then $c_1(T_1)Cc_2(T_1) - \{c_1(T_1), c_2(T_1)\}$ (or $a_1(T_1)Aa_2(T_1) - \{a_1(T_1), a_2(T_1)\}$) does not intersect T_2. For, otherwise, we may modify C (or A) by replacing $c_1(T_1)Cc_2(T_1)$ (or $a_1(T_1)Aa_2(T_1)$) with an induced path in T_1 from $c_1(T_1)$ to $c_2(T_1)$ (or from $a_1(T_1)$ to $a_2(T_1)$). The new A and C do not affect (a), (b) and (c) but enlarge $L(A,C)$, contradicting (d).

Because of the disjoint paths Y and Z in H, $(H, z_i, y_1, z_{3-i}, y_2)$ is not 3-planar. By (1) $A - \{y_1, z_i\} \neq \emptyset$. Hence, since $H - \{y_2, z_1, z_2\}$ is 2-connected, $L(A,C) \neq \emptyset$. Thus, since $(J, z_i, v_1, \ldots, v_k, y_1, z_{3-i}, y_2)$ is 3-planar (by (8)) and $J(A,C)$ does not intersect $A-\{y_1, z_i\}$ (by (6)), one of the following holds: There exist $(A\cup C)$-bridges T_1, T_2 of H such that $T_1\cup T_2\subseteq L(A,C)$, $z_iAa_2(T_1)$ properly contains $z_iAa_1(T_2)$, and $c_1(T_1)Cy_1$ properly contains $c_2(T_2)Cy_1$; or there exists an $(A\cup C)$-bridge T of H such that $T\subseteq L(A,C)$ and $T\cup a_1(T)Aa_2(T)\cup c_1(T)Cc_2(T)$ has disjoint paths from $a_1(T), a_2(T)$ to $c_2(T), c_1(T)$, respectively. In either case, we have (9).

(10) $r_1, r_2 \in V(tCy_1)$ for all choices of R_1, R_2 in (9), or $r_1, r_2 \in V(z_iCs)$ for all choices of R_1, R_2 in (9).

For, suppose there exist R_1, R_2 such that $r_1\in V(z_iCs)$ and $r_2\in V(tCy_1)$, or $r_1\in V(sCt)-\{s,t\}$, or $r_2\in V(sCt)-\{s,t\}$. Let $A' := z_iAr_2'\cup R_2\cup r_2Cy_1$ and $C' := z_iCr_1\cup R_1\cup r_1'Ay_1$. We may assume A', C' are induced paths in H (by taking induced paths in $H[A']$ and $H[C']$). Note that A', B, C' satisfy (a), and $J(A,C)\subseteq J(A',C')$. However, because of P and Q, $J(A',C')$ intersects both $A'-\{z_i,y_1\}$ and $C'-\{z_i,y_1\}$, contradicting (b) (via (6)) and completing the proof of (10).

If $r_1, r_2\in V(z_iCs)$ for all choices of R_1, R_2 in (9) then we choose such R_1, R_2 that z_iAr_1' and z_iCr_2 are maximal, and let $z' := r_1'$ and $z'' = r_2$; otherwise, define $z' = z'' = z_i$. Similarly, if $r_1, r_2 \in V(tCy_1)$ for all choices of R_1, R_2 in (9), then we choose such R_1, R_2 that y_1Ar_2' and y_1Cr_1 are maximal, and let $y' := r_2'$ and $y'' = r_1$; otherwise, define $y' = y'' = y_1$. By (10), z_i, z', y', y_1 occur on A in order, and z_i, z'', s, t, y'', y_1 occur on C in order.

Note that H has a path W from some $y\in V(B)\cup V(P-s)\cup V(Q-t)$ to some $w\in V(z_iAz'-\{z',z_i\})\cup V(z_iCz''-\{z'',z_i\})\cup V(y'Ay_1-\{y',y_1\})\cup V(y''Cy_1-\{y'',y_1\})$ such that W is internally disjoint from K. For, otherwise, $(H, z_i, y_1, z_{3-i}, y_2)$ is 3-planar, contradicting the existence of the disjoint paths Y and Z. By (6), $w\notin V(A)$. If $w\in V(z_iAz'-\{z',z_i\})\cup V(y'Ay_1-\{y',y_1\})$ then we can find the desired P, Q. So assume $w\in V(z_iCz''-\{z'',z_i\})\cup V(y''Cy_1-\{y'',y_1\})$. By $(*)$ and (1), $y\notin V(B-y_2)$ and $y\notin V(P\cup Q)$. This forces $y=y_2$, which is impossible as $N_H(y_2)=\{w_2\}$. ∎

Remark. Note from the proof of Lemma 3.3.3 that the conclusions (ii) and (iii) hold for those paths A, B, C that satisfy (a), (b), (c) and (d).

3.4 Finding TK_5

In this section, we prove Theorem 3.1.1. Let G be a 5-connected nonplanar graph and let $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. Let $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$ be distinct and let $G' := G - \{y_2v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$.

We may assume that $G' - x_1x_2$ has an induced path L from x_1 to x_2 such that $y_1, y_2 \notin V(L)$, $(G - y_2) - L$ is 2-connected, and $w_1, w_2, w_3 \in V(L)$; for otherwise, the conclusion of Theorem 3.1.1 follows from Lemma 3.2.2. Hence, $G' - x_1x_2$ has an induced path X from x_1 to x_2 such that $y_1 \notin V(X)$, $w_1y_2, w_3y_2 \in E(X)$, and $G' - X = G - X$ is 2-connected. Hence, $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3)$ is a 9-tuple.

We may assume that there exist $z_i \in V(x_iXy_2) - \{x_i, y_2\}$ for $i \in [2]$ such that $H := G' - (X - \{y_2, z_1, z_2\})$ has disjoint paths Y, Z from y_1, z_1 to y_2, z_2, respectively; for, otherwise, the conclusion of Theorem 3.1.1 follows from Lemma 3.3.1. We choose such Y, Z so that z_1Xz_2 is maximal. Then $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3, z_1, z_2)$ is an 11-tuple.

By Lemma 3.3.2 and by symmetry, we may assume that

(1) for $i \in [2]$, H has no path through z_i, z_{3-i}, y_1, y_2 in order (so $y_1z_i \notin E(G)$),

and that there exist independent paths A, B, C in H with A and C from z_1 to y_1, and B from y_2 to z_2. See Figure 1.

Let $J(A, C)$ denote the $(A \cup C)$-bridge of H containing B, and $L(A, C)$ denote the union of $(A \cup C)$-bridges of H intersecting both $A - \{y_1, z_1\}$ and $C - \{y_1, z_1\}$. We may choose A, B, C such that the following are satisfied in the order listed:

(a) A, B, C are induced paths in H,

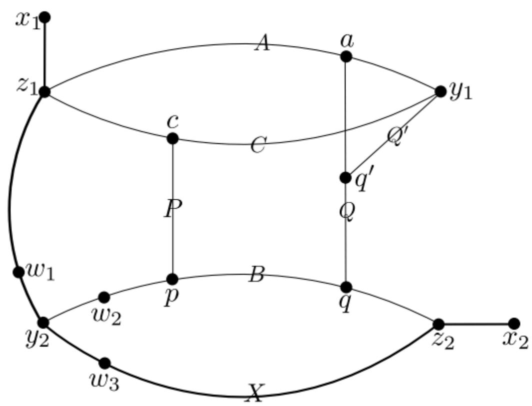

Figure 1: An intermediate structure 1

(b) whenever possible $J(A,C) \subseteq L(A,C)$,

(c) $J(A,C)$ is maximal, and

(d) $L(A,C)$ is maximal.

By Lemma 3.3.3 and its proof (see the remark at the end of Section 4), we may assume that

$$z_2x_2 \in E(X)$$

and that there exist disjoint paths P, Q in H from $p, q \in V(B - y_2)$ to $c \in V(C) - \{y_1, z_1\}, a \in V(A) - \{y_1, z_1\}$, respectively, and internally disjoint from $A \cup B \cup C$. By symmetry between A and C, we assume that y_2, p, q, z_2 occur on B in order. We further choose A, B, C, P, Q so that

(2) qBz_2 is minimal, then pBz_2 is maximal, and then $aAy_1 \cup cCz_1$ is minimal.

Let B' denote the union of B and the B-bridges of H not containing $A \cup C$. Note that all paths in H from $A \cup C$ to B' and internally disjoint from B' must have an end in B. For convenience, let

$$K := A \cup B' \cup C \cup P \cup Q.$$

Then

(3) H has no path from $aAy_1 - a$ to $z_1Cc - c$ and internally disjoint from K.

For, suppose S is a path in H from some vertex $s \in V(aAy_1 - a)$ to some vertex $s' \in V(z_1Cc - c)$ and internally disjoint from K. Then $z_2Bq \cup Q \cup aAz_1 \cup z_1Cs' \cup S \cup sAy_1 \cup y_1Cc \cup P \cup pBy_2$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1).

We proceed by proving a number of claims from which Theorem 3.1.1 will follow. Our intermediate goal is to prove (12) that H contains a path from y_1 to $Q - a$ and internally disjoint from K. However, the claims leading to (12) will also be useful when we later consider structure of G near z_1.

(4) $B' - y_2$ has no cut vertex contained in $qBz_2 - z_2$ and, hence, for any $q^* \in V(B') - \{y_2, q\}$, $B' - y_2$ has independent paths P_1, P_2 from z_2 to q, q^*, respectively.

Suppose $B' - y_2$ contains a cut vertex u with $u \in V(qBz_2 - z_2)$. Choose u so that uBz_2 is minimal. Since $H - \{y_2, z_1\}$ is 2-connected, there is a path S in H from some $s' \in V(uBz_2 - u)$ to some $s \in V(A \cup C \cup P \cup Q) - \{p, q\}$ and internally disjoint from K. By the minimality of uBz_2, the u-bridge of $B' - y_2$ containing uBz_2 has independent paths R_1, R_2 from z_2 to s', u, respectively. By the minimality of qBz_2 in (2), S is disjoint from $(P \cup Q \cup A \cup C) - \{z_1, y_1\}$. If $s = z_1$ then $(R_1 \cup S) \cup A \cup (y_1Cc \cup P \cup pBy_2)$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1). So $s = y_1$. Then $(z_1Aa \cup Q \cup qBu \cup R_2) \cup (R_1 \cup S) \cup (y_1Cc \cup P \cup pBy_2)$ is a path in H through z_1, z_2, y_1, y_2 in order, contradicting (1).

Hence, $B' - y_2$ has no cut vertex contained in $qBz_2 - z_2$. Thus, the second half of (4) follows from Menger's theorem.

(5) We may assume that G' has no path from $aAy_1 - a$ to z_1Xz_2 and internally disjoint from $K \cup X$, and no path from $cCy_1 - c$ to $z_1Xz_2 - z_1$ and internally disjoint from $K \cup X$.

For, suppose S is a path in G' from some $s \in V(aAy_1 - a) \cup V(cCy_1 - c)$ to some $s' \in V(z_1Xz_2)$ and internally disjoint from $K \cup X$, such that $s' \neq z_1$ if $s \in V(cCy_1 - c)$. If $s' = z_1$ then $s \in V(aAy_1 - a)$; so $z_2Bq \cup Q \cup aAz_1 \cup S \cup sAy_1 \cup y_1Cc \cup P \cup pBy_2$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1). If $s' = z_2$ then $s = y_1$ by (2); so $(z_1Aa \cup Q \cup qBz_2) \cup S \cup y_1Cc \cup P \cup pBy_2$ is a path in H through z_1, z_2, y_1, y_2 in order, contradicting (1). Hence, $s' \in V(z_1Xz_2) - \{z_1, z_2\}$.

Suppose $s' \in V(z_1Xy_2 - z_1)$. Let P_1, P_2 be the paths in (4) with $q^* = p$. If $s \in V(aAy_1 - a)$ then $z_2x_2 \cup z_2Xy_2 \cup (P_2 \cup P \cup cCy_1) \cup (P_1 \cup Q \cup aAz_1 \cup z_1Xx_1) \cup (y_1As \cup S \cup s'Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. If $s \in V(cAy_1 - c)$ then $z_2x_2 \cup z_2Xy_2 \cup (P_2 \cup P \cup cCz_1 \cup z_1Xx_1) \cup (P_1 \cup Q \cup aAy_1) \cup (y_1Cs \cup S \cup s'Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Now assume $s' \in V(z_2Xy_2 - z_2)$. If $s \in V(aAy_1 - a)$, then $z_1Xx_1 \cup z_1Xy_2 \cup C \cup (z_1Aa \cup Q \cup qBz_2 \cup z_2x_2) \cup (y_1As \cup S \cup s'Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. If $s \in V(cCy_1 - c)$, then $z_1Xx_1 \cup z_1Xy_2 \cup A \cup (z_1Cc \cup P \cup pBz_2 \cup z_2x_2) \cup (y_1Cs \cup S \cup s'Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. This completes the proof of (5).

Denote by $L(A)$ (respectively, $L(C)$) the union of $(A \cup C)$-bridges of H not intersecting C (respectively, A). Let $C' = C \cup L(C)$. The next four claims concern paths from $x_1Xz_1 - z_1$ to other parts of G'. We may assume that

(6) $N(x_1Xz_1 - \{x_1, z_1\}) \subseteq V(C') \cup \{x_1, z_1\}$, and that G' has no disjoint paths from $s_1, s_2 \in V(x_1Xz_1 - z_1)$ to $s'_1, s'_2 \in V(C)$, respectively, and internally disjoint from $K \cup X$ such that $s'_2 \in V(cCy_1 - c)$, x_1, s_1, s_2, z_1 occur on X in order, and z_1, s'_1, s'_2, y_1 occur on C in order.

First, suppose $N(x_1Xz_1 - \{x_1, z_1\}) \not\subseteq V(C') \cup \{x_1, z_1\}$. Then there exists a path S in G' from some $s \in V(x_1Xz_1) - \{x_1, z_1\}$ to some $s' \in V(A \cup B' \cup P \cup Q) - \{c, y_1, y_2, z_1, z_2\}$ and internally disjoint from $K \cup X$. If $s' \in V(A) - \{z_1, y_1\}$ then $y_1Cc \cup P \cup pBy_2$, $S \cup s'Aa \cup Q \cup qBz_2$ contradict the choice of Y, Z. If $s' \in V(Q-a)$ then $y_1Cc \cup P \cup pBy_2$, $S \cup s'Qq \cup qBz_2$ contradict the choice of Y, Z. If $s' \in V(P-c)$ then let P_1, P_2 be the paths in (4) with $q^* = p$; now $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAy_1) \cup (P_2 \cup pPs' \cup S \cup sXx_1) \cup (C \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. If $s' \in V(B') - \{y_2, p, q\}$ then let P_1, P_2 be the paths in (4) with $q^* = s'$; now $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAy_1) \cup (P_2 \cup S \cup sXx_1) \cup (C \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Now assume G' has disjoint paths S_1, S_2 from $s_1, s_2 \in V(x_1Xz_1 - z_1)$ to $s_1', s_2' \in V(C)$, respectively, and internally disjoint from $K \cup X$ such that $s_2' \in V(cCy_1 - c)$, x_1, s_1, s_2, z_1 occur on X in order, and z_1, s_1', s_2', y_1 occur on C in order. Let P_1, P_2 be the paths in (4) with $q^* = p$. Then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAy_1) \cup (P_2 \cup P \cup cCs_1' \cup S_1 \cup s_1Xx_1) \cup (y_1Cs_2' \cup S_2 \cup s_2Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. This completes the proof of (6).

(7) For any path W in G' from x_1 to some $w \in V(K) - \{y_1, z_1\}$ and internally disjoint from $K \cup X$, we may assume $w \in V(A \cup C) - \{y_1, z_1\}$. (Note that such W exists as G is 5-connected and $G' - X$ is 2-connected.)

For, let W be a path in G' from x_1 to $w \in V(K) - \{y_1, z_1\}$ and internally disjoint from $K \cup X$, such that $w \notin V(A \cup C) - \{z_1, y_1\}$. Then $w \neq y_2$ as $N_{G'}(y_2) = \{w_1, w_2, w_3, x_1, x_2\}$.

Suppose $w \in V(B' - q)$. Let P_1, P_2 be the paths in (4) with $q^* = w$. Then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAy_1) \cup (P_2 \cup W) \cup (C \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

So assume $w \notin V(B' - q)$. Let P_1, P_2 be the paths in (4) with $q^* = p$. If $w \in V(P - c)$ then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAy_1) \cup (P_2 \cup pPw \cup W) \cup (C \cup z_1Xy_2) \cup$

$G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. If $w \in V(Q-a)$ then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQw \cup W) \cup (P_2 \cup P \cup cCy_1) \cup (A \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. This completes the proof of (7).

(8) We may assume that G' has no path from $x_1Xz_1 - x_1$ to y_1 and internally disjoint from $K \cup X$.

For, suppose that R is a path in G' from some $x \in V(x_1Xz_1 - x_1)$ to y_1 and internally disjoint from $K \cup X$. Then $x \neq z_1$; as otherwise $z_2Bq \cup Q \cup aAz_1 \cup R \cup y_1Cc \cup P \cup pBy_2$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1). Let P_1, P_2 be the paths in (4) with $q^* = p$. We use W from (7). If $w \in V(A) - \{z_1, y_1\}$ then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAw \cup W) \cup (P_2 \cup P \cup cCy_1) \cup (R \cup xXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. If $w \in V(C) - \{z_1, y_1\}$ then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAy_1) \cup (P_2 \cup P \cup cCw \cup W) \cup (R \cup xXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. This completes the proof of (8).

(9) If G' has a path from $x_1Xz_1 - \{x_1, z_1\}$ to $cCy_1 - c$ and internally disjoint from $K \cup X$, then we may assume that

- $w \in V(C) - \{y_1, z_1\}$ for any choice of W in (7), and
- G' has no path from x_2 to $C - \{y_1, z_1\}$ and internally disjoint from $K \cup X$.

Let S be a path in G' from some $s \in V(x_1Xz_1) - \{x_1, z_1\}$ to $V(cCy_1 - c)$ and internally disjoint from $K \cup X$. Since X is induced in $G' - x_1x_2$, $G'[H - \{y_2, z_1, z_2\} + s]$ is 2-connected. Hence, since $N(x_1Xz_1 - \{x_1, z_1\}) \subseteq V(C') \cup \{x_1, z_1\}$ (by (6)), G' has independent paths S_1, S_2 from s to distinct $s_1, s_2 \in V(C) - \{z_1, y_1\}$ and internally disjoint from $K \cup X$. Because of S, we may assume that z_1, s_1, s_2, y_1 occur on C in this order and $s_2 \in V(cCy_1 - c)$.

Suppose we may choose the W in (7) with $w \in V(A) - \{z_1, y_1\}$. Let P_1, P_2 be the paths in (4) with $q^* = p$. Then $z_2x_2 \cup z_2Xy_2 \cup sXx_1 \cup sXy_2 \cup (P_2 \cup P \cup cCs_1 \cup S_1) \cup$

$(S_2 \cup s_2Cy_1 \cup y_1x_2) \cup (P_1 \cup Q \cup aAw \cup W) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices s, x_1, x_2, y_2, z_2.

Now assume that S' is a path in G' from x_2 to some $s' \in V(C) - \{y_1, z_1\}$ and internally disjoint from $K \cup X$. Then $S_1 \cup S_2 \cup S' \cup (C - z_1)$ contains independent paths S'_1, S'_2 which are from s to y_1, x_2, respectively (when $s' \in V(z_1Cs_2) - \{s_2, z_1\}$), or from s to c, x_2, respectively (when $s' \in V(s_2Cy_1 - y_1)$). If S'_1, S'_2 end at y_1, x_2, respectively, then $sXx_1 \cup sXy_2 \cup S'_1 \cup S'_2 \cup (y_1Aa \cup Q \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices s, x_1, x_2, y_1, y_2. So assume that S'_1, S'_2 end at c, x_2, respectively. Let P_1, P_2 be the paths in (4) with $q^* = p$. Then $sXx_1 \cup sXy_2 \cup z_2x_2 \cup z_2Xy_2 \cup (S'_1 \cup P \cup P_2) \cup S'_2 \cup (P_1 \cup Q \cup aAy_1 \cup y_1x_1) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices s, x_1, x_2, y_2, z_2. This completes the proof of (9).

The next two claims deal with $L(A)$ and $L(C)$. First, we may assume that

(10) $L(A) \cap A \subseteq z_1Aa$.

For any $(A \cup C)$-bridge R of H contained in $L(A)$, let $z(R), y(R) \in V(R \cap A)$ such that $z(R)Ay(R)$ is maximal. Suppose for some $(A \cup C)$-bridge R_1 of H contained in $L(A)$, we have $y(R_1)Az(R_1) \not\subseteq z_1Aa$. Let $R_1, \ldots, R_m$ be a maximal sequence of $(A \cup C)$-bridges of H contained in $L(A)$, such that for each $i \in \{2, \ldots, m\}$, R_i contains an internal vertex of $\bigcup_{j=1}^{i-1} z(R_j)Ay(R_j)$ (which is a path). Let $a_1, a_2 \in V(A)$ such that $\bigcup_{j=1}^{m} z(R_j)Ay(R_j) = a_1Aa_2$. By (c), $J(A, C)$ does not intersect $a_1Aa_2 - \{a_1, a_2\}$; so $a_1, a_2 \in V(aAy_1)$. By (d), G' has no path from $a_1Aa_2 - \{a_1, a_2\}$ to C and internally disjoint from $K \cup X$. Hence by (5), $\{a_1, a_2, x_1, x_2, y_2\}$ is a cut in G. Thus, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a_1, a_2, x_1, x_2, y_2\}$, $P \cup Q \cup B' \cup C \cup X \subseteq G_1$, and $a_1Aa_2 \cup \left(\bigcup_{j=1}^{m} R_j\right) \subseteq G_2$.

Let $z \in V(G_2) - \{a_1, a_2, x_1, x_2, y_2\}$ and assume z_1, a_1, a_2, y_1 occur on A in order. Since G is 5-connected, $G_2 - y_2$ contains four independent paths R_1, R_2, R_3, R_4 from z to x_1, x_2, a_1, a_2, respectively. Now $R_1 \cup R_2 \cup (R_3 \cup a_1Az_1 \cup z_1Xy_2) \cup (R_4 \cup a_2Ay_1) \cup (y_1Cc \cup$

$P \cup pBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z. This completes the proof of (10).

(11) We may assume that if R is an $(A \cup C)$-bridge of H contained in $L(C)$ and $R \cap (cCy_1 - c) \neq \emptyset$ then $|V(R) - V(C)| = 1$ and $N(R - C) = \{c_1, c_2, s_1, s_2, y_2\}$, with $c_1Cc_2 = c_1c_2$ and $s_1s_2 = s_1Xs_2 \subseteq z_1Xx_1$.

For any $(A \cup C)$-bridge R in $L(C)$, let $z(R), y(R) \in V(C \cap R)$ such that $z(R)Cy(R)$ is maximal. Let R_1 be an $(A \cup C)$-bridge of H contained in $L(C)$ such that $R_1 \cap (cCy_1 - c) \neq \emptyset$.

Let $R_1, \ldots, R_m$ be a maximal sequence of $(A \cup C)$-bridges of H contained in $L(C)$, such that for each $i \in \{2, \ldots, m\}$, R_i contains an internal vertex of $\bigcup_{j=1}^{i-1} z(R_j)Cy(R_j)$ (which is a path). Let $c_1, c_2 \in V(C)$ such that $c_1Cc_2 = \bigcup_{j=1}^{m} z(R_j)Cy(R_j)$, with z_1, c_1, c_2, y_1 on C in order. So $c_2 \in V(cCy_1 - y_1)$ and, hence, $c_1 \in V(cCy_1 - y_1)$ by (c) and the existence of P. Let $R' = \bigcup_{j=1}^{m} R_j \cup c_1Cc_2$.

By (c), G' has no path from $c_1Cc_2 - \{c_1, c_2\}$ to $V(B' \cup P \cup Q) \cup \{z_1\}$ and internally disjoint from $K \cup X$. By (d), G' has no path from $c_1Cc_2 - \{c_1, c_2\}$ to $A - \{y_1, z_1\}$ and internally disjoint from $K \cup X$.

If $N(x_2) \cap V(R' - \{c_1, c_2\}) \neq \emptyset$ then by (5) and (9), $N(R' - \{c_1, c_2\}) = \{x_1, x_2, y_2, c_1, c_2\}$. Let $z \in V(R') - \{x_1, x_2, c_1, c_2\}$. Since G is 5-connected, R' has independent paths W_1, W_2, W_3, W_4 from z to x_1, x_2, c_2, c_1, respectively. Now $W_1 \cup W_2 \cup (W_3 \cup c_2Cy_1) \cup (W_4 \cup c_1Cz_1 \cup z_1Xy_2) \cup (y_1Aa \cup Q \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z.

So we may assume $N(x_2) \cap V(R' - \{c_1, c_2\}) = \emptyset$. Since G is 5-connected, it follows from (5) that there exist distinct $s_1, s_2 \in V(x_1Xz_1 - z_1) \cap N(R' - \{c_1, c_2\})$. Choose s_1, s_2 such that s_1Xs_2 is maximal and assume that x_1, s_1, s_2, z_1 occur on X in this order. By (6), $\{c_1, c_2, s_1, s_2, y_2\}$ is a 5-cut in G; so G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{c_1, c_2, s_1, s_2, y_2\}$ and $R' \cup c_1Cc_2 \cup s_1Xs_2 \subseteq G_2$. By (6) again,

$(G_2 - y_2, c_1, c_2, s_1, s_2)$ is planar (since G is 5-connected). If $|V(G_2)| \geq 7$ then by Lemma 2.3.8, (i) or (ii) or (iii) holds. So we may assume that $|V(G_2)| = 6$, and we have the assertion of (11).

We may assume that

(12) H has a path Q' from y_1 to some $q' \in V(Q - a)$ and internally disjoint from K.

First, suppose that $y_1 \in V(J(A, C))$. Then, H has a path Q' from y_1 to some $q' \in V(P - c) \cup V(Q - a) \cup V(B)$ internally disjoint from K. We may assume $q' \in V(P - c) \cup V(B)$; for otherwise, $q' \in V(Q - a)$ and the claim holds. If $q' \in V(P - c) \cup V(y_2 Bq - q)$ then $(P - c) \cup (y_2 Bq - q) \cup Q'$ contains a path Q'' from y_1 to y_2; so $z_1 X x_1 \cup z_1 X y_2 \cup C \cup (z_1 A a \cup Q \cup q B z_2 \cup z_2 x_2) \cup Q'' \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. Hence, we may assume $q' \in V(qBz_2 - q)$. Let P_1, P_2 be the paths in (4) with $q^* = q'$. Then $z_2 x_2 \cup z_2 X y_2 \cup (P_1 \cup Q \cup aAz_1 \cup z_1 X x_1) \cup (P_2 \cup Q') \cup (y_1 C c \cup P \cup p B y_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Thus, we may assume that $y_1 \notin V(J(A, C))$. Note that $y_1 \notin V(L(A))$ (by (10)) and $y_1 \notin V(L(C))$ (by (8) and (11)). Hence, since $y_1 y_2 \notin E(G)$ and G is 5-connected, y_1 is contained in some $(A \cup C)$-bridge of H, say D_1, with $D_1 \subseteq L(A, C)$ and $D_1 \neq J(A, C)$. Note that $|V(D_1)| \geq 3$ as A and C are induced paths. For any $(A \cup C)$-bridge D of H with that $D \subseteq L(A, C)$ and $D \neq J(A, C)$, let $a(D) \in V(A) \cap V(D)$ and $c(D) \in V(C) \cap V(D)$ such that $z_1 A a(D)$ and $z_1 C c(D)$ are minimal.

Let $D_1, \ldots, D_k$ be a maximal sequence of $(A \cup C)$-bridges of H with $D_i \subseteq L(A, C)$ (so $D_i \neq J(A, C)$) for $i \in [k]$, such that, for each $i \in [k-1]$, $D_{i+1} \cap (A \cup C)$ is not contained in $\bigcup_{j=1}^{i} (c(D_j) C y_1 \cup a(D_j) A y_1)$, and $D_{i+1} \cap (A \cup C)$ is not contained in $\bigcap_{j=1}^{i} (z_1 C c(D_j) \cup z_1 A a(D_j))$. Note that for any $i \in [k]$, $\bigcup_{j=1}^{i} a(D_j) A y_1$ and $\bigcup_{j=1}^{i} c(D_j) C y_1$ are paths. So let $a_i \in V(A)$ and $c_i \in V(C)$ such that $\bigcup_{j=1}^{i} a(D_j) A y_1 = a_i A y_1$ and $\bigcup_{j=1}^{i} c(D_j) C y_1 = c_i C y_1$. Let $S_i = a_i C y_1 \cup c_i C y_1 \cup \left(\bigcup_{j=1}^{i} D_j \right)$.

Next, we claim that for any $l \in [k]$ and for any $r_l \in V(S_l) - \{a_l, c_l\}$ there exist three independent paths A_l, C_l, R_l in S_l from y_1 to a_l, c_l, r_l, respectively. This is clear when $l = 1$; note that if $a_l = y_1$, or $c_l = y_1$, or $r_l = y_1$ then A_l, or C_l, or R_l is a trivial path. Now assume that the assertion is true for some $l \in [k-1]$. Let $r_{l+1} \in V(S_{l+1}) - \{a_{l+1}, c_{l+1}\}$. When $r_{l+1} \in V(S_l) - \{a_l, c_l\}$ let $r_l := r_{l+1}$; otherwise, let $r_l \in V(D_{l+1})$ with $r_l \in V(a_l A y_1 - a_l) \cup V(c_l C y_1 - c_l)$. By induction hypothesis, there are three independent paths A_l, C_l, R_l in S_l from y_1 to a_l, c_l, r_l, respectively. If $r_{l+1} \in V(S_l) - \{a_l, c_l\}$ then $A_{l+1} := A_l \cup a_l A a_{l+1}, C_{l+1} := C_l \cup c_l C c_{l+1}, R_{l+1} := R_l$ are the desired paths in S_{l+1}. If $r_{l+1} \in V(D_{l+1}) - V(A \cup C)$ then let P_{l+1} be a path in D_{l+1} from r_l to r_{l+1} and internally disjoint from $A \cup C$; we see that $A_{l+1} := A_l \cup a_l A a_{l+1}, C_{l+1} := C_l \cup c_l C c_{l+1}, R_{l+1} := R_l \cup P_{l+1}$ are the desired paths in S_{l+1}. So we may assume by symmetry that $r_{l+1} \in V(a_{l+1} A a_l - a_{l+1})$. Let Q_{l+1} be a path in D_{l+1} from r_l to a_{l+1} and internally disjoint from $A \cup C$. Now $R_{l+1} := A_l \cup a_l A r_{l+1}, C_{l+1} := C_l \cup c_l C c_{l+1}, A_{l+1} := R_l \cup Q_{l+1}$ are the desired paths in S_{l+1}.

We claim that $J(A, C)$ has no vertex in $(a_k A y_1 \cup c_k C y_1) - \{a_k, c_k\}$. For, suppose there exists $r \in V(J(A, C))$ such that $r \in V(a_k A y_1 - a_k) \cup V(c_k C y_1 - c_k)$. Then let A_k, C_k, R_k be independent (induced) paths in S_k from y_1 to a_k, c_k, r, respectively. Let A', C' be obtained from A, C by replacing $a_k A y_1, c_k C y_1$ with A_k, C_k, respectively. We see that $J(A', C')$ contains $J(A, C)$ and r, contradicting (c).

Therefore, $a \in V(z_1 A a_k)$ and $c \in V(z_1 C c_k)$. Moreover, no $(A \cup C)$-bridge of H in $L(A)$ intersects $a_k A y_1 - a_k$ (by (10)). Let S'_k be the union of S_k and all $(A \cup C)$-bridges of H contained in $L(C)$ and intersecting $c_k C y_1 - c_k$. Then by (5) and (11), $N(S'_k - \{a_k, c_k\}) - \{a_k, c_k, x_2, y_2\} \subseteq V(x_1 X z_1)$. Since G is 5-connected, $N(S'_k - \{a_k, c_k\}) - \{a_k, c_k, x_2, y_2\} \neq \emptyset$.

We may assume that $N(S'_k - \{a_k, c_k\}) - \{y_2, x_2, a_k, c_k\} \neq \{x_1\}$. For, otherwise, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a_k, c_k, x_1, x_2, y_2\}$ and $X \cup P \cup Q \subseteq G_1$, and $S'_k \subseteq G_2$. Clearly, $|V(G_1)| \geq 7$. Since G is 5-connected and $y_1 y_2 \notin E(G)$,

$|V(G_2)| \geq 7$. Hence, the assertion follows from Lemma 2.3.9.

Thus, we may let $z \in N(S'_k - \{a_k, c_k\}) - \{a_k, c_k, x_1, x_2, y_2\}$ such that x_1Xz is maximal. Then $z \neq z_1$. For otherwise, let $r \in V(S'_k) - \{a_k, c_k\}$ such that $rz_1 \in E(G)$. Let $r' = r$ if $r \in V(S_k)$ and, otherwise, let $r' \in V(c_kCy_1 - c_k)$ with $r'r \in E(G)$ (which exists by (11)). Let A_k, C_k, R_k be independent (induced) paths in S_k from y_1 to a_k, c_k, r', respectively. Now $z_2Bq \cup Q \cup aAz_1 \cup (z_1rr' \cup R_k) \cup C_k \cup c_kCc \cup P \cup pBy_2$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1).

Let C^* be the subgraph of G induced by the union of $x_1Xz - x_1$ and the vertices of $L(C) - C$ adjacent to $c_kCy_1 - c_k$ (each of which, by (11), has exactly two neighbors on C and exactly two on x_1Xz_1). Clearly, C^* is connected. Let $G_z = G[x_1Xz \cup S'_k + x_2]$ and let G'_z be the graph obtained from $G_z - \{x_1, x_2\}$ by contracting C^* to a new vertex c^*.

Note that G'_z has no disjoint paths from a_k, c_k to c^*, y_1, respectively; as otherwise, such paths, $c_kCc \cup P \cup pBy_2$, and $a_kAa \cup Q \cup qBz_2$ give two disjoint paths in H which would contradict the choice of Y, Z. Hence, by Lemma 2.3.1, there exists a collection $\mathcal{A}$ of subsets of $V(G'_z) - \{a_k, c_k, c^*, y_1\}$ such that $(G'_z, \mathcal{A}, a_k, c_k, c^*, y_1)$ is 3-planar. We choose $\mathcal{A}$ so that each member of $\mathcal{A}$ is minimal and, subject to this, $|\mathcal{A}|$ is minimal.

We claim that $\mathcal{A} = \emptyset$. For, let $T \in \mathcal{A}$. By (10), $T \cap V(L(A)) = \emptyset$. Moreover, $T \cap V(L(C)) = \emptyset$; for otherwise, by (11), $c^* \in N(T)$ and $|N(T) \cap V(C)| = 2$; so by (11) again (and since C is induced in H), $(G'_z, \mathcal{A} - \{T\}, a_k, c_k, c^*, y_1)$ is 3-planar, contradicting the choice of $\mathcal{A}$. Thus, $G[T]$ has a component, say T', such that $T' \subseteq L(A, C)$. Hence, for any $t \in V(T')$, $L(A, C)$ has a path from t to $aAy_1 - y_1$ (respectively, $cCy_1 - y_1$) and internally disjoint from $A \cup C$. Since G is 5-connected, $\{x_1, x_2\} \cap N(T') \neq \emptyset$. Therefore, for some $i \in [2]$, G' contains a path from x_i to $aAy_1 - y_1$ as well as a path from x_i to $cCy_1 - y_1$, both internally disjoint from $K \cup X$. However, this contradicts (9).

Hence, $(G'_z, a_k, c_k, c^*, y_1)$ is planar. So by (6) and (11), $(G_z - x_2, a_k, c_k, z, x_1, y_1)$ is planar. By (9) and (10), $N(x_2) \cap V(S_k) \subseteq V(a_k A y_1)$. Therefore, since $(G_z - x_2) - a_k A y_1$ is connected (by (10)), (G_z, a_k, c_k, z, x_2) is planar.

We claim that $\{a_k, c_k, z, x_2, y_2\}$ is a 5-cut in G. For, otherwise, by (7) and (9), G' has a path S_1 from x_1 to $z_1 C c_k - \{z_1, c_k\}$ and internally disjoint from $K \cup X$. However, G' has a path S_2 from z to $c_k X y_1 - c_k$ and internally disjoint from $K \cup X$. Now S_1, S_2 contradict the second part of (6).

Hence, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a_k, c_k, z, x_2, y_2\}$, $B' \cup P \cup Q \cup X \subseteq G_1$, and $G_z \subseteq G_2$. Clearly, $|V(G_i)| \geq 7$ for $i \in [2]$. So (i) or (ii) or (iii) follows from Lemma 2.3.8.

Now that we have established (12), the remainder of this proof will make heavy use of Q'. Our next goal is to obtain structure around z_1, which is done using claims (13) – (17). We may assume that

(13) $x_1 z_1 \in E(X)$, $w \in V(A) - \{y_1, z_1\}$ for any choice of W in (7), and G' has no path from x_2 to $(A \cup C) - y_1$ and internally disjoint from $K \cup Q' \cup X$.

Let P_1, P_2 be the paths in (4) with $q^* = p$. Suppose $x_1 z_1 \notin E(X)$. Let $x_1 s \in E(X)$. By (6), G has a path S from s to some $s' \in V(C) - \{y_1, z_1\}$ and internally disjoint from $K \cup Q' \cup X$ (as $Q' \subseteq J(A, C)$). Hence, $z_2 x_2 \cup z_2 X y_2 \cup (P_1 \cup qQq' \cup Q') \cup (P_2 \cup P \cup cCs' \cup S \cup sx_1) \cup (A \cup z_1 X y_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Now suppose W is a path in (7) ending at $w \in V(C) - \{y_1, z_1\}$. Then $z_2 x_2 \cup z_2 X y_2 \cup (P_1 \cup qQq' \cup Q') \cup (P_2 \cup P \cup cCw \cup W) \cup (A \cup z_1 X y_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Finally, suppose G' has a path S from x_2 to some $s \in V(A \cup C) - \{y_1\}$ and internally disjoint from $K \cup Q' \cup X$. If $s \in V(A - y_1)$ then $z_1 x_1 \cup z_1 X y_2 \cup C \cup (z_1 A s \cup S) \cup (Q' \cup q'Qq \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. If $s \in V(C - y_1)$ then $z_1 x_1 \cup z_1 X y_2 \cup A \cup (z_1 C s \cup S) \cup (Q' \cup q'Qq \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

(14) We may assume that G' has no path from y_2Xz_2 to $(A \cup C) - y_1$ and internally disjoint from $K \cup Q' \cup X$, and no path from $y_2Xz_1 - z_1$ to $A - z_1$ and internally disjoint from $K \cup Q' \cup X$.

First, suppose S is a path in G' from some $s \in V(y_2Xz_2)$ to some $s' \in V(A \cup C) - \{y_1\}$ and internally disjoint from $K \cup Q' \cup X$. Then $s \neq y_2$ as $N_{G'}(y_2) = \{w_1, w_2, w_3, x_1, x_2\}$. If $s' \in V(C - y_1)$ then $z_1x_1 \cup z_1Xy_2 \cup A \cup (z_1Cs' \cup S \cup sXx_2) \cup (Q' \cup q'Qq \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. If $s' \in V(A - y_1)$ then $z_1x_1 \cup z_1Xy_2 \cup C \cup (z_1As' \cup S \cup sXx_2) \cup (Q' \cup q'Qq \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Now suppose S is a path in G' from $s \in V(y_2Xz_1 - z_1)$ to $s' \in V(A - z_1)$ and internally disjoint from $K \cup Q' \cup X$. Let P_1, P_2 be the paths in (4) with $q^* = p$. Then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQq' \cup Q') \cup (P_2 \cup P \cup cCz_1 \cup z_1x_1) \cup (y_1As' \cup S \cup sXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

(15) We may assume that

- $J(A, C) \cap (z_1Cc - c) = \emptyset$,
- any path in $J(A, C)$ from $A - \{y_1, z_1\}$ to $(P - c) \cup (Q - a) \cup (Q' - y_1) \cup B$ and internally disjoint from $K \cup Q'$ must end on $(Q \cup Q') - q$, and
- for any $(A \cup C)$-bridge D of H with $D \neq J(A, C)$, if $V(D) \cap V(z_1Cc - c) \neq \emptyset$ and $u \in V(D) \cap V(z_1Ay_1 - z_1)$ then $J(A, C) \cap (z_1Au - \{z_1, u\}) = \emptyset$.

First, suppose there exists $s \in V(J(A, C)) \cap V(z_1Cc - c)$. Then H has a path S from s to some $s' \in V(P - c) \cup V(Q - a) \cup V(Q' - y_1) \cup V(B - y_2)$ and internally disjoint from $K \cup Q'$. If $s' \in V(Q' - y_1) \cup V(Q - a) \cup V(z_2Bp - p)$ then $S \cup (Q' - y_1) \cup (Q - a) \cup (z_2Bp - p)$ contains a path S' from s to z_2; so $S' \cup sCz_1 \cup A \cup y_1Cc \cup P \cup pBy_2$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1). Hence, $s' \in V(P - c) \cup V(y_2Bp - y_2)$ and, by (2), $s = z_1$. Let P_1, P_2 be the paths in (4) with $q^* = p$ (if $s' \in V(P - c)$)

or $q^* = s'$ (if $s' \in V(y_2Bp) - \{p, y_2\}$). Then $S \cup (P - c) \cup P_2$ contains a path S' from z_1 to z_2. Let W, w be given as in (7). By (13), $w \in V(A) - \{y_1, z_1\}$. Now $z_2x_2 \cup z_2Xy_2 \cup z_1x_1 \cup z_1Xy_2 \cup S' \cup (P_1 \cup Q \cup aAw \cup W) \cup (C \cup y_1x_2) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_2, z_1, z_2.

Now suppose S is path in $J(A, C)$ from $s \in V(A - \{y_1, z_1\})$ to $s' \in V(P - c) \cup V(B - q)$ and internally disjoint from $K \cup Q'$. Since $N_{G'}(y_2) = \{w_1, w_2, w_3, x_1, x_2\}$, $s' \neq y_2$. Let P_1, P_2 be the paths in (4) with $q^* = p$ (if $s' \in V(P - c)$) or $q^* = s'$ (if $s' \in V(B - q)$). Let S' be a path in $P_2 \cup S \cup (P - c)$ from s to z_2. Let W, w be given as in (7). By (13), $w \in V(A) - \{y_1, z_1\}$. Hence, $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQq' \cup Q') \cup (S' \cup sAw \cup W) \cup (C \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Finally, suppose D is some $(A \cup C)$-bridge of H with $D \neq J(A, C)$, $v \in V(D) \cap V(z_1Cc - c)$, and $u \in V(D) \cap V(z_1Ay_1 - z_1)$. Then D has a path T from v to u and internally disjoint from $K \cup Q'$. If there exists $s \in V(J(A, C)) \cap V(z_1Au - \{z_1, u\})$ then $J(A, C)$ has a path S from s to some $s' \in V(Q - a)$ and internally disjoint from K. Now $z_2Bq \cup qQs' \cup S \cup sAz_1 \cup z_1Cv \cup T \cup uAy_1 \cup y_1Cc \cup P \cup pBy_2$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1).

(16) We may assume $L(A) = \emptyset$.

Suppose $L(A) \neq \emptyset$. For each $(A \cup C)$-bridge R of H contained in $L(A)$, let $a_1(R), a_2(R) \in V(R \cap A)$ with $a_1(R)Aa_2(R)$ maximal. Let $R_1, \dots, R_m$ be a maximal sequence of $(A \cup C)$-bridges of H contained in $L(A)$, such that for $i = 2, \dots, m$, R_i contains an internal vertex of $\bigcup_{j=1}^{i-1}(a_1(R_j)Aa_2(R_j))$ (which is a path). Let $a_1, a_2 \in V(A)$ such that $\bigcup_{j=1}^{m} a_1(R_j)Aa_2(R_j) = a_1Aa_2$. Let $L = \bigcup_{j=1}^{m} R_j$.

By (c), $J(A, C) \cap (a_1Aa_2 - \{a_1, a_2\}) = \emptyset$. By (d), $L(A, C) \cap (a_1Aa_2 - \{a_1, a_2\}) = \emptyset$. By (10), $a_1, a_2 \in V(z_1Aa)$. So $z_1 \notin N(L \cup a_1Aa_2 - \{a_1, a_2\})$. Hence by (14), $V(z_1Xz_2 - y_2) \cap N(L \cup a_1Aa_2 - \{a_1, a_2\}) = \emptyset$. By (13), $x_2 \notin N(L \cup a_1Aa_2 - \{a_1, a_2\})$.

Thus, $\{a_1, a_2, x_1, y_2\}$ is a cut in G separating L from X, which is a contradiction (since G is 5-connected).

(17) $z_1c \in E(C)$, $z_1y_2 \in E(G)$, and z_1 has degree 5 in G.

Let C^* be the union of z_1Cc and all $(A \cup C)$-bridges of H intersecting $z_1Cc - c$. By (15), $V(C^* \cap J(A, C)) = \{c\}$.

Suppose (17) fails. If $C^* = z_1Cc$ then, since A, C are induced paths and $L(A) = \emptyset$ (by (16)), $z_1y_2 \in E(G)$ and $z_1Cc \neq z_1c$; so any vertex of $z_1Cc - \{c, z_1\}$ would have degree 2 in G (by (15)), a contradiction. So $C^* - z_1Cc \neq \emptyset$. Since $G' - X$ is 2-connected, $(C^* - z_1Cc) \cap (A - z_1) \neq \emptyset$ by (c) (and since $J(A.C) \cap \cap (zCc - c) = \emptyset$ by (15)). Moreover, if $|V(z_1Cc)| \geq 3$ then there is a path in C^* from $z_1Cc - \{c, z_1\}$ to $A - z_1$ and internally disjoint from $A \cup C$.

Let $a^* \in V(A \cap C^*)$ with a^*Ay_1 minimal, and let $u \in V(z_1Xy_2)$ with uXy_2 minimal such that u is a neighbor of $(C^* - c) \cup (z_1Aa^* - a^*)$.

We may assume that $\{a^*, c, u, x_1, y_2\}$ is a 5-cut in G. First, note, by (15), that $J(A, C) \cap ((z_1Aa^* - a^*) \cup (z_1Cc - c)) = \emptyset$ (in particular, $a^* \in V(z_1Aa)$). Hence, if $u = z_1$ then it is clear from (d), (13) and (14) that $\{a^*, c, u, x_1, y_2\}$ is a 5-cut in G. So we may assume $u \neq z_1$. Then G' contains a path T from u to $u' \in V(A - z_1)$ and internally disjoint from $A \cup cCy_1 \cup P \cup Q \cup Q' \cup B'$. Suppose $\{a^*, c, u, x_1, y_2\}$ is not a 5-cut in G. Then by (d), (13) and (14), G' has a path R from $r \in V(z_1Xu - u)$ to $r' \in V(P - c) \cup V(Q - a) \cup V(Q' - y_1) \cup V(B')$ and internally disjoint from $K \cup X$. Note that $r' \neq y_2$ as $N_{G'}(y_2) = \{w_1, w_2, w_3, x_1, x_2\}$. If $r' \in V(B' - q)$ then let P_1, P_2 be the paths in (4) with $q^* = r'$; now $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQq' \cup Q') \cup (P_2 \cup R \cup rXx_1) \cup (y_1Au' \cup T \cup uXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2. If $r' \in V(P - c)$ then let P_1, P_2 be the paths in (4) with $q^* = p$; now $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQq' \cup Q') \cup (P_2 \cup pPr' \cup R \cup rXx_1) \cup (y_1Au' \cup T \cup uXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2. Now assume

$r' \in V(Q-a) \cup V(Q'-y_1)$. Then $(Q-a) \cup (Q'-y_1) \cup R$ contains a path R' from r to q. Let P_1, P_2 be the paths in (4) with $q^* = p$; now $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup R' \cup rXx_1) \cup (P_2 \cup P \cup cCy_1) \cup (y_1Au' \cup T \cup uXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2.

Thus, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a^*, c, u, x_1, y_2\}$, $uXx_2 \cup P \cup Q \subseteq G_1$, and $C^* \cup z_1Cc \cup z_1Aa^* \subseteq G_2$. Suppose $G_2 - y_2$ contains disjoint paths T_1, T_2 from u, x_1 to a^*, c, respectively. Let P_1, P_2 be the paths in (4) with $q^* = p$. Then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQq' \cup Q') \cup (P_2 \cup P \cup T_2) \cup (y_1Aa^* \cup T_1 \cup uXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. So we may assume that such T_1, T_2 do not exist. Then by Lemma 2.3.1, $(G_2 - y_2, u, x_1, a^*, c)$ is planar (as G is 5-connected). If $|V(G_2)| \geq 7$ then, by Lemma 2.3.8, (i) or (ii) or (iii) holds. Hence, we may assume that $|V(G_2)| = 6$ and, hence, we have (17).

We have now forced a structure around z_1. Next, we study the structure of $G'[B' \cup y_2Xz_2]$ to complete the proof of Theorem 3.1.1. We may assume that

(18) $(G'[B' \cup y_2Xz_2], p, q, z_2, y_2)$ is 3-planar.

For, otherwise, by Lemma 2.3.1, $G'[B' \cup y_2Xz_2]$ has disjoint paths R_1, R_2 from q, p to y_2, z_2, respectively. Now $z_1x_1 \cup z_1Xy_2 \cup A \cup (z_1Cc \cup P \cup R_2 \cup z_2x_2) \cup (R_1 \cup qQq' \cup Q') \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. So we may assume (18).

Since G is 5-connected, G is $(5, V(K \cup Q' \cup y_2Xx_2 \cup z_1x_1))$-connected. Recall that $w_1y_2 \in E(x_1Xy_2)$. Then w_1y_2 and w_1Xz_1 are independent paths in G from w_1 to y_2, z_1, respectively. So by Lemma 2.3.4, G has five independent paths Z_1, Z_2, Z_3, Z_4, Z_5 from w_1 to z_1, y_2, z_3, z_4, z_5, respectively, and internally disjoint from $K \cup Q' \cup y_2Xx_2 \cup z_1x_1$, where $z_3, z_4, z_5 \in V(K \cup Q' \cup y_2Xx_2 \cup z_1x_1)$. Note that we may assume $Z_2 = w_1y_2$. Hence, Z_1, Z_2, Z_3, Z_4, Z_5 are paths in G'. By the fact that X is induced, by (14), and by (5) and (17), $z_3, z_4, z_5 \in V(P) \cup V(Q - a) \cup V(Q') \cup V(B' - y_2)$. Recall that $L(A) = \emptyset$ from (16), and recall W and w from (7) and (13).

(19) We may assume that at least two of Z_3, Z_4, Z_5 end in $B' - y_2$.

First, suppose at least two of Z_3, Z_4, Z_5 end on P. Without loss of generality, let c, z_3, z_4, p occur on P in this order. Let P_1, P_2 be the paths in (4) with $q^* = p$. Then $(Z_1 \cup z_1x_1) \cup Z_2 \cup z_2x_2 \cup z_2Xy_2 \cup (Z_4 \cup z_4Pp \cup P_2) \cup (Z_3 \cup z_3Pc \cup cCy_1 \cup y_1x_2) \cup (P_1 \cup Q \cup aAw \cup W) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_2, z_2.

Now assume at least two of Z_3, Z_4, Z_5 are on $Q \cup Q'$, say Z_3 and Z_4. Then $Z_3 \cup Z_4 \cup Q \cup Q'$ contains two independent paths Z_3', Z_4' from w_1 to z', q, respectively, where $z' \in \{a, y_1\}$. Hence $(Z_1 \cup z_1x_1) \cup Z_2 \cup (Z_3' \cup z'Ay_1) \cup (Z_4' \cup qBz_2 \cup z_2x_2) \cup (y_2Bp \cup P \cup cCy_1) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_1, y_2.

So we may assume that $z_3 \in V(B') - \{p, q\}$, and hence $Z_3 = w_1z_3$. Suppose none of Z_4, Z_5 ends in $B' - y_2$. Then we may assume $z_4 \in V(P - p)$. Let P_1, P_2 be the paths in (4) with $q^* = z_3$. Then $(Z_1 \cup z_1x_1) \cup Z_2 \cup z_2x_2 \cup z_2Xy_2 \cup (Z_3 \cup P_2) \cup (P_1 \cup Q \cup aAw \cup W) \cup (Z_4 \cup z_4Pc \cup cCy_1 \cup y_1x_2) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_2, z_2.

(20) We may assume that

- w_1 has at most one neighbor in B' that is in qBz_2 or separated from y_2Bp in $G'[B' \cup y_2Xz_2]$ by a 2-cut contained in qBz_2, and
- w_1 has at most one neighbor in B' that is in $y_2Bp - y_2$ or separated from qBz_2 in $G'[B' \cup y_2Xz_2]$ by a 2-cut contained in y_2Bp.

Suppose there exist distinct $v_1, v_2 \in N(w_1) \cap V(B')$ such that for $i \in [2]$, $v_i \in V(qBz_2)$ or $G'[B' \cup y_2Xz_2]$ has a 2-cut contained in qBz_2 and separating v_i from y_2Bp. Then, since $(G'[B' \cup y_2Xz_2], p, q, z_2, y_2)$ is 3-planar (by (18)) and $H - y_2$ is 2-connected, $G'[B' + w_1] - y_2Bp$ contains independent paths S_1, S_2 from w_1 to q, z_2, respectively.

Now $w_1Xx_1 \cup w_1y_2 \cup (S_1 \cup qQq' \cup Q') \cup (S_2 \cup z_2x_2) \cup (y_1Cc \cup P \cup pBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_1, y_2.

Now suppose there exist distinct $v_1, v_2 \in N(w_1) \cap V(B')$ such that for $i \in [2]$, $v_i \in V(y_2Bp)$ or $G'[B' \cup y_2Xz_2]$ has a 2-cut contained in y_2Bp and separating v_i from qBz_2. Then, since $(G'[B' \cup y_2Xz_2], p, q, z_2, y_2)$ is 3-planar (by (18)) and $H - y_2$ is 2-connected, $G'[B' + w_1] - (qBz_2 - z_2)$ has independent paths S_1, S_2 from w_1 to p, z_2, respectively. Now $w_1Xx_1 \cup w_1y_2 \cup z_2x_2 \cup z_2Xy_2 \cup S_2 \cup (S_1 \cup P \cup cCy_1 \cup y_1x_2) \cup (z_2Bq \cup Q \cup aAw \cup W) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_2, z_2.

(21) $G'[B' \cup y_2Xz_2]$ has a 2-separation (B_1, B_2) such that $N(w_1) \cap V(B' - y_2) \subseteq V(B_1)$, $pBq \subseteq B_1$, and $y_2Xz_2 \subseteq B_2$.

Let $z \in N(w_1) \cap V(B')$ be arbitrary. If there exists a path S in $B' - (pBy_2 \cup (qBz_2 - z_2))$ from z_2 to z then $z_2x_2 \cup z_2Xy_2 \cup (z_2Bq \cup qQq' \cup Q') \cup (S \cup zw_1 \cup w_1Xx_1) \cup (y_1Cc \cup P \cup pBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. So we may assume that such path S does not exist. Then, since $(G'[B' \cup y_2Xz_2], p, q, z_2, y_2)$ is 3-planar (by (18)) and $G' - X$ is 2-connected, $z \in V(y_2Xp \cup qBz_2)$ (in which case let $B'_z = z$ and $B''_z = G'[B' \cup y_2Xz_2]$), or $G'[B' \cup y_2Xz_2]$ has a 2-separation (B'_z, B''_z) such that $B'_z \cap B''_z \subseteq y_2Bp \cup qBz_2 \cup y_2Xz_2$, $z \in V(B'_z - B''_z)$ and $z_2 \in V(B''_z - B'_z)$.

We claim that we may assume that w_1 has exactly two neighbors in B', say v_1, v_2, such that $v_1 \in V(y_2Bp - y_2)$ or $G'[B' \cup y_2Xz_2]$ has a 2-cut contained in y_2Bp and separating v_1 from qBz_2, and $v_2 \in V(qBz_2 - z_2)$ or $G'[B' \cup y_2Xz_2]$ has a 2-cut contained in qBz_2 and separating v_2 from y_2Bp. This follows from (20) if for every choice of z, $B'_z \cap B''_z \subseteq y_2Bp$ or $B'_z \cap B''_z \subseteq qBz_2$. So we may assume that there exists $v \in N(w_1) \cap V(B')$ such that $pBq \subseteq B'_v$ and we choose v and (B'_v, B''_v) with B'_v maximal. If $pBq \subseteq B'_z$ for all choices of z then, by (18), we have (21). Thus, we may assume that there exists $z \in N(w_1) \cap V(B')$ such that $pBq \not\subseteq B'_z$ for any choice of (B'_z, B''_z). Then $B'_z \cap B''_z \subseteq y_2Bp$ or $B'_z \cap B''_z \subseteq qBz_2$. First, assume $B'_z \cap B''_z \subseteq qBz_2$.

Then by the maximality of B'_v, $B'-y_2Bp$ has independent paths T_1, T_2 from z_2 to q, z, respectively. Hence, $z_2x_2 \cup z_2Xy_2 \cup (T_1 \cup qQq' \cup Q') \cup (T_2 \cup zw_1 \cup w_1Xx_1) \cup (y_1Cc \cup P \cup pBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. Now assume $B'_z \cap B''_z \subseteq y_2Bp$. Then by (20), for any $t \in N(w_1) \cap V(B'_v)$, $t \notin V(y_2Bp - y_2)$ and $G'[B' \cup y_2Xz_2]$ has no 2-cut contained in y_2Bp and separating t from qBz_2. If for every choice of $t \in N(w_1) \cap V(B'_v)$, we have $t \in V(qBz_2 - z_2)$ or $G'[B' \cup y_2Xz_2]$ has a 2-cut contained in qBz_2 and separating t from y_2Bp then the claim follows from (20). Hence, we may assume that t can be chosen so that $t \notin V(qBz_2 - z_2)$ and $G'[B' \cup y_2Xz_2]$ has no 2-cut contained in qBz_2 and separating t from y_2Bp. Then, by (18) and 2-connectedness of $G' - X$, $G[B' + w_1] - (qBz_2 - z_2)$ has independent paths S_1, S_2 from w_1 to p, z_2, respectively. Now $w_1Xx_1 \cup w_1y_2 \cup z_2x_2 \cup z_2Xy_2 \cup S_2 \cup (S_1 \cup P \cup cCy_1 \cup y_1x_2) \cup (z_2Bq \cup Q \cup aAw \cup W) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_2, z_2.

Thus, we may assume that $Z_3 = w_1v_1$, $Z_4 = w_1v_2$, and Z_5 ends at some $v_3 \in V(P \cup Q \cup Q') - \{a, p, q\}$. Suppose $v_3 \in V(P - p)$. Let P_1, P_2 be the paths in (4) with $q^* = v_1$. Then $w_1Xx_1 \cup w_1y_2 \cup z_2x_2 \cup z_2Xy_2 \cup (w_1v_1 \cup P_2) \cup (Z_5 \cup v_3Pc \cup cCy_1 \cup y_1x_2) \cup (P_1 \cup Q \cup aAw \cup W) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_2, z_2.

Now assume $v_3 \in V(Q \cup Q') - \{a, q\}$. Then $(B' - y_2Bp) \cup Z_5 \cup Q \cup Q' \cup (A - z_1) \cup w_1v_2$ has independent paths R_1, R_2 from w_1 to y_1, z_2, respectively. So $w_1Xx_1 \cup w_1y_2 \cup R_1 \cup (R_2 \cup z_2x_2) \cup (y_1Cc \cup P \cup pBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_1, y_2. This completes the proof of (21).

By (21), let $V(B_1 \cap B_2) = \{t_1, t_2\}$ with $t_1 \in V(y_2Bp)$ and $t_2 \in V(qBz_2)$. Choose $\{t_1, t_2\}$ so that B_2 is minimal. Then we may assume that $(G'[B_2 + x_2], t_1, t_2, x_2, y_2)$ is 3-planar. For, otherwise, by Lemma 2.3.1, $G'[B_2 + x_2]$ contains disjoint paths T_1, T_2 from t_1, t_2 to x_2, y_2, respectively. Then $z_1x_1 \cup z_1Xy_2 \cup A \cup (z_1Cc \cup P \cup pBt_1 \cup T_1) \cup (Q' \cup q'Qq \cup qBt_2 \cup T_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices

x_1, x_2, y_1, y_2, z_1.

Suppose there exists $ss' \in E(G)$ such that $s \in V(z_1 X w_1 - w_1)$ and $s' \in V(B_2) - \{t_1, t_2\}$. Then $s' \notin V(X)$, as X is induced in $G' - x_1x_2$. By (19), (20) and (21), we may assume that $B_1 - qBt_2$ contains a path R from z_3 to p. By the minimality of B_2 and 2-connectedness of $H - y_2$, $(B_2 - t_1) - (y_2 X z_2 - z_2)$ contains independent paths R_1, R_2 from z_2 to s', t_2, respectively. Now $z_2x_2 \cup z_2Xy_2 \cup (R_1 \cup s's \cup sXx_1) \cup (R_2 \cup t_2Bq \cup qQq' \cup Q') \cup (y_1Cc \cup P \cup R \cup z_3w_1y_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Thus, we may assume that ss' does not exist. Since G is 5-connected, $\{t_1, t_2, y_2, x_2\}$ is not a cut. So H has a path T from some $t \in V(y_2Xx_2) - \{y_2, x_2\}$ to some $t' \in V(P \cup Q \cup Q' \cup A \cup C) - \{p, q\}$ and internally disjoint from $K \cup Q'$. By (14), $t' \notin V(A \cup C) - \{y_1\}$.

If $t' \in V(P - p)$ then $z_1x_1 \cup z_1Xy_2 \cup A \cup (z_1Cc \cup cPt' \cup T \cup tXx_2) \cup (Q' \cup q'Qq \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. So we assume $t' \in V(Q \cup Q') - \{a, q\}$.

If $q \neq q'$ or $t' \in V(Q')$ then $(T \cup Q \cup Q') - q$ has a path Q^* from t to y_1; now $z_1x_1 \cup z_1Xy_2 \cup A \cup (z_1Cc \cup P \cup pBz_2 \cup z_2x_2) \cup (Q^* \cup sXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. So assume $q = q'$ and $t' \in V(Q) - \{a, q\}$. Then $z_1x_1 \cup z_1Xy_2 \cup C \cup (z_1Aa \cup aQt' \cup T \cup tXx_2) \cup (Q' \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. ∎

CHAPTER IV

3-VERTICES IN K_4^-

4.1 Main Result

In this section, we prove the following theorem.

Theorem 4.1.1 *Let G be a 5-connected nonplanar graph and $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. Then one of the following holds:*

(i) G contains a TK_5 in which x_1 is not a branch vertex.

(ii) $G - x_1$ contains K_4^-, or G contains K_4^- in which x_1 is of degree 2.

(iii) x_2, y_1, y_2 may be chosen so that for any distinct $z_0, z_1 \in N(x_1) - \{x_2, y_1, y_2\}$, $G - \{x_1v : v \notin \{z_0, z_1, x_2, y_1, y_2\}\}$ contains TK_5.

Similar to our discussion in Section 3.1, we show the relation between Theorem 4.1.1 and case (b) in Section 2.2.

Let H be a 5-connected nonplanar graph not containing K_4^-. If case (b) in Section 2.2 occurs, then there is a connected subgraph M of H such that $G := H/M$ is 5-connected and nonplanar. Furthermore, there exists $\{x_1, x_2, y_1, y_2\} \subseteq V(G)$ such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ with $y_1y_2 \notin E(G)$ and x_1 is the vertex representing the contraction of M.

Let P be a path in $H[V(M) \cup \{y_1, y_2\}]$ from y_1 to y_2 and Q be a path in $H[V(M) \cup \{x_2\}]$ from x_2 to some vertex $v \in V(P) - \{y_1, y_2\}$ independent from P. It is easy to see that P and Q gives three independent paths from v to x_2, y_1, y_2, respectively. By Lemma 2.3.4, there are five independent paths S_1, S_2, S_3, S_4, S_5 in $H[V(M) \cup$

$\{x_2, y_1, y_2, z_0, z_1\}]$ from v to x_2, y_1, y_2, z_0, z_1, respectively, where $z_0, z_1 \in N_G(x_1) - \{x_2, y_1, y_2\}$.

Now we may assume that one of the three results in Theorem 4.1.1 holds. If (i) holds, i.e. G contains a TK_5 in which x_1 is not a branch vertex, then a TK_5 in H can be easily derived from the one in G.

If (ii) holds, then either H itself contains a K_4^- (and furthermore, H contains a TK_5 by J. Ma and X. Yu's result) or it can be reduced to case (a) in Section 2.2.

If (iii) holds, by the existence of the five independent paths S_1, S_2, S_3, S_4, S_5 in $H[V(M) \cup \{x_2, y_1, y_2, z_0, z_1\}]$ from v to x_2, y_1, y_2, z_0, z_1, respectively, then H contains a TK_5.

4.2 Non-separating paths

Note that condition (iii) in Lemma 2.3.8, Lemma 2.3.9 and Lemma 2.3.10 that G has a 5-separation (G'_1, G'_2) such that $V(G'_1 \cap G'_2) = \{a, a_1, a_2, a_3, a_4\}$ and G'_2 is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding a and the edges ab_i for $i \in [4]$. This condition implies that G contains a K_4^- in which a is of degree 2. So in this chapter we only need the weaker versions of these results.

Lemma 4.2.1 *Let G be a 5-connected nonplanar graph and let (G_1, G_2) be a 5-separation in G. Suppose $|V(G_i)| \geq 7$ for $i \in [2]$, $a \in V(G_1 \cap G_2)$, and $(G_2 - a, V(G_1 \cap G_2) - \{a\})$ is planar. Then one of the following holds:*

(i) G contains a TK_5 in which a is not a branch vertex.

(ii) $G - a$ contains K_4^-, or G contains a K_4^- in which a is of degree 2.

Lemma 4.2.2 *Let G be a 5-connected graph and (G_1, G_2) be a 5-separation in G. Suppose that $|V(G_i)| \geq 7$ for $i \in [2]$ and $G[V(G_1 \cap G_2)]$ contains a triangle aa_1a_2a. Then one of the following holds:*

(*i*) *G contains a* TK_5 *in which* a *is not a branch vertex.*

(*ii*) $G - a$ *contains* K_4^-, *or* G *contains a* K_4^- *in which* a *is of degree 2.*

(*iii*) *For any distinct* $u_1, u_2, u_3 \in N(a) - \{a_1, a_2\}$, $G - \{av : v \notin \{a_1, a_2, u_1, u_2, u_3\}\}$ *contains* TK_5.

Lemma 4.2.3 *Let* G *be a* 5*-connected nonplanar graph and* $a \in V(G)$ *such that* $G - a$ *is planar. Then one of the following holds:*

(*i*) *G contains a* TK_5 *in which* a *is not a branch vertex.*

(*ii*) $G - a$ *contains* K_4^-, *or* G *contains a* K_4^- *in which* a *is of degree 2.*

Let G be a 5-connected nonplanar graph and $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. To prove Theorem 4.1.1, we need to find a path in G satisfying certain properties (see (*iii*) and (*iv*) of Lemma 4.2.5). As a first step, we prove the following

Lemma 4.2.4 *Let* G *be a* 5*-connected nonplanar graph and* $x_1, x_2, y_1, y_2 \in V(G)$ *be distinct such that* $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ *and* $y_1y_2 \notin E(G)$. *Let* $z_0, z_1 \in N(x_1) - \{x_2, y_1, y_2\}$ *be distinct. Then one of the following holds:*

(*i*) *G contains a* TK_5 *in which* x_1 *is not a branch vertex.*

(*ii*) $G - x_1$ *contains* K_4^-, *or* G *contains a* K_4^- *in which* x_1 *is of degree 2.*

(*iii*) *There exist* $i \in \{0, 1\}$ *and an induced path* X *in* $G - x_1$ *from* z_i *to* x_2 *such that* $(G - x_1) - X$ *is a chain of blocks from* y_1 *to* y_2, $z_{1-i} \notin V(X)$, *and one of* y_1, y_2 *is contained in a nontrivial block of* $(G - x_1) - X$.

Proof. We may assume $G - x_1$ contains disjoint paths X, Y from z_1, y_1 to x_2, y_2, respectively. For, otherwise, since G is 5-connected, it follows from Lemma 2.3.1 that $(G - x_1, z_1, y_1, x_2, y_2)$ is planar; so (*i*) or (*ii*) holds by Lemma 4.2.3.

Hence $(G - x_1) - X$ contains a chain of blocks from y_1 to y_2, say B. We may assume that $(G - x_1) - X$ is a chain of blocks from y_1 to y_2. For otherwise, we may apply Lemma 3.2.1 to conclude that G has a 5-separation (G_1, G_2) such that $x_1 \in V(G_1 \cap G_2)$, $B + \{x_1, x_2, z_1\} \subseteq G_1$, $|V(G_2)| \geq 7$, and $(G_2 - x_1, V(G_1 \cap G_2) - \{x_1\})$ is planar. If $|V(G_1)| \geq 7$ then (i) or (ii) follows from Lemma 4.2.1. So assume $|V(G_1)| \leq 6$. Since $y_1y_2 \notin E(G)$, $|V(G_1)| = 6$ and $|V(B)| = 3$. Let $V(B) = \{y_1, y_2, v\}$. Since G is 5-connected and $y_1y_2 \notin E(G)$, $y_1, y_2 \in V(G_1 \cap G_2) = N(v)$. Hence, $G[\{v, x_1, x_2, y_1\}] - x_1x_2$ is a K_4^- in which x_1 is of degree 2, and (ii) holds.

We may further assume that $z_0 \notin V(X)$. For, suppose $z_0 \in V(X)$. Since G is 5-connected and X is induced in $G - x_1$, every vertex of X has at least two neighbors in $(G - x_1) - X$. Hence, $(G - x_1) - z_0Xx_2$ is also a chain of blocks from y_1 to y_2. So we can simply use z_0Xx_2 as X.

Let B_1, B_2 be the blocks in $(G - x_1) - X$ containing y_1, y_2, respectively. If one of B_1, B_2 is nontrivial, then (iii) holds. So we may assume that $|V(B_1)| = |V(B_2)| = 2$. Since X is induced and G is 5-connected, there exists $z \in N(x_2) - (\{x_1, y_1, y_2\} \cup V(X))$, and y_1 and y_2 each have at least two neighbors on $X - x_2$. Let Z be a path in $(G - x_1) - X - \{y_1, y_2\}$ from z_0 to z. Then y_1 and y_2 are each contained in a nontrivial block of $(G - x_1) - Z$. So $(G - x_1) - Z$ contains a chain of blocks, say B, from y_1 to y_2, and the blocks in $(G - x_1) - Z$ containing y_1, y_2 are nontrivial. Thus, we may apply Lemma 3.2.1 to G, Z and B. If (ii) of Lemma 3.2.1 holds, we have (iii). So assume (i) of Lemma 3.2.1 holds. Then, as in the second paragraph of this proof, (i) or (ii) follows from Lemma 4.2.1. ∎

We may assume that (iii) of Lemma 4.2.4 holds and parts (iii) and (iv) of the next lemma give more detailed structure of G. We refer the reader to Figure 2 for (iii) of Lemma 4.2.5, and Figure 3 for (iv) of Lemma 4.2.5.

Lemma 4.2.5 *Let G be a 5-connected nonplanar graph and $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. Let $z_0, z_1 \in N(x_1) -$*

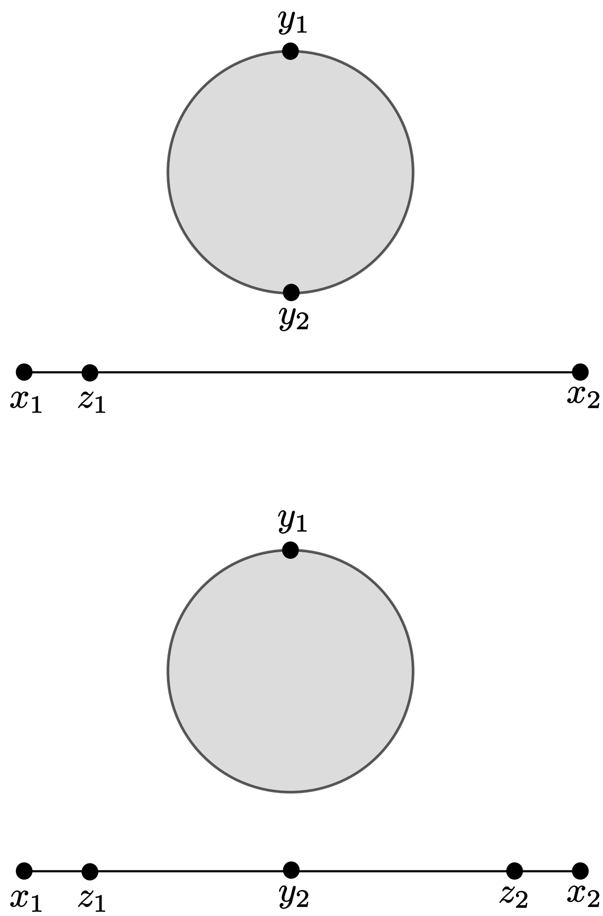

Figure 2: Structure of G in (iii) of Lemma 4.2.5.

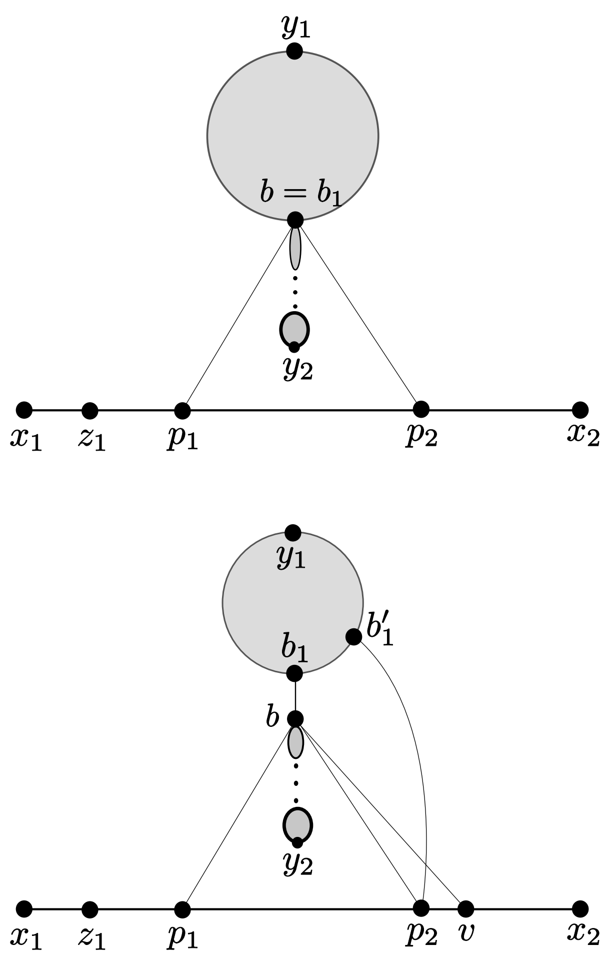

Figure 3: Structure of G in (iv) of Lemma 4.2.5.

$\{x_2, y_1, y_2\}$ be distinct and let $G' := G - \{x_1x : x \notin \{x_2, y_1, y_2, z_0, z_1\}\}$. Then one of the following holds:

(i) G' contains TK_5, or G contains a TK_5 in which x_1 is not a branch vertex.

(ii) $G - x_1$ contains K_4^-, or G contains a K_4^- in which x_1 is of degree 2.

(iii) The notation of z_0, z_1 may be chosen so that $(G - x_1) - x_2y_2$ has an induced path X from z_1 to x_2 such that $z_0, y_1 \notin V(X)$, and $(G - x_1) - X$ is 2-connected.

(iv) The notation of z_0, z_1 may be chosen so that there exists an induced path X in $G - x_1$ from z_1 to x_2 such that $z_0 \notin V(X)$, $(G - x_1) - X$ is a chain of blocks $B_1, \ldots, B_k$ from y_1 to y_2 with B_1 nontrivial, $z_0 \in V(B_1)$ when z_1 has at least two neighbors in B_1, and $(G - x_1) - x_2y_2$ has a 3-separation (Y_1, Y_2) such that $V(Y_1 \cap Y_2) = \{b, p_1, p_2\}$, z_1, p_1, p_2, x_2 occur on X in this order, $Y_1 = G[B_1 \cup z_1Xp_1 \cup p_2Xx_2 + b]$, $p_1Xp_2 + y_2 \subseteq Y_2$, and p_1, p_2 each have at least two neighbors in $Y_2 - B_1$. Moreover, if $b \notin V(B_1)$ then $V(B_2) = \{b_1, b\}$ with $b_1 \in V(B_1)$, and there exists some $j \in [2]$ such that p_{3-j} has a unique neighbor b_1' in B_1, b has a unique neighbor v in X such that $vp_{3-j} \in E(X) - E(p_1Xp_2)$, $vb_1 \notin E(G)$ and $p_jb \notin E(G)$.

Proof. We begin our proof by applying Lemma 4.2.4 to G, x_1, x_2, y_1, y_2. If (i) or (ii) of Lemma 4.2.4 holds then assertion (i) or (ii) of this lemma holds. So we may assume that (iii) of Lemma 4.2.4 holds. Then $(G - x_1) - x_2y_2$ has an induced path X from z_1 to x_2 such that $z_0, y_1 \notin V(X)$, $(G - x_1) - X$ has a nontrivial block B_1 containing y_1, and y_1 is not a cut vertex of $(G - x_1) - X$. (Note that we are not requiring the stronger condition that $y_2 \notin V(X)$ or $(G - x_1) - X$ be a chain of blocks.) We choose such a path X that

(1) B_1 is maximal,

(2) subject to (1), whenever possible, $(G - x_1) - X$ has a chain of blocks from y_1 to y_2 and containing B_1, and

(3) subject to (2), the component H of $(G - x_1) - X$ containing B_1 is maximal.

Let $\mathcal{C}$ be the set of all components of $(G - x_1) - X$ different from H. Then

(4) $\mathcal{C} = \emptyset$, and if $y_2 \notin V(X)$ then $H = (G - x_1) - X$ and H is a chain of blocks from y_1 to y_2 and containing B_1.

First, suppose $\mathcal{C} = \emptyset$. Then $H = (G - x_1) - X$. Suppose $y_2 \notin V(X)$. Then H has a chain of blocks, say B, from y_1 to y_2 and containing B_1. By Lemma 3.2.1, (4) holds, or G has a 5-separation (G_1, G_2) such that $x_1 \in V(G_1 \cap G_2)$, $B + \{x_1, x_2, z_1\} \subseteq G_1$, $|V(G_2)| \geq 7$ and $(G_2 - x_1, V(G_1 \cap G_2) - \{x_1\})$ is planar. Thus we may assume the latter. Since $y_1 y_2 \notin E(G)$, $|V(B)| \geq 3$. So $|V(G_1)| \geq 6$. If $|V(G_1)| = 6$ then, since $y_1 y_2 \notin E(G)$ and G is 5-connected, $y_1, y_2, z_1 \in V(G_1 \cap G_2)$ and there exists $v \in V(G_1) - V(G_2)$ such that $N(v) = V(G_1 \cap G_2)$; now $G[\{v, x_1, x_2, y_1\}] - x_1 y_1$ is a K_4^- in which x_1 is of degree 2, and (ii) holds. So we may assume $|V(G_1)| \geq 7$. Then (i) or (ii) follows from Lemma 4.2.1 again.

Now suppose $\mathcal{C} \neq \emptyset$. For each $D \in \mathcal{C}$, let $u_D, v_D \in V(X)$ be the neighbors of D in $G - x_2 y_2$ with $u_D X v_D$ maximal such that z_1, u_D, v_D, x_2 occur on X in this order. Define a new graph $G_{\mathcal{C}}$ such that $V(G_{\mathcal{C}}) = \mathcal{C}$, and two components $C, D \in \mathcal{C}$ are adjacent in $G_{\mathcal{C}}$ if $u_C X v_C - \{u_C, v_C\}$ contains a neighbor of D or $u_D X v_D - \{u_D, v_D\}$ contains a neighbor of C.

Note that, for any component $\mathcal{D}$ of $G_{\mathcal{C}}$, $\bigcup_{D \in V(\mathcal{D})} u_D X v_D$ is a subpath of X. Since G is 5-connected, there exist $y \in V(H)$ and $C \in V(\mathcal{D})$ with $N(y) \cap V(u_C X v_C - \{u_C, v_C\}) \neq \emptyset$.

If $y \neq y_1$ then let Q be an induced path in $G[C + \{u_C, v_C\}] - x_2 y_2$ from u_C to v_C, and let X' be obtained from X by replacing $u_C X v_C$ with Q. Then B_1 is contained in a block of $(G - x_1) - X'$, and y_1 is not a cut vertex of $(G - x_1) - X'$. Moreover, if

$(G-x_1)-X$ has a chain of blocks from y_1 to y_2 then so does $(G-x_1)-X'$. However, the component of $(G-x_1)-X'$ containing B_1 is larger than H, contradicting (3).

So we may assume that $y = y_1$ for all choices of y and C. Let $uXv := \bigcup_{D\in V(\mathcal{D})} u_D X v_D$. Since G is 5-connected, $y_2 \in V(\bigcup_{D\in V(\mathcal{D})} D) \cup V(uXv - \{u,v\})$ and G has a separation (G_1, G_2) such that $V(G_1\cap G_2) = \{u, v, x_1, x_2, y_1\}$, $G_1 := G[\bigcup_{D\in V(\mathcal{D})} D \cup uXv + \{x_1, x_2, y_1\}]$, and $B_1 \cup z_1Xu \cup vXx_2 \subseteq G_2$. Clearly, $|V(G_i)| \geq 7$ for $i \in [2]$. Since $G[\{x_1, x_2, y_1\}] \cong K_3$, (i) or (ii) follows from Lemma 4.2.2. This completes the proof of (4).

Let $\mathcal{B}$ be the set of all B_1-bridges of H. For each $D \in \mathcal{B}$, let $b_D \in V(D) \cap V(B_1)$ and $u_D, v_D \in V(X)$ be the neighbors of D in $G - x_2y_2$ with u_DXv_D maximal. Define a new graph $G_{\mathcal{B}}$ such that $V(G_{\mathcal{B}}) = \mathcal{B}$, and two B_1-bridges $C, D \in \mathcal{B}$ are adjacent in $G_{\mathcal{B}}$ if $u_CXv_C - \{u_C, v_C\}$ contains a neighbor of $D - b_D$ or $u_DXv_D - \{u_D, v_D\}$ contains a neighbor of $C - b_C$. Note that, for any component $\mathcal{D}$ of $G_{\mathcal{B}}$, $\bigcup_{D\in V(\mathcal{D})} u_DXv_D$ is a subpath of X, whose ends are denoted by $u_{\mathcal{D}}, v_{\mathcal{D}}$. We let $S_{\mathcal{D}} := \{b_D : D \in V(\mathcal{D})\} \cup (N(u_{\mathcal{D}}Xv_{\mathcal{D}} - \{u_{\mathcal{D}}, v_{\mathcal{D}}\}) \cap V(B_1))$. We may assume that

(5) for any component $\mathcal{D}$ of $G_{\mathcal{B}}$, $|S_{\mathcal{D}}| \leq 2$ and $y_2 \in \left(\bigcup_{D\in V(\mathcal{D})} V(D)\right) \cup V(u_{\mathcal{D}}Xv_{\mathcal{D}}) - (\{u_{\mathcal{D}}, v_{\mathcal{D}}\} \cup S_{\mathcal{D}})$.

First, we may assume $|S_{\mathcal{D}}| \leq 2$. For, suppose $|S_{\mathcal{D}}| \geq 3$. Then there exist $D \in V(\mathcal{D})$, $r_1, r_2 \in V(u_DXv_D) - \{u_D, v_D\}$, and distinct $r_1', r_2' \in V(B_1)$ such that for $i \in [2]$, $r_ir_i' \in E(G)$ or $r_i' \in V(D_i)$ for some $D_i \in V(\mathcal{D}) - \{D\}$. (To see this, we choose $D \in V(\mathcal{D})$ such that there is a maximum number of vertices in B_1 from which G has a path to $u_DXv_D - \{u_D, v_D\}$ and internally disjoint from $B_1 \cup D \cup X$. If this number is at most 1, we can show that $|S_{\mathcal{D}}| \leq 2$.) Let $R_i = r_ir_i'$ if $r_ir_i' \in E(G)$; and otherwise let R_i be a path in $G[D_i + r_i]$ from r_i to r_i' and internally disjoint from X. Let Q denote an induced path in $G[D + \{u_D, v_D\}] - b_D - x_2y_2$ between u_D and v_D, and let X' be obtained from X by replacing u_DXv_D with Q. Clearly, the block of

$(G-x_1)-X'$ containing y_1 contains B_1 as well as the path $R_1 \cup r_1Xr_2 \cup R_2$. Note that $y_1 \neq b_D$ (as y_1 is not a cut vertex in H). Moreover, if $y_1 = r_i'$ for some $i \in [2]$ then D_i is not defined and $r_ir_i' \in E(G)$. So y_1 is not a cut vertex of $(G-x_1)-X'$. Thus, X' contradicts the choice of X, because of (1).

Now assume $y_2 \notin \bigcup_{D \in V(\mathcal{D})} V(D) \cup V(u_{\mathcal{D}}Xv_{\mathcal{D}}) - (\{u_{\mathcal{D}}, v_{\mathcal{D}}\} \cup S_{\mathcal{D}})$. Then $S_{\mathcal{D}} \cup \{u_{\mathcal{D}}, v_{\mathcal{D}}, x_1\}$ is a cut in G; so $|S_{\mathcal{D}}| = 2$ (as G is 5-connected). Let $S_{\mathcal{D}} = \{p, q\}$. Then G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{p, q, u_{\mathcal{D}}, v_{\mathcal{D}}, x_1\}$, $B_1 \cup z_1Xu_{\mathcal{D}} \cup v_{\mathcal{D}}Xx_2 \subseteq G_1$, and G_2 contains $u_{\mathcal{D}}Xv_{\mathcal{D}}$ and the B_1-bridges of H contained in $\mathcal{D}$. If $(G_2 - x_1, u_{\mathcal{D}}, p, v_{\mathcal{D}}, q)$ is planar then, since $|V(G_i)| \geq 7$ for $i \in [2]$, the assertion of this lemma follows from Lemma 4.2.1. So we may assume that $(G_2 - x_1, u_{\mathcal{D}}, p, v_{\mathcal{D}}, q)$ is not planar. Then by Lemma 2.3.1, $G_2 - x_1$ contains disjoint paths S, T from $u_{\mathcal{D}}, p$ to $v_{\mathcal{D}}, q$, respectively.

We apply Lemma 3.2.1 to $G_2 - x_1$ and $\{u_{\mathcal{D}}, v_{\mathcal{D}}, p, q\}$. If (i) of Lemma 3.2.1 holds then from the separation in $G_2 - x_1$, we derive a 5-separation (G_1', G_2') in G such that $x_1 \in V(G_1' \cap G_2')$, $B_1 \cup T + x_1 \subseteq G_1'$, $|V(G_2')| \geq 7$, and $(G_2' - x_1, V(G_1' \cap G_2') - \{x_1\})$ is planar. So (i) or (ii) follows from Lemma 4.2.1. We may thus assume that (ii) of Lemma 3.2.1 holds. Thus, there is an induced path S' in $G_2 - x_1$ from $u_{\mathcal{D}}$ to $v_{\mathcal{D}}$ such that $(G_2 - x_1) - S'$ is a chain of blocks from p to q. Now let X' be obtained from X by replacing $u_{\mathcal{D}}Xv_{\mathcal{D}}$ with S'. Then y_1 is not a cut vertex of $(G-x_1)-X'$, and the block of $(G-x_1)-X'$ containing y_1 contains B_1 and $(G_2 - x_1) - S'$, contradicting (1). This completes the proof of (5).

We may also assume that

(6) for any B_1-bridge D of H, $y_2 \notin V(u_DXv_D) - \{u_D, v_D\}$.

For, suppose $y_2 \in V(u_DXv_D) - \{u_D, v_D\}$ for some B_1-bridge D of H. Choose X and D so that, subject to (1)-(3), u_DXv_D is maximal.

We claim that $\{D\}$ is a component of $G_{\mathcal{B}}$. For, otherwise, by the maximality of

u_DXv_D, there exists a B_1-bridge C of H such that $N(C) \cap V(u_DXv_D - \{u_D, v_D\}) \neq \emptyset$. Let T be an induced path in $G[D + \{u_D, v_D\}] - b_D - x_2y_2$ from u_D to v_D. By replacing u_DXv_D with T we obtain a path X' from X such that y_1 is not a cut vertex in $(G - x_1) - X'$, B_1 is contained in a block of $(G - x_1) - X'$, and $(G - x_1) - X'$ has a chain of blocks from y_1 to y_2 and containing B_1, contradicting the choice of X (in (2) as $y_2 \in V(X)$).

Hence, by (5), $V(G_{\mathcal{B}}) = \{D\}$. If G has an edge from $u_DXv_D - \{u_D, v_D\}$ to $B_1 - y_1$ or if y_1 has two neighbors, one on $u_DXy_2 - u_D$ and one on $v_DXy_2 - v_D$, then let X' be obtained from X by replacing u_DXv_D with an induced path in $G[D + \{u_D, v_D\}] - b_D - x_2y_2$ from u_D to v_D. In the former case, $(G - x_1) - X'$ has a chain of blocks from y_1 to y_2 and containing B_1, contradicting (2). In the latter case, $(G - x_1) - X'$ has a cycle containing $\{y_1, y_2\}$. So by Lemmas 3.2.1 and 4.2.1, (i) or (ii) holds, or there is an induced path X^* in $G - x_1$ from z_1 to x_2 such that $y_1, y_2 \notin V(X^*)$ and $(G - x_1) - X^*$ is 2-connected, and (iii) holds.

Therefore, we may assume $N(u_DXv_D - \{u_D, v_D\}) \cap V(B_1) = \{y_1\}$, and $N(y_1) \cap V(u_DXv_D - \{u_D, v_D\}) \subseteq V(u_DXy_2)$ or $N(y_1) \cap V(u_DXv_D - \{u_D, v_D\}) \subseteq V(v_DXy_2)$. Let $L = G[D \cup u_DXv_D]$ and let $L' = G[L + y_1]$.

Suppose L has disjoint paths from u_D, b_D to v_D, y_2, respectively. We may apply Lemma 3.2.1 to L and $\{u_D, v_D, b_D, y_2\}$. If L has an induced path S from u_D to v_D such that $L - S$ is a chain of blocks from b_D to y_2 then let X' be obtained from X by replacing u_DXv_D with S; now $(G - x_1) - X'$ is a chain of blocks from y_1 to y_2 and containing B_1, contradicting (2). So we may assume that L has a 4-separation as given in (i) of Lemma 3.2.1. Thus G has a 5-separation (G_1, G_2) such that $x_1 \in V(G_1 \cap G_2)$, $|V(G_i)| \geq 2$ for $i \in [2]$, and $(G_2 - x_1, V(G_1 \cap G_2) - \{x_1\})$ is planar. Hence, (i) or (ii) follows from Lemma 4.2.1.

Thus, we may assume that such disjoint paths do not exist in L. By Lemma 2.3.1, there exists a collection $\mathcal{A}$ of subsets of $V(L) - \{b_D, u_D, v_D, y_2\}$ such that $(L, \mathcal{A}, u_D, b_D, v_D, y_2)$

is 3-planar.

We now show that $(L' - y_1v_D, u_D, b_D, v_D, y_2, y_1)$ is planar (when $N(y_1) \cap V(u_DXv_D - \{u_D, v_D\}) \subseteq V(u_DXy_2)$), or $(L' - y_1u_D, u_D, b_D, v_D, y_1, y_2)$ is planar (when $N(y_1) \cap V(u_DXv_D - \{u_D, v_D\}) \subseteq V(v_DXy_2)$). Since the arguments for these two cases are the same, we consider only the case $N(y_1) \cap V(u_DXv_D - \{u_D, v_D\}) \subseteq V(u_DXy_2)$. Since G is 5-connected, for each $A \in \mathcal{A}$, $\{x_1, y_1\} \subseteq N(A)$ and $|N_L(A)| = 3$; and since $N(y_1) \cap V(u_DXv_D - \{u_D, v_D\}) \subseteq V(u_DXy_2)$, $|N_L(A) \cap V(X)| = 2$. For each such A, let $a_1, a_2 \in N_L(A) \cap V(X)$ and let $a \in N_L(A) - V(X)$. If $(G[A \cup N_L(A) \cup \{y_1\}], a_1, a, a_2, y_1)$ is planar, for any choice $A \in \mathcal{A}$, then $(L' - y_1v_D, u_D, b_D, v_D, y_2, y_1)$ is planar. So we may assume that, for some choice of A, $(G[A \cup N_L(A) \cup \{y_1\}], a_1, a, a_2, y_1)$ is not planar. (Note that $G[A \cup N_L(A) \cup \{y_1\}]$ is $(4, N_L(A) \cup \{y_1\})$-connected.) Hence, by Lemma 2.3.1, $G[A \cup N_L(A) \cup \{y_1\}]$ contains disjoint paths from a_1, a to a_2, y_1, respectively. So we can apply Lemma 3.2.1 to $G[A \cup N_L(A) \cup \{y_1\}]$ and $\{a, a_1, a_2, y_1\}$. If (i) of Lemma 3.2.1 occurs then G has a 5-separation (G_1, G_2) such that $x_1 \in V(G_1 \cap G_2)$, $|V(G_i)| \geq 5$ for $i \in [2]$, and $(G_2 - x_1, V(G_1 \cap G_2) - \{x_1\})$ is planar; so (i) or (ii) follows from Lemma 4.2.1. Hence, we may assume that (ii) of Lemma 3.2.1 occurs. Then $G[A \cup N_L(A) \cup \{y_1\}]$ has an induced path S from a_1 to a_2 such that $G[A \cup N_L(A) \cup \{y_1\}] - S$ is a chain of blocks from y_1 to a. Let X' be obtained from X by replacing a_1Xa_2 with S. Then the block of $(G - x_1) - X'$ containing y_1 contains B_1 and $G[A \cup N_L(A) \cup \{y_1\}] - S$, and y_1 is not a cut vertex in $(G - x_1) - X'$, contradicting (1).

Hence, G has a 6-separation (G_1, G_2) with $V(G_1 \cap G_2) = \{b_D, u_D, v_D, x_1, y_1, y_2\}$ and $G_2 - x_1 = L' - y_1v_D$ (or $G_2 - x_1 = L' - y_1u_D$). Since $(L' - y_1v_D, u_D, b_D, v_D, y_2, y_1)$ (or $(L' - y_1u_D, u_D, b_D, v_D, y_1, y_2)$) is planar and $|V(G_2)| \geq 8$, the assertion follows from Lemma 2.3.12 (and then Lemma 4.2.1). This completes the proof of (6).

If $y_2 \in V(X)$ then by (4), (5) and (6), H is 2-connected; so (iii) holds. Thus we may assume $y_2 \notin V(X)$. Then by (4), H is a chain of blocks from y_1 to y_2 and containing B_1, which we denote as $B_1 \dots B_k$. We may assume $k \geq 2$; as otherwise, (iii) holds. Let $y_1 \in V(B_1) - V(B_2)$, $y_2 \in V(B_k) - V(B_{k-1})$, and $b_i \in V(B_i) \cap V(B_{i+1})$ for $i \in [k-1]$. Note that

- if z_1 has at least two neighbors in B_1 then $z_0 \in V(B_1)$.

For, suppose z_1 has at least two neighbors in B_1 and $z_0 \notin V(B_1)$. Let $w \in V(X)$ with wXx_2 minimal such that w is a neighbor of $\bigcup_{i=2}^{k} B_i - b_1$ in $G - x_2y_2$. Recall that $z_0 \notin V(X)$. Let W be an induced path in $G[(\bigcup_{i=2}^{k} B_i) + w - b_1] - x_2y_2$ from z_0 to w, and let $X' = W \cup wXx_2$. Then, since y_1 is not a cut vertex of H, y_1 is not a cut vertex of $(G - x_1) - X'$. However, the block of $(G - x_1) - X'$ containing y_1 contains $B_1 + z_1$, contradicting (1).

We further choose X so that, subject to (1), (2) and (3),

(7) B_k is maximal.

Let $q_1, q_2 \in V(X)$ be the neighbors of $\bigcup_{i=2}^{k} B_i - b_1$ in $G - x_2y_2$ with q_1Xq_2 maximal, and assume that z_1, q_1, q_2, x_2 occur on X in this order. We may assume that

(8) there exists $b_1' \in V(B_1 - b_1)$ such that $N(q_1Xq_2 - \{q_1, q_2\}) \cap V(B_1 - b_1) = \{b_1'\}$.

For, otherwise, by (5), $N(q_1Xq_2 - \{q_1, q_2\}) \cap V(B_1 - b_1) = \emptyset$. Hence, (iv) holds with $b = b_1$, $p_1 = q_1$, and $p_2 = q_2$.

Thus G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b_1, b_1', q_1, q_2, x_1, y_2\}$, $G_1 = G[(B_1 \cup z_1Xq_1 \cup q_2Xx_2) + \{x_1, y_2\}]$ and G_2 contains $\bigcup_{i=2}^{k} B_i$ and q_1Xq_2. Note that $xy \notin E(G_2)$ for any pair of $\{x, y\} \subseteq \{b_1, b_1', q_1, q_2\}$, and $x_2y_2 \notin E(G_2)$. We may assume that

(9) there exists a collection $\mathcal{A}$ of subsets of $V(G_2 - x_1) - \{b_1, b_1', q_1, q_2\}$ such that $(G_2 - x_1, \mathcal{A}, b_1, q_1, b_1', q_2)$ is 3-planar.

For, otherwise, by Lemma 2.3.1, $G_2 - x_1$ has disjoint paths S, S' from b_1, q_1 to b_1', q_2, respectively. We may choose S' to be induced and let X' be obtained from X by replacing $q_1 X q_2$ with S'. Then $B_1 \cup S$ is contained in a block of $(G - x_1) - X'$. Thus, by (1), $y_1 = b_1'$ and y_1 is a cut vertex of $(G - x_1) - X'$.

Suppose $G_2 - x_1$ is $(4, \{b_1, b_1', q_1, q_2\})$-connected. Applying Lemma 3.2.1 (and then Lemma 4.2.1) to $G_2 - x_1$ and $\{q_1, q_2, b_1, b_1'\}$, we may assume that there is an induced path S^* in $G_2 - x_1$ from q_1 to q_2 such that $(G_2 - x_1) - S^*$ is a chain of blocks. Let X^* be obtained from X by replacing $q_1 X q_2$ with S^*. Then B_1 is properly contained in a block of $(G - x_1) - X^*$, and y_1 is not a cut vertex of $(G - x_1) - X^*$. This contradicts (1).

Thus, $G_2 - x_1$ is not $(4, \{b_1, b_1', q_1, q_2\})$-connected. Since G is 5-connected and y_2 is the only vertex in $V(G_2) - \{b_1, b_1', q_1, q_2, x_1\}$ adjacent to x_2, $G_2 - x_1$ has a 3-cut T separating y_2 from $\{b_1, b_1', q_1, q_2\}$. Choose T so that the component J of $(G_2 - x_1) - T$ containing y_2 is maximal. Let G_2' be obtained from $G_2 - J$ by adding an edge between every pair of vertices in T. Then $G_2' - x_1$ is $(4, \{b_1, b_1', q_1, q_2\})$-connected, and the paths S, S' also give rise to disjoint paths in $G_2' - x_1$ from b_1, q_1 to b_1', q_2, respectively. Hence by applying Lemma 3.2.1 (and then Lemma 4.2.1) to $G_2' - x_1$ and $\{q_1, q_2, b_1, b_1'\}$, we find an induced path S'' in $G_2' - x_1$ from q_1 to q_2 such that $(G_2' - x_1) - S''$ is a chain of blocks from b_1 to b_1'. Note that S'' gives rise to an induced path S^* in G_2 by replacing $S'' \cap G_2'[T]$ with an induced path in $G_2[J + T]$. Let X^* be obtained from X by replacing $q_1 X q_2$ with S^*. Then B_1 is properly contained in a block of $(G - x_1) - X^*$. Since $y_2 \notin V(X)$, $b_1' \notin T \cup V(J)$. Hence, y_1 is not a cut vertex in $(G - x_1) - X^*$. Thus, we have a contradiction to (1) which completes the proof of (9).

We may assume that, for any choice of $\mathcal{A}$ in (9),

(10) $\mathcal{A} \neq \emptyset$.

For, otherwise, $G_2 - x_1$ has no cut of size at most 3 separating y_2 from $\{b_1, b_1', q_1, q_2\}$. Hence, G_2 is $(5, \{b_1, b_1', q_1, q_2, x_1\})$-connected and $(G_2 - x_1, b_1, q_1, b_1', q_2)$ is planar. We may assume that $G_2 - x_1$ is a plane graph with b_1, q_1, b_1', q_2 incident with its outer face.

If y_2 is also incident with the outer face of $G_2 - x_1$ then (i) or (ii) holds by applying Lemma 2.3.12 (and then Lemma 4.2.1) to $G_2 - x_1$ and $\{b_1, b_1', q_1, q_2, x_1, y_2\}$. So assume that y_2 is not incident with the outer face of $G_2 - x_1$. Then by Lemma 2.3.7, the vertices of $G_2 - x_1$ cofacial with y_2 induce a cycle C_{y_2} in $G_2 - x_1$, and $G_2 - x_1$ contains paths P_1, P_2, P_3 from y_2 to $\{b_1, b_1', q_1, q_2\}$ such that $V(P_i \cap P_j) = \{y_2\}$ for $1 \leq i < j \leq 3$, and $|V(P_i \cap C_{y_2})| = |V(P_i) \cap \{b_1, b_1', q_1, q_2\}| = 1$ for $i \in [3]$. Let $K = C_{y_2} \cup P_1 \cup P_2 \cup P_3$.

If P_1, P_2, P_3 end at q_1, b_1 (or b_1'), q_2, respectively, then let Q be a path in B_1 from y_1 to b_1 (or b_1'); now $K \cup (x_1z_1 \cup z_1Xq_1) \cup (x_1x_2 \cup x_2Xq_2) \cup (x_1y_1 \cup Q) \cup x_1y_2$ is a TK_5 in G'. For the remaining cases, let Q_1, Q_2 be independent paths in B_1 from y_1 to b_1', b_1, respectively. If P_1, P_2, P_3 end at b_1, q_1, b_1', respectively, then $K \cup Q_1 \cup Q_2 \cup (y_1x_1z_1 \cup z_1Xq_1) \cup y_1x_2y_2$ is a TK_5 in G'. If P_1, P_2, P_3 end at b_1, q_2, b_1', respectively then $K \cup Q_1 \cup Q_2 \cup (y_1x_2 \cup x_2Xq_2) \cup y_1x_1y_2$ is a TK_5 in G'. This proves (10).

By (10) and the 5-connectedness of G, we may let $\mathcal{A} = \{A\}$ and $y_2 \in A$. Moreover, $|N(A) - \{x_1, x_2\}| = 3$. Choose $\mathcal{A}$ so that

(11) A is maximal.

Then

(12) $b_1' \notin N(A)$, and we may assume that $N(b') \cap V(B_k - b_{k-1}) = \emptyset$ for any $b' \in N(b_1') \cap V(q_1Xq_2)$, and $|N(A) \cap V(q_1Xq_2)| = 2$.

Suppose $b_1' \in N(A)$. Then $A \cap V(q_1Xq_2 - \{q_1, q_2\}) \neq \emptyset$. Hence, $|N(A) \cap V(q_1Xq_2)| \geq 2$. Since $y_2 \in A$ and $y_2 \notin V(X)$, $|N(A) \cap V(B_i)| \geq 1$ for some $2 \leq i \leq k$, a contradiction as $|N(A) - \{x_1, x_2\}| = 3$.

Now suppose there exist $b' \in N(b_1') \cap V(q_1Xq_2)$ and $b'' \in N(b') \cap V(B_k - b_{k-1})$. Then B_k has independent paths P_2, P_2' from y_2 to b_{k-1}, b'', respectively. Let P_1, P_1'

be independent paths in B_1 from y_1 to b_1, b_1', respectively, and let P be a path in $\bigcup_{j=2}^{k-1} B_j$ from b_1 to b_{k-1}. Then $(b'Xz_1 \cup z_1x_1) \cup b'Xx_2 \cup (b'b_1' \cup P_1') \cup (b'b'' \cup P_2') \cup (P_1 \cup P \cup P_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices b', x_1, x_2, y_1, y_2.

Finally, assume $|N(A) \cap V(q_1Xq_2)| \leq 1$. Then, since $B_k - b_{k-1}$ has at least two neighbors on q_1Xq_2 (as G is 5-connected), B_k is 2-connected and $V(B_k - b_{k-1}) \not\subseteq A$. Hence, $|N(A) \cap V(B_k)| \geq 2$. Let $q_1', q_2' \in N(B_k - b_{k-1}) \cap V(X)$ such that $q_1'Xq_2'$ is maximal. Then there exists $b' \in N(b_1') \cap V(q_1'Xq_2' - \{q_1', q_2'\})$; otherwise $V(B_k \cup q_1'Xq_2') - \{b_{k-1}, q_1', q_2'\}$ contradicts the choice of A in (11). Since G is 5-connected and $(G_2 - x_1, \mathcal{A}, b_1, q_1, b_1', q_2)$ is 3-planar, b' has a neighbor b'' in $B_k - b_{k-1}$, a contradiction. So $|N(A) \cap V(q_1Xq_2)| \geq 2$. Indeed $|N(A) \cap V(q_1Xq_2)| = 2$, since $(G - x_1) - X$ is connected, $y_2 \notin V(X)$ and $|N(A) - \{x_1, x_2\}| = 3$. This concludes the proof of (12).

Since $|N(A) \cap V(q_1Xq_2)| = 2$ (by (12)), there exists $2 \leq l \leq k-1$ such that $b_l \in N(A)$ and $\bigcup_{j=l+1}^{k} V(B_j) \subseteq A$. Note that $N(A) \cap V(q_1Xq_2) \neq \{q_1, q_2\}$, as b_1' has a neighbor in $q_1Xq_2 - \{q_1, q_2\}$. We may assume that

(13) there exists $i \in [2]$ such that $q_i \in N(A)$ and $N(q_i) \cap V(G_2 - x_1) \subseteq A \cup N(A)$.

For, suppose otherwise. Then for $i \in [2]$, $q_i \notin N(A)$ or $N(q_i) \cap V(G_2 - x_1) \not\subseteq A \cup N(A)$. Hence, $G_2[\bigcup_{j=2}^{l} B_j + \{q_1, q_2\} - b_1]$ contains an induced path P from q_1 to q_2.

We may assume $b_1' \neq y_1$. For, suppose $b_1' = y_1$. Since G is 5-connected, there exists $t \in [2]$ such that $G[\bigcup_{j=l+1}^{k} V(B_j) \cup q_1Xq_2 + y_1] - \{b_l, q_{3-t}\}$ has independent paths P_1, P_2 from y_2 to y_1, q_t, respectively. If q_t has a neighbor $s \in V(B_1)$ then let S be a path in B_1 from s to y_1; now $G[\{x_1, x_2, y_1, y_2\}] \cup (x_1z_1 \cup z_1Xq_1 \cup P \cup q_2Xx_2) \cup (q_ts \cup S) \cup P_2 \cup P_1$ is a TK_5 in G' with branch vertices q_t, x_1, x_2, y_1, y_2. So assume that q_t has no neighbor in B_1. Then we may assume $q_t \notin \{z_1, x_2\}$ and $q_tx_2 \notin E(X)$; for otherwise, $\{b_1, q_{3-t}, x_1, x_2, y_1\}$ is a 5-cut in G containing the triangle $x_1x_2y_1x_1$, and the assertion follows from Lemma 4.2.2. Now let $vq_t \in E(X) - E(q_1Xq_2)$. Then

$G[B_1 + v]$ has independent paths R_1, R_2 from v to y_1, b_1, respectively. Let R be a path in $G[\bigcup_{j=2}^{l} B_j + q_{3-t}]$ from b_1 to q_{3-t}. Then $G[\{x_1, x_2, y_1, y_2\}] \cup R_1 \cup (vq_t \cup P_2) \cup (R_2 \cup R \cup (X - (q_1Xq_2 - q_{3-t})) \cup x_1z_1) \cup P_1$ is a TK_5 in G' with branch vertices v, x_1, x_2, y_1, y_2.

Let $t_1, t_2 \in V(X - x_2) \cap N(B_k - b_{k-1})$ with t_1Xt_2 maximal. We claim that $G[B_k \cup t_1Xt_2] - b_{k-1}$ is 2-connected. For, suppose not. Then $G[B_k \cup t_1Xt_2]$ has a 2-separation (L_1, L_2) such that $b_{k-1} \in V(L_1 \cap L_2)$ and $t_1Xt_2 \subseteq L_1$. Now $V(L_1 \cap L_2) \cup \{x_1, x_2\}$ is a 4-cut in G, a contradiction.

Let X' be obtained from X by replacing q_1Xq_2 with P. Then $(G - x_1) - X'$ has a chain of blocks from y_1 to y_2, in which B_1 is a block containing y_1, and the block containing y_2 contains $(B_k - b_{k-1}) \cup t_1Xt_2$ (whose size is larger than B_k). Since $b'_1 \neq y_1$, y_1 is not a cut vertex. This contradicts the choice of X for (7) (subject to (1), (2) and (3)). So we have (13).

Then $q_{3-i} \notin N(A)$, and $x_2 \neq q_i$ (otherwise $N(A) \cup \{x_1\}$ would be a 4-cut in G). Let $a \in N(A) - \{x_1, x_2, q_i, b_l\}$. Then $a \in V(X)$ and $\{a, b_1, b'_1, b_l, q_{3-i}, x_1\}$ is a 6-cut in G. So G has a 6-separation (G'_1, G'_2) such that $V(G'_1 \cap G'_2) = \{a, b_1, b'_1, b_l, q_{3-i}, x_1\}$ and $G'_2 := G_2 - (A \cup \{q_i\})$. Note that $(G'_2 - x_1, b_1, b_l, a, b'_1, q_{3-i})$ is planar. If $|V(G'_2)| \geq 8$ then we may apply Lemma 2.3.12 to (G'_1, G'_2) and conclude, with help from Lemma 4.2.1, that (i) or (ii) holds. So assume $|V(G'_2)| = 6$ or $|V(G'_2)| = 7$. Note that $G - x_1$ has a separation (Y_1, Y_2) such that $V(Y_1 \cap Y_2) = \{a, b_l, q_i\}$, Y_1 is induced in G by the union of $B_1 \cup G'_2$ and $(X - x_1) - (q_iXa - \{a, q_i\})$, and $aXq_i + y_2 \subseteq Y_2$.

Case 1. $|V(G'_2)| = 6$.

Then $l = 2$ and $b_2q_{3-i}, aq_{3-i}, ab'_1 \in E(G)$. We claim that $b_2q_i \notin E(G)$. For, suppose $b_2q_i \in E(G)$. Let P be a path in $\bigcup_{j=3}^{k-1} B_j$ from b_2 to b_{k-1}. Since G is 5-connected, $B_k - b_{k-1}$ has at least two neighbors on q_iXa. We may choose $a_1a_2 \in E(G)$ with $a_1 \in q_iXa - q_i$ and $a_2 \in V(B_k - b_{k-1})$. Let Q_1, Q_2 be independent paths in B_k from y_2 to b_{k-1}, a_2, respectively, and P_1, P_2 be independent paths in Y_1 from y_1 to

b_1, b_1', respectively. Now $G[\{x_1, x_2, y_1, y_2\}] \cup (b_2q_1 \cup q_1Xz_1 \cup z_1x_1) \cup (b_2q_2 \cup q_2Xx_2) \cup (P \cup Q_1) \cup (b_2b_1 \cup P_1) \cup (P_2 \cup b_1'a \cup aXa_1 \cup a_1a_2 \cup Q_2)$ is a TK_5 in G' with branch vertices b_2, x_1, x_2, y_1, y_2.

We also claim that $ab_1 \notin E(G)$. For, otherwise, let P be an induced path in $G[\bigcup_{j=3}^k B_j + q_i]$ from q_i to b_2. Let X' be obtained from X by replacing q_iXq_{3-i} with $P \cup b_2q_{3-i}$. Then, in $(G - x_1) - X'$, there is a block containing both B_1 and a, and y_1 is not a cut vertex. This contradicts (1).

If $q_{3-i}b_1 \notin E(G)$ then (iv) holds with $b = b_2$, $p_j = q_i$, $p_{3-j} = a$, and $v = q_{3-i}$. So we may assume $q_{3-i}b_1 \in E(G)$. We consider two cases: $x_2 \neq q_{3-i}$ and $x_2 = q_{3-i}$.

First, suppose $x_2 \neq q_{3-i}$. Note that $q_{3-i} \neq x_1$. Since G is 5-connected, x_2 has at least one neighbor in $B_1 - b_1'$. Thus, $G[B_1 + x_2]$ has independent paths P_1, P_2 from b_1 to x_2, b_1', respectively. If $G[Y_2 + x_2]$ contains a path P from q_i to x_2 and containing $\{a, b_2\}$ then $G[\{b_1, b_2, q_{3-i}\}] \cup P_1 \cup (P_2 \cup ab_1') \cup aq_{3-i} \cup P \cup (x_2x_1z_1 \cup z_1Xq_1) \cup x_2Xq_2$ is a TK_5 in G' with branch vertices $a, b_1, b_2, q_{3-i}, x_2$. Thus, it remains to prove the existence of P. Note that $G[Y_2 + x_2]$ is $(4, \{a, b_2, p_i, x_2\})$-connected. First, consider the case when $G[Y_2 + x_2]$ has disjoint paths from b_2, x_2 to a, q_i, respectively. Then by Lemma 3.2.1 and then Lemma 4.2.1, (i) or (ii) holds, or there is a path S in $G[Y_2 + x_2]$ from a to b_2 such that $G[Y_2 + x_2] - S$ is a chain of blocks from q_i to x_2. Now the existence of P follows from the fact that Y_2 is 2-connected. So assume $G[Y_2 + x_2]$ has no disjoint paths from b_2, x_2 to a, q_i, respectively. By Lemma 2.3.1, $(G[Y_2 + x_2], b_2, x_2, a, q_i)$ is planar. If $|V(G[Y_2 + x_2])| \geq 6$ then the assertion of the lemma follows from Lemma 4.2.1. So $|V(G[Y_2 + x_2])| = 5$. If $ab_2 \in E(G)$ then $G[\{q_i, a, b_2, y_2\}] \cong K_4^-$; and if $ab_2 \notin E(G)$ then $G[\{q_i, a, x_1, y_2\}]$ contains a K_4^- in which x_1 is of degree 2. So (ii) holds.

Now suppose $x_2 = q_{3-i}$. Then we may assume that $b_1' \neq y_1$, for otherwise $G[\{a, x_1, x_2, y_1\}]$ contains a K_4^- in which x_1 is of degree 2, and (ii) holds. Thus B_1 has independent paths P_1, P_2 from b_1 to y_1, b_1', respectively. If Y_2 has a cycle C

containing $\{a, b_2, y_2\}$, then $C \cup G[\{a, b_1, b_2, q_{3-i}\}] \cup (P_2 \cup b_1'a) \cup (P_1 \cup y_1x_1y_2) \cup y_2x_2$ is a TK_5 in G' with branch vertices $a, b_1, b_2, q_{3-i}, y_2$. So we may assume that the cycle C in Y_2 does not exist. Since Y_2 is 2-connected, it follows from Lemma 2.3.5 that Y_2 has 2-cuts S_u, for $u \in \{a, b_2, y_2\}$, separating u from $\{a, b_2, y_2\} - \{u\}$. Since G is 5-connected, we see that S_{y_2} separates $\{q_i, y_2\}$ from $\{a, b_2\}$. Hence, $d_G(b_2) = 5$ and $x_1b_2 \in E(G)$. Now $G[\{b_1, b_2, x_1, x_2\}]$ contains a K_4^- in which x_1 is of degree 2, and (ii) holds.

Case 2. $|V(G_2')| = 7$.

Let $z \in V(G_2') - \{a, b_1, b_l, b_1', q_{3-i}, x_1\}$. Suppose $z \notin V(X)$. Then $b_1'a \in E(G)$. Since G is 5-connected and B_1 is a block of H, $zb_1' \notin E(G)$ and $za, zq_{3-i}, zb_l, zb_1, zx_1 \in E(G)$. We may assume $b_1'q_{3-i} \notin E(G)$, as otherwise, $G[\{a, b_1', q_{3-i}, z\}]$ contains K_4^- and (ii) holds. Thus, $G[B_1 + q_{3-i}]$ has independent paths P_1, P_2 from b_1 to b_1', q_{3-i}, respectively. Note $b_1b_l \in E(G)$ by the maximality of A in (11). In $G[A \cup \{a, b_l, q_i\}]$ we find independent paths Q_1, Q_2 from b_l to q_i, a, respectively. Now $G[\{a, b_1, b_l, q_{3-i}, z\}] \cup (P_1 \cup b_1'a) \cup P_2 \cup Q_2 \cup (q_2Xx_2 \cup x_2x_1z_1 \cup z_1Xq_1 \cup Q_1)$ is a TK_5 in G' with branch vertices a, b_1, b_l, q_{3-i}, z.

So we may assume $z \in V(X)$. Then $b_1b_l, q_{3-i}b_l \in E(G)$. We may assume $b_1a, b_1z \notin E(G)$. For, suppose $b_1a \in E(G)$ or $b_1z \in E(G)$. Let X' be obtained from X by replacing q_1Xq_2 with b_lq_{3-i} and a path in $Y_2 - a$ from b_l to q_i. Then, $B_1 + a$ or $B_1 + z$ is contained in a block of $(G - x_1) - X'$, and y_1 is not a cut vertex of $(G - x_1) - X'$, contradicting (1).

Hence, zb_1', zb_l, $zx_1 \in E(G)$ and $q_{3-i} \neq x_1$. We may assume $x_1q_{3-i} \notin E(G)$; as otherwise, $G[\{b_l, q_{3-i}, x_1, z\}]$ contains a K_4^- in which x_1 is of degree 2, and (ii) holds. Note that $b_1'a \in E(G)$ by the maximality of A in (11). Let $q \in N(q_{3-i}) \cap V(B_1 - b_1)$, and let P_1, P_2 be independent paths in B_1 from b_1' to b_1, q, respectively. Let Q_1, Q_2 be independent paths in Y_2 from a to b_l, q_i, respectively. Then $G[\{a, b_l, b_1', q_{3-i}, z\}] \cup (P_1 \cup b_1b_l) \cup (P_2 \cup qq_{3-i}) \cup Q_1 \cup (Q_2 \cup q_1Xz_1 \cup z_1x_1x_2 \cup x_2Xq_2)$ is a TK_5 in G' with branch vertices a, b_l, b'_1, q_{3-i}, z. ∎

4.3 Two special cases

We need to consider the conclusions of Lemma 4.2.5. (i) and (ii) of Lemma 4.2.5 are desired cases. Lemma 2.3.6 can be used to deal with (iii) of Lemma 4.2.5 when $y_2 \notin V(X)$. So it remains to consider (iii) of Lemma 4.2.5 when $y_2 \in V(X)$ and (iv) of Lemma 4.2.5.

We will use the notation in Lemma 4.2.5. See Figures 2 and 3. In particular, X is an induced path in $(G - x_1) - x_2y_2$ from z_1 to x_2 and $G' := G - \{x_1x : x \notin \{x_2, y_1, y_2, z_0, z_1\}\}$. Also recall from in (iv) of Lemma 4.2.5 the the separation (Y_1, Y_2) and the vertices $p_j, p_{3-j}, v, b, b_1, b'_1$. Let z_2 be the neighbor of x_2 on X.

For any vertex $x \in V(G)$ and $S \subseteq G$, we use $e(x, S)$ to denote the number of edges of G from x to S.

First , we need some structural information on Y_2.

Lemma 4.3.1 *Suppose (iv) of Lemma 4.2.5 holds. Then Y_2 has independent paths from y_2 to b, p_1, p_2, respectively, and, for $i \in [2]$, Y_2 has a path from b to p_{3-i} and containing $\{y_2, p_i\}$. Moreover, one of the following holds:*

(i) G' contains TK_5, or G contains a TK_5 in which x_1 is not a branch vertex.

(ii) $G - x_1$ contains K_4^-, or G contains a K_4^- in which x_1 is of degree 2.

(iii) If $e(p_i, B_1 - b_1) \geq 1$ for some $i \in [2]$ then Y_2 has a path through b, p_i, y_2, p_{3-i} in order, and $Y_2 - b_1$ has a cycle containing $\{p_1, p_2, y_2\}$. If $b \neq b_1$ and $i = 2$ with $p_iv \in E(X)$ and $vb, vx_1 \in E(G)$ then Y_2 has a cycle containing $\{b, p_i, y_2\}$.

Proof. Since G is 5-connected, Y_2 is $(3, \{b, p_1, p_2\})$-connected. So by Menger's theorem, Y_2 has independent paths from y_2 to b, p_1, p_2, respectively.

Next, let $i \in [2]$, and consider the graph $Y'_2 := Y_2 + \{t, tb, tp_{3-i}\}$, which is 2-connected. If Y'_2 has a cycle C containing $\{b, t, y_2\}$ then $C - t$ is a path in Y_2 from

b to p_{3-i} and containing $\{y_2, p_i\}$. So suppose such a cycle C does not exist. Then by Lemma 2.3.5, Y_2' has a 2-cut T separating y_2 from $\{p_i, t\}$ and $\{p_i, t\} \cap T = \emptyset$. However, $T \cup \{x_1, x_2\}$ is a 4-cut in G, a contradiction.

We now show that (i) holds or the first part of (iii) holds. Suppose $e(p_i, B_1 - b_1) \geq 1$. Let S denote a path in Y_2 from b to p_{3-i} and containing $\{p_i, y_2\}$.

We may assume that S must go through b, p_i, y_2, p_{3-i} in order. For, suppose S goes through b, y_2, p_i, p_{3-i} in this order. Since $e(p_i, B_1 - b_1) \geq 1$, $G[B_1 + p_i]$ has independent paths P_1, P_2 from y_1 to b_1, p_i, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup S \cup P_2 \cup ((X - (p_1 X p_2 - \{p_1, p_2\})) \cup x_1 z_1) \cup (P_1 \cup b_1 b)$ is a TK_5 in G' with branch vertices p_i, x_1, x_2, y_1, y_2, and (i) holds.

Note that $Y_2 - b_1$ is 2-connected. For, suppose not. Then $b = b_1$ and $Y_2 - b_1$ has a 1-separation (Y_{21}, Y_{22}) such that $|V(Y_{21} - Y_{22}) \cap \{p_1, p_2, y_2\}| \leq 1$. Since each of $\{p_1, p_2, y_2\}$ has at least two neighbors in $Y_2 - b_1$, $(V(Y_{21} - Y_{22}) \cap \{p_1, p_2, y_2\}) \cup \{b, x_1\} \cup V(Y_{21} \cap Y_{22})$ is a cut in G of size at most 4, a contradiction. Thus $Y_2 - b_1$ is 2-connected.

Now suppose no cycle in $Y_2 - b_1$ contains $\{p_1, p_2, y_2\}$. Then, (i) or (ii) or (iii) of Lemma 2.3.5 holds. We use the notation in Lemma 2.3.5 (with p_1, p_2, y_2 playing the roles of y_1, y_2, y_3 there). If (i) of Lemma 2.3.5 occurs then let $S = \{a_1, a_1'\}$, $a_2 = a_3 = a_1$, and $a_2' = a_3' = a_1'$; if (ii) or (iii) of Lemma 2.3.5 occurs let $S_{p_j} = \{a_j, a_j'\}$ for $j \in [2]$ and let $S_{y_2} = \{a_3, a_3'\}$. Let A, A' denote the components of $(Y_2 - b_1) - (D_{p_1} \cup D_{p_2} \cup D_{y_2})$ such that $a_j \in V(A)$ and $a_j' \in V(A')$ for $j \in [3]$. Note that if (ii) of Lemma 2.3.5 occurs and $A \neq A'$, then either $A = a_3$ and $\{a_1', a_2', a_3'\} \subseteq V(A')$, or $A' = a_3'$ and $\{a_1, a_2, a_3\} \subseteq V(A)$.

Since $Y_2 - b_1$ is 2-connected, there exist paths S_1, S_2, S_3 in $D_{p_1}, D_{p_2}, D_{y_2}$, respectively, with S_j from a_j to a_j' for $j \in [3]$, $p_j \in V(S_j)$ for $j \in [2]$, and $y_2 \in V(S_3)$. Since G is 5-connected, $b \in V(D_{y_2})$ or $b = b_1$ has a neighbor in D_{y_2}. Hence, $G[D_{y_2} + b]$ contains a path T_3 from b to some $t \in V(S_3) - \{a_3, a_3'\}$ and internally disjoint from S_3. By symmetry, we may assume $t \in V(y_2 S_3 a_3)$. Let T_1 be a path in A from a_i to

a_{3-i}, and T_2 be a path in A' from a_i' to a_3'. Then $T_3 \cup tS_3a_3' \cup T_2 \cup S_i \cup T_1 \cup a_{3-i}S_{3-i}p_{3-i}$ is a path from b to p_{3-i} through y_2, p_i in order. This is a contradiction as we have assumed that such a path S does not exist.

Next, we prove that (i) or (ii) holds or the second part of (iii) holds. Suppose $b \neq b_1$, $p_2v \in E(p_2Xx_2)$, and $vb, vx_1 \in E(G)$. Suppose Y_2 has no cycle containing $\{b, p_2, y_2\}$. Then (i) or (ii) or (iii) of Lemma 2.3.5 holds. We use the notation in Lemma 2.3.5 (with b, p_2, y_2 playing the roles of y_1, y_2, y_3 there, respectively). So there is a 2-cut $S_{y_2} = \{a_3, a_3'\}$ in Y_2 such that $Y_2 - S_{y_2}$ has a component D_{y_2} with $y_2 \in V(D_{y_2})$ and $b, p_2 \notin V(D_{y_2}) \cup S_{y_2}$. Since G is 5-connected, $p_1 \in V(D_{y_2})$. Note that $Y_2 - D_{y_2}$ is $(4, \{a_3, a_3', b, p_2\})$-connected.

Suppose $(Y_2 - D_{y_2}, a_3, b, a_3', p_2)$ is not planar. Then by Lemma 2.3.1, $Y_2 - D_{y_2}$ contains disjoint paths from a_3, b to a_3', p_i, respectively. By Lemma 3.2.1, we may assume that $Y_2 - D_{y_2}$ has an induced path S from b to p_2 such that $(Y_2 - D_{y_2}) - S$ is a chain of blocks from a_3 to a_3'; for otherwise, we may apply Lemma 4.2.1 to show that (i) or (ii) holds. Thus $Y_2 - D_{y_2}$ has a path S_1 from a_3 to a_3' and containing $\{b, p_2\}$ (as Y_2 is 2-connected). Let S_2 be a path in $G[D_{y_2} + \{a_3, a_3'\}]$ from a_3 to a_3' through y_2. Then $S_1 \cup S_2$ is a cycle containing $\{b, p_2, y_2\}$, a contradiction.

So we may assume $(Y_2 - D_{y_2}, a_3, b, a_3', p_2)$ is planar. Hence, $bp_2 \notin E(G)$. If $|V(Y_2 - D_{y_2})| \geq 6$ then (i) or (ii) follows from Lemma 4.2.1 (by considering the 5-cut $\{a_3, a_3', b, p_i, x_1\}$).

Now suppose $|V(Y_2 - D_{y_2})| = 5$. Let $t \in V(Y_2 - D_{y_2}) - \{a_3, a_3', b, p_2\}$. Since G is 5-connected, $ta_3, ta_3', tb, tp_2, tx_1 \in E(G)$. By symmetry between a_3 and a_3', we may assume $a_3' \in V(X)$. Then $a_3'p_2 \in E(G)$. If $ba_3' \in E(G)$ then $G[\{a_3', b, p_2, t\}] \cong K_4^-$, and (ii) holds. So assume $ba_3' \notin E(G)$. Then, since G is 5-connected, $ba_3, bx_1 \in E(G)$. Now $G[\{a_3, b, t, x_1\}]$ contains K_4^- in which x_1 is of degree 2, and (ii) holds.

So $|V(Y_2 - D_{y_2})| = 4$ and, hence, (i) of Lemma 2.3.5 occurs. Moreover, $V(D_b) = \{b\}$ and $V(D_{p_2}) = \{p_2\}$. We claim that $D := G[D_{y_2} + \{a_3, a_3', x_1\}] + \{c, cx_1, cy_2\}$ has a cycle C containing $\{c, a_3, a_3'\}$; for otherwise, by Lemma 2.3.5, $D - c$ has a 2-cut either separating a_3 from $\{x_1, y_2, a_3', p_1\}$ or separating a_3' from $\{x_1, y_2, a_3, p_1\}$, contradicting the 5-connectedness of G. Let Q be a path in $G[B_1 + \{b, p_2\}]$ from b to p_2. Now $a_3ba_3'p_2a_3 \cup Q \cup (C - c) \cup (x_1v \cup vXx_2 \cup x_2y_2) \cup vb \cup vp_2$ is a TK_5 in G with branch vertices a_3, a_3', b, p_2, v. ∎

The next two results provide information on $e(z_i, B_1)$ for $i \in [2]$ in the case when $y_2 \notin V(X)$.

Lemma 4.3.2 *Suppose (iv) of Lemma 4.2.5 holds with $b \neq b_1$. Then one of the following holds:*

(i) G' contains TK_5, or G contains a TK_5 in which x_1 is not a branch vertex.

(ii) $G - x_1$ contains K_4^-, or G contains a K_4^- in which x_1 is of degree 2.

(iii) $e(z_i, B_1) \geq 2$ for $i \in [2]$.

Proof. Recall the notation from (iv) of Lemma 4.2.5. In particular, $v \in V(X) - V(p_1Xp_2)$. Suppose $e(z_i, B_1) \leq 1$ for some $i \in [2]$.

Case 1. $v \in V(z_1Xp_1 - p_1)$; so $p_1v \in E(X)$.

In this case, $e(z_1, Y_2) \leq 2$ (with equality only if $z_1 = v$). Hence, $e(z_1, B_1) \geq 2$, since G is 5-connected. Thus, $e(z_2, B_1) \leq 1$. Indeed, since $\{x_1, x_2, p_1, b\}$ cannot be a cut in G, $e(z_2, B_1) = 1$ and $z_2 = p_2$. By Lemma 4.3.1, Y_2 has a path Q from b to p_1 and containing $\{y_2, z_2\}$.

Suppose b, z_2, y_2, p_1 occur on Q in this order. If $b_1' \in N(z_2)$ then let P_1, P_2 be independent paths in $G[B_1 + x_2]$ from b_1' to y_1, x_2, respectively; now $G[\{x_1, x_2, y_2\}] \cup z_2x_2 \cup (z_2Qb \cup bv \cup vXz_1 \cup z_1x_1) \cup z_2Qy_2 \cup b_1'z_2 \cup (b_1'p_1 \cup p_1Qy_2) \cup (P_1 \cup y_1x_1) \cup P_2$ is a TK_5 in G' with branch vertices b_1', x_1, x_2, y_2, z_2. So assume $b_1' \notin N(z_2)$. Let P_1, P_2 be independent paths in $G[B_1 + z_2]$ from y_1 to b_1', z_2, respectively. Then

$G[\{x_1, x_2, y_1, y_2\}] \cup z_2x_2 \cup (z_2Qb \cup bv \cup vXz_1 \cup z_1x_1) \cup z_2Qy_2 \cup P_2 \cup (y_2Qp_1 \cup p_1b_1' \cup P_1)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

So assume that b, y_2, z_2, p_1 must occur on Q in this order. Then, by Lemma 4.3.1, we may assume $e(z_2, B_1 - b_1) = 0$. Since G is 5-connected and $p_2 = z_2$, $b_1z_2 \in E(G)$; as otherwise, $\{b, p_1, x_1, x_2\}$ would be a cut in G. Let P_1, P_2 be independent paths in $G[B_1 + x_2]$ from b_1 to y_1, x_2, respectively. Then $G[\{x_1, x_2, y_2\}] \cup z_2x_2 \cup (z_2Qp_1 \cup p_1Xz_1 \cup z_1x_1) \cup z_2Qy_2 \cup (b_1b \cup bQy_2) \cup b_1z_2 \cup (P_1 \cup y_1x_1) \cup P_2$ is a TK_5 in G' with branch vertices b_1, x_1, x_2, y_2, z_2.

Case 2. $v \in V(p_2Xx_2 - p_2)$; so $p_2v \in E(X)$.

Since $\{b, p_2, x_1, x_2\}$ cannot be a cut in G, $e(z_1, B_1) \geq 1$. We consider two cases.

Subcase 2.1. $e(z_1, B_1) = 1$.

Then $z_1 = p_1$. By Lemma 4.3.1, Y_2 has a path Q from b to p_2 and containing $\{z_1, y_2\}$.

Suppose b, z_1, y_2, p_2 occur on Q in this order. If $b_1' \in N(z_1)$ then $x_2 \neq v$ as $\{x_1, x_2, b_1, b_1'\}$ is not a cut in G; so $e(x_2, B_1 - y_1) \geq 1$. Let P_1, P_2 be independent paths in $G[B_1+x_2]$ from b_1' to y_1, x_2, respectively. Then $G[\{x_1, x_2, y_2\}] \cup z_1x_1 \cup (z_1Qb \cup bv \cup vXx_2) \cup z_1Qy_2 \cup b_1'z_1 \cup (b_1'p_2 \cup p_2Qy_2) \cup (P_1 \cup y_1x_1) \cup P_2$ is a TK_5 in G' with branch vertices b_1', x_1, x_2, y_2, z_1. Hence, assume $b_1' \notin N(z_1)$. Then let P_1, P_2 be independent paths in $G[B_1 + z_1]$ from y_1 to b_1', z_1, respectively; now $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup (z_1Qb \cup bv \cup vXx_2) \cup z_1Qy_2 \cup P_2 \cup (y_2Qp_2 \cup p_2b_1' \cup P_1)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

So we may assume b, y_2, z_1, p_2 must occur on Q in this order. Hence, by Lemma 4.3.1, we may assume $e(p_1, B_1 - b_1) = 0$; so $b_1 \in N(z_1)$ as $\{b, p_2, x_1, x_2\}$ is not a cut in G. Then $e(x_2, B_1 - y_1) \geq 1$; otherwise, $x_2 = v$, and $\{b_1, b_1', x_1, x_2\}$ would be a cut in G. Let P_1, P_2 be independent paths in $G[B_1 + x_2]$ from b_1 to y_1, x_2, respectively. Then $G[\{x_1, x_2, y_2\}] \cup z_1x_1 \cup (z_1Qp_2 \cup p_2Xx_2) \cup z_1Qy_2 \cup b_1z_1 \cup (b_1b \cup bQy_2) \cup (P_1 \cup y_1x_1) \cup P_2$ is a TK_5 in G' with branch vertices b_1, z_1, x_1, x_2, y_2.

Subcase 2.2. $e(z_1, B_1) \geq 2$.

Then $e(z_2, B_1) \leq 1$. Hence, $z_2 = p_2$ or $z_2 = v$. Suppose $z_2 = p_2$. Then $x_2 = v$; so $x_1 v \in E(G)$. Hence, by (iii) of Lemma 4.3.1, Y_2 has a cycle C containing $\{b, z_2, y_2\}$. Let P_1, P_2 be independent paths in B_1 from y_1 to b_1, b_1', respectively. Now $C \cup x_2y_2 \cup x_2z_2 \cup x_2b \cup y_1x_2 \cup y_1x_1y_2 \cup (P_1 \cup b_1b) \cup (P_2 \cup b_1'z_2)$ is a TK_5 in G' with branch vertices b, x_2, y_1, y_2, z_2.

So we may assume $z_2 = v$. Since $e(z_2, B_1) = 1$, $x_1v \in E(G)$. Hence, by (iii) of Lemma 4.3.1, Y_2 has a cycle C containing $\{b, p_2, y_2\}$. Let P_1, P_2 be independent paths in $G[B_1 + x_2]$ from x_2 to b_1, b_1', respectively. Note that P_1, P_2 exist since x_2 has at least two neighbors in B_1. Then $C \cup z_2b \cup z_2p_2 \cup z_2x_1y_2 \cup x_2y_2 \cup x_2z_2 \cup (P_1 \cup b_1b) \cup (P_2 \cup b_1'p_2)$ is a TK_5 in G' with branch vertices b, p_2, x_2, y_2, z_2. ∎

Lemma 4.3.3 *Suppose $y_2 \notin V(X)$. Then one of the following holds:*

(i) G' contains TK_5, or G contains a TK_5 in which x_1 is not a branch vertex.

(ii) $G - x_1$ contains K_4^-, or G contains K_4^- in which x_1 is of degree 2.

(iii) There exists $i \in [2]$ such that $e(z_i, B_1 - b_1) \geq 2$ and $e(z_{3-i}, B_1 - b_1) \geq 1$.

Proof. Suppose (iii) fails. First, assume $b \neq b_1$; so (iv) of Lemma 4.2.5 occurs. Then by Lemma 4.3.2, we have, for $i \in [2]$, $e(z_i, B_1 - b_1) = 1$ and $b_1z_i \in E(G)$. Let P_1, P_2 be independent paths in B_1 from y_1 to b_1, b_1', respectively. Recall, from (iv) of Lemma 4.2.5, the role of $j \in [2]$ and the vertices p_{3-j}, v. Since b_1' is the only neighbor of p_{3-j} in B_1, $p_{3-j} \notin \{z_1, z_2\}$. Let Q be a path in $Y_2 - \{z_1, z_2\}$ from b to p_{3-j} through y_2. Then $G[\{x_1, x_2, y_1, y_2\}] \cup b_1z_1x_1 \cup b_1z_2x_2 \cup (b_1b \cup bQy_2) \cup P_1 \cup (y_2Qp_{3-j} \cup p_{3-j}b_1' \cup P_2)$ is a TK_5 in G' with branch vertices b_1, x_1, x_2, y_1, y_2.

So we may assume $b = b_1$. Then, for $i \in [2]$, $e(z_i, B_1 - b_1) \geq 1$ as $\{b, p_{3-i}, x_1, x_2\}$ is not a cut in G. Hence, since (iii) fails, $e(z_i, B_1 - b_1) = 1$ for $i \in [2]$. For $i \in [2]$, let $z_i' \in N(z_i) \cap V(B_1)$. Since G is 5-connected, $z_1 = p_1$.

Case 1. $z_2 \neq p_2$.

Then, since G is 5-connected, $z_2x_1, z_2b \in E(G)$. First, assume that there is no edge from $p_2Xz_2 - z_2$ to $B_1 - b$. Then G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b, x_1, x_2, z_1, z_2\}$, $B_1 \subseteq G_1$, and $Y_2 \subseteq G_2$. Clearly, $|V(G_i)| \geq 7$ for $i \in [2]$. Since $x_1x_2z_2x_1$ is a triangle in G, the assertion of the lemma follows from Lemma 4.2.2.

Hence, we may assume that there exists $uu' \in E(G)$ with $u \in V(p_2Xz_2 - z_2)$ and $u' \in V(B_1 - b)$. Suppose, for some choice of uu', $u' \neq z_1'$ and $B_1 - b$ contains independent paths P_1, P_2 from y_1 to z_1', u', respectively. By Lemma 4.3.1 (since $e(p_1, B_1 - b_1) = 1$), Y_2 contains a path Q from b to p_2 through p_1, y_2 in order. Now $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup (z_1Qb \cup bz_2x_2) \cup (z_1z_1' \cup P_1) \cup z_1Qy_2 \cup (P_2 \cup u'u \cup uXp_2 \cup p_2Qy_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Therefore, we may assume that for any choice of uu', $u' = z_1'$ or the paths P_1, P_2 do not exist. Since B_1 is 2-connected, B_1 has a 2-separation (B', B'') such that $b \in V(B' \cap B'')$, $y_1 \in V(B')$ and $z_1', u' \in V(B'')$ for all $u' \in N(p_2Xz_2 - z_2)$. Here, if $u' = z_1'$ for all $u' \in N(p_2Xz_2 - z_2)$, we let $B' = B_1$ and $B'' = \{b, z_1'\}$. Thus G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = V(B' \cap B'') \cup \{x_1, x_2, z_2\}$, $B' \subseteq G_1$ and $B'' \cup Y_2 \subseteq G_2$. Clearly, $|V(G_2)| \geq 7$.

If $|V(G_1)| \geq 7$ then the assertion of the lemma follows from Lemma 4.2.2 (as $x_1x_2z_2x_1$ is a triangle in G). So assume $|V(G_1)| \leq 6$. Then, since G is 5-connected, $z_2y_1 \in E(G)$. So $G[\{x_1, x_2, y_1, z_2\}] - x_1y_1 \cong K_4^-$ in which x_1 is of degree 2, and *(ii)* holds.

Case 2. $z_2 = p_2$.

We may assume $z_i' \neq y_1$ for $i \in [2]$. For, otherwise, G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b, p_{3-i}, x_1, x_2, y_1\}$, $B_1 \subseteq G_1$ and $Y_2 \subseteq G_2$. Clearly, $|V(G_i)| \geq 7$ for $i \in [2]$. Since $G[\{x_1, x_2, y_1\}] \cong K_3$, the assertion of the lemma follows from Lemma 4.2.2.

Note that $z_1' \neq z_2'$ as otherwise $\{b, x_1, x_2, z_1'\}$ would be a cut in G. Let $K = G[B_1 + \{x_2, z_1, z_2\}]$. Suppose K contains disjoint paths Z_1, Z_2 from z_1, z_2 to x_2, y_1, respectively. By Lemma 4.3.1, let C be a cycle in $Y_2 - b_1$ containing $\{y_2, z_1, z_2\}$. Then $G[\{x_1, x_2, y_2\}] \cup C \cup z_1x_1 \cup z_2x_2 \cup (Z_2 \cup y_1x_1) \cup Z_1$ is a TK_5 in G' with branch vertices x_1, x_2, y_2, z_1, z_2.

So we may assume that such Z_1, Z_2 do not exist. Then by Lemma 2.3.1, there exists a collection $\mathcal{A}$ of pairwise disjoint subsets of $V(K) - \{x_2, y_1, z_1, z_2\}$ such that $(K, \mathcal{A}, z_1, z_2, x_2, y_1)$ is 3-planar. Since G is 5-connected, either $\mathcal{A} = \emptyset$ or $|\mathcal{A}| = 1$. When $|\mathcal{A}| = 1$ let $\mathcal{A} = \{A\}$; then $b_1 \in A$. We choose $\mathcal{A}$ so that $|\mathcal{A}|$ is minimal and, subject to this, $|A|$ is minimal when $\mathcal{A} = \{A\}$. Note that if A exists then $|A| \geq 2$ (by the minimality of $|\mathcal{A}|$ and $|A|$). Moreover, $|N_K(A)| = 3$ as $N_K(A) \cup \{b_1, x_1\}$ is not a cut in G.

We may assume if $\mathcal{A} \neq \emptyset$ then $\{x_2, z_1, z_2\} \cap N_K(A) = \emptyset$. For, suppose there exists $u \in \{x_2, z_1, z_2\} \cap N_K(A)$. Let $S := (N_K(A) \cup \{x_1, x_2, z_1, z_2\}) - \{u\}$ if $u \in \{z_1, z_2\}$ and let $S := N_K(A) \cup \{x_1, x_2, z_1, z_2\}$ if $u = x_2$. Then S is a cut in G separating $B_1 - A$ from Y_2. Since G is 5-connected, $|S| = 5$ if $u \in \{z_1, z_2\}$ and $|S| = 6$ if $u = x_2$. Therefore, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = S$, $B_1 - A \subseteq G_1$, and $Y_2 \subseteq G_2$. Note that $(G_1 - x_1, S - \{x_1\})$ is planar. Clearly, $|V(G_2)| \geq 7$. Since $y_1 \notin \{z_1', z_2'\}$, $|V(G_1)| \geq 7$ if $|S| = 5$ and $|V(G_1)| \geq 8$ if $|S| = 6$. Thus, if $|S| = 5$ then the assertion of the lemma follows from Lemma 4.2.1, and if $|S| = 6$ then the assertion of the lemma follows from Lemma 2.3.12 and then Lemma 4.2.1.

If $\mathcal{A} = \emptyset$ let $K^* = K$; otherwise, let K^* be the graph obtained from K by deleting A and adding new edges joining every pair of distinct vertices in $N_K(A)$. Since B_1 is 2-connected and G is 5-connected, $K' := K^* - \{x_2, z_1, z_2\}$ is a 2-connected planar graph. Take a plane embedding of K' and let D denote its outer cycle. Let $t \in V(D)$ such that $t \in N(x_2)$ and tDz_2' is minimal.

When $\mathcal{A} \neq \emptyset$, $N_K(A) \not\subseteq V(D)$; as otherwise, if we write $N_K(A) = \{s_1, s_2, s_3\} \subseteq$

$V(D)$ with $s_2 \in V(s_1Ds_3)$, then $\{b_1, s_1, s_3, x_1\}$ is a cut in G, a contradiction. Further, if $\mathcal{A} \neq \emptyset$ and if we write $N_K(A) = \{a, a_1, a_2\}$ with $a \in N_K(A) - V(tDz_1')$, then, by the minimality of $\mathcal{A}$ and A, $G[A \cup N_K(A)]$ contains disjoint paths P_1, P_2 from a, a_2 to b_1, a_1, respectively. If $\mathcal{A} = \emptyset$ let $Q = tDz_1'$, $P_1 = a = a_1 = a_2 = b_1$ and $P_2 = \emptyset$. If $\mathcal{A} \neq \emptyset$ let $Q = tDz_1'$ if $a_1a_2 \notin E(tDz_1')$; and otherwise let $Q = (tDz_1' - a_1a_2) \cup P_2$. Note that Q is a path in B_1.

Suppose $K' - (tDz_1' - z_2')$ has independent paths S_1, S_2 from y_1 to $z_2', \{a, a_1, a_2\}$, respectively, and internally disjoint from $\{a, a_1, a_2\}$. We may assume the notation is chosen so that $a \in V(S_2)$. For $i \in [2]$, let $S_i' = S_i$ if $a_1a_2 \notin E(S_i)$; and otherwise let S_i' be obtained from S_i by replacing a_1a_2 with P_2. By Lemma 4.3.1, let Q_1, Q_2 be independent paths in Y_2 from y_2 to z_2, b, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup (z_2'Qz_1' \cup z_1'z_1x_1) \cup (z_2'Qt \cup tx_2) \cup (z_2'z_2 \cup Q_1) \cup S_1' \cup (S_2' \cup P_2 \cup Q_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2'.

So we may assume that such S_1, S_2 do not exist. Then by planarity, K' has a cut $\{s_1, s_2, s_3\}$ separating y_1 from $\{a, z_2'\}$, with $s_1 \in V(z_2'Dz_1')$ and $s_3 \in V(tDz_2')$. Clearly, $\{s_1, s_2, s_3\}$ is also a cut in B_1 separating y_1 from $\{z_2'\} \cup A$. Denote by M the $\{s_1, s_2, s_3\}$-bridge of B_1 containing y_1. If $V(M) - \{s_1, s_2, s_3\} = \{y_1\}$ then $s_1 = z_1'$ and $s_3 = t$; now $G[\{t, x_1, x_2, y_1\}]$ contains a K_4^- in which x_1 is of degree 2, and *(ii)* holds. So assume $|V(M) - \{s_1, s_2, s_3\}| \geq 2$. Then G has a 6-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{s_1, s_2, s_3, x_1, x_2, z_1\}$, $G_2 = G[M + \{z_1, x_1, x_2\}]$, and $(G_2 - x_1, z_1, s_1, s_2, s_3, x_2)$ is planar. Now $|V(G_i)| \geq 8$ for $i \in [2]$; so the assertion follows from Lemma 2.3.12 and then Lemma 4.2.1. ∎

4.4 Substructure

In this section, we derive a substructure in G by finding five paths A, B, C, Y, Z in $H := G[B_1 + \{z_1, z_2\}]$. The paths Y, Z are found in the following lemma.

Lemma 4.4.1 *Suppose $y_2 \in V(X)$ (see (iii) of Lemma 4.2.5), or $y_2 \notin V(X)$ and*

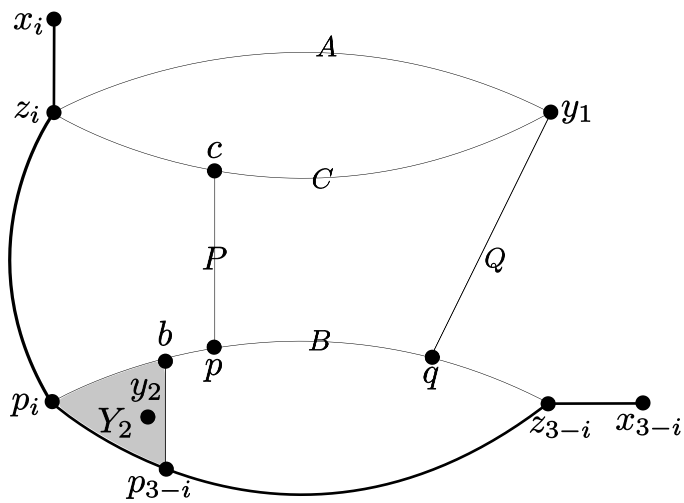

Figure 4: An intermediate structure 2

$e(z_i, B_1) \geq 2$ for some $i \in [2]$ (see (iv) of Lemma 4.2.5). Let $b_1 \in N(y_2) \cap V(B_1)$ if $y_2 \in V(X)$, and let $\{b_1\} = V(B_1) \cap V(B_2)$ if $y_2 \notin V(X)$. Then one of the following holds:

(i) G' contains TK_5 or G contains a TK_5 in which x_1 is not a branch vertex.

(ii) $G - x_1$ contains K_4^-, or G contains a K_4^- in which x_1 is of degree 2.

(iii) H contains disjoint paths Y, Z from y_1, z_1 to b_1, z_2, respectively.

Proof. Suppose (*iii*) fails. Then by Lemma 2.3.1, there exists a collection $\mathcal{A}$ of subsets of $V(H) - \{b_1, y_1, z_1, z_2\}$ such that $(H, \mathcal{A}, b_1, z_1, y_1, z_2)$ is 3-planar.

Since B_1 is 2-connected, $|N_H(A) \cap \{z_1, z_2\}| \leq 1$ for all $A \in \mathcal{A}$. Let $\mathcal{A}' = \{A \in \mathcal{A} : |\{z_1, z_2\} \cap N_H(A)| = 0\}$ and $\mathcal{A}'' = \{A \in \mathcal{A} : |\{z_1, z_2\} \cap N_H(A)| = 1\}$. Let $p(H, \mathcal{A})$ be the graph obtained from H by deleting A (for each $A \in \mathcal{A}$) and adding new edges joining every pair of distinct vertices in $N_H(A)$. Since G is 5-connected and B_1 is 2-connected, $p(H, \mathcal{A}) - \{z_1, z_2\}$ is 2-connected and we may assume that it is drawn in the plane with outer cycle D, such that for each $A \in \mathcal{A}''$, the edges joining the vertices in $N_H(A) - \{z_1, z_2\}$ occur on D.

For each $j \in [2]$, let $t_j \in V(D)$ such that H has a path from z_j to t_j and internally disjoint from $p(H, \mathcal{A})$, and subject to this, t_2, b_1, t_1, y_1 occur on D in clockwise order, and t_2Dt_1 is maximal. When $e(z_1, B_1) \geq 2$, let $t_1' \in V(b_1Dt_1)$ with $t_1'Dt_1$ maximal such that H has independent paths R_1, R_1' from z_1 to t_1, t_1', respectively, and internally disjoint from $p(H, \mathcal{A})$. When $e(z_2, B_2) \geq 2$, let $t_2' \in V(t_2Db_1)$ with t_2Dt_2' maximal such that H has independent paths R_2, R_2' from z_2 to t_2, t_2', respectively, and internally disjoint from $p(H, \mathcal{A})$.

Next we define vertices y_{21}, y_{22} and paths Q_1, Q_2, Q_3. If $y_2 \in V(X)$, then let $p_1 = p_2 = b = y_2$, let $Q_j := y_2$ for $j \in [3]$, and let $y_{21}, y_{22} \in N(y_2) \cap V(D)$ such that $t_2', y_{22}, y_{21}, t_1'$ occur on D in clockwise order and $y_{22}Dy_{21}$ is maximal. If $y_2 \notin V(X)$ and both $e(z_1, B_1) \geq 2$ and $e(z_2, B_2) \geq 2$, then let $y_{21} = y_{22} = b_1$ and, by Lemma 4.3.1, let Q_1, Q_2, Q_3 be independent paths in Y_2 from y_2 to p_1, p_2, b, respectively. Now assume $y_2 \notin V(X)$ and $e(z_{3-i}, B_1) = 1$. Then $z_{3-i} = p_{3-i}$ and, by Lemma 4.3.1, Y_2 has a path Q_{3-i}^* through b, z_{3-i}, y_2, p_i in order. Let $R_{3-i}' := b_1b \cup bQ_{3-i}^*z_{3-i}$, $t_{3-i}' := b_1$, $Q_{3-i} := y_2Q_{3-i}^*z_{3-i}$, and $Q_i := p_iQ_{3-i}^*y_2$, Let R_{3-i} be a path in H from z_{3-i} to t_{3-i} and internally disjoint from $p(H, \mathcal{A})$. (Note that in this final case, R_{3-i} and R_{3-i}' are independent, and Q_3, y_{21} and y_{22} are not defined.)

Let $\mathcal{A}_1 = \{A \in \mathcal{A} : z_1 \in N_H(A) \text{ or } N_H(A) \subseteq V(b_1Dy_1)\}$, $\mathcal{A}_2 = \{A \in \mathcal{A} : z_2 \in N_H(A) \text{ or } N_H(A) \subseteq V(y_1Db_1)\}$, and $A_j = \bigcup_{A \in \mathcal{A}_j} A$ for $j \in [2]$. Let $F_1 := G'[V(x_1z_1 \cup z_1Xp_1) \cup A_1 \cup V(b_1Dy_1)]$ and $F_2 := G'[V(x_2Xp_2) \cup A_2 \cup V(y_1Db_1)]$. Write $b_1Dy_1 = v_1 \dots v_m$ and $x_1z_1 \cup z_1Xp_1 = v_{m+1} \dots v_n$ with $v_1 = b_1$, $v_m = y_1$, $v_{m+1} = x_1$, and $v_n = p_1$. Write $y_1Db_1 = u_1 \dots u_k$ and $p_2Xx_2 = u_{k+1} \dots u_l$ such that $u_1 = y_1$, $u_k = b_1$, $u_{k+1} = p_2$ and $u_l = x_2$. We may assume that

(1) $(F_1, v_1, \dots, v_n)$ and $(F_2, u_1, \dots, u_l)$ are planar.

We only prove that $(F_1, v_1, \dots, v_n)$ is planar; the argument for $(F_2, u_1, \dots, u_l)$ is similar. Suppose $(F_1, v_1, \dots, v_n)$ is not planar. Then by Lemma 2.3.2, there exist

$1 \leq q < r < s < t \leq n$ such that F_1 contains disjoint paths S_1, S_2 from v_q, v_r to v_s, v_t, respectively. By the definition of F_1 (and since X is induced), we see that $r \leq m$ and $s \geq m+1$. Note that y_1Dt_2, $t_2'Dv_q$, v_rDy_1 give rise to independent paths T_1, T_2, T_3, respectively, in B_1 with the same ends. Hence, $G[\{x_1, x_2, y_1, y_2\}] \cup z_2x_2 \cup (z_2Xp_2 \cup Q_2) \cup (R_2 \cup T_1) \cup (R_2' \cup T_2 \cup S_1 \cup v_sXz_1 \cup z_1x_1) \cup (T_3 \cup S_2 \cup v_tXp_1 \cup Q_1)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. This completes the proof of (1).

We may also assume that

(2) $N_H(x_2) \subseteq V(F_2 + x_1)$.

For, suppose there exists $a \in N_H(x_2) - V(F_2 + x_1)$. If $a \notin A$ for all $A \in \mathcal{A}$ let $a' = a$ and $S = a$; and if $a \in A$ for some $A \in \mathcal{A}$ then let $a' \in N_H(A)$ and S be a path in $G[A + a']$ from a to a'.

First, we may choose a and a' so that $a' \notin V(t_1Dy_1 - y_1)$ and no 2-cut of B_1 separating a from y_1Dt_2 is contained in t_1Dy_1. For, otherwise, let T_1, T_2, T_3 be independent paths in B_1 corresponding to $t_2'Dt_1', t_1Da', y_1Dt_2$, respectively. Then $G[\{x_1, x_2, y_2\}] \cup z_1x_1 \cup z_2x_2 \cup (R_1' \cup T_1 \cup R_2') \cup (z_1Xp_1 \cup Q_1) \cup (z_2Xp_2 \cup Q_2) \cup (R_1 \cup T_2 \cup S \cup ax_2) \cup (R_2 \cup T_3 \cup y_1x_1)$ is a TK_5 in G' with branch vertices x_1, x_2, y_2, z_1, z_2.

Suppose that $p(H, \mathcal{A}) - t_1Dt_2 - \{z_1, z_2\}$ has a path T from a' to t_1'. Then T, t_1Dt_2 give rise to independent paths T_1, T_2, respectively, in B_1. So $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup (z_1Xp_1 \cup Q_1) \cup (R_1 \cup t_1T_2y_1) \cup (R_1' \cup T_1 \cup S \cup ax_2) \cup (y_1T_2t_2 \cup R_2 \cup z_2Xp_2 \cup Q_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

So we may assume that such T does not exist. By planarity, there is a cut $\{s_1, s_2\}$ in B_1 separating t_1' from $N_H(x_2) - V(F_2 + x_1)$, with $s_1, s_2 \in V(t_1Dt_2)$. Since $\{s_1, s_2\} \not\subseteq V(t_1Dy_1)$ and $a \notin V(F_2 + x_1)$, we may let $s_1 \in V(t_1Dy_1 - y_1)$ and $s_2 \in V(y_1Dt_2 - y_1)$. Let M be the $\{s_1, s_2\}$-bridge of B_1 containing y_1. We choose $\{s_1, s_2\}$ so that M is minimal (subject to just the property that $s_1 \in V(t_1Dy_1 - y_1)$ and $s_2 \in V(y_1Dt_2 - y_1)$).

Since $\{s_1, s_2, x_1, x_2\}$ cannot be a cut in G, there exists $vv' \in E(G)$ with $v' \in V(M) - \{s_1, s_2\}$ and $v \in V(z_j X p_j - z_j)$ for some $j \in [2]$. By minimality, M has independent paths P_1, P_2 from y_1 to s_{3-j}, v', respectively. Let T_1 be a path in $B_1 - (M - s_j)$ corresponding to $t_2' D t_1'$, and T_2 be a path in $B_1 - (M - s_j)$ corresponding to $t_1 D s_1$ (when $j = 2$) or $s_2 D t_2$ (when $j = 1$). Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-j} x_{3-j} \cup (z_{3-j} X p_{3-j} \cup Q_{3-j}) \cup (R'_{3-j} \cup T_1 \cup R'_j \cup z_j x_j) \cup (R_{3-j} \cup T_2 \cup P_1) \cup (P_2 \cup v'v \cup vXp_j \cup Q_j)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

We may assume

(3) $N(z_1 X p_1 - z_1) \cap V(B_1) \not\subseteq V(F_1)$ or $N(z_2 X p_2 - z_2) \cap V(B_1) \not\subseteq V(F_2)$.

For, suppose $N(z_j X p_j - z_j) \cap V(B_1) \subseteq V(F_j)$ for $j \in [2]$. If $y_2 \in V(X)$ then by (1) and (2), $G - x_1$ is planar; so the assertion of this lemma follows from Lemma 4.2.3. Hence, we may assume $y_2 \notin V(X)$. By (1) and (2), $b = b_1$, and $(G[B_1 \cup z_1 X p_1 \cup p_2 X x_2], p_1, b, p_2, x_2)$ is planar. So G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b, p_1, p_2, x_1, x_2\}$ and $G_2 = G[(B_1 \cup z_1 X p_1 \cup x_2 X p_2) + x_1]$. Clearly, $|V(G_j)| \geq 7$ for $j \in [2]$. Hence, the assertion of this lemma follows from Lemma 4.2.1.

Since the rest of the argument is the same for the two cases in (3), we will assume

(4) $N(z_2 X p_2 - z_2) \cap V(B_1) \not\subseteq V(F_2)$ (and, hence, $e(z_2, B_1) \geq 2$).

Let $vv' \in E(G)$ with $v \in V(B_1 - F_2)$ and $v' \in V(z_2 X p_2 - z_2)$. Let $v'' = v$ and $S = v$ if $v \notin A$ for all $A \in \mathcal{A}$; otherwise, let $v \in A \in \mathcal{A}$ and $v'' \in N_H(A)$ such that $v'' \notin V(F_2)$, and let S be a path in $G[A + v'']$ from v to v''.

Suppose $(p(H, \mathcal{A}) - \{z_1, z_2\}) - t_2' D t_1'$ has independent paths P_1, P_2 from y_1 to t_1, v'', respectively. Then $P_1, P_2, t_2' D t_1'$ give rise to independent paths P_1', P_2', T in B_1, respectively (with the same ends). Now $G[\{x_1, x_2, y_1, y_2\}] \cup z_1 x_1 \cup (R_1 \cup P_1') \cup (z_1 X p_1 \cup Q_1) \cup (R_1' \cup T \cup R_2' \cup z_2 x_2) \cup (P_2' \cup S \cup vv' \cup v' X p_2 \cup Q_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

So we may assume that such P_1, P_2 do not exist in $p(H, \mathcal{A})$. Then by planarity and the existence of $t_1 D y_1$, $p(H, \mathcal{A}) - \{z_1, z_2\}$ has a cut $\{s_1, s_2\}$, separating y_1 from $\{v'', t_1\}$, with $s_1 \in V(t_2' D t_1')$ and $s_2 \in V(t_1 D y_1)$. Clearly, $\{s_1, s_2\}$ is also a cut in B_1. Denote by M_v, M_y the $\{s_1, s_2\}$-bridges of B_1 containing $\{v'', t_1\}$, y_1, respectively. We choose $\{s_1, s_2\}$ so that M_y is minimal. Since v is arbitrary, $N(z_2 X p_2 - z_2) \cap V(B_1 - F_2) \subseteq V(M_v)$. We choose vv' with $v' X x_2$ minimal.

We may assume

(5) $y_{22} \in V(M_v)$ (when defined) and, for any $q \in V(p_2 X v' - v')$, $N(q) \cap V(M_y - \{s_1, s_2\}) = \emptyset$.

Suppose (5) fails. Recall that y_{22} is defined only when $y_2 \in V(X)$, or when $y_2 \notin V(X)$ and both $e(z_1, B_1) \geq 2$ and $e(z_2, B_2) \geq 2$. If y_{22} is defined and $y_{22} \notin V(M_v)$ let $q = b$, $q' = y_{22}$, and $Q' = q'q \cup Q_3$; and if y_{22} is defined and $y_{22} \in V(M_v)$ let $q \in V(p_2 X v' - v')$, $q' \in N(q) \cap V(M_y - \{s_1, s_2\})$, and $Q' = q'q \cup qXp_2 \cup Q_2$.

Since B_1 is 2-connected, there exists $j \in [2]$ such that $M_v - s_{3-j}$ contains disjoint paths T_1, T_2 from $\{t_1, t_1'\}$ to $\{v'', s_j\}$. Note that $R_1 \cup R_1' \cup T_1 \cup T_2$ contains independent paths T_1', T_2' from z_1 to v'', s_j, respectively. If M_y contains independent paths S_1, S_2 from y_1 to q', s_j, then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1 x_1 \cup (z_1 X p_1 \cup Q_1) \cup (T_1' \cup S \cup vv' \cup v' X x_2) \cup (T_2' \cup S_2) \cup (Q' \cup S_1)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. So we may assume S_1, S_2 do not exist in M_y; hence M_y has a cut vertex c that separates y_1 from $\{q', s_j\}$.

By the minimality of M_y and the existence of $y_1 D s_1$, $c \in V(y_1 D t_2' - t_2')$; so we must have $j = 1$. Denote by C_q, C_y the c-bridges of M_y containing $\{q', s_1\}$, y_1, respectively, and choose c with C_y minimal. Then $N(p_2 X v' - v') \cap V(C_y - \{c, s_2\}) = \emptyset$.

We may assume that there exist $uu' \in E(G)$ with $u \in V(z_1 X p_1 - z_1)$ and $u' \in V(C_y) - \{c, s_2\}$. For, otherwise, by (1) and (2), there exists $z \in V(v' X x_2)$ such that $\{c, s_2, x_1, x_2, z\}$ is a cut in G, and G has a separation (G_1, G_2) such that $V(G_1 \cap$

$G_2) = \{c, s_2, x_1, x_2, z\}$, $M_v \cup z_1Xz \subseteq G_1$, $M_y \subseteq G_2$, and $(G_2 - x_1, \{c, s_2, x_2, z\})$ is planar. Clearly, $|V(G_1)| \geq 7$. If $|V(G_2)| \geq 7$ then the assertion of the lemma follows from Lemma 4.2.1. If $|V(G_2)| = 6$ then $z = z_2$ and $y_1z_2 \in E(G)$; now $G[\{x_1, x_2, y_1, z_2\}] - x_1z_2 \cong K_4^-$ in which x_1 is of degree 2, and (ii) holds.

By the minimality of M_y and C_y, $C_y - s_2$ has independent paths U_1, U_2 from y_1 to c, u', respectively. In $M_v - s_1$, we find a path T from t_1 to v''. Let X^* be an induced path in $G - x_1$ from z_1 to x_2 such that $V(X') \subseteq V(R_1 \cup T \cup S \cup vv' \cup v'Xx_2)$. Now $U_1 \cup U_2 \cup (C_q - s_1) \cup u'u \cup uXp_1 \cup Q_1 \cup Q_2 \cup p_2Xq \cup qq'$ is contained in $(G - x_1) - X^*$ and contains a cycle through y_1 and y_2. Hence by Lemma 3.2.1 and Lemma 4.2.1, we may assume that $G - x_1$ contains an induced path X' from z_1 to x_2 such that $y_1, y_2 \notin V(X')$ and $(G - x_1) - X'$ is 2-connected. So the assertion of this lemma follows from Lemma 2.3.6. This proves (5).

We may assume $N(z_1Xp_1 - z_1) \cap V(M_y - \{s_1, s_2\}) \neq \emptyset$. For, otherwise, by (5), G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{s_1, s_2, v', x_1, x_2\}$, $G_2 := G[v'Xx_2 \cup M_y + x_1]$m and $(G_2 - x_1, s_1, s_2, x_2, v')$ is planar. Clearly, $|V(G_1)| \geq 7$. If $|V(G_2)| \geq 7$ then the assertion of this lemma follows from Lemma 4.2.1. So assume $|V(G_2)| = 6$. Then $v' = z_2$ and $y_1z_2 \in E(G)$. So $G[\{x_1, x_2, y_1, z_2\}] - x_1z_2 \cong K_4^-$ in which x_1 is of degree 2, and (ii) holds.

So there exists $uu' \in E(G)$ with $u' \in V(z_1Xp_1 - z_1)$ and $u \in V(M_y) - \{s_1, s_2\}$. Hence, $e(z_1, B_1) \geq 2$; so y_{21}, y_{22}, Q_3 are defined. Let P_u be a path in M_y from u to some $u_D \in V(s_2Ds_1) - \{s_1, s_2\}$ and internally disjoint from $V(D)$ (by minimality of M_y), and P_v be a path in M_v from v'' to some $v_D \in V(s_1Ds_2)$ and internally disjoint from $V(D)$. By the definition of F_2, we may choose v_D so that $v_D \notin V(s_1Dy_{22})$.

We may assume $v_D \in V(t_1'Dy_1 - t_1')$. For, suppose $v_D \in V(y_{22}Dt_1' - y_{22})$. Let T_1, T_2, T_3 be independent paths in B_1 corresponding to t_1Dy_1, v_DDt_1', y_1Dy_{22}, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup (z_1Xp_1 \cup Q_1) \cup (R_1 \cup T_1) \cup (R_1' \cup T_2 \cup P_v \cup S \cup vv' \cup v'Xx_2) \cup (T_3 \cup y_{22}b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Next, we consider the location of u_D. Suppose $u_D \in V(t_2' D s_1 - s_1)$. Let T_1, T_2, T_3 be independent paths in B_1 corresponding to $y_1 D t_2$, $t_2' D u_D$, $y_{21} D y_1$, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_2 x_2 \cup (z_2 X p_2 \cup Q_2) \cup (R_2 \cup T_1) \cup (R_2' \cup T_2 \cup P_u \cup uu' \cup u' X z_1 \cup z_1 x_1) \cup (T_3 \cup y_{21} b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Now suppose $u_D \in V(s_2 D y_1)$. Let T_1, T_2, T_3 be independent paths in B_1 corresponding to $y_1 D t_2$, $t_2' D t_1'$, $u_D D y_1$, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_2 x_2 \cup (z_2 X p_2 \cup Q_2) \cup (R_2 \cup T_1) \cup (R_2' \cup T_2 \cup R_1' \cup z_1 x_1) \cup (T_3 \cup P_u \cup uu' \cup u' X p_1 \cup Q_1)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

So we may assume $u_D \in V(y_1 D t_2' - t_2')$. Let T_1, T_2, T_3 be independent paths in B_1 corresponding to $y_1 D u_D$, $t_2' D t_1'$, $v_D D y_1$, respectively. Thus, $(G - x_1) - (R_1' \cup T_2 \cup R_2' \cup z_2 x_2)$ contains the cycle $T_1 \cup P_u \cup uu' \cup u' X p_1 \cup Q_1 \cup Q_2 \cup p_2 X v' \cup vv' \cup S \cup P_v \cup T_3$. Hence, by Lemma 3.2.1 and Lemma 4.2.1, we may assume that $G - x_1$ contains a path X' from z_1 to x_2 such that $y_1, y_2 \notin V(X')$ and $(G - x_1) - X'$ is 2-connected. So the assertion of this lemma follows from Lemma 2.3.6. ∎

We now prove the existence of three paths A, B, C in $H := G[B_1 + \{z_1, z_2\}]$.

Lemma 4.4.2 *Let $b_1 \in N(y_2) \cap V(B_1)$ when $y_2 \in V(X)$, and let $\{b_1\} = V(B_1) \cap V(B_2)$ when $y_2 \notin V(X)$. Then one of the following holds:*

(i) *G' contains TK_5, or G contains a TK_5 in which x_1 is not a branch vertex.*

(ii) *$G - x_1$ contains K_4^-, or G contains a K_4^- in which x_1 is of degree 2.*

(iii) *There exists $i \in [2]$ such that H contains independent paths A, B, C, with A and C from z_i to y_1 and B from b_1 to z_{3-i}.*

Proof. If $y_2 \notin V(X)$ then by Lemma 4.3.1, let Q_1, Q_2, Q_3 be independent paths in Y_2 from y_2 to p_1, p_2, b, respectively. Moreover, if $y_2 \in V(X)$ then let $Q_1 = Q_2 = Q_3 = y_2$. We may assume that

(1) for $i \in [2]$, H has no path through z_{3-i}, z_i, y_1, b_1 in order.

For, if H has a path S through z_{3-i}, z_i, y_1, b_1 in order. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_i x_i \cup (z_i X p_i \cup Q_i) \cup z_i S y_1 \cup (z_i S z_{3-i} \cup z_{3-i} x_{3-i}) \cup (y_1 S b_1 \cup b_1 b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i.

We may also assume that

(2) for $i \in [2]$ with $e(z_i, B_1 - b_1) \geq 2$, H has a 2-separation (F_i', F_i'') such that $b_1 \in V(F_i')$, $z_i \in V(F_i' - F_i'')$ and $\{y_1, z_{3-i}\} \subseteq V(F_i'' - F_i')$.

Suppose $i \in [2]$ and $e(z_i, B_1 - b_1) \geq 2$. Let K be obtained from H by duplicating z_i and y_1 with copies z_i' and y_1', respectively. So in K, y_1 and y_1' are not adjacent, but have the same set of neighbors, namely $N_H(y_1)$; and the same holds for z_i and z_i'.

Suppose K contains disjoint paths A', B', C' from $\{z_i, z_i', b_1\}$ to $\{y_1, y_1', z_{3-i}\}$, with $z_i \in V(A'), z_i' \in V(C')$ and $b_1 \in V(B')$. If $z_{3-i} \notin V(B')$ then, after identifying y_1 with y_1' and z_i with z_i', we obtain from $A' \cup B' \cup C'$ a path in H from z_{3-i} to b_1 through z_i, y_1 in order, contradicting (1). Hence $z_{3-i} \in V(B')$, and we get the desired paths for (iii) from $A' \cup B' \cup C'$, by identifying y_1 with y_1' and z_i with z_i'.

So we may assume that such A', B', C' do not exist. Then K has a separation (K', K'') such that $|V(K' \cap K'')| \leq 2$, $\{z_i, z_i', b_1\} \subseteq V(K')$ and $\{y_1, y_1', z_{3-i}\} \subseteq V(K'')$. Since $H - z_{3-i}$ is 2-connected, $z_{3-i} \notin V(K' \cap K'')$.

We claim that $z_i, z_i' \notin V(K' \cap K'')$. For, if exactly one of z_i, z_i' is in $V(K' \cap K'')$ then, since z_i, z_i' have the same set of neighbors in K, $V(K' \cap K'') - \{z_i, z_i'\}$ is a cut in H separating $\{z_{3-i}, y_1\}$ from $\{z_i, b_1\}$, a contradiction. Now assume $\{z_i, z_i'\} = V(K' \cap K'')$. Then z_i is a cut vertex in H separating b_1 from $\{y_1, z_{3-i}\}$, a contradiction.

We may assume that $y_1, y_1' \notin V(K' \cap K'')$. First, suppose exactly one of y_1, y_1' is in $V(K' \cap K'')$. Then, since y_1, y_1' have the same set of neighbors in K, $V(K' \cap K'') - \{y_1, y_1'\}$ is a cut in H separating $\{z_{3-i}, y_1\}$ from $\{z_i, b_1\}$, a contradiction. Now assume $\{y_1, y_1'\} = V(K' \cap K'')$. Then y_1 is a cut vertex in H separating z_{3-i} from $\{b_1, z_i\}$. This implies that $N(z_{3-i}) \cap V(B_1) = \{y_1\}$; so $y_2 \notin V(X)$ and $z_{3-i} = p_{3-i}$.

We may assume $i = 2$; for otherwise, $G[\{x_1, x_2, y_1, z_2\}] - x_1z_2 \cong K_4^-$ in which x_1 is of degree 2, and (ii) holds. Then $z_1 = p_1$, and G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b, p_2, x_1, x_2, y_1\}$ and $G_2 = G[B_1 \cup x_2Xp_2 + \{x_1, b\}]$. Note that $x_1x_2y_1x_1$ is a triangle and $|V(G_j)| \geq 7$ for $j \in [2]$. So the assertion of this lemma follows from Lemma 4.2.2.

Thus, since B_1 is 2-connected, $|V(K' \cap K'')| = 2$. Let $V(K' \cap K'') = \{s, t\}$, and let F_i' (respectively, F_i'') be obtained from K' (respectively, K'') by identifying z_i' with z_i (respectively, y_1' with y_1). Then (F_i', F_i'') gives the desired 2-separation in H, completing the proof of (2).

We now consider three cases.

Case 1. $e(z_i, B_1 - b_1) \geq 2$ for $i \in [2]$.

For $i \in [2]$, let $V(F_i' \cap F_i'') = \{s_i, t_i\}$ as in (2). Let Z_1, B_1' denote the $\{s_1, t_1\}$-bridges of F_1' containing z_1, b_1, respectively, and let Y_1, Z_2 denote the $\{s_1, t_1\}$-bridges of F_1'' containing y_1, z_2, respectively.

Suppose $Y_1 \neq Z_2$, and suppose $Z_1 \neq B_1'$ or $b_1 \in \{s_1, t_1\}$. Let $b_1 = s_1$ if $b_1 \in \{s_1, t_1\}$. Then Z_1 has independent paths S_1, T_1 from z_1 to s_1, t_1, respectively. Moreover, Z_2 has independent paths S_2, T_2 from z_2 to s_1, t_1, respectively, $B_1' - t_1$ has a path P from s_1 to b_1, and Y_1 has independent paths S_3, T_3 from y_1 to s_1, t_1, respectively. So $x_1z_1 \cup (z_1Xp_1 \cup Q_1) \cup x_1y_2 \cup (z_2Xp_2 \cup Q_2) \cup z_2x_2x_1 \cup (T_2 \cup T_1) \cup S_1 \cup S_2 \cup (S_3 \cup y_1x_1) \cup (P \cup b_1b \cup Q_3)$ is a TK_5 in G' with branch vertices s_1, x_1, y_2, z_1, z_2.

Thus, we may assume that $Y_1 = Z_2$, or $Z_1 = B_1'$ and $b_1 \notin \{s_1, t_1\}$. First, suppose $Y_1 \neq Z_2$. Then $Z_1 = B_1'$ and $b_1 \notin \{s_1, t_1\}$, and hence $B_1' - \{s_1, t_1\}$ has a path from z_1 to b_1. Since H is 2-connected, $Y_1 \cup Z_2$ has two independent paths from y_1 to z_2. However, this contradicts the existence of the separation (F_2', F_2'').

So $Y_1 = Z_2$. Thus, by symmetry, we may assume $t_2 \in V(Y_1) - \{s_1, t_1\}$. Suppose $b_1 \notin \{s_1, t_1\}$ and $B_1' = Z_1$. Then $s_2 \in V(B_1') - \{s_1, t_1\}$. Moreover, $\{s_2, t_2\}$ separates s_1 from t_1 in H; for otherwise, either t_2 separates z_2 from $\{b_1, y_1, z_1\}$ in H, or t_2

separates y_1 from $\{b_1, z_1, z_2\}$ in H, a contradiction. Thus, we may assume that in H, $\{s_2, t_2\}$ separates $\{b_1, s_1, z_2\}$ from $\{t_1, y_1, z_1\}$. However, this contradicts the existence of Y, Z.

Therefore, $B_1' \neq Z_1$ or $b_1 \in \{s_1, t_1\}$. If $b_1 \notin \{s_1, t_1\}$ then $B_1' \neq Z_1$; so $s_2 \in \{s_1, t_1\}$ (because of (F_2', F_2'')), and we may assume $s_2 = s_1$. If $b_1 \in \{s_1, t_1\}$ then we may assume that $b_1 = s_1$; so $s_2 = s_1$ or, in Z_1, s_2 separates s_1 from $\{t_1, z_1\}$. Let Y_1', Z_2' be the t_2-bridges of $Y_1 - \{s_1, t_1\}$ containing y_1, z_2, respectively. Again, because of the existence of (F_2', F_2''), t_1 has no neighbor in $Z_2' - t_2$. Hence, by the existence of Y, Z, s_1 has a neighbor in $Y_1' - t_2$; and, thus, $s_2 = s_1$ and $G[Y_1' + \{s_1, t_1\}]$ has disjoint paths S_1, T_1 from s_1, t_1 to y_1, t_2, respectively. Let S_2, T_2 be independent paths in $G[Z_2' + s_1]$ from z_2 to s_1, t_2, respectively, and S, T be independent paths in Z_1 from z_1 to s_1, t_1, respectively. Let P be a path in $B_1' - t_1$ from s_1 to b_1. Then $x_1z_1 \cup (z_1Xp_1 \cup Q_1) \cup x_1y_2 \cup (z_2Xp_2 \cup Q_2) \cup z_2x_2x_1 \cup (T_2 \cup T_1 \cup T) \cup S \cup (S_1 \cup y_1x_1) \cup S_2 \cup (P \cup b_1b \cup Q_3)$ is a TK_5 in G' with branch vertices s_1, x_1, y_2, z_1, z_2.

Case 2. $e(z_2, B_1 - b_1) \geq 2$.

If $y_2 \in V(X)$ then $e(z_1, B_1 - b_1) \geq 2$, and if $y_2 \notin V(X)$ then, by Lemma 4.3.3, $e(z_1, B_1 - b_1) \geq 1$. In view of Case 1, we may assume $e(z_1, B_1 - b_1) = 1$; so $z_1 = p_1$ and $y_2 \notin V(X)$. Note that if $b \neq b_1$ then, by Lemma 4.3.2, we may assume $z_1b_1 \in E(G)$; so $b_1 \in V(F_2' \cap F_2'')$. By Lemma 4.3.1, we may assume that Y_2 has a path Q from p_2 to b_1 through y_2, z_1 in this order.

For convenience, let $F' := F_2'$, $F'' := F_2''$, $s := s_2$ and $t := t_2$. So $b_1, z_2 \in V(F')$ and $y_1, z_1 \in V(F'')$. We choose (F', F'') so that F'' is minimal. Let z_1' denote the unique neighbor of z_1 in $B_1 - b_1$.

Subcase 2.1. $N(z_2Xp_2 - z_2) \cap V(F'' - \{z_1, s, t\}) \not\subseteq \{z_1'\}$.

Let $uu' \in E(G)$, with $u \in V(F'') - \{z_1, z_1', s, t\}$ and $u' \in V(z_2Xp_2 - z_2)$. Note that F' contains a path S from z_2 to b such that $|V(S) \cap \{s, t\}| \leq 1$. Moreover, if there exists $r \in \{s, t\}$ such that $r \in V(S)$ for all such path S, then $b_1 = r$.

If $(F'' - z_1) - S$ contains independent paths T_1, T_2 from y_1 to z_1', u, respectively, then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup z_1Qy_2 \cup (z_1Qb \cup bb_1 \cup S \cup z_2x_2) \cup (z_1z_1' \cup T_1) \cup (T_2 \cup uu' \cup u'Xp_2 \cup p_2Qy_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

So we may assume that such T_1, T_2 do not exist. Hence, there is a cut vertex c in $(F''-z_1)-S$ separating y_1 from $\{u, z_1'\}$. Denote by M_1, M_2 the $(\{c\} \cup (V(S) \cap \{s,t\}))$-bridges of $F'' - z_1$ containing y_1, $\{u, z_1'\}$, respectively. We may choose c so that M_1 is minimal. Then $N(z_2Xp_2 - z_2) \cap V(F'') \subseteq V(M_2)$ (as uu' was chosen arbitrarily).

Since G is 5-connected, $\{s,t\} \subseteq V(M_1)$ (as otherwise $\{c, x_1, x_2\} \cup (\{s,t\} \cap V(M_1))$ would be a cut in G), and M_1 contains independent paths R_1, R_2, R_3 from y_1 to c, s, t, respectively. Since B_1 is 2-connected, $\{s,t\} \cap V(M_2) \neq \emptyset$ and there exist choices of u and $r \in \{s,t\} \cap V(M_2)$ such that M_2 contains disjoint paths R_4, R_5 from $\{z_1', u\}$ to $\{c, r\}$. Thus, $R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5$ contains independent paths from y_1 to z_1', u, respectively. By the non-existence of T_1 and T_2, $r \in V(S)$ for every choice of S. Hence, $b_1 = r$. So $\{s,t\} \cap V(M_2) = \{r\}$, and $V(S) \cap \{s,t\} = \{r\}$ for every choice of S. Without loss of generality, we may assume that $r = t$.

We further choose uu' so that $u'Xp_2$ is maximal. Suppose $N(u'Xp_2 - u') \cap V(F' - \{s,t\}) = \emptyset$. Then G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{s, t, u', x_1, x_2\}$ and $G_2 = G[F' \cup x_2Xu' + x_1]$. Clearly, $|V(G_1)| \geq 7$. Since $e(z_2, B_1 - b_1) \geq 2$, $|V(G_2)| \geq 7$. If $(G_2 - x_1, x_2, s, t, u')$ is planar then the assertion of this lemma follows from Lemma 4.2.1. Hence, we may assume, by Lemma 2.3.1, that $G_2 - x_1$ contains disjoint paths X_1, X_2 from u', x_2 to s, t, respectively. Let X_3 be a path in $M_2 - t$ from z_1' to c. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup z_1Qy_2 \cup (z_1Qb \cup bb_1 \cup X_2) \cup (z_1z_1' \cup X_3 \cup R_1) \cup (R_2 \cup X_1 \cup u'Xp_2 \cup p_2Qy_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

So assume that there exists $ww' \in E(G)$ with $w' \in V(u'Xp_2 - u')$ and $w \in V(F' - \{s,t\})$. Let S_1 be a path in $F' - t$ from w to s and S_2 be a path in $M_2 - t$ from z_1' to u. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup z_1Qy_2 \cup (z_1Qb \cup bb_1 \cup R_3) \cup (z_1z_1' \cup S_2 \cup$

$uu' \cup u'Xx_2) \cup (R_2 \cup S_1 \cup ww' \cup w'Xp_2 \cup p_2Qy_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Subcase 2.2. $N(z_2Xp_2 - z_2) \cap V(F'' - \{z_1, s, t\}) \subseteq \{z_1'\}$.

Then $\{s, t, x_1, x_2, z_1'\}$ is a 5-cut in G separating F'' from $F' \cup Y_2$. Since G is 5-connected, $F'' - z_1$ has independent paths T_1, T_2, T_3 from y_1 to s, t, z_1', respectively.

Let $F_g := (F'' - z_1) + \{g, gs, gt\}$, where g is a new vertex. Since G is 5-connected and we are in Subcase 2.2, F_g has no 2-cut separating y_1 from $\{g, z_1'\}$. Hence, by Lemma 2.3.5, there is a cycle in F_g containing $\{g, y_1, z_1'\}$ and, after removing g from this cycle, we get a path R in $F'' - z_1$ from s to t and containing $\{y_1, z_1'\}$.

Let $x = p_2$ if $N(z_2Xp_2 - z_2) \cap V(F'' - \{z_1, s, t\}) = \emptyset$ and, otherwise, let $x \in N(z_1') \cap N(z_2Xp_2 - z_2)$ with xXz_2 minimal.

We may assume that $N(xXp_2 - x) \cap V(B_1 - \{b_1, z_1'\}) = \emptyset$. For, otherwise, there exists $rr' \in E(G)$ such that $r \in V(B_1) - \{b_1, z_1'\}$ and $r' \in V(xXp_2 - x)$. Then $r \in V(F')$ and $x \neq p_2$; so $xz_1' \in E(G)$. Note that F' has disjoint paths from $\{s, t\}$ to $\{b_1, r\}$, which, combined with T_1, T_2, gives independent paths P_1, P_2 in $B_1 - z_1'$ from y_1 to b_1, r, respectively. Hence, in $(G - x_1) - (z_1z_1'x \cup xXx_2)$, $\{y_1, y_2\}$ is contained in the cycle $P_1 \cup P_2 \cup r'Xp_2 \cup Q_2 \cup Q_3 \cup bb_1$. Hence, by Lemma 3.2.1 and Lemma 4.2.1, we may assume that $G - x_1$ has a path X' from z_1 to x_2 such that $y_1, y_2 \notin V(X)$, and $(G - x_1) - X'$ is 2-connected. Thus, the assertion of this lemma follows from Lemma 2.3.6.

We may assume $b = b_1$. For, suppose $b \neq b_1$. Then, using the notation from (*iv*) of Lemma 4.2.5, $v \in V(p_2Xx_2 - p_2)$ and $b_1' \in V(B_1 - b_1)$. Let P_1, P_2 be independent paths in B_1 from y_1 to b_1, b_1', respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup z_1Qy_2 \cup (z_1Qb \cup bb_1 \cup P_1) \cup (z_1Qb \cup bv \cup vXx_2) \cup (P_2 \cup b_1'p_2 \cup p_2Qy_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Therefore, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b_1, s, t, x, x_1, x_2\}$ and $G_2 = G[F' \cup xXx_2 + x_1]$. Let $G_2' = G_2 + \{r, rs, rt\}$, where r is a new vertex.

We may assume that $(G_2' - x_1, \mathcal{A}, b_1, x, x_2, r)$ is 3-planar for some collection $\mathcal{A}$ of subsets of $V(G_2' - x_1) - \{b_1, x, x_2, r\}$. For, otherwise, by Lemma 2.3.1, $G_2' - x_1$ contains disjoint paths R_1, R_2 from b_1, x to x_2, r, respectively. Let $R = T_2 \cup (R_2 - r)$ if $R_2 - r$ ends at t, and $R = T_1 \cup (R - r)$ otherwise. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup z_1Qy_2 \cup (z_1Qb_1 \cup R_1) \cup (z_1z_1' \cup T_3) \cup (R \cup xXp_2 \cup p_2Qy_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

We choose $\mathcal{A}$ to be minimal and define J, s', t' as follows. If $\mathcal{A} = \emptyset$ then after relabeling of s, t (if necessary), we may assume $(G_2' - x_1, b_1, x, x_2, s, t)$ is planar and let $J = G_2$, $s' = s$ and $t' = t$. Now assume $\mathcal{A} \neq \emptyset$. Then, by the minimality of $\mathcal{A}$ and 5-connectedness of G, $\mathcal{A}$ has a unique member, say A, such that $r \in N(A)$ and $\{s, t\} \subseteq A$ and, moreover, $G'[A \cup \{s', t'\}]$ is connected, where $N(A) \cap V(F') = \{r, s', t'\}$. Let J denote the $\{s', t', x_1\}$-bridge of G_2' containing $\{b_1, x, x_2\}$. We may assume, after suitable labeling of s', t', $(J - x_1, b_1, x, x_2, s', t')$ is planar.

Suppose $b_1 \in \{s', t'\}$. Then G has a 5-separation (L_1, L_2) such that $V(L_1 \cap L_2) = \{s', t', x, x_1, x_2\}$ and $L_2 = J$. If $|V(J)| \geq 7$ then the assertion of this lemma follows from Lemma 4.2.1. So assume $|V(J)| \leq 6$. Since $e(z_2, B_1 - b_1) \geq 2$, there exists $v \in N(z_2) \cap V(F' - \{s', t', z_2\})$. Since G is 5-connected, $vx_1, vx_2 \in E(G)$. Hence, $G[\{v, x_1, x_2, z_2\}]$ contains a K_4^- in which x_1 is of degree 2.

Thus, we may assume that $b_1 \notin \{s', t'\}$. Then G has a 6-separation (L_1, L_2) such that $V(L_1 \cap L_2) = \{b_1, s', t', x, x_1, x_2\}$ and $L_2 = J$. If $|V(J)| \geq 8$ then the assertion of this lemma follows from Lemmas 2.3.12 and 4.2.1.

So assume $|V(J)| \leq 7$. By planarity of J and 2-connectedness of B_1, $z_2t' \notin E(G)$. Thus, since $e(z_2, B_1 - b_1) \geq 2$, $z_2s' \in E(G)$ and there exists $v \in V(J - \{s', t', x, x_2, z_2\}$ such that $z_2v \in E(G)$. So $|V(J)| = 7$ and $z_2 = x$. By the minimality of F', $vt' \in E(G)$; and by the 2-connectedness of B_1, $vs', vb_1 \in E(G)$. We may assume $x_2v \notin E(G)$, as otherwise $G[\{s', v, x_2, z_2\}]$ (and, hence, $G - x_1$) contains a K_4^- and (ii) holds. Thus, $vx_1 \in E(G)$ as G is 5-connected. Moreover, $z_2 = p_2$ as otherwise,

$z_2x_1 \not E(G)$ (as G is 5-connected) and $G[\{s', v, x_1, z_2\}] - x_1s' \cong K_4^-$ in which x_1 is of degree 2; so (ii) holds.

If $F'' - z_1$ has independent paths P_1, P_2 from t' to s', z_1', respectively, and if Y_2 has a cycle C containing $\{p_1, p_2, y_2\}$ then $G[\{b_1, t', v\}] \cup z_2v \cup (z_2s' \cup P_1) \cup C \cup (z_1z_1' \cup P_2) \cup (z_1x_1v)$ is a TK_5 in G with branch vertices b_1, t', v, z_1, z_2. So we may assume P_1, P_2 do not exist, or C does not exist.

First, suppose P_1, P_2 do not exist in $F'' - z_1$. Then $F'' - z_1$ has 1-separation (L_1, L_2) such that $t' \in V(L_1 - L_2)$ and $\{s', z_1'\} \subseteq V(L_2)$. Since G is 5-connected, $|V(L_1)| = 2$ and $x_1t' \in E(G)$. Now $G[\{b_1, t', v, x_1\}] - x_1b_1 \cong K_4^-$ in which x_1 is of degree 2, and (ii) holds.

Now assume C does not exist. Then by Lemma 2.3.5, Y_2 has 2-cuts S_b, S_z such that b_1 is a in component D_b of $Y_2 - S_b$, $p_1 = z_1$ is in a component D_z of $Y_2 - S_z$, and $V(D_b) \cap (V(D_z) \cup S_z \cup \{p_2\}) = \emptyset = V(D_z) \cap (V(D_b) \cup S_b \cup \{p_2\})$. If $y_2 \notin V(D_b)$ then $S_b \cup \{t', v\}$ is a cut in G, a contradiction. So $y_2 \in V(D_b)$. Then $y_2 \in V(D_z)$. Then $S_z \cup \{x_1, z_1'\}$ is a cut in G, a contradiction.

Case 3. $e(z_2, B_1 - b_1) \leq 1$.

If $y_2 \in V(X)$ then, since G is 5-connected, $e(z_1, B_1 - b_1) \geq 2$ and $e(z_2, B_1 - b_1) = 1$. If $y_2 \notin V(X)$ then, by Lemma 4.3.3, $e(z_2, B_1 - b_1) = 1$ and $e(z_1, B_1 - b_1) \geq 2$.

For convenience, let $F' := F_1'$, $F'' := F_1''$, $s := s_1$ and $t := t_1$. Then $b_1, z_1 \in V(F')$ and $y_1, z_2 \in V(F'') - V(F')$. We choose (F', F'') so that F'' is minimal. Let z_2' denote the unique neighbor of z_2 in $B_1 - b_1$. Note that if $z_2 \neq p_2$ then $z_2b_1 \in E(G)$. By (iii) of Lemma 4.3.1, $G[Y_2 + b_1 + p_2Xz_2]$ contains a path Q from p_1 to b_1 through y_2, p_2 in order.

Subcase 3.1. $N(z_1Xp_1 - z_1) \cap V(F'' - \{z_2, s, t\}) \not\subseteq \{z_2'\}$.

Let $uu' \in E(G)$ with $u' \in V(z_1Xp_1 - z_1)$ and $u \in V(F'' - \{s, t, z_2, z_2'\})$. Since B_1 is 2-connected, F' contains a path S from z_1 to b_1 such that $|V(S) \cap \{s, t\}| \leq 1$.

Suppose $(F'' - z_2) - S$ contains independent paths S_1, S_2 from y_1 to z_2', u, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_2x_2 \cup z_2Qy_2 \cup (z_2Qb_1 \cup S \cup z_1x_1) \cup (z_2z_2' \cup S_1) \cup (S_2 \cup uu' \cup u'Xp_1 \cup p_1Qy_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

So we may assume that such S_1, S_2 do not exist in $(F'' - z_2) - S$ for any choice of S and any choice of u. Hence, $(F'' - z_2) - S$ has a cut vertex c which separates y_1 from $N(z_1Xp_1 - z_1) \cup \{z_2'\}$. Denote by M_1, M_2 the $(\{c\} \cup (\{s,t\} \cap V(S)))$-bridges of $F'' - z_2$ containing y_1, $(N(z_1Xp_1 - z_1) \cap V(F'' - \{s, t, z_2\})) \cup \{z_2'\}$, respectively. Since G is 5-connected, $\{s,t\} \subseteq V(M_1)$ (to avoid the cut $\{c, x_1, x_2\} \cup (V(S) \cap \{s,t\})$) and M_1 contains independent paths R_1, R_2, R_3 from y_1 to c, s, t, respectively. Since B_1 is 2-connected, $\{s,t\} \cap V(M_2) \neq \emptyset$, say $t \in V(M_2)$. Note that M_2 contains disjoint paths T_1, T_2 from $\{z_2', u\}$ to $\{c, t\}$. Now $R_1 \cup R_3 \cup T_1 \cup T_2$ contains independent paths from y_1 to z_2', u, respectively, which avoids s and uses t. So by the nonexistence of S_1, S_2, $t \in V(S)$ for any choice of S, which implies $b_1 = t$.

Choose uu' so that $u'Xp_1$ is maximal. Since $\{x_1, u', s, t\}$ cannot be a cut in G separating F' from $F'' \cup Y_2 \cup p_2Xx_2$, there exists $ww' \in E(G)$ such that $w \in V(F' - \{s, t, z_1\})$ and $w' \in V(u'Xp_1 - u') \cup V(p_2Xx_2)$.

Suppose $w' \in V(u'Xp_1 - u')$. Let P_1 be a path in $F' - \{z_1, t\}$ from w to s and P_2 be a path in $M_2 - t$ from z_2' to u. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_2x_2 \cup z_2Qy_2 \cup (z_2Qb_1 \cup R_3) \cup (z_2z_2' \cup P_2 \cup uu' \cup u'Xz_1 \cup z_1x_1) \cup (R_2 \cup P_1 \cup ww' \cup w'Xp_1 \cup p_1Qy_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Now assume $w' \in V(p_2Xx_2)$. Then $F' - t$ contains a path W from z_1 to w. Hence $X' := W \cup ww' \cup w'Xx_2$ is a path in $G - x_1$ from z_1 to x_2 such that in $(G - x_1) - X'$, $\{y_1, y_2\}$ is contained in a cycle (which is contained in $(Y_2 - p_2) \cup p_1Xu' \cup u'u \cup M_2 \cup (M_1 - s)$). Hence by Lemma 3.2.1 and Lemma 4.2.1, we may assume that X' is such that $y_1, y_2 \notin V(X)$, and $(G - x_1) - X'$ is 2-connected. Thus, the assertion of this lemma follows from Lemma 2.3.6.

Subcase 3.2. $N(z_1Xp_1 - z_1) \cap V(F'' - \{z_2, s, t\}) \subseteq \{z_2'\}$.

First, we show that $\{s, t, x_1, x_2, z_2'\}$ is a 5-cut in G separating $F'' - z_2$ from $F' \cup Y_2 \cup X$. For, otherwise, there exists $ww' \in E(G)$ with $w \in V(F'') - \{s, t\}$ and $w' \in V(p_2 X z_2 - z_2)$. Let P_1, P_2 be independent paths in F' from z_1 to r, b_1, respectively, with $r \in \{s, t\}$. Without loss of generality, we may assume $r = s$. By the minimality of F'', $F'' - t$ has independent paths R_1, R_2 from y_1 to s, w, respectively. Now $G[\{x_1, x_2, y_1, y_2\}] \cup z_1 x_1 \cup (z_1 X p_1 \cup Q_1) \cup (P_1 \cup R_1) \cup (P_2 \cup b_1 z_2 x_2) \cup (R_2 \cup ww' \cup w' X p_2 \cup Q_2)$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_1.

Hence, since G is 5-connected, $F'' - z_2$ contains independent paths T_1, T_2, T_3 from y_1 to s, t, z_2', respectively, and $F'' - z_2$ has no 2-cut separating y_1 from $\{s, t, z_2'\}$. Let $F_g := (F'' - z_2) + \{g, gs, gt\}$, where g is a new vertex. Then by Lemma 2.3.5, F_g has a cycle containing $\{g, y_1, z_2'\}$. Thus, we may assume by symmetry that $F'' - z_2$ has a path S from s to t and through y_1, z_2' in order.

We may assume $N(x_2) \cap V(F' - \{s, t\}) = \emptyset$. For, suppose there exists $x_2^* \in N(x_2) \cap V(F' - \{s, t\})$. Since B_1 is 2-connected, F' contains independent paths R_1, R_2 from z_1 to x_2^*, r, respectively, for some $r \in \{s, t\}$. (This can be done by considering whether or not z_1 and x_2^* are contained in the same $\{s, t\}$-bridge of F'.) Let $T = T_1$ if $r = s$, and $T = T_2$ if $r = t$. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1 x_1 \cup (z_1 X p_1 \cup Q_1) \cup (R_1 \cup x_2^* x_2) \cup (R_2 \cup T) \cup (Q_2 \cup p_2 X z_2 \cup z_2 z_2' \cup T_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Let $x = p_1$ if $N(z_2') \cap V(z_1 X p_1 - z_1) = \emptyset$, and otherwise let $x \in N(z_2') \cap V(z_1 X p_1 - z_1)$ with $z_1 X x$ minimal.

Suppose $z_2' x_2 \in E(G)$. Then we may assume $x_1 z_2 \notin E(G)$; for otherwise, $G[\{x_1, x_2, z_2, z_2'\}] - x_1 z_2' \cong K_4^-$ in which x_1 is of degree 2, and (ii) holds. Hence, $z_2 = p_2$, and $\{b_1, s, t, x, x_1\}$ is a 5-cut in G separating $F' \cup z_1 X x$ from $F'' \cup Y_2$. Since G is 5-connected, $b_1 \notin \{s, t\}$. Let (G_1, G_2) be a 5-separation in G such that $V(G_1 \cap G_2) = \{b_1, s, t, x, x_1\}$ and $G_2 = G[F' \cup z_1 X x + x_1]$. Clearly, $|V(G_i)| \geq 7$ for $i \in [2]$. If $(G_2 - x_1, b_1, x, s, t)$ is planar then the assertion of this lemma follows from

Lemma 4.2.1. So we may assume that this is not the case. Then by Lemma 2.3.1, $G_2 - x_1$ has disjoint paths S_1, S_2 from s, t to b_1, x, respectively. Now $z_2z_2'x_2z_2 \cup y_1x_2 \cup y_1Sz_2' \cup (y_1Ss \cup S_1 \cup b_1Qz_2) \cup y_2Qz_2 \cup (y_2Qp_1 \cup p_1Xx \cup S_2 \cup tSz_2') \cup y_2x_2 \cup y_2x_1y_1$ is a TK_5 in G' with branch vertices x_2, z_2, z_2', y_1, y_2.

Now assume $z_2'x_2 \notin E(G)$. Then x_2 has a neighbor in $F'' - \{y_1, z_2'\}$. Let r be a new vertex. We may assume that $(F'' + \{r, rs, rt\}) - z_2$ has disjoint paths S_1, S_2 from r, z_2' to x_2, y_1, respectively. For, suppose such paths S_1, S_2 do not exist. Then by Lemma 2.3.1, there exists a collection $\mathcal{A}$ of disjoint subsets of $F_2'' - \{x_2, y_1, z_2\}$ such that $(F'' + \{r, rs, rt\}) - z_2, r, y_1, x_2, z_2')$ is 3-planar. By the minimality of F'', we may assume $(F'' - z_2, s, t, y_1, x_2, z_2')$ is planar. Thus, since z_2' is the only neighbor of z_2 in $F'' - F'$, G has a 5-separation (G_1', G_2') with $V(G_1' \cap G_2') = \{s, t, x_1, x_2, z_2\}$ and $G_2' - x_1 = F''$. Moreover, $(G_2' - x_1, s, t, x_2, z_2)$ is planar. Since $|V(G_j')| \geq 7$ for $j \in [2]$, the assertion of this lemma follows from Lemma 4.2.1.

Without loss of generality, let $rs \in S_1$. If $F' - t$ has independent paths P_1, P_2 from z_1 to s, b_1, respectively, then $G[\{x_1, x_2, y_2\}] \cup z_1x_1 \cup (P_1 \cup (S_1 - r)) \cup (z_1Xp_1 \cup p_1Qy_2) \cup (z_2z_2' \cup S_2 \cup y_1x_1) \cup z_2x_2 \cup z_2Qy_2 \cup (z_2Qb_1 \cup P_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_2, z_1, z_2. So we may assume that such P_1, P_2 do not exist in $F' - t$.

Thus F' has a 2-separation (F_1, F_2) such that $t \in V(F_1 \cap F_2)$, $z_1 \in V(F_1 - F_2)$ and $\{b_1, s\} \subseteq V(F_2 - F_1)$. Choose this separation so that F_1 is minimal. Let $s' \in V(F_1 \cap F_2) - \{t\}$. Since $\{s', t, z_1, x_1\}$ cannot be a cut in G, $V(F_1) = \{s', t', z_1\}$ or there exists $zz' \in E(G)$ such that $z \in V(z_1Xp_1 - z_1) \cup V(p_2Xz_2 - z_2)$ and $z' \in V(F_1) - \{z_1, s', t\}$.

First, assume $V(F_1) = \{s', t', z_1\}$. Then $z_1 = p_1$ as G is 5-connected. By (iii) of Lemma 4.3.1, let Q' be a path in Y_2 from p_2 to b_1 and through y_2, p_1 in order, and let C be a cycle in $Y_2 - b_1$ containing $\{p_1, p_2, y_2\}$. Let $C' := Q' \cup p_2Xz_2 \cup z_2b_1$ If $z_2 \neq p_2$; and let $C' := C$ if $z_2 = p_2$. If $F' - \{b_1, t, z_1\}$ has a path S from s' to s then $x_1x_2y_2x_1 \cup z_1x_1 \cup z_2x_2 \cup C' \cup (z_1s' \cup S \cup S_1) \cup (z_2z_2' \cup S_2 \cup y_1x_1)$ is a TK_5 in G' with branch vertices x_1, x_2, y_2, z_1, z_2. So we may assume such S does not exist.

Then F' has a separation (F_1', F_2'') such that $V(F_1' \cap F_1'') = \{b_1, t\}$, $\{s', z_1\} \subseteq V(F_1')$ and $s \in V(F_1'') - \{b_1, t\}$. Since G is 5-connected, $\{b_1, t, x_1, z_1\}$ is not a cut in G, and $F_1' - \{b_1, t, z_1\}$ has a path S' from s' to some $z \in N(p_2 X z_2 - z_2)$. Let $z' \in N(z) \cap V(p_2 X z_2 - z_2)$. Let S be a path in $F_2 - t$ from s to b_1. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1 x_1 \cup Q_1 \cup (z_1 s' \cup S' \cup zz' \cup z' X x_2) \cup (z_1 t \cup T_2) \cup (T_1 \cup S \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Thus, we may assume that $zz' \in E(G)$ such that $z \in V(z_1 X p_1 - z_1) \cup V(p_2 X z_2 - z_2)$ and $z' \in V(F_1) - \{z_1, s', t\}$.

Suppose $z \in V(x X p_1 - x)$. Let $X' = z_1 X x \cup x z_2' z_2 x_2$. Then, $T_1 \cup T_2 \cup (F' - z_1) \cup zz' \cup z X p_1 \cup Y_2$ is contained in $G - X'$ and has a cycle containing $\{y_1, y_2\}$. Hence, by Lemma 3.2.1 and then Lemma 4.2.1, we may assume that $G - x_1$ has an induced path X'' from z_1 to x_2 such that $y_1, y_2 \notin V(X'')$ and $G - X''$ is 2-connected. Then the assertion of this lemma follows from Lemma 2.3.6.

Now suppose $z \in V(p_2 X z_2 - z_2)$. By the minimality of F_1, $F_1 - t$ has independent paths L_1, L_2 from z_1 to s', z', respectively. In $F_2 \cup (F'' - z_2)$, we find independent paths L_1', L_2' from y_1 to s', b_1, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1 x_1 \cup (z_1 X p_1 \cup Q_1) \cup (L_1 \cup L_1') \cup (L_2 \cup z' z \cup z X x_2) \cup (L_2' \cup b_1 b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Hence, we may assume $z \in V(z_1 X x - z_1)$ for all such zz'. Choose such z with $z_1 X z$ is maximal. Since $\{s', t, x_1, z\}$ cannot be a cut in G, there exists $uu' \in E(G)$ such that $u \in V(z_1 X z - \{z_1, z\})$ and $u' \in V(F_2) - \{s', t\}$. Let P_1 be a path in $F_1 - \{s', z_1\}$ from z' to t, and P_2 be a path in $F_2 - t$ from u' to b_1. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_2 x_2 \cup (z_2 z_2' \cup T_3) \cup (z_2 X p_2 \cup p_2 Q y_2) \cup (z_2 Q b_1 \cup P_2 \cup u' u \cup u X z_1 \cup z_1 x_1) \cup (T_2 \cup P_1 \cup z' z \cup z X p_1 \cup p_1 Q y_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. ∎

4.5 *Finding* TK_5

Recall the notation from Lemma 4.2.5 and the previous section. In particular, $H := G[B_1 + \{z_1, z_2\}]$, $G' := G - \{x_1x : x \notin \{x_2, y_1, y_2, z_0, z_1\}\}$, $b_1 \in N(y_2) \cap V(B_1)$ if $y_2 \in V(X)$, and $b_1 \in V(B_1 \cap B_2)$ if $y_2 \notin V(X)$. Our objective is to find TK_5 in G' using the structural information on H produced in the previous sections. By Lemma 4.3.1,

(A1) Y_2 has independent paths Q_1, Q_2, Q_3 from y_2 to p_1, p_2, b, respectively.

Note that if $y_2 \in V(X)$ then $e(z_1, B_1 - b_1) \geq 2$ and $e(z_2, B_1 - b_1) \geq 1$. Thus, by Lemma 4.3.3, we may assume that there exists $i \in [2]$ for which $e(z_i, B_1 - b_1) \geq 2$ and $e(z_{3-i}, B_1 - b_1) \geq 1$. (Moreover, by Lemma 4.3.2, $e(z_{3-i}, B_1) = 1$ only if $b = b_1$ and, hence, $z_{3-i} = p_{3-i}$.) Then by Lemma 4.3.1,

(A2) Y_2 has a path T from b to p_i through p_{3-i}, y_2 in order, respectively.

By Lemma 4.4.1, we may assume that

(A3) H has disjoint paths Y, Z from y_1, z_1 to b_1, z_2, respectively.

By Lemma 4.4.2, we may assume that

(A4) H has independent paths A, B, C, with A, C from z_i to y_1, and B from b_1 to z_{3-i}.

Let $J(A, C)$ denote the $(A \cup C)$-bridge of H containing B, and $L(A, C)$ denote the union of all $(A \cup C)$-bridges of H with attachments on both A and C. We may choose A, B, C such that the following are satisfied in the order listed:

(a) A, B, C are induced paths in H,

(b) whenever possible, $J(A, C) \subseteq L(A, C)$,

(c) $J(A, C)$ is maximal, and

(d) $L(A, C)$ is maximal.

We refer the reader to Figure 4 for an illustration. We may assume that

(A5) for any $j \in [2]$, H contains no path from z_j to b_1 and through z_{3-j}, y_1 in order.

For, suppose H does contain a path, say R, from z_j to b_1 and through z_{3-j}, y_1 in order. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-j}x_{3-j} \cup (z_{3-j}Xp_{3-j} \cup Q_{3-j}) \cup (z_{3-j}Rz_j \cup z_jx_j) \cup z_{3-j}Ry_1 \cup (y_1Rb_1 \cup b_1b \cup Q_3)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-j}$. Thus, we may assume (A5).

Since B_1 is 2-connected and $e(z_{3-i}, B_1 - b_1) \geq 1$, H has disjoint paths P, Q from $p, q \in V(B)$ to $c, a \in V(A \cup C) - \{z_i\}$, respectively, and internally disjoint from $A \cup B \cup C$. By symmetry between A and C, we may assume that b_1, p, q, z_{3-i} occur on B in order. By (A5), $c \neq y_1$. We choose such P, Q that the following are satisfied in order listed:

(A6) qBz_{3-i} is minimal, pBz_{3-i} is maximal, the subpath of $(A \cup C) - z_i$ between a and y_1 is minimal, and the subpath of $(A \cup C) - z_i$ between c and $y_1$is maximal.

Let B' denote the union of B and the B-bridges of H not containing $A \cup C$. Note that all paths in H from $A \cup C$ to B' and internally disjoint from B' must have an end in B. We may assume that

(A7) if $e(z_{3-i}, B_1) \geq 2$ then, for any $q^* \in V(B' - q)$, B' has independent paths from z_{3-i} to q, q^*, respectively.

For, suppose $e(z_{3-i}, B_1) \geq 2$ and for some $q^* \in V(B' - q)$, B' has no independent paths from z_{3-i} to q, q^*, respectively. Then $q \neq z_{3-i}$, and B' has a 1-separation (B'_1, B'_2) such that $q, q^* \in V(B'_2)$ and $z_{3-i} \in V(B'_1) - V(B'_2)$. Note that $b_1 \in V(B'_2)$. Choose (B'_1, B'_2) with B'_1 minimal, and let $z \in V(B'_1 \cap B'_2)$. Since $e(z_{3-i}, B_1) \geq 2$,

$|V(B_1')| \geq 3$; so H has a path R from some $s \in V(B_1' - z)$ to some $t \in V(A \cup C \cup P \cup Q)$ and internally disjoint from $A \cup B \cup C \cup P \cup Q$.

By the choice of P, Q in (A6), we see that $t = z_i$. Let S be a path in B_1' from z_{3-i} to s, respectively. Now $S \cup R \cup A \cup y_1Cc \cup P \cup pBb_1$ is a path contradicting (A5). Hence

We will show that we may assume $a = y_1$ (see (3)), derive structural information about G' and H (see (4)–(7)), and will consider whether or not $z_i \in V(J(A, C))$ (see Case 1 and Case 2). First, we may assume that

(1) $N(y_1) \cap V(z_jXp_j - z_j) = \emptyset$ for $j \in [2]$.

For, suppose there exists $s \in N(y_1) \cap V(z_jXp_j - z_j)$ for some $j \in [2]$. If $j = 3 - i$ then, using the paths Q_1, Q_2, Q_3 from (A1), we see that $G[\{x_1, x_2, y_1, y_2\}] \cup z_ix_i \cup (z_iXp_i \cup Q_i) \cup A \cup (z_iCc \cup P \cup pBz_{3-i} \cup z_{3-i}x_{3-i}) \cup (y_1s \cup sXp_{3-i} \cup Q_{3-i})$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i.

So assume $j = i$. Suppose $e(z_{3-i}, B_1) = 1$. Then $z_{3-i} = p_{3-i}$. Recall the path T from (A2). Note that $z_{3-i}Tb \cup bb_1 \cup A \cup B \cup C \cup P \cup Q$ contains independent paths S_1, S_2 from z_{3-i} to z_i, y_1, respectively. Hence $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup (S_1 \cup z_ix_i) \cup S_2 \cup (y_1s \cup sXp_i \cup p_iTy_2)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

Now assume $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be independent paths from (A7) with $q^* = p$. Then $P_1 \cup P_2 \cup A \cup B \cup C \cup P \cup Q$ contains independent paths S_1, S_2 from z_{3-i} to z_i, y_1, respectively. Now $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup (S_1 \cup z_ix_i) \cup S_2 \cup (y_1s \cup sXp_i \cup Q_i)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. This proves (1).

We may assume

(2) $y_1 \in V(J(A, C))$.

For, suppose $y_1 \notin V(J(A,C))$. By (1) and 5-connectedness of G, $y_1 \in V(D_1)$ for some $(A \cup C)$-bridge D_1 of H with $D_1 \neq J(A,C)$. Thus, let $D_1, \ldots, D_k$ be a maximal sequence of $(A \cup C)$-bridges of H with $D_j \neq J(A,C)$ for $j \in [k]$, such that, for each $l \in [k-1]$,

$$D_{l+1} \text{ has a vertex not in } \bigcup_{j \in [l]} (c_j C y_1 \cup a_j A y_1) \text{ and a vertex not in } \bigcap_{j \in [l]} (z_i C c_j \cup z_i A a_j),$$

where for each $j \in [k]$, $a_j \in V(D_j \cap A)$ and $c_j \in V(D_j \cap C)$ such that $a_j A y_1$ and $c_j C y_1$ are maximal. Let $S_l := \bigcup_{j \in [l]} (D_j \cup a_j A y_1 \cup c_j C y_1)$.

We claim that for any $l \in [k]$ and for any $r_l \in V(S_l) - \{a_l, c_l\}$, S_l has three independent paths A_l, C_l, R_l from y_1 to a_l, c_l, r_l, respectively. This is obvious for $l = 1$ (if $a_l = y_1$, or $c_l = y_1$, or $r_l = y_1$ then A_l, or C_l, or R_l is a trivial path). Now assume $k \geq 2$ and the claim holds for some $l \in [k-1]$. Let $r_{l+1} \in V(S_{l+1}) - \{a_{l+1}, c_{l+1}\}$. When $r_{l+1} \in V(S_l) - \{a_l, c_l\}$ let $r_l := r_{l+1}$; otherwise, let $r_l \in V(a_l A y_1 - a_l) \cup V(c_l C y_1 - c_l)$ with $r_l \in V(D_{l+1})$. By assumption, S_l has independent paths A_l, C_l, R_l from y_1 to a_l, c_l, r_l, respectively. If $r_{l+1} \in V(S_l) - \{a_l, c_l\}$ then $A_{l+1} := A_l \cup a_l A a_{l+1}, C_{l+1} := C_l \cup c_l C c_{l+1}, R_{l+1} := R_l$ are the desired paths in S_{l+1}. If $r_{l+1} \in V(D_{l+1}) - V(A \cup C)$ then let P_{l+1} be a path in D_{l+1} from r_l to r_{l+1} internally disjoint from $A \cup C$; we see that $A_{l+1} := A_l \cup a_l A a_{l+1}, C_{l+1} := C_l \cup c_l C c_{l+1}, R_{l+1} := R_l \cup P_{l+1}$ are the desired paths in S_{l+1}. So we may assume by symmetry that $r_{l+1} \in V(a_{l+1} A a_l - a_{l+1})$. Let Q_{l+1} be a path in D_{l+1} from r_l to a_{l+1} internally disjoint from $A \cup C$. Now $R_{l+1} := A_l \cup a_l A r_{l+1}, C_{l+1} := C_l \cup c_l C c_{l+1}, A_{l+1} := R_l \cup Q_{l+1}$ are the desired paths in S_{l+1}.

Hence, by (c), $J(A,C)$ does not intersect $(a_k A y_1 \cup c_k C y_1) - \{a_k, c_k\}$. Since G is 5-connected, $\{a_k, c_k, x_1, x_2\}$ cannot be a cut in G separating S_k from $X \cup J(A,C)$. So there exists $ss' \in E(G)$ such that $s \in V(S_k) - \{a_k, c_k\}$ and $s' \in V(z_1 X p_1 \cup z_2 X p_2)$. By the above claim, let A_k, C_k, R_k be independent paths in S_k from y_1 to a_k, c_k, s, respectively; so $s' \notin \{z_1, z_2\}$ by (c).

Suppose $s' \in V(z_{3-i}Xp_{3-i} - z_{3-i})$. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_ix_i \cup (z_iXp_i \cup Q_i) \cup (z_iCc \cup P \cup pBz_{3-i} \cup z_{3-i}x_{3-i}) \cup (z_iAa_k \cup A_k) \cup (R_k \cup ss' \cup s'Xp_{3-i} \cup Q_{3-i})$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i.

So we may assume $s' \in V(z_iXp_i - z_i)$. Suppose $e(z_{3-i}, B_1) = 1$. Then $z_{3-i} = p_{3-i}$. Recall the path T from (A2). Note that $z_{3-i}Tb \cup bb_1 \cup z_iAa_k \cup z_iCc_k \cup P \cup Q \cup B$ contains independent paths S_1, S_2 from z_{3-i} to z_i, v, respectively, for some $v \in \{a_k, c_k\}$. Let $S = A_k$ if $v = a_k$, and $S = C_k$ if $v = c_k$. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup (S_1 \cup z_ix_i) \cup (S_2 \cup S) \cup (R_k \cup ss' \cup s'Xp_i \cup p_iTy_2)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

Hence, we may assume $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be independent paths from (A7) with $q^* = p$. Then, $P_1 \cup P_2 \cup z_iAa_k \cup z_iCc_k \cup P \cup Q \cup B$ contains independent paths S_1, S_2 from z_{3-i} to z_i, v, respectively, for some $v \in \{a_k, c_k\}$. Let $S = A_k$ if $v = a_k$, and $S = C_k$ if $v = c_k$. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup (S_1 \cup z_ix_i) \cup (S_2 \cup S) \cup (R_k \cup ss' \cup s'Xp_i \cup Q_i)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. This completes the proof of (2).

For convenience, we let $K := A \cup B \cup C \cup P \cup Q$. We claim that

(3) $a = y_1$

Suppose $a \neq y_1$. By (2), $J(A, C)$ has a path S from y_1 to some vertex $s \in V(P \cup Q \cup B) - \{c, a\}$ and internally disjoint from K. By (A6), $s \notin V(Q \cup qBz_{3-i})$. So $s \in V(P \cup b_1Bq - q)$. Let $R = aAz_i$ and $R' = C$ if $a \in V(A)$; and $R = aCz_i$ and $R' = A$ if $a \in V(C)$. Also, let $S' = S \cup sBb_1$ if $s \in V(B)$, and $S' = S \cup sPp \cup pBb_1$ if $s \in V(P)$. Then $z_{3-i}Bq \cup Q \cup R \cup R' \cup S'$ is a path contradicting (A5).

Before we distinguish cases according to whether or not $z_i \in V(J(A, C))$, we derive further information about G'. We may assume that

(4) for any path W in G' from x_i to some $w \in V(K) - \{z_i, y_1\}$ and internally disjoint from K, we have $w \in V(A) - \{z_i, y_1\}$.

To see this, suppose $w \notin V(A) - \{z_i, y_1\}$. First, assume $e(z_{3-i}, B_1) = 1$. Then $z_{3-i} = p_{3-i}$. Recall the path T from (A2). So $z_{3-i}Tb_1 \cup B \cup (C - z_i) \cup W \cup P \cup Q$ contains independent paths S_1, S_2 from z_{3-i} to x_i, y_1, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup S_1 \cup S_2 \cup (A \cup z_iXp_i \cup p_iTy_2)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

Thus, we may assume $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be independent paths in B' from (A7) with $q^* = p$. So $P_1 \cup P_2 \cup B \cup (C - z_i) \cup W \cup P \cup Q$ contains independent paths S_1, S_2 from z_{3-i} to x_i, y_1, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup S_1 \cup S_2 \cup (A \cup z_iXp_i \cup Q_i)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. This completes the proof of (4).

Since G is 5-connected and $z_0 \in V(B_1)$ when $e(z_1, B_1) \geq 2$ (by (iv) of Lemma 4.2.5), it follows from (4) that

G' has a path W from x_i to $w \in V(A) - \{y_1, z_i\}$ and internally disjoint from K.

Hence, $|V(A)| \geq 3$ and $|V(C)| \geq 3$. Since A and C are induced paths in H,

$$y_1z_i \notin E(G).$$

We may assume that

(5) G' has no path from $z_{3-i}Xp_{3-i} - y_2$ to $(A \cup C) - y_1$ and internally disjoint from K, G' has no path from $z_iXp_i - z_i$ to $(A \cup cCy_1) - \{z_i, c\}$ and internally disjoint from K, and if $i = 1$ then G' has no path from x_{3-i} to $(A \cup C) - y_1$ and internally disjoint from K.

First, suppose S is a path in G' from some $s \in V(z_{3-i}Xp_{3-i} - y_2)$ to some $s' \in V(A \cup C) - \{y_1\}$. Then $A \cup C \cup S$ contains independent paths S_1, S_2 from z_i to y_1, s, respectively. Hence, $G[\{x_1, x_2, y_1, y_2\}] \cup z_ix_i \cup (z_iXp_i \cup Q_i) \cup S_1 \cup (S_2 \cup sXz_{3-i} \cup z_{3-i}x_{3-i}) \cup (Q \cup qBb_1 \cup b_1b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i.

Now assume that S is a path in G' from some $s \in V(z_iXp_i - z_i)$ to some $s' \in V(A \cup cCy_1) - \{z_i, c\}$ and internally disjoint from K. Let $S' = y_1As'$ if $s' \in V(A)$, and $S' = y_1Cs'$ if $s' \in V(cCy_1)$. If $e(z_{3-i}, B_1) = 1$ then $z_{3-i} = p_{3-i}$ and, using the path T from (A2), we see that $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup (z_{3-i}Bq \cup Q) \cup (z_{3-i}Tb_1 \cup b_1Bp \cup P \cup cCz_i \cup z_ix_i) \cup (S' \cup S \cup sXp_i \cup p_iTy_2)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. So assume $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be independent paths from (A7) with $q^* = p$. Now $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup (P_1 \cup Q) \cup (P_2 \cup P \cup cCz_i \cup z_ix_i) \cup (S' \cup S \cup sXp_i \cup Q_i)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

Now suppose $i = 1$ and S is a path in G' from x_{3-i} to some $s \in V(A \cup C) - \{y_1\}$ and internally disjoint from K. If $s \in V(A - y_1)$, then $G[\{x_1, x_2, y_1, y_2\}] \cup z_ix_i \cup (z_iXp_i \cup Q_i) \cup C \cup (z_iAs \cup S) \cup (Q \cup qBb_1 \cup b_1b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i. So assume $s \in V(C - y_1)$. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_ix_i \cup (z_iXp_i \cup Q_i) \cup A \cup (z_iCs \cup S) \cup (Q \cup qBb_1 \cup b_1b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i. This completes the proof of (5).

(6) We may assume that

(6.1) any path in $J(A, C)$ from $A - \{z_i, y_1\}$ to $(P \cup Q \cup B) - \{c, y_1\}$ and internally disjoint from K must end on Q,

(6.2) if an $(A \cup C)$-bridge of H contained in $L(A, C)$ intersects $z_iCc - c$ and contains a vertex $z \in V(A - z_i)$ then $J(A, C) \cap (z_iAz - \{z_i, z\}) = \emptyset$, and

(6.3) $J(A, C) \cap (z_iCc - \{z_i, c\}) = \emptyset$, and any path in $J(A, C)$ from z_i to $(P \cup Q \cup B) - \{c, y_1\}$ and internally disjoint from K must end on $(P - c) \cup b_1Bp$.

To prove (6.1), let S be a path in $J(A, C)$ from $s \in V(A) - \{z_i, y_1\}$ to $s' \in V(P \cup B) - \{c, q, y_1\}$ and internally disjoint from K. Note that $s' \notin V(qBz_{3-i} - q)$ by (A6).

Suppose $e(z_{3-i}, B_1) = 1$. Then $z_{3-i} = p_{3-i}$ and we use the path T from (A2). Let S' be a path in $(P - c) \cup (b_1Bq - q)$ from b_1 to s'. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup$

$z_{3-i}Ty_2 \cup (z_{3-i}Tb_1 \cup S' \cup S \cup sAw \cup W) \cup (z_{3-i}Bq \cup Q) \cup (C \cup z_iXp_i \cup p_iTy_2)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. So we may assume $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be the paths from (A7), with $q^* = p$ when $s' \in V(P)$ and $q^* = s'$ when $s' \in V(B)$. So $P_1 \cup P_2 \cup B \cup S \cup Q$ contains independent paths S_1, S_2 from z_{3-i} to s, y_1, respectively. Now $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup (S_1 \cup sAw \cup W) \cup S_2 \cup (C \cup z_iXp_i \cup Q_i)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

To prove (6.2), let D be a path contained in $L(A, C)$ from $z' \in V(z_iCc - c)$ to $z \in V(A - z_i)$ and internally disjoint from K. Suppose there exists $s \in V(J(A, C)) \cap V(z_iAz - \{z_i, z\})$. By (6.1), $J(A, C)$ has a path S from s to some $s' \in V(Q - y_1)$ and internally disjoint from K. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_ix_i \cup (z_iXp_i \cup Q_i) \cup (z_iAs \cup S \cup s'Qq \cup qBz_{3-i} \cup z_{3-i}x_{3-i}) \cup (z_iCz' \cup D \cup zAy_1) \cup (y_1Cc \cup P \cup pBb_1 \cup b_1b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i.

To prove (6.3), let S be a path in $J(A, C)$ from $s \in V(z_iCc - c)$ to $s' \in V(P \cup Q \cup B) - \{c, y_1\}$ and internally disjoint from K. Suppose $s' \in V(Q \cup z_{3-i}Bp) - \{p, y_1\}$. Then $(S \cup Q \cup pBz_{3-i}) - \{p, y_1\}$ contains a path S' from s to z_{3-i}. So $G[\{x_1, x_2, y_1, y_2\}] \cup z_ix_i \cup (z_iXp_i \cup Q_i) \cup (z_iCs \cup S' \cup z_{3-i}x_{3-i}) \cup A \cup (y_1Cc \cup P \cup pBb_1 \cup b_1b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i. Thus, we may assume $s' \in V(P - c) \cup V(b_1Bp)$. By (A6), $s = z_i$. This proves (6).

Denote by $L(A)$ (respectively, $L(C)$) the union of all $(A \cup C)$-bridges of H whose intersection with $A \cup C$ is contained in A (respectively, C).

(7) $L(A) = \emptyset$, and $L(C) \cap C \subseteq z_iCc$.

Suppose $L(A) \neq \emptyset$, and let R_1 be an $(A \cup C)$-bridge of H contained in $L(A)$. We construct a maximal sequence $R_1, \ldots, R_m$ of $(A \cup C)$-bridges of H contained in $L(A)$, such that for $2 \leq i \leq m$, R_i has a vertex internal to $\bigcup_{j=1}^{i-1} l_jAr_j$ (which is a path), where $l_j, r_j \in V(R_j \cap A)$ with l_jAr_j maximal. Let $a_1, a_2 \in V(A)$ such that $\bigcup_{j=1}^{m} l_jAr_j = a_1Aa_2$. By (c), $J(A, C) \cap (a_1Aa_2 - \{a_1, a_2\}) = \emptyset$; by (d) and the maximality of

$R_1, \ldots, R_m$, $L(A,C)$ has no path from $a_1Aa_2 - \{a_1, a_2\}$ to $(A - a_1Aa_2) \cup (C - \{y_1, z_i\})$; and by (5), $(z_1Xp_1 \cup z_2Xp_2) - \{a_1, a_2, z_i\}$ contains no neighbor of $(\bigcup_{j=1}^m R_j \cup a_1Aa_2) - \{a_1, a_2\}$. Hence, $\{a_1, a_2, x_1, x_2\}$ is a 4-cut in G, a contradiction. Therefore, $L(A) = \emptyset$.

Now assume $L(C) \cap C \not\subseteq z_iCc$, and let R_1 be an $(A \cup C)$-bridge of H contained in $L(C)$ such that $R_1 \cap (cCy_1 - c) \neq \emptyset$. We construct a maximal sequence $R_1, \ldots, R_m$ of $(A \cup C)$-bridges of H contained in $L(C)$ such that for $2 \leq i \leq m$, R_i has a vertex internal to $\bigcup_{j=1}^{i-1} l_jCr_j$ (which is a path), where $l_j, r_j \in V(R_j \cap C)$ with l_jCr_j maximal. Let $c_1, c_2 \in V(C)$ such that $\bigcup_{j=1}^m l_jCr_j = c_1Cc_2$. By the existence of P and (c), $c_1, c_2 \in cCy_1$; by (c), $J(A,C) \cap (c_1Cc_2 - \{c_1, c_2\}) = \emptyset$; by (d) and the maximality of $R_1, \ldots, R_m$, $L(A,C) \cap (c_1Cc_2 - \{c_1, c_2\}) = \emptyset$; and by (5) and the maximality of $R_1, \ldots, R_m$, $z_1Xp_1 \cup z_2Xp_2$ contains no neighbor of $(\bigcup_{j=1}^m R_j \cup c_1Cc_2) - \{c_1, c_2\}$. Hence, $\{c_1, c_2, x_1, x_2\}$ is a 4-cut in G, a contradiction. Therefore, $L(C) \cap C \subseteq z_iCc$. This proves (7).

Let F be the union of all $(A \cup C)$-bridges of H different from $J(A,C)$ and intersecting $z_iCc - c$. When $F \neq \emptyset$, let $a^* \in V(F \cap A)$ with a^*Ay_1 minimal, and let r be the neighbor of $(F \cup z_iAa^* \cup z_iCc) - \{a^*, c\}$ on $z_iXp_i - z_i$ with rXp_i minimal.

Case 1. $z_i \in V(J(A,C))$.

By (6.3), $J(A,C)$ contains a path S from z_i to some $s \in V(P - c) \cup V(b_1Bp)$ and internally disjoint from K.

Subcase 1.1. $F \neq \emptyset$.

Suppose $r \neq z_i$. Then by (5) and the definition of r, G' has a path R from r to $r' \in V(z_iCc) - \{z_i, c\}$ and internally disjoint from $K \cup X$, and by (6.3), R is disjoint from $J(A,C)$. First, assume $e(z_{3-i}, B_1) = 1$. Then $z_{3-i} = p_{3-i}$ and we use the path T from (A2). Note that $S \cup P \cup pBb_1$ contains a path S' from z_i to b_1. Hence, $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup (z_{3-i}Tb \cup bb_1 \cup S' \cup z_ix_i) \cup (z_{3-i}Bq \cup Q) \cup (y_1Cr' \cup R \cup rXp_i \cup p_iTy_2)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. So assume $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be independent paths from (A7) with $q^* = p$. So $P_1 \cup P_2 \cup B \cup S \cup (P - c) \cup Q$ contains independent paths S_1, S_2 from z_{3-i} to z_i, y_1, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup (S_1 \cup z_ix_i) \cup S_2 \cup (y_1Cr' \cup R \cup rXp_i \cup Q_i)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

So $r = z_i$ and, hence, $\{a^*, c, x_1, x_2, z_i\}$ is a 5-cut in G. Thus, $i = 2$ by (5). Let $F^* := G[F \cup z_iAa^* \cup z_iCc + \{x_1, x_2\}]$

Suppose $F^* - x_1$ has disjoint paths S_1, S_2 from x_i, z_i to c, a^*, respectively. If $e(z_{3-i}, B_1) = 1$ then $z_{3-i} = p_{3-i}$ and, using the path T from (A2), we see that $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup (z_{3-i}Tb \cup bb_1 \cup b_1Bp \cup P \cup S_1) \cup (z_{3-i}Bq \cup Q) \cup (y_1Aa^* \cup S_2 \cup z_iXp_i \cup p_iTy_2)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. Now assume $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be independent paths from (A7) with $q^* = p$. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup (P_1 \cup Q) \cup (P_2 \cup P \cup S_1) \cup (y_1Aa^* \cup S_2 \cup z_iXp_i \cup Q_i)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

Thus, we may assume that such S_1, S_2 do not exist. Then by Lemma 2.3.1, $(F^* - x_1, x_i, z_i, c, a^*)$ is planar. If $|V(F^*)| \geq 7$, then the assertion of Theorem 4.1.1 follows from Lemma 4.2.1. So assume $|V(F^*)| = 6$. Let $z \in V(F^* - x_1) - \{x_i, z_i, c, a^*\}$. Then $G[\{x_i, z_i, z, c\}] \cong K_4^-$, and (ii) of Theorem 4.1.1 holds (as $i = 2$ in this case).

Subcase 1.2. $F = \emptyset$.

Then $L(C) = \emptyset$ by (7). Also, $L(A) = \emptyset$ by (7). Hence, by (4) and the comment preceding (5), $W = x_iw$ with $w \in V(A) - \{z_i, y_1\}$.

We may assume that $J(A, C) \cap (A - \{z_i, y_1\}) = \emptyset$. For, otherwise, let $t \in V(J(A, C)) \cap V(A - \{z_i, y_1\})$. By (6.1), $J(A, C)$ contains a path T from t to $t' \in V(Q - y_1)$ and internally disjoint from K, and T must be internally disjoint from S. Note that $(S \cup P \cup b_1Bp) - c$ contains a path S' from z_i to b_1 and internally disjoint from $T \cup Q \cup z_{3-i}Bq$. If $e(z_{3-i}, B_1) = 1$ then $z_{3-i} = p_{3-i}$ and, using the path T from (A2), we see that $G[\{x_1, x_2, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup z_ix_i \cup (z_iXp_i \cup p_iTy_2) \cup (z_{3-i}Tb \cup bb_1 \cup S') \cup (C \cup y_1x_{3-i}) \cup (z_{3-i}Bq \cup qQt' \cup T \cup tAw \cup wx_i)$ is a TK_5 in G' with branch vertices x_1, x_2, y_2, z_1, z_2. So assume that $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be independent paths from (A7) with $q^* = p$. So $P_1 \cup P_2 \cup B \cup S \cup (P - c) \cup (Q - y_1) \cup T$ contains independent paths S_1, S_2 from z_{3-i} to z_i, t, respectively. Then $G[\{x_1, x_2, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup z_ix_i \cup (z_iXp_i \cup Q_i) \cup S_1 \cup (C \cup y_1x_{3-i}) \cup (S_2 \cup tAw \cup wx_i)$ is a TK_5 in G' with branch vertices x_1, x_2, y_2, z_1, z_2.

By (A5), $J := J(A, C) \cup C$ contains no disjoint paths from z_i, y_1 to z_{3-i}, b_1, respectively. Hence by Lemma 2.3.1, there exists a collection $\mathcal{L}$ of subsets of $V(J) - \{b_1, y_1, z_1, z_2\}$ such that $(J, \mathcal{L}, z_i, y_1, z_{3-i}, b_1)$ is 3-planar. We choose $\mathcal{L}$ so that each $L \in \mathcal{L}$ is minimal and, subject to this, $|\mathcal{L}|$ is minimal.

We claim that for each $L \in \mathcal{L}$, $L \cap V(L(A, C)) = \emptyset$. For suppose there exists $L \in \mathcal{L}$ such that $L \cap V(L(A, C)) \neq \emptyset$. Then, since G is 5-connected, $|N_J(L) \cap V(C)| \geq 2$. Assume for the moment that $N_J(L) \subseteq V(C)$. Then, since $L(C) = \emptyset$ and $J(A, C) \cap (A - \{z_i, y_1\}) = \emptyset$, $L \subseteq V(C)$. However, since C is an induced path in G, we see that $(J, \mathcal{L} - \{L\}, z_i, y_1, z_{3-i}, b_1)$ is 3-planar, contradicting the choice of $\mathcal{L}$. Thus, let $N_J(L) = \{t_1, t_2, t_3\}$ such that $t_1, t_2 \in V(C)$ and $t_3 \notin V(C)$. Then $J(A, C)$ contains a path R from t_3 to B and internally disjoint from $B \cup C$. Let $t \in L \cap V(L(A, C))$. By the minimality of L, $G[L + \{t_1, t_2, t_3\}]$ contains disjoint paths T_1, T_2 from t_1, t to t_2, t_3, respectively. We may choose T_1 to be induced, and let $C' := z_iCt_1 \cup T_1 \cup t_2Cy_1$. Then A, B, C' satisfy (a), but $J(A, C') \subseteq L(A, C')$ (because of T_2), contradicting (2) (as $J(A, C) \cap (A - \{z_i, y_1\}) = \emptyset$).

Because of the existence of Y, Z in (A3), there are disjoint paths R_1, R_2 in $L(A, C)$ from $r_1, r_2 \in V(A)$ to $r_1', r_2' \in V(C)$ such that z_i, r_1, r_2, y_1 occur on A in order and z_i, r_2', r_1', y_1 occur on C in order. Let $A' = z_iAr_1 \cup R_1 \cup r_1'Cy_1$ and $C' = z_iCr_2' \cup R_2 \cup r_2Ay_1$. Let $t_1, t_2 \in V(C - \{z_i, y_1\}) \cap V(J(A, C))$ with t_1Ct_2 maximal, and assume that z_i, t_1, t_2, y_1 occur on C in this order. By the planarity of $(J, z_i, y_1, z_{3-i}, b_1)$ and by (6.3), $t_1 = c$.

Then either $t_1Ct_2 \subseteq z_iCr_2'$ for all choices of R_1 and R_2, or $t_1Ct_2 \subseteq r_1'Cy_1$ for all choices of R_1 and R_2; for otherwise, $J(A', C') \subseteq L(A', C')$, and A', B, C' contradict the choice of A, B, C in (b). Moreover, since $F = \emptyset$, $t_1Ct_2 \subseteq z_iCr_2'$ for all choices of R_1 and R_2. Choose R_1, R_2 so that z_iAr_1 and z_iCr_2' are minimal. Since G is 5-connected, $\{r_1, r_2', x_1, y_1\}$ cannot be a cut in G. So by (5), G' has a path R from x_2 to some $v \in V(r_1Ay_1 - \{r_1, y_1\}) \cup V(r_2'Cy_2 - \{r_2', y_1\})$ and internally disjoint from K.

First, assume $i = 1$. If $v \in V(r_1Ay_1) - \{r_1, y_1\}$ then $G[\{x_1, x_2, y_1, y_2\}] \cup z_ix_i \cup C \cup (z_iXp_i \cup Q_i) \cup (zAv \cup R) \cup (Q \cup qBz_{3-i} \cup Q_{3-i})$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i. If $v \in V(r_2'Cy_1) - \{r_2', y_1\}$ then $G[\{x_1, x_2, y_1, y_2\}] \cup z_ix_i \cup A \cup (z_iXp_i \cup Q_i) \cup (z_iCv \cup R) \cup (Q \cup qBz_{3-i} \cup Q_{3-i})$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i.

Hence, we may assume $i = 2$. If $e(z_{3-i}, B_1) = 1$ then $z_{3-i} = p_{3-i}$ and, using the path T from (A2), we see that $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup (z_{3-i}Bq \cup Q) \cup (z_{3-i}Tb_1 \cup b_1Bp \cup P \cup cCr_2' \cup R_2 \cup r_2Av \cup R) \cup (y_1Cr_1' \cup R_1 \cup r_1Az_i \cup z_iXp_i \cup p_iTy_2)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. So assume $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be independent paths from (A7) with $q^* = p$. Now $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup (P_1 \cup Q) \cup (P_2 \cup P \cup cCr_2' \cup R_2 \cup r_2Av \cup R) \cup (y_1Cr_1' \cup R_1 \cup r_1Az_i \cup z_iXp_i \cup Q_i)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

Case 2. $z_i \notin V(J(A, C))$.

Then $F \neq \emptyset$ as the degree of z_i in G' is at least 5. So a^* and r are defined.

Subcase 2.1. $r \neq z_i$, and G' contains a path S from some $s \in V(z_iXr) - \{z_i, r\}$ to some $s' \in V(P \cup Q \cup B') - \{y_1, c\}$ and internally disjoint from $A \cup B' \cup C \cup P \cup Q \cup X$.

Note that $s' \in V(B)$ if $s' \in V(B')$. First, assume $s' \in V(Q - y_1) \cup V(pBz_{3-i} - p)$. Then $S \cup (Q - y_1) \cup (pBz_{3-i} - p)$ has a path S' from s to z_{3-i}. By (5), let R be a path in G' from r to some $r' \in V(z_iCc) - \{z_i, c\}$ and internally disjoint from $A \cup C \cup J(A, C) \cup X$. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_ix_i \cup (z_iXs \cup S' \cup z_{3-i}x_{3-i}) \cup A \cup (z_iCr' \cup R \cup rXp_i \cup Q_i) \cup (y_1Cc \cup P \cup pBb_1 \cup b_1b \cup Q_3)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_i.

Hence, we may assume $s' \in V(P - c) \cup V(b_1Bp)$. Since $F \neq \emptyset$ and B_1 is 2-connected, $a^* \neq z_i$; so G' has a path R' from r to some $r' \in V(z_iAa^* - z_i)$ and internally disjoint from $A \cup cCy_1 \cup J(A, C) \cup X$.

Suppose $e(z_{3-i}, B_1) = 1$. Then $z_{3-i} = p_{3-i}$ and we use the path T from (A2). Note that $(P - c) \cup Q \cup B \cup z_{3-i}Tb \cup bb_1$ contains independent paths S_1, S_2 from z_{3-i} to s', y_1, respectively. So $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup z_{3-i}Ty_2 \cup (S_1 \cup S \cup sXz_i \cup z_ix_i) \cup S_2 \cup (y_1Ar' \cup R' \cup rXp_i \cup p_iTy_2)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

Now assume $e(z_{3-i}, B_1) \geq 2$. Let P_1, P_2 be independent paths from (A7) with $q^* = p$ if $s' \in P$ and $q^* = s'$ if $s' \in V(pBb_1)$. So $P_1 \cup P_2 \cup B \cup S \cup P \cup Q$ contains independent paths S_1, S_2 from z_{3-i} to s, y_1, respectively. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_{3-i}x_{3-i} \cup (z_{3-i}Xp_{3-i} \cup Q_{3-i}) \cup S_2 \cup (S_1 \cup sXz_i \cup z_ix_i) \cup (y_1Ar' \cup R' \cup rXp_i \cup Q_i)$ is a TK_5 in G' with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$.

Subcase 2.2. $r = z_i$, or G' contains no path from $z_iXr - \{z_i, r\}$ to $(P \cup Q \cup B') - \{y_1, c\}$ and internally disjoint from $A \cup B' \cup C \cup P \cup Q \cup X$.

Then by (5), (6.2) and (6.3), $\{a^*, c, r, x_1, x_2\}$ is a 5-cut in G. Hence, since G is 5-connected, $i = 2$ by (5). Therefore, G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a^*, c, r, x_1, x_2\}$ and $G_2 = G[F \cup z_2Cc \cup z_2Aa^* \cup x_2Xr + x_1]$.

Suppose $G_2 - x_1$ contains disjoint paths S_1, S_2 from r, x_2 to a^*, c, respectively. If $e(z_1, B_1) = 1$ then $z_1 = p_1$ and, using the path T from (A2) with $i = 2$, we see that $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup z_1Ty_2 \cup (z_1Bq \cup Q) \cup (z_1Tb_1 \cup b_1Bp \cup P \cup S_2) \cup (y_1Aa^* \cup S_1 \cup rXp_2 \cup p_2Ty_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. So assume $e(z_1, B_1) \geq 2$. Let P_1, P_2 be independent paths from (A7) with $q^* = p$. Then $G[\{x_1, x_2, y_1, y_2\}] \cup z_1x_1 \cup (z_1Xp_1 \cup Q_1) \cup (P_1 \cup Q) \cup (P_2 \cup P \cup S_2) \cup (y_1Aa^* \cup S_1 \cup rXp_2 \cup Q_2)$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Thus, we may assume that such S_1, S_2 do not exist in $G_2 - x_1$. Then by Lemma 2.3.1, $(G_2 - x_1, r, x_2, a^*, c)$ is planar. If $|V(G_2)| \geq 7$ then the assertion of Theorem 4.1.1 follows from Lemma 4.2.1. So assume $|V(G_2)| \leq 6$. If $r = z_2$ and there exists

$z \in V(G_2) - \{a^*, c, x_1, x_2, z_2\}$ then $za^*, zc, zx_1, zx_2, zz_2 \in E(G)$ (as G is 5-connected); so $G[\{c, x_2, z, z_2\}]$ contains K_4^- and (ii) of Theorem 4.1.1 holds. Hence, we may assume that $r \neq z_2$ or $V(G_2) = \{a^*, c, x_1, x_2, z_2\}$. Then, $z_2x_1, z_2c \in E(G)$ and $L(C) = \emptyset$ (by (7)).

Recall that $y_1z_2 \notin E(G)$; so $G[\{x_1, x_2, y_1, z_2\}] \cong K_4^-$. We complete the proof of Theorem 4.1.1 by proving (iv) for this new K_4^-. Let $z_0', z_1' \in N(x_1) - \{x_2, y_1, z_2\}$ be distinct and let $G'' := G - \{x_1v : v \notin \{x_2, y_1, z_0', z_1', z_2\}\}$.

Suppose $z_1' \in V(J(A, C)) - V(A \cup C)$ or $z_1' \in V(Y_2)$ or $z_1' \in V(X)$. Then $(J(A, C) \cup Y_2 \cup X \cup x_2y_2 \cup bb_1) - (A \cup C)$ contains a path from z_1' to x_2. Hence, $G - x_1$ contains an induced path X' from z_1' to x_2 such that $A \cup C$ is a cycle in $(G - x_1) - X'$ and $\{y_1, z_2\} \subseteq V(A \cup C)$. So by Lemma 3.2.1, we may assume that X' is chosen so that $y_1, y_2 \notin V(X')$ and $(G - x_1) - X'$ is 2-connected. Then by Lemma 2.3.6, G'' contains TK_5 (which uses $G[\{x_1, x_2, z_2, y_1\}]$ and x_1z_1').

So assume $z_1' \in V(L(A, C) - J(A, C)) \cup V(A \cup C)$ (as $L(A) = L(C) = \emptyset$). In fact, $z_1' \in V(C) - \{z_2, y_1\}$. For otherwise, $(W \cup L(A, C) \cup A) - C$ contains an induced path X' from z_1' to x_2, where W comes from (4) and the remark preceding (5). Then $(G - x_1) - X'$ contains $C \cup Q \cup qBb_1 \cup (X - \{x_1, x_2\}) \cup Y_2$, which has a cycle containing $\{y_1, z_2\}$. By Lemma 3.2.1, we may assume that X' is chosen so that $y_1, y_2 \notin V(X')$ and $(G - x_1) - X'$ is 2-connected. Now the assertion of Theorem 4.1.1 follows from Lemma 2.3.6.

If $z_1' \in V(J(A, C))$, then there is a path P' in $J(A, C)$ from z_1' to some $p' \in V(B)$ and internally disjoint from $A \cup B \cup C$. So $G[\{x_1, x_2, y_1, z_2\}] \cup z_1'x_1 \cup z_1'Cz_2 \cup z_1'Cy_1 \cup (P' \cup p'Bb_1 \cup b_1b \cup Q_3 \cup y_2x_2) \cup A$ is a TK_5 in G'' with branch vertices x_1, x_2, y_1, z_2, z_1'.

Thus, we may assume that $z_1' \notin V(J(A, C))$. So there is a path A' in $L(A, C)$ from z_1' to some $a' \in V(A)$ and internally disjoint from $J(A, C) \cup A \cup C$. Recall the path W from (4) and the remark preceding (5). Now $G[\{x_1, x_2, y_1, z_2\}] \cup z_1'x_1 \cup z_1'Cz_2 \cup z_1'Cy_1 \cup (A' \cup a'Aw \cup W) \cup (Q \cup qBb_1 \cup b_1b \cup Q_3 \cup Q_2 \cup p_2Xz_2)$ is a TK_5 in G'' with branch vertices x_1, x_2, y_1, z_2, z_1'. ∎

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VITA

Dawei He was born in Hefei, Anhui province, China in 1987. He got his B.S. and M.S. in Mathematics at East China Normal University in 2008 and 2011, respectively. He has been starting working with professor Xingxing Yu as a Ph.D. student in Mathematics at Georgia Institute of Technology since Spring 2012. His research interests include combinatorics, graph theory, and algorithms, etc. During the process of pursuing his Ph.D. in Mathematics, he also obtained his M.S. in Computational Science and Engineering at Georgia Institute of Technology in Fall 2015.