SUBDIVISIONS OF COMPLETE GRAPHS

A Dissertation
Presented to
The Academic Faculty

By

Yan Wang

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in
Algorithms, Combinatorics and Optimization

School of Mathematics
Georgia Institute of Technology

August 2017

Copyright © Yan Wang 2017
SUBDIVISIONS OF COMPLETE GRAPHS

Approved by:

Dr. Xingxing Yu, Advisor
School of Mathematics
Georgia Institute of Technology

Dr. Richard Peng
School of Computer Science
Georgia Institute of Technology

Dr. Prasad Tetali
School of Mathematics
Georgia Institute of Technology

Dr. Robin Thomas
School of Mathematics
Georgia Institute of Technology

Dr. Lutz Warnke
School of Mathematics
Georgia Institute of Technology

Date Approved: April 10, 2017
To my parents.
ACKNOWLEDGEMENTS

Firstly, I would like to express my sincere gratitude to my advisor, Prof. Xingxing Yu, for the continuous support of my Ph.D. study and related research, for his patience, vision and foresight, and immense knowledge. His guidance helped me overcome the obstacles in the research throughout my Ph.D. study.

Besides my advisor, I would like to thank Prof. Richard Peng, Prof. Robin Thomas, Prof. Prasad Tetali and Prof. Lutz Warnke for serving on my thesis committee, and Prof. Ken-ichi Kawarabayashi for being my thesis reader.

My sincere thanks also goes to the ACO program at Georgia Institute of Technology, especially to the ACO Director, Prof. Robin Thomas, who provided me the opportunity to join this excellent program, to learn and work with top researchers in graph theory and combinatorics in the world, and to attend conferences and workshops.

I would like to thank my co-authors Dawei He and Xingxing Yu. Also, I am fortunate to make many friends who gave me help and support during these years.

Last but not least, I would like to express my heartfelt gratitude to my parents and my girlfriend Yawen Liao. Without their unconditional support, encouragement and understanding, I would not be able to pursue my dream.
# TABLE OF CONTENTS

**Acknowledgments** .......................................................... iv

**Summary** ........................................................................ vi

**Chapter 1: Introduction** .................................................... 1
  1.1 Graph preliminaries .................................................... 1
  1.2 Main results ................................................................... 4

**Chapter 2: \( K_5 \)-subdivisions in 5-connected nonplanar graphs** ........ 6
  2.1 Introduction ................................................................... 6
  2.2 Related Problems ........................................................ 8
  2.3 Proof sketch of Theorem 1.2.1 ..................................... 10
  2.4 Previous results .......................................................... 11
  2.5 Obstruction to three paths ......................................... 15
  2.6 Quadruples and special structure .............................. 22
  2.7 Interactions between quadruples ............................ 47
  2.8 Proof of Theorem 1.2.1 ........................................... 59

**References** ...................................................................... 67

**Vita** .................................................................................. 68
SUMMARY

A subdivision of a graph $G$, also known as a topological $G$ and denoted by $TG$, is a graph obtained from $G$ by replacing certain edges of $G$ with internally vertex-disjoint paths. This dissertation studies a problem in structural graph theory regarding subdivisions of a complete graph in graphs.

In this dissertation, we focus on $TK_5$, or subdivisions of $K_5$. A well known theorem of Kuratowski in 1932 states that a graph is planar if, and only if, it does not contain a subdivision of $K_5$ or $K_{3,3}$. Wagner proved in 1937 that if a graph other than $K_5$ does not contain any subdivision of $K_{3,3}$ then it is planar or it admits a cut of size at most 2. Kelmans and, independently, Seymour conjectured in the 1970s that if a graph does not contain any subdivision of $K_5$ then it is planar or it admits a cut of size at most 4. In this dissertation, we give a proof of the Kelmans-Seymour conjecture.
CHAPTER 1
INTRODUCTION

1.1 Graph preliminaries

We begin with some basic notation and terminology for graphs. A (simple) graph $G$ is an ordered pair $(V(G), E(G))$ where $V(G)$ is a set and $E(G)$ is a set of 2-element subsets of $V(G)$. A vertex is an element of $V(G)$ and an edge is an element of $E(G)$. A graph is finite if it contains finite number of vertices. In this dissertation, we only focus on finite graphs.

Given a graph $G$, an edge $\{u,v\}$ of $G$ can also be written as $uv$. Two vertices $u, v$ of $G$ are adjacent in $G$ if $uv \in E(G)$. A vertex $u$ is a neighbor of a vertex $v$ in $G$ if $u$ is adjacent to $v$. For any $u \in V(G)$, the neighborhood of $u$ is the set of neighbors of $u$ in $G$, denoted as $N_G(u)$. The degree of a vertex $u$ is the size of its neighborhood, denoted as $\text{deg}_G(u)$. When understood, the reference to $G$ may be dropped. The maximum degree $\Delta(G)$ of a graph $G$ is the maximum of degree of a vertex in $G$. The minimum degree $\delta(G)$ of a graph $G$ is the minimum of degree of a vertex in $G$. The average degree $d(G)$ of a graph $G$ is the average of degree of a vertex in $G$. A vertex $v$ of $G$ is incident to an edge $e$ of $G$ if $v \in e$.

A complete graph on $n$ vertices, denoted as $K_n$ is the graph of $n$ vertices such that every pair of vertices are adjacent. A graph $G = (V, E)$ is called $r$-partite if $V$ admits a partition into $r$ classes such that every edge is adjacent to two vertices in different classes: vertices in the same partition class must not be adjacent. Instead of “2-partite” one usually says bipartite. An $r$-partite graph in which every two vertices from different partition classes are adjacent is called complete. Moreover, $K_4^-$ is the graph obtained from $K_4$ with a single edge removed and $K_{3,3}$ is the complete bipartite graph with two partitions of size 3.

Given two graphs $S$ and $G$, we say $S$ is a subgraph of $G$ if $V(S) \subseteq V(G)$ and $E(S) \subseteq E(G)$, denoted as $S \subseteq G$. We may view $S \subseteq V(G)$ as a subgraph of $G$ with vertex set
and for $S \subseteq G$, the subgraph of $G$ induced by $V(S)$, denoted as $G[S]$, is the graph with $V(G[S]) = V(S)$ and $E(G[S]) = \{uv \in E(G) : u, v \in V(S)\}$. For $S \subseteq G$ let $N_G(S) = \{x \in V(G) \setminus V(S) : N_G(x) \cap V(S) \neq \emptyset\}$. When understood, the reference to $G$ may be dropped.

For $S \subseteq E(G)$, $G - S$ denotes the graph obtained from $G$ by deleting all edges in $S$; and for $K, L \subseteq G$, $K - L$ denotes the graph obtained from $K$ by deleting $V(K \cap L)$ and all edges of $K$ incident with $V(K \cap L)$.

A separation in a graph $G$ consists of a pair of subgraphs $G_1, G_2$ of $G$, denoted as $(G_1, G_2)$, such that $E(G_1) \cup E(G_2) = E(G)$, $E(G_1 \cap G_2) = \emptyset$, and $E(G_1) \cup (V(G_1) \setminus V(G_2)) \neq \emptyset \neq E(G_2) \cup (V(G_2) \setminus V(G_1))$. The order of this separation is $|V(G_1) \cap V(G_2)|$.

A path is a non-empty graph $P = (V(P), E(P))$ where $V(P)$ consists of distinct vertices $v_0, v_1, \ldots, v_n$ and $E(P) = \{v_0v_1, v_1v_2, \ldots, v_{n-1}v_n\}$. The length of a path is the number of edges it contains. Given a path $P$ in a graph and $x, y \in V(P)$, $xPy$ denotes the subpath of $P$ between $x$ and $y$ (inclusive). The ends of the path $P$ are the vertices of the minimum degree in $P$, and all other vertices of $P$ (if any) are its internal vertices. A path $P$ with ends $u$ and $v$ (or an $u$-$v$ path) is also said to be from $u$ to $v$ or between $u$ and $v$. A collection of paths are said to be independent if no vertex of any path in this collection is an internal vertex of any other path in the collection. The distance between two vertices $u$ and $v$ in a graph $G$ is the minimum length of a $u$-$v$ path in $G$.

A cycle is a non-empty graph $C = (V(C), E(C))$ where $V(C)$ consists of distinct
vertices $v_0, v_1, \ldots, v_n$ and $E(C) = \{v_0v_1, v_1v_2, \ldots, v_nv_n, v_nv_0\}$. The length of a cycle is the number of edges it contains. The girth of a graph $G$, denoted as $g(G)$, is the minimum length of a cycle contained in $G$.

Let $G$ be a graph. Let $K \subseteq G$, $S \subseteq V(G)$, and $T$ a collection of 2-element subsets of $V(K) \cup S$. Then $K + (S \cup T)$ denotes the graph with vertex set $V(K) \cup S$ and edge set $E(K) \cup T$, and if $T = \{\{x, y\}\}$ we write $K + xy$ instead of $K + \{\{x, y\}\}$.

For any positive integer $k$, let $[k] := \{1, \ldots, k\}$. A 3-planar graph $(G, \mathcal{A})$ consists of a graph $G$ and a set $\mathcal{A} = \{A_1, \ldots, A_k\}$ of pairwise disjoint subsets of $V(G)$ (possibly $\mathcal{A} = \emptyset$ when $k = 0$) such that

(a) for distinct $i, j \in [k]$, $N(A_i) \cap A_j = \emptyset$,

(b) for $i \in [k]$, $|N(A_i)| \leq 3$, and

(c) if $p(G, \mathcal{A})$ denotes the graph obtained from $G$ by (for each $i$) deleting $A_i$ and adding edges joining every pair of distinct vertices in $N(A_i)$, then $p(G, \mathcal{A})$ may be drawn in a closed disc $D$ with no pair of edges crossing such that, for each $A_i$ with $|N(A_i)| = 3$, $N(A_i)$ induces a facial triangle in $p(G, \mathcal{A})$.

If, in addition, $b_1, \ldots, b_n$ are vertices of $G$ such that $b_i \notin A_j$ for any $i \in [n]$ and $j \in [k]$ and $b_1, \ldots, b_n$ occur on the boundary of the disc $D$ in that cyclic order, then we say that $(G, \mathcal{A}, b_1, \ldots, b_n)$ is 3-planar. If there is no need to specify $\mathcal{A}$, we will simply say that $(G, b_1, \ldots, b_n)$ is 3-planar. If there is no need to specify the order of $b_1, \ldots, b_n$ then we simply say that $(G, \{b_1, \ldots, b_n\})$ is 3-planar. When $\mathcal{A} = \emptyset$, we say that $(G, b_1, \ldots, b_n)$ and $(G, \{b_1, \ldots, b_n\})$ are planar. An apex graph is a graph that can be made planar by the removal of a single vertex.

Given a graph $F$, an $F$-subdivision or a subdivision of $F$ is a graph $H$ obtained from $F$ by replacing edges of $F$ with paths through new vertices of degree 2, denoted as $TF$. If $G$ contains an $F$-subdivision as a subgraph, we say $F$ is a topological minor of $G$ and $G$ contains $TF$. Furthermore, the vertices in $TF$ that correspond to the vertices of $F$ are said
to be its branch vertices. In particular, $TK_5$ denotes a subdivision of $K_5$, and the vertices in a $TK_5$ of degree four are its branch vertices.

A (proper) $k$-coloring of a graph $G = (V, E)$ is a map $c : V \rightarrow [k]$ such that $c(v) \neq c(w)$ whenever $v$ and $w$ are adjacent. The chromatic number $\chi(G)$ of $G$ is the smallest integer $k$ such that $G$ has a $k$-coloring.

For additional notations and background on graph theory, the readers are referred to Diestel’s text [1].

1.2 Main results

This dissertation studies the structural aspect of subdivisions of complete graphs in graphs. In the next chapter, we focus on $TK_5$, or subdivisions of $K_5$. A well known theorem of Kuratowski [2] in 1932 states that a graph is planar if, and only if, it does not contain a subdivision of $K_5$ or $K_{3,3}$. Wagner [3] proved in 1937 that if a graph other than $K_5$ does not contain any subdivision of $K_{3,3}$ then it is planar or it admits a cut of size at most 2. Kelmans [4] and, independently, Seymour [5] conjectured in the 1970s that if a graph does not contain any subdivision of $K_5$ then it is planar or it admits a cut of size at most 4.

Kelmans-Seymour conjecture is related to other problems in graph theory. A simple application of Euler’s formula for planar graphs shows that, for $n \geq 3$, if an $n$-vertex graph has at least $3n - 5$ edges then it must be nonplanar and, hence, contains $TK_5$ or $TK_{3,3}$. Dirac [6] conjectured that for $n \geq 3$, if an $n$-vertex graph has at least $3n - 5$ edges then it must contain $TK_5$. This conjecture was also reported by Erdős and Hajnal [7]. Kelmans [4] showed that a minimal counterexample to Dirac’s conjecture must be 5-connected. Kézdy and McGuiness [8] showed that a minimal counterexample to Dirac’s conjecture must be 5-connected and contains $\overline{K_4}$ (obtained from the complete graph $K_4$ by deleting an edge). After some partial results in [9, 10, 11, 12], Dirac’s conjecture was proved by Mader [13], where he also showed that every 5-connected $n$-vertex graph with at least $3n - 6$ edges contains $TK_5$ or $\overline{K_4}$. We note that the Kelmans-Seymour conjecture implies Mader’s
One of the motivations for us to work on the Kelmans-Seymour conjecture was the following conjecture of Hajós (see e.g., [14]) that graphs containing no $TK_5$ are 4-colorable. This conjecture, if true, would generalize the Four Color Theorem. It is known that Hajós’ conjecture holds for graphs with large girth (see Kühn and Osthus [15]). Let $G$ be a possible counterexample to Hajós’ conjecture with $|V(G)|$ minimum. Then the Kelmans-Seymour conjecture implies that $G$ has connectivity at most 4. By a standard coloring argument, it is easy to show that $G$ must be 3-connected. It is shown in [16] that $G$ must be 4-connected. It is further shown in [17] that for every 4-cut $T$ of $G$, $G - T$ has exactly two components. The work in [16, 17] suggests that $G$ should be “close” to being 5-connected.

Hajós actually made a more general conjecture in the 1950s: For any positive integer $k$, every graph containing no $TK_{k+1}$ is $k$-colorable. This is easy to verify for $k \leq 3$ (see [18]), and disproved in [19] for $k \geq 6$. However, it remains open for $k = 4$ and $k = 5$. Thomassen [14] pointed out connections between Hajós’ conjecture and Ramsey numbers, maximum cuts, and perfect graphs. We refer the reader to [14] for other work and references related to Hajós’ conjecture and topological minors.

In fact, Erdős and Fajtlowicz [20] showed that the above general Hajós’ conjecture for $k \geq 6$ fails for almost all graphs. Let $H(n) := \max\{\chi(G)/\sigma(G) : G \text{ is a graph with } |V(G)| = n\}$, where $\chi(G)$ denotes the chromatic number of $G$ and $\sigma(G)$ denotes the largest $t$ such that $G$ contains $TK_t$. Erdős and Fajtlowicz [20] showed that $H(n) = \Omega(\sqrt{n}/\log n)$, and conjectured that $H(n) = \Theta(\sqrt{n}/\log n)$. This conjecture was verified by Fox, Lee and Sudakov [21], by studying $\sigma(G)$ in terms of independence number $\alpha(G)$.

In Chapter 2, we give a proof of the Kelmans-Seymour conjecture by proving the following

**Theorem 1.2.1** Every 5-connected non-planar graph contains $TK_5$.

This is joint work with Dawei He and Xingxing Yu. We also discuss several related results and problems.
CHAPTER 2

$K_5$-SUBDIVISIONS IN 5-CONNECTED NONPLANAR GRAPHS

In this chapter, we study $K_5$-subdivisions in 5-connected nonplanar graphs. A well known theorem of Kuratowski in 1932 states that a graph is planar if, and only if, it does not contain a subdivision of $K_5$ or $K_{3,3}$. Wagner proved in 1937 that if a graph other than $K_5$ does not contain any subdivision of $K_{3,3}$ then it is planar or it admits a cut of size at most 2. Kelmans and, independently, Seymour conjectured in the 1970s that if a graph does not contain any subdivision of $K_5$ then it is planar or it admits a cut of size at most 4. In this chapter, we give a proof of the Kelmans-Seymour conjecture. We also discuss several related results and problems.

This is joint work with Dawei He and Xingxing Yu.

2.1 Introduction

In 1930, Kuratowski [2] prove the following well known result.

Theorem 2.1.1 A graph is planar if, and only if, it does not contain $TK_5$ or $TK_{3,3}$.

A simple application of Euler’s formula for planar graphs shows that, for $n \geq 3$, if an $n$-vertex graph has at least $3n - 5$ edges then it must be nonplanar and, hence, contains $TK_5$ or $TK_{3,3}$. Dirac [6] conjectured that for $n \geq 3$, if an $n$-vertex graph has at least $3n - 5$ edges then it must contain $TK_5$. This conjecture was also reported by Erdős and Hajnal [7]. Kelmans [4] showed that a minimal counterexample to Dirac’s conjecture must be 5-connected. Kézdy and McGuiness [8] showed that a minimal counterexample to Dirac’s conjecture must be 5-connected and contains $K_4^-$ (obtained from the complete graph $K_4$ by deleting an edge). After some partial results in [9, 10, 11, 12], Dirac’s conjecture was proved by Mader [13], where he also showed that every 5-connected $n$-vertex graph with
at least $3n - 6$ edges contains $TK_5$ or $K_4^-$.  

Seymour [5] (also see [13, 12]) and, independently, Kelmans [4] made the following.

**Conjecture 2.1.2** Every 5-connected nonplanar graph contains $TK_5$.

Thus, the Kelmans-Seymour conjecture implies Mader’s theorem. This conjecture is also related to several interesting problems, which we will discuss later.

He, Wang and Yu [22, 23, 24] produced lemmas needed for proving this Kelmans-Seymour conjecture, and in this dissertation we are now ready to prove the following.

**Theorem 1.2.1** Every 5-connected non-planar graph contains $TK_5$.

The starting point of our work is the following result of Ma and Yu [25, 26]: Every 5-connected nonplanar graph containing $K_4^-$ has a $TK_5$. This result, combined with the result of Kézdy and McGuiness [8] on minimal counterexamples to Dirac’s conjecture, gives an alternative proof of Mader’s theorem. Also using this result, Aigner-Horev [27] proved that every 5-connected nonplanar apex graph contains $TK_5$. A simpler proof of Aigner-Horev’s result using discharging argument was obtained by Ma, Thomas and Yu, and, independently, by Kawarabayashi, see [28].

The reminder of this chapter is organized as follows. In the next section, we discuss several related problems. We give a brief sketch of the proof of Theorem 1.2.1 in Section 2.3. We will need a number of results from [22, 23, 24], which are given in Section 2.4. In Section 2.5, we derive a simplified version of a result on disjoint paths from [29, 30, 31], which will be used several times in Section 2.6. For each subgraph $T$ of $H$ with $v \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, we will associate to it a quadruple $(T, S_T, A, B)$, where, roughly, $A \cap B = \emptyset$, $H - S_T = A \cup B$, and $H$ has no edge between $A$ and $B$. (A precise definition of a quadruple is given in Section 2.6.) In Section 2.6, we prove some basic properties of quadruples, and take care of two special cases involving quadruples (using disjoint paths results from Section 2.5). In Section 2.7, we take care of other cases involving quadruples. We complete the proof of Theorem 1.2.1 in Section 2.8.
2.2 Related Problems

Theorem 1.2.1 implies that if a graph contains no $TK_5$ then it is planar, or admits a cut of size at most 4. This is a step towards a more useful structural description of the class of graphs containing no $TK_5$. There is a nice result for graphs containing no $TK_{3,3}$ due to Wagner [3]: Every such graph is planar, or is a $K_5$, or admits a cut of size at most 2.

Mader [13] conjectured that every simple graph with minimum degree at least 5 and no $K_4^-$ contains $TK_5$, and he also asked the following.

**Question 2.2.1** Does every simple graph on $n \geq 4$ vertices with more than $12(n-2)/5$ edges contain $K_4^-$, $K_{2,3}$, or $TK_5$?

In a recent paper [28], it is shown that an affirmative answer to Question 2.2.1 implies the Kelmans-Seymour conjecture. As an independent approach to resolve the Kelmans-Seymour conjecture, Kawarabayashi, Ma and Yu planned to find a contractible cycle in a 5-connected nonplanar graph containing no $K_4^-$ or $K_{2,3}$, and then use such a cycle to find a $TK_5$ by applying augmenting path arguments. This plan (if successful), combined with the results in [26, 28], would give an alternative (and cleaner) solution to the Kelmans-Seymour conjecture.

One of the motivations for us to work on the Kelmans-Seymour conjecture was the following conjecture of Hajós (see e.g., [14]) which, if true, would generalize the Four Color Theorem.

**Conjecture 2.2.2** Graphs containing no $TK_5$ are 4-colorable.

It is known that Conjecture 2.2.2 holds for graphs with large girth (see Kühn and Osthus [15]). Let $G$ be a possible counterexample to Conjecture 2.2.2 with $|V(G)|$ minimum. Then our result on the Kelmans-Seymour conjecture implies that $G$ has connectivity at most 4. By a standard coloring argument, it is easy to show that $G$ must be 3-connected. It is shown in [16] that $G$ must be 4-connected. It is further shown in [17] that for every 4-cut
T of G, G − T has exactly two components. The work in [16, 17] suggests that G should be “close” to being 5-connected.

Hajós actually made a more general conjecture in the 1950s: For any positive integer k, every graph containing no TK_k+1 is k-colorable. This is easy to verify for k ≤ 3 (see [18]), and disproved in [19] for k ≥ 6. However, it remains open for k = 4 (Conjecture 2.2.2) and k = 5. Thomassen [14] pointed out connections between Hajós’ conjecture and Ramsey numbers, maximum cuts, and perfect graphs. We refer the reader to [14] for other work and references related to Hajós’ conjecture and topological minors.

In fact, Erdős and Fajtlowicz [20] showed that the above general Hajós’ conjecture for k ≥ 6 fails for almost all graphs. Let H(n) := max{χ(G)/σ(G) : G is a graph with |V(G)| = n}, where χ(G) denotes the chromatic number of G and σ(G) denotes the largest t such that G contains TK_t. Erdős and Fajtlowicz [20] showed that H(n) = Ω(√n/log n), and conjectured that H(n) = Θ(√n/log n). This conjecture was verified by Fox, Lee and Sudakov [21], by studying σ(G) in terms of independence number α(G). The following conjecture of Fox, Lee and Sudakov [21] is interesting.

**Conjecture 2.2.3** There is a constant c > 0 such that every graph G with χ(G) = k satisfies σ(G) ≥ c√k log k.

A key idea in [25, 26, 22, 23, 24] for finding TK_5 in graphs containing K_4^- is to find a nonseparating path in a graph that avoids two given vertices. Let G be a 5-connected nonplanar graph and x_1, x_2, y_1, y_2 ∈ V(G) such that {x_1, x_2, y_1, y_2} induces a K_4^- in which x_1, x_2 are of degree 3. We used an induced path X in G between x_1 and x_2 such that G − X is 2-connected and {y_1, y_2} ∉ V(X), and in certain cases we need X to contain a special edge at x_1 (for example, in Section 2.8, x_1 = x is the special vertex representing the contraction of M). If we could find such X that G − X is 3-connected then our proofs would have been much simpler. This is related to the following conjecture of Lovász [32].

**Conjecture 2.2.4** There exists an integer valued function f(k) such that for any f(k)-connected graph G and for any A ⊆ V(G) with |A| = 2, there exist vertex disjoint sub-
graphs $G_1, G_2$ of $G$ such that $V(G_1) \cup V(G_2) = V(G)$, $G_1$ is a path between the vertices in $A$, and $G_2$ is $k$-connected.

A classical result of Tutte [33] implies $f(1) = 3$. That $f(2) = 5$ was proved by Kriesell [34] and, independently, by Chen, Gould and Yu [35]. Despite much effort from the research community, Conjecture 2.2.4 remains open for $k \geq 3$. Variations of Conjecture 2.2.4 for $k = 2$ are used in [25, 26, 22, 23, 24] to resolve the Kelmans-Seymour conjecture. An edge version of Conjecture 2.2.4 was conjectured by Kriesell and proved by Kawarabayashi et al. [36]. Thomassen [37] conjectured a statement that is more general than Conjecture 2.2.4 by allowing $|A| \geq 2$ and requiring $A \subseteq V(G_1)$ and $G_1$ be $k$-connected.

2.3 Proof sketch of Theorem 1.2.1

We now briefly describe the process for proving Theorem 1.2.1. For a more detailed version, we recommend the reader to read Section 2.8 first, which should also give motivation to some of the technical lemmas listed in Sections 2.4, 2.5, 2.6 and 2.7.

Suppose $G$ is a 5-connected non-planar graph not containing $K_4^-$. We fix a vertex $v \in V(G)$, and let $M$ be a maximal connected subgraph of $G$ such that $v \in V(M)$, $G/M$ (the graph obtained from $G$ by contracting $M$) is nonplanar, $G/M$ contains no $K_4^-$, and $G/M$ is 5-connected (i.e., $M$ is contractible). Note that $V(M) = \{v\}$ is possible. Let $x$ denote the vertex of $H := G/M$ resulting from the contraction of $M$. Then, for each subgraph $T$ of $H$ with $v \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, $H/T$ is planar, or $H/T$ contains $K_4^-$, or $H/T$ is not 5-connected. If, for some $T$, $H/T$ is planar or contains $K_4^-$ then we can find a $TK_5$ in $G$ using results from [22, 23, 24]. Thus, in this dissertation, our main work is to deal with the final case: for any $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, $H/T$ is nonplanar, $H/T$ contains no $K_4^-$, and $H/T$ is not 5-connected. In this case, there exists $S_T \subseteq V(H)$ such that $V(T) \subseteq S_T$, $|S_T| = 5$ or $|S_T| = 6$, and $H - S_T$ is not connected. We will be using such cuts to divide the graph into smaller parts and use them to find a special $TK_5$ in $H$. The reason to also include the case $T \cong K_3$ is to avoid the situation
when $T \cong K_2$, $|S_T| = 5$, and $H - S_T$ has exactly two components, one of which is trivial. This does not cause problem when $T \cong K_3$, as the graph $H$ would then contain $K_4^-$, and we could use results from [22, 23, 24].

2.4 Previous results

In this section, we list a number of previous results which we will use as lemmas in our proof of Theorem 1.2.1. We begin with the main result of [25, 26].

**Lemma 2.4.1** Every 5-connected nonplanar graph containing $K_4^-$ has a TK$_5$.

We also need the main result of [23] to take care of the case when the vertex $x$ in $H = G/M$ (see Section 2.3) is a degree 2 vertex in a $K_4^-$ (which is $y_2$ in the lemma below).

**Lemma 2.4.2** Let $G$ be a 5-connected nonplanar graph and $\{x_1, x_2, y_1, y_2\} \subseteq V(G)$ such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ with $y_1y_2 \notin E(G)$. Then one of the following holds:

(i) $G$ contains a TK$_5$ in which $y_2$ is not a branch vertex.

(ii) $G - y_2$ contains $K_4^-$. 

(iii) $G$ has a 5-separation $(G_1, G_2)$ such that $V(G_1 \cap G_2) = \{y_2, a_1, a_2, a_3, a_4\}$, and $G_2$ is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding $y_2$ and the edges $y_2b_i$ for $i \in \{4\}$.

(iv) For $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$, $G - \{y_2v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$ contains TK$_5$.

To deal with conclusion (iii) of Lemma 2.4.2, we need Proposition 1.3 from [22] in which $a$ plays the role of $y_2$ in Lemma 2.4.2.

**Lemma 2.4.3** Let $G$ be a 5-connected nonplanar graph, $(G_1, G_2)$ a 5-separation in $G$, $V(G_1 \cap G_2) = \{a, a_1, a_2, a_3, a_4\}$ such that $G_2$ is the graph obtained from the edge-disjoint
union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding $a$ and the edges $ab_i$, $i \in [4]$. Suppose $|V(G_1)| \geq 7$. Then, for any $u_1, u_2 \in N(a) - \{b_1, b_2, b_3\}$, $G - \{av : v \not\in \{b_1, b_2, b_3, u_1, u_2\}\}$ contains $TK_5$.

Next we list a few results from [22, 23, 24]. For convenience, we state their versions from [24]. First, we need Theorem 1.1 in [24] to take care of the case when the vertex $x$ in $H = G/M$ (see Section 2.3) is a degree 3 vertex in a $K_4^-$ (which is $x_1$ in the lemma below).

**Lemma 2.4.4** Let $G$ be a 5-connected nonplanar graph and $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \not\in E(G)$. Then one of the following holds:

(i) $G$ contains a $TK_5$ in which $x_1$ is not a branch vertex.

(ii) $G - x_1$ contains $K_4^-$, or $G$ contains a $K_4^-$ in which $x_1$ is of degree 2.

(iii) $x_2, y_1, y_2$ may be chosen so that for any distinct $z_0, z_1 \in N(x_1) - \{x_2, y_1, y_2\}$, $G - \{x_1v : v \not\in \{x_2, y_1, y_2, z_0, z_1\}\}$ contains $TK_5$.

When applying the next three lemmas, the vertex $a$ will correspond to the vertex $x$ in $H = G/M$ in Section 2.3. The following result is Lemma 2.7 in [24], which deals with 5-separations with an apex side.

**Lemma 2.4.5** Let $G$ be a 5-connected nonplanar graph and let $(G_1, G_2)$ be a 5-separation in $G$. Suppose $|V(G_i)| \geq 7$ for $i \in [2]$, $a \in V(G_1 \cap G_2)$, and $(G_2 - a, V(G_1 \cap G_2) - \{a\})$ is planar. Then one of the following holds:

(i) $G$ contains a $TK_5$ in which $a$ is not a branch vertex.

(ii) $G - a$ contains $K_4^-$, or $G$ contains a $K_4^-$ in which $a$ is of degree 2.

The next result is Lemma 2.8 in [24], which will be used to take care of 5-cuts containing the vertices of a triangle.
Lemma 2.4.6 Let $G$ be a 5-connected graph and $(G_1, G_2)$ be a 5-separation in $G$. Suppose that $|V(G_i)| \geq 7$ for $i \in [2]$ and $G[V(G_1 \cap G_2)]$ contains a triangle $aa_1a_2a$. Then one of the following holds:

(i) $G$ contains a $TK_5$ in which $a$ is not a branch vertex.

(ii) $G - a$ contains $K_4^-$, or $G$ contains a $K_4^-$ in which $a$ is of degree 2.

(iii) For any distinct $u_1, u_2, u_3 \in N(a) - \{a_1, a_2\}$, $G - \{av : v \notin \{a_1, a_2, u_1, u_2, u_3\}\}$ contains $TK_5$.

The following is Lemma 2.9 in [24].

Lemma 2.4.7 Let $G$ be a graph, $A \subseteq V(G)$, and $a \in A$ such that $|A| = 6$, $|V(G)| \geq 8$, $(G - a, A - \{a\})$ is planar, and $G$ is $(5, A)$-connected. Then one of the following holds:

(i) $G - a$ contains $K_4^-$, or $G$ contains a $K_4^-$ in which the degree of $a$ is 2.

(ii) $G$ has a 5-separation $(G_1, G_2)$ such that $a \in V(G_1 \cap G_2)$, $|V(G_2)| \geq 7$, $A \subseteq V(G_1)$, and $(G_2 - a, V(G_1 \cap G_2) - \{a\})$ is planar.

We need Theorem 1.4 in [22]. This will be used to show that, for a quadruple $(T, S_T, A, B)$ in $H = G/M$ with $x \in V(T)$ (see Section 2.3), $x$ has a neighbor in $A$ (which corresponds to $G_1 - G_2$ in the statement).

Lemma 2.4.8 Let $G$ be a 5-connected graph and $x \in V(G)$, and let $(G_1, G_2)$ be a 6-separation in $G$ such that $x \in V(G_1 \cap G_2)$, $G[V(G_1 \cap G_2)]$ contains a triangle $xx_1x_2x$, $|V(G_i)| \geq 7$ for $i \in [2]$. Moreover, assume that $(G_1, G_2)$ is chosen so that, subject to $\{x, x_1, x_2\} \subseteq V(G_1 \cap G_2)$ and $|V(G_i)| \geq 7$ for $i \in [2]$, $G_1$ is minimal. Let $V(G_1 \cap G_2) = \{x, x_1, x_2, v_1, v_2, v_3\}$. Then $N(x) \cap V(G_1 - G_2) \neq \emptyset$, or one of the following holds:

(i) $G$ contains a $TK_5$ in which $x$ is not a branch vertex.

(ii) $G$ contains $K_4^-$. 

13
(iii) There exists $x_3 \in N(x)$ such that for any distinct $y_1, y_2 \in N(x) - \{x_1, x_2, x_3\}$, $G - \{x : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}$ contains $TK_5$.

(iv) For some $i \in [2]$ and some $j \in [3]$, $N(x_i) \subseteq V(G_1 - G_2) \cup \{x, x_3 - i\}$, and any three independent paths in $G_1 - x$ from $\{x_1, x_2\}$ to $v_1, v_2, v_3$, respectively, with two from $x_i$ and one from $x_3 - i$, must contain a path from $x_3 - i$ to $v_j$.

We remark that conclusion (iv) in Lemma 2.4.8 will be dealt with in Section 2.6, using a result on disjoint paths from [29, 30, 31]. We also need Proposition 4.1 from [22] to deal with the case when $H/T$ is planar (see Section 2.3) for some $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$.

**Lemma 2.4.9** Let $G$ be a 5-connected nonplanar graph, $x \in V(G)$, $T \subseteq G$ such that $x \in V(T)$, $T \cong K_2$ or $T \cong K_3$, $G/T$ is 5-connected and planar. Then $G - T$ contains $K_4^-$.

We conclude this section with three additional results, first of which is a result of Seymour [38]; equivalent versions are proved in [39, 40, 41].

**Lemma 2.4.10** Let $G$ be a graph and let $s_1, s_2, t_1, t_2 \in V(G)$ be distinct. Then either $G$ contains disjoint paths from $s_1$ to $t_1$ and from $s_2$ to $t_2$, or $(G, s_1, s_2, t_1, t_2)$ is 3-planar.

The second result is due to Perfect [42].

**Lemma 2.4.11** Let $G$ be a graph, $u \in V(G)$, and $A \subseteq V(G - u)$. Suppose there exist $k$ independent paths from $u$ to distinct $a_1, \ldots, a_k \in A$, respectively, and internally disjoint from $A$. Then for any $n \geq k$, if there exist $n$ independent paths $P_1, \ldots, P_n$ in $G$ from $u$ to $n$ distinct vertices in $A$ and internally disjoint from $A$ then $P_1, \ldots, P_n$ may be chosen so that $a_i \in V(P_i)$ for $i \in [k]$.

The third result is due to Watkins and Mesner [43].
Lemma 2.4.12 Let $G$ be a 2-connected graph and let $y_1, y_2, y_3$ be three distinct vertices of $G$. Then $G$ has no cycle containing $\{y_1, y_2, y_3\}$ if, and only if, one of the following holds:

(i) There exists a 2-cut $S$ in $G$ and there exist pairwise disjoint subgraphs $D_{y_i}$ of $G - S$, $i \in [3]$, such that $y_i \in V(D_{y_i})$ and each $D_{y_i}$ is a union of components of $G - S$.

(ii) There exist 2-cuts $S_{y_i}$ in $G$, $i \in [3]$, and pairwise disjoint subgraphs $D_{y_i}$ of $G$, such that $y_i \in V(D_{y_i})$, each $D_{y_i}$ is a union of components of $G - S_{y_i}$, there exists $z \in S_{y_1} \cap S_{y_2} \cap S_{y_3}$, and $S_{y_1} - \{z\}, S_{y_2} - \{z\}, S_{y_3} - \{z\}$ are pairwise disjoint.

(iii) There exist pairwise disjoint 2-cuts $S_{y_i}$ in $G$ and pairwise disjoint subgraphs $D_{y_i}$ of $G - S_{y_i}$, $i \in [3]$, such that $y_i \in V(D_{y_i})$, $D_{y_i}$ is a union of components of $G - S_{y_i}$, and $G - V(D_{y_1} \cup D_{y_2} \cup D_{y_3})$ has precisely two components, each containing exactly one vertex from $S_{y_i}$ for $i \in [3]$.

2.5 Obstruction to three paths

In order to deal with (iv) of Lemma 2.4.8, we need a result of Yu [29, 30, 31], which characterizes graphs $G$ in which any three disjoint paths from $\{a, b, c\} \subseteq V(G)$ to $\{a', b', c'\} \subseteq V(G)$ must contain a path from $b$ to $b'$. The objective of this section is to derive a much simpler version of that characterization by imposing extra conditions on $G$. This result will be used several times in the proofs of Lemmas 2.6.4 and 2.6.6. To state the result from [29, 30, 31], we need to describe rungs and ladders.

Let $G$ be a graph, $\{a, b, c\} \subseteq V(G)$, and $\{a', b', c'\} \subseteq V(G)$. Suppose $\{a, b, c\} \neq \{a', b', c'\}$, and assume that $G$ has no separation $(G_1, G_2)$ such that $|V(G_1 \cap G_2)| \leq 3$, $\{a, b, c\} \subseteq V(G_1)$, and $\{a', b', c'\} \subseteq V(G_2)$. We say that $(G, (a, b, c), (a', b', c'))$ is a rung if one of the following holds:

1. $b = b'$ or $\{a, c\} = \{a', c'\}$.

2. $a = a'$ and $(G - a, c, c', b', b)$ is 3-planar, or $c = c'$ and $(G - c, a, a', b', b)$ is 3-planar.
(3) \( \{a, b, c\} \cap \{a', b', c'\} = \emptyset \) and \((G, a', b', c', c, b, a)\) is 3-planar.

(4) \( \{a, b, c\} \cap \{a', b', c'\} = \emptyset \), \(G\) has a 1-separation \((G_1, G_2)\) such that (i) \( \{a, a', b, b'\} \subseteq V(G_1)\), \(\{c, c'\} \subseteq V(G_2)\), and \((G_1, a, a', b', b)\) is 3-planar, or (ii) \( \{c, c', b, b'\} \subseteq V(G_1)\), \(\{a, a'\} \subseteq V(G_2)\), and \((G_1, c, c', b', b)\) is 3-planar.

(5) \( \{a, b, c\} \cap \{a', b', c'\} = \emptyset \), and \(G\) has a separation \((G_1, G_2)\) such that \(V(G_1 \cap G_2) = \{z, b\}\) (or \(V(G_1 \cap G_2) = \{z, b'\}\)), and (i) \((G, a, a', b', b)\) is 3-planar, \(\{a, a', b, b'\} \subseteq V(G_1)\), \(\{c, c'\} \subseteq V(G_2)\), and \((G_1, c, c', z, b)\) (or \((G_2, c, c', b', z)\)) is 3-planar, or (ii) \((G, c, c', b', b)\) is 3-planar, \(\{c, c', b, b'\} \subseteq V(G_1)\), \(\{a, a'\} \subseteq V(G_2)\), and \((G_2, a, a', z, b)\) (or \((G_2, a, a', b', z)\)) is 3-planar.

(6) \( \{a, b, c\} \cap \{a', b', c'\} = \emptyset \), and there are pairwise edge disjoint subgraphs \(G_a, G_c, M\) of \(G\) such that \(G = G_a \cup G_c \cup M\), \(V(G_a \cap M) = \{u, w\}\), \(V(G_c \cap M) = \{p, q\}\), \(V(G_a \cap G_c) = \emptyset\), and (i) \(\{a, a', b\} \subseteq V(G_a)\), \(\{c, c', b\} \subseteq V(G_c)\), and \((G_a, a, a', b, w, u)\) and \((G_c, c, c', b, p, q)\) are 3-planar, or (ii) \(\{a, a', b\} \subseteq V(G_a)\), \(\{c, c', b'\} \subseteq V(G_c)\), \((G_a, b, a, a', w, u)\) and \((G_c, b', c, c', p, q)\) are 3-planar.

(7) \( \{a, b, c\} \cap \{a', b', c'\} = \emptyset \), and there are pairwise edge disjoint subgraphs \(G_a, G_c, M\) of \(G\) such that \(G = G_a \cup G_c \cup M\), \(V(G_a \cap M) = \{b, b'\}\), \(V(G_c \cap M) = \{b, b', p\}\), \(V(G_a \cap G_c) = \{b, b'\}\), \(\{a, a'\} \subseteq V(G_a)\), \(\{c, c'\} \subseteq V(G_c)\), and \((G_a, a, a', b', w, b)\) and \((G_c, c', c, b, p, b')\) are 3-planar.

Let \(L\) be a graph and let \(R_1, \ldots, R_m\) be edge disjoint subgraphs of \(L\) such that

(i) \((R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))\) is a rung for each \(i \in [m]\),

(ii) \(V(R_i \cap R_j) = \{x_i, v_i, y_i\} \cap \{x_{j-1}, v_{j-1}, y_{j-1}\}\) for \(i, j \in [m]\) with \(i < j\),

(iii) for any distinct \(i, j \in [m]\), if \(x_i = x_j\) then \(x_k = x_i\) for all \(i \leq k \leq j\), if \(v_i = v_j\) then \(v_k = v_i\) for all \(i \leq k \leq j\), and if \(y_i = y_j\) then \(y_k = y_i\) for all \(i \leq k \leq j\), and

(iv) \(L = (\bigcup_{i=1}^m R_i) + S\), where \(S\) consists of those edges of \(L\) each of which has both ends in \(\{x_i, v_i, y_i\}\) for some \(i \in [m]\).

Then \((L, (x_0, v_0, y_0), (x_m, v_m, y_m))\) is a ladder with rungs \((R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))\), \(i \in [m]\), or simply, a ladder along \(v_0 \ldots v_m\).
By the definition of a rung, we see that a ladder $(L, (x_0, v_0, y_0), (x_m, v_m, y_m))$ has three disjoint paths from \( \{x_0, v_0, y_0\} \) to \( \{x_m, v_m, y_m\} \).

For a sequence \( W \), the reduced sequence of \( W \) is the sequence obtained from \( W \) by removing all but one consecutive identical elements. For example, the reduced sequence of \( aaabcca \) is \( abca \). We can now state the main result in [31].

**Lemma 2.5.1** Let \( G \) be a graph, \( \{a, b, c\} \subseteq V(G) \), and \( \{a', b', c'\} \subseteq V(G) \) such that \( \{a, b, c\} \neq \{a', b', c'\} \). Assume that, for any \( T \subseteq V(G) \) with \( |T| \leq 3 \), every component of \( G - T \) contains some element of \( \{a, b, c\} \cup \{a', b', c'\} \). Then any three disjoint paths in \( G \) from \( \{a, b, c\} \) to \( \{a', b', c'\} \) must include one from \( b \) to \( b' \) if, and only if, one of the following statements holds:

(i) \( G \) has a separation \( (G_1, G_2) \) of order at most 2 such that \( \{a, b, c\} \subseteq V(G_1) \) and \( \{a', b', c'\} \subseteq V(G_2) \).

(ii) \( (G, (a, b, c), (a', b', c')) \) is a ladder.

(iii) \( G \) has a separation \( (J, L) \) such that \( V(J \cap L) = \{w_0, \ldots, w_n\} \), \( (J, w_0, \ldots, w_n) \) is 3-planar, \( \{a, b, c\} \cup \{a', b', c'\} \subseteq V(L) \), \( (L, (a, b, c), (a', b', c')) \) is a ladder along a sequence \( v_0 \ldots v_m \), where \( v_0 = b \), \( v_m = b' \), and \( w_0 \ldots w_n \) is the reduced sequence of \( v_0 \ldots v_m \).

We may view \( (ii) \) as a special case of \( (iii) \) by letting \( J \) be a subgraph of \( L \). In the applications of Lemma 2.5.1 in this paper, we will consider rungs and ladders in a 5-connected graph without \( TK_5 \). With such extra conditions, the rungs have much simpler structure, as given in the next two lemmas.

**Lemma 2.5.2** Let \( G \) be a 5-connected graph and \( (R, R') \) a separation in \( G \) such that \( |V(R')| \geq 8 \), \( V(R \cap R') = \{a, b\} \cup \{a', b', c'\}, a \neq b \), and \( a', b', c' \) are pairwise distinct. Let \( R^* \) be obtained from \( R \) by adding the new vertex \( c \) and joining \( c \) to each neighbor of \( a \) in \( R \) with an edge, and assume \( (R^*, (a, b, c), (a', b', c')) \) is a rung. Then \( b = b' \), \( V(R) = \{a, b, a', c'\} \) and \( E(R) = \{aa', ac'\} \).
Proof. Since \( a \) and \( c \) have the same set of neighbors in \( R^* \) and \((R^*, (a, b, c), (a', b', c'))\) is a rung, it follows from the definition of a rung that \((R^*, (a, b, c), (a', b', c'))\) is of type (1) or (2). Then, since \( G \) is 5-connected, \( V(R) = \{a, b\} \cup \{a', b', c'\} \).

Suppose \((R^*, (a, b, c), (a', b', c'))\) is of type (2). By symmetry, we may assume that \( c = c' \) and \((G-c, a, a', b', b)\) is 3-planar. Then \( ab \notin E(G) \) or \( a'b \notin E(G) \). Hence, \( \{a', b, c\} \) or \( \{a, b', c\} \) would be a cut in \( R^* \) separating \( \{a, b, c\} \) from \( \{a', b', c'\} \), a contradiction.

So \((R^*, (a, b, c), (a', b', c'))\) is of type (1). Then, since \( R^* \) has no separation of order at most 3 separating \( \{a, b, c\} \) from \( \{a', b', c'\} \), we deduce that \( a \neq a', c \neq c' \), and \( E(R) = \{aa', ac'\} \).

Note that the conclusion of Lemma 2.5.2 is a special case of (i) of the next lemma.

Lemma 2.5.3 Let \( G \) be a 5-connected graph and \((R, R')\) a separation in \( G \) such that \(|V(R')| \geq 8\), \( V(R \cap R') = \{a, b, c\} \cup \{a', b', c'\}, \{a, b, c\} \neq \{a', b', c'\}, \) and \((R, (a, b, c), (a', b', c'))\) is a rung. Then \( G \) contains \( TK_5 \) or \( K_4^- \), or one of the following holds:

(i) \( b = b' \).

(ii) \( \{a, c\} = \{a', c'\}, V(R) = \{a, c, b, b'\}, \) and \( E(R) = \{bb'\} \).

(iii) \( V(R) - (\{a, b, c\} \cup \{a', b', c'\}) = \{v\} \) and \( N(v) = \{a, b, c\} \cup \{a', b', c'\}, \) and either \( a = a' \) and \( E(R - v) = \{bb', cc'\} \) or \( c = c' \) and \( E(R - v) = \{bb', aa'\} \).

(iv) \( \{a, b, c\} \cap \{a', b', c'\} = \emptyset, V(R) - \{a, a', b, b', c, c'\} = \{v\}, N(v) = \{a, a', b, b', c, c'\}, \) and \( E(R - v) = \{aa', bb', cc'\} \).

Proof. Without loss of generality, let \( A, B, C \) be disjoint paths in \( R \) from \( a, b, c \) to \( a', b', c' \), respectively. First, we consider the case when \( \{a, b, c\} \cap \{a', b', c'\} \neq \emptyset \). If \( b = b' \) then (i) holds; so we may assume \( b \neq b' \). If \( a = a' \) and \( c = c' \) then, since \( G \) is 5-connected, \( V(R) = \{a, b, b'\} \); so \( bb' \in E(R) \) (because of the paths \( A, B, C \)), and we have (ii). Thus by symmetry between \( \{a, a'\} \) and \( \{c, c'\} \), we may assume \( c \neq c' \). Suppose \( a = a' \). Then by the definition of a rung, \( R - a \) has no disjoint paths from \( b, c \) to
connecting.

Hence, \((R, R')\) holds.

Suppose (4) holds for \((R, (a, b, c), (a', b', c'))\). By symmetry, assume that \(R\) has a 1-separation \((G_1, G_2)\) such that \(\{a, a', b, b'\} \subseteq V(G_1), \{c, c'\} \subseteq V(G_2)\), and \((G_1, a, a', b', b)\) is 3-planar. Let \(V(G_1 \cap G_2) = \{v\}\). Since \(G\) is 5-connected, \((G_1, a, a', b', b)\) is planar and \(V(G_2) = \{v, c, c'\}\). Moreover, \(vc, vc', cc' \in E(G)\); for otherwise \(R\) would have a separation \((R_1, R_2)\) such that \(\{a, b, c\} \subseteq V(R_1), \{a', b', c'\} \subseteq V(R_2)\), and \(V(R_1 \cap R_2) \in \{\{a, b, c\}, \{a', b', c\}, \{a, b, v\}\}\). If \(|V(G_1)| \geq 7\) then the assertion follows from Lemma 2.4.5, using the separation \((R, R')\). If \(|V(G_1)| = 6\) then let \(t \in V(G_1) - \{a, a', b, b', v\}\); now \(N(t) = \{a, a', b, b', v\}\) and \(|(N(v) - \{a, b, c\}) \cap N(t)| \geq 2\) (since \(G\) is 5-connected), and hence \(R\) (and therefore \(G\)) contains \(K_5\). So we may assume \(V(G_1) = \{a, a', b, b', v\}\). Then \(va' \in E(G)\); otherwise \(N(v) = \{a, b, b', c, c'\}\) and, hence, \(ab' \notin E(G)\) (as \((G_1, a, a', b', b)\) is planar), which implies that \(\{a, b', c'\}\) is a cut in \(R\) separating \(\{a, b, c\}\) from \(\{a', b', c'\}\), a contradiction. Similarly, \(va, vb, vb' \in E(G)\). Then by planarity of \((G_1, a, a', b', b)\), we have \(ab', ba' \not\in E(G)\). So \(aa', bb' \in E(G)\) as \(\{c, v, b'\}\) and \(\{a, v, c\}\) are not 3-cuts in \(R\) separating \(\{a, b, c\}\) from \(\{a', b', c'\}\). Thus we have \((iv)\).
Suppose (5) holds for \((R, (a, b, c), (a', b', c'))\), and assume by symmetry that \((R, a, a', b', b)\) is 3-planar, and \(R\) has a separation \((G_1, G_2)\) such that \(V(G_1 \cap G_2) = \{z, b\}, \{a, a', b, b'\} \subseteq V(G_1), \{c, c'\} \subseteq V(G_2)\), and \((G_2, c, c', z, b)\) is 3-planar. Since \(G\) is 5-connected, \(V(G_2) = \{b, z, c, c'\}\). Then \(cz, cc' \in E(G)\) as otherwise, \(\{a, b, c'\}\) or \(\{a, b, z\}\) would be a 3-cut in \(R\) separating \(\{a', b', c'\}\). Hence, since \((G_2, b, z, c', c)\) is planar, \(bc' \notin E(G)\). Since \((R, a, a', b', b)\) is 3-planar, \((G_1, a, a', b', b)\) is 3-planar. Thus, the separation \((G_1, G_2 - b)\) shows that \((R, (a, b, c), (a', b', c'))\) is of type (4); so we may assume that \((iv)\) holds by the argument in the previous paragraph.

Now suppose (6) holds for \((R, (a, b, c), (a', b', c'))\), and, by symmetry, assume that there are pairwise edge disjoint subgraphs \(G_a, G_c, M\) of \(R\) such that \(R = G_a \cup G_c \cup M, V(G_a \cap M) = \{u, w\}, V(G_c \cap M) = \{p, q\}, V(G_a \cap G_c) = \emptyset, \{a, a', b'\} \subseteq V(G_a), \{c, c', b\} \subseteq V(G_c)\), and \((G_a, a, a', b', w, u)\) and \((G_c, c', c, b, p, q)\) are 3-planar. Since \(G\) is 5-connected, we have \(V(M) = \{p, q, u, w\}\), and \((G_a, a, a', b', w, u)\) and \((G_c, c', c, b, p, q)\) are planar. We may assume that \(\vert V(G_c) - \{b, c, c', p, q\} \vert \leq 1\) and \(\vert V(G_a) - \{a, a', b', u, w\} \vert \leq 1\), as otherwise the assertion follows from Lemma 2.4.5 with the separation \((G_c, G_a \cup M \cup R')\) or \((G_a, G_c \cup M \cup R')\). If there exists \(v \in V(G_c) - \{b, c, c', p, q\}\) then, since \(G\) is 5-connected, \(N(v) = \{b, c, c', p, q\}\) and \(\vert (N(p) - \{u, w\}) \cap \{b, c, c', q\} \vert \geq 2\); so \(R\) (and hence \(G\)) contains \(K^r_{14}\). Thus we may assume \(V(G_c) = \{b, c, c', p, q\}\). Since \(G\) is 5-connected, \(p\) and \(q\) each have at least five neighbors in \(G_c \cup M\). Hence, since \((G_c, b, c, c', q, p)\) is planar, \(N(p) = \{u, w, b, c, q\}\) and \(N(q) = \{u, w, c, c', p\}\); so \(G[\{p, q, u, w\}]\) (and hence \(G\)) contains \(K^r_{14}\).

Suppose (7) holds for \((R, (a, b, c), (a', b', c'))\). Then there are pairwise edge disjoint subgraphs \(G_a, G_c, M\) of \(R\) such that \(R = G_a \cup G_c \cup M, V(G_a \cap M) = \{b, b', w\}, V(G_c \cap M) = \{b, b', p\}, V(G_a \cap G_c) = \{b, b'\}, \{a, a'\} \subseteq V(G_a), \{c, c'\} \subseteq V(G_c),\) and \((G_a, a, a', b', w, b)\) and \((G_c, c', c, b, p, b')\) are 3-planar. Since \(G\) is 5-connected, we have \(V(M) = \{b, b', p, w\}\), and \((G_a, a, a', b', w, b)\) and \((G_c, c', c, b, p, b')\) are actually planar. If \(\vert V(G_c) \vert \geq 7\) then the assertion follows from Lemma 2.4.5 with the separation \((G_c, G_a \cup
Thus we may assume $|V(G_c)| \leq 6$. If there exists $q \in V(G_c) - \{b, b', c, c', p\}$ then $N(q) = \{b, b', c, c', p\}$ (as $G$ is 5-connected); therefore, since $(G_c, c', c, b, p, b')$ is planar, $N(p) \subseteq \{b, b', w, q\}$, a contradiction. Thus $V(G_c) = \{b, b', c, c', p\}$ and, hence, $N(p) = \{b, b', c, c', w\}$. Similarly, by considering $G_a$, we may assume $N(w) = \{a, a', b, b', p\}$. Thus $G[\{b, b', p, w\}]$ (and hence $G$) contains $K_4^-$.

Finally, assume that (3) holds for $(R, (a, b, c), (a', b', c'))$. So $(R, a', b', c', c, b, a)$ is planar (as $G$ is 5-connected), and we may assume that $R$ is embedded in a closed disc with no edge crossings such that $a, b, c, c', b', a'$ occur on the boundary of the disc in clockwise order. We apply the discharging method. For convenience, let $A = \{a, b, c, a', b', c'\}$, $F(R)$ denote the set of faces of $R$, and $f_{\infty}$ denote the outer face of $R$ (which is incident with all vertices in $A$). For each $f \in F(R)$, let $d_R(f)$ denote the number of incidences of the edges of $R$ with $f$, and $\partial f$ denote the set of vertices of $R$ incident with $f$. For $x \in V(R) \cup F(R)$, let $\sigma(x) = d_R(x) - 4$ be the charge of $x$. Note that $R$ is connected as in $R$ there is no separation $(R_1, R_2)$ of order at most 3 such that $\{a, b, c\} \subseteq V(R_1)$ and $\{a', b', c'\} \subseteq V(R_2)$. Hence, by Euler’s formula, $\sum_{x \in V(R) \cup F(R)} \sigma(x) = -8$.

We redistribute charges according to the following rule: For each $v \in V(R) - A$, $v$ sends $1/2$ to each $f \in F(R)$ that is incident with $v$ and has $d_R(f) = 3$. Let $\tau(x)$ denote the new charge for all $x \in V(R) \cup F(R)$. Then

$$\sum_{x \in V(R) \cup F(R)} \tau(x) = \sum_{x \in V(R) \cup F(R)} \sigma(x) = -8.$$  

Note that we may assume $K_4^- \not\subseteq G$. Thus, each $v \in V(R) - A$ is incident with at most $\lfloor d_R(v)/2 \rfloor$ faces $f \in F(R)$ with $d_R(f) = 3$; so $\tau(v) \geq 0$ (as $d_R(v) \geq 5$). Moreover, for $f \in F(R)$, $\tau(f) \geq 0$ unless $d_R(f) = 3$ and $f$ is incident with at least two vertices in $A$.

Since $R$ has no separation $(R_1, R_2)$ of order at most 3 such that $\{a, b, c\} \subseteq V(R_1)$ and $\{a', b', c'\} \subseteq V(R_2)$, we see that $\{a, b, c\}$ and $\{a', b', c'\}$ are independent in $R$. Moreover, since $(R, a, a', b', c, b)$ is planar, $ab', ac', ba', bc', ca', cb' \not\in E(R)$, and $d_R(v) \geq 2$ for
Let $v \in A$. Hence, $bb' \notin E(R)$; otherwise, since $G$ is 5-connected, $V(R) = A$ (to avoid 4-cuts \{a, a', b, b'\} and \{b, b', c, c'\}), which in turn would force $d_R(v) \leq 1$ for some $v \in A$.

Therefore, $d_R(f_\infty) \geq 10$, and if $f \in F(R)$ with $d_R(f) = 3$ and $|\partial f \cap A| \geq 2$ then $\partial f \cap A = \{a, a'\}$ or $\partial f \cap A = \{c, c'\}$. Hence,

$$
\sum_{x \in V(R) \cup F(R)} \tau(x) \geq \sum_{v \in V(R)} \tau(v) + \sum_{f \in F(R), |\partial f \cap A| \geq 2} \tau(f)
\geq \sum_{v \in A} (d_R(v) - 4) + (d_R(f_\infty) - 4) + \sum_{d_R(f) = 3, |\partial f \cap A| \geq 2} (d_R(f) - 4)
\geq (-12) + (10 - 4) + (-1) \times 2
= -8.
$$

Thus, all the inequalities above hold with equality. In particular, $d_R(f_\infty) = 10$, $d(x) = 2$ for $x \in A$, and there exist $u, v \in V(R) - A$ such that $uad' u$ and $vcc'v$ are triangles and $aa'ub'vcvbuca$ is the outer walk of $R$. Since $G$ is 5-connected and $(R, a, b, c, c', b', a')$ is planar, $V(R) = A \cup \{u, v\}$ and $uv \in E(R)$. Hence, $G[\{b, b', u, v\}] \cong K^-_4$, a contradiction.

\[ \blacksquare \]

### 2.6 Quadruples and special structure

As mentioned in Section 2.3, we need to deal with 5-connected graphs in which every edge or triangle at a given vertex is contained in a cut of size 5 or 6. Thus, for convenience, we introduce the following concept of quadruple.

Let $G$ be a graph. For $x \in V(G)$, let $Q_x$ denote the set of all quadruples $(T, S_T, A, B)$, such that

1. $T \subseteq G$, $x \in V(T)$, and $T \cong K_2$ or $T \cong K_3$,
2. $S_T$ is a cut of $G$ with $V(T) \subseteq S_T$, $A$ is a nonempty union of components of $G - S_T$, and $B = G - A - S_T \neq \emptyset$,
3. if $T \cong K_3$ then $5 \leq |S_T| \leq 6$, and
(4) if $T \cong K_2$ then $|S_T| = 5$, $|V(A)| \geq 2$, and $|V(B)| \geq 2$.

The purpose of this section is to derive useful properties of quadruples, in particular, those $(T, S_T, A, B)$ that minimize $|V(A)|$. We begin with a few simple properties, first of which gives a bound on $|V(A)|$.

**Lemma 2.6.1** Let $G$ be a 5-connected graph, $x \in V(G)$, and $(T, S_T, A, B) \in Q_x$. Then $G$ contains $K_4^-$, or $|V(A)| \geq 5 \leq |V(B)|$.

**Proof.** Suppose there exists $(T, S_T, A, B) \in Q_x$ such that $|V(A)| \leq 4$ or $|V(B)| \leq 4$. We choose such $(T, S_T, A, B) \in Q_x$ with $|V(A)|$ minimum. Then $|V(A)| \leq 4$. Let $\delta$ denote the minimum degree of $A$, and let $u \in V(A)$ such that $u$ has degree $\delta$ in $A$.

We may assume $\delta \geq 1$. For, suppose $\delta = 0$. If $T \cong K_3$ then, since $G$ is 5-connected, $|N(u) \cap S_T| \geq 5$; so $G[T + u]$ contains $K_4^-$. Hence we may assume $T \cong K_2$. Then $|V(A)| \geq 2$. In fact, by the minimality of $|V(A)|$, $|V(A)| = 2$ and $A$ consists of two isolated vertices. Now $G[A \cup T]$ contains $K_4^-$.

**Case 1.** $\delta = 1$.

Then $|N(u) \cap S_T| \geq 4$. Let $v$ be the unique neighbor of $u$ in $A$. Since $|V(A)| \leq 4$ and $G$ is 5-connected, $|N(v) \cap S_T| \geq 2$. We may assume $|N(u) \cap N(v) \cap S_T| \leq 1$; for, otherwise, $G[S_T \cup \{u, v\}]$ contains $K_4^-$. Suppose $|N(v) \cap S_T| \geq 3$ or $N(u) \cap N(v) \cap S_T = \emptyset$. Then $|S_T| = 6$ and, hence, $T \cong K_3$. Therefore, $|N(u) \cap V(T)| \geq 2$ or $|N(v) \cap V(T)| \geq 2$; so $G[T + u]$ or $G[T + v]$ contains $K_4^-$. Hence, we may assume that $|N(v) \cap S_T| \leq 2$ and $|N(u) \cap N(v) \cap S_T| = 1$. Then, since $|V(A)| \leq 4$ and $G$ is 5-connected, $|N(v) \cap S_T| = 2$, $|N(v) \cap V(A)| = 3$, and $|V(A)| = 4$. Let $v_1, v_2 \in V(A) - \{u, v\}$, and let $w \in N(u) \cap N(v) \cap S_T$. Since $G$ is 5-connected, $|N(v_i) \cap S_T| \geq 3$ for $i \in [2]$.

We may assume $w \notin V(T)$; for, if $w \in V(T)$ then $|V(T) \cap N(u)| \geq 2$ or $|V(T) \cap N(v)| \geq 2$, and $G[T + \{u, v\}]$ contains $K_4^-$. We may also assume $w \notin N(v_i)$ for $i \in [2]$,
as otherwise $G[\{u, v, w, v_i\}]$ contains $K^-_4$.

If $v_1v_2 \notin E(G)$ then $|N(v_i) \cap S_T| \geq 4$ for $i \in [2]$; so $|N(v_i) \cap V(T)| \geq 2$ for $i \in [2]$ (since $w \notin N(v_i)$ and $w \notin V(T)$), and hence, $G[T + \{v_1, v_2\}]$ contains $K^-_4$. So assume $v_1v_2 \in E(G)$. Since $G$ is 5-connected and $w \notin N(v_i)$ for $i \in [2]$, there exists $w' \in N(v_1) \cap N(v_2) \cap S_T$. Now $G[\{v, v_1, v_2, w'\}]$ contains $K^-_4$.

**Case 2.** $\delta \geq 2$.

If $|V(A)| = 3$ then $A \cong K_3$ and, since $G$ is 5-connected, $|N(a) \cap S_T| \geq 3$ for all $a \in V(A)$; hence, since $|S_T| \leq 6$, $G[V(A) \cup S_T]$ contains $K^-_4$. So assume $|V(A)| = 4$. We may further assume that $A$ is a cycle as otherwise $A$ contains $K^-_4$. Moreover, we may assume that for any $st \in E(A)$, $|N(s) \cap N(t) \cap S_T| \leq 1$; for otherwise $G[\{s, t\} \cup S_T]$ contains $K^-_4$. Let $A = uvwr$.

Suppose $T \cong K_2$. Then for any $st \in E(A)$, $(N(s) \cup N(t)) - V(A) = S_T$ and $|N(s) \cap N(t) \cap S_T| = 1$. Let $S_T = \{x_1, x_2, x_3, x_4, x_5\}$ and, without loss of generality, let $N(u) \cap A = \{x_1, x_2, x_3\}$ and $N(v) \cap A = \{x_3, x_4, x_5\}$. Since $(N(w) \cup N(r)) - V(A) = S_T$, $wx_3 \in E(G)$ or $rx_3 \in E(G)$. Then $G[\{u, v, w, x_3\}] \cong K^-_4$ or $G[\{r, u, v, x_3\}] \cong K^-_4$.

Now assume $T \cong K_3$. Let $S_T = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ such that $V(T) = \{x_1, x_2, x_3\}$. We may assume $|N(a) \cap V(T)| \leq 1$ for each $a \in V(A)$, for, otherwise, $G[T + a]$ contains $K^-_4$. Hence, let $x_4, x_5 \in N(u)$, $x_5, x_6 \in N(v)$, and $x_6, x_4 \in N(w)$. Note that $N(r) \cap \{x_4, x_6\} \neq \emptyset$. If $x_4 \in N(r)$ then $G[\{u, w, r, x_4\}] \cong K^-_4$, and if $x_6 \in N(r)$ then $G[\{v, w, r, x_6\}] \cong K^-_4$.

Next, we show that if a graph $G$ has no contractible edge or triangle at some vertex $x$ then every edge of $G$ at $x$ is associated with a quadruple in $Q_x$.

**Lemma 2.6.2** Let $G$ be a 5-connected graph and $x \in V(G)$. Suppose for any $T \subseteq G$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, $G/T$ is not 5-connected. Then for any $ax \in E(G)$, there exists $(T', S_{T'}, C, D) \in Q_x$ such that $\{a, x\} \subseteq V(T')$.

**Proof.** Let $T_1 = ax$. By assumption, $G/T_1$ is not 5-connected. So there exists a 5-cut $S_{T_1}$ in
$G$ with $V(T_1) \subseteq S_{T_1}$. We may assume that $G - S_{T_1}$ has a trivial component; for otherwise, let $C$ be a component of $G - S_{T_1}$ and $D = (G - S_{T_1}) - C$. Then $(T_1, S_{T_1}, C, D) \in Q_x$ is the desired quadruple.

So let $y \in V(G)$ such that $y$ is a component of $G - S_{T_1}$. Let $T_2 := G[T_1 + y] \cong K_3$. By assumption, $G/T_2$ is not 5-connected. So there exists a cut $S_{T_2}$ in $G$ such that $V(T_2) \subseteq S_{T_2}$ and $|S_{T_2}| \in \{5, 6\}$. Let $C$ be a component of $G - S_{T_2}$ and $D = (G - S_{T_2}) - C$. Then $(T_2, S_{T_2}, C, D) \in Q_x$ is the desired quadruple. \hfill \blacksquare

We now show that if $(T, S_T, A, B)$ is chosen to minimize $|V(A)|$ then we may assume $T \cong K_3$.

**Lemma 2.6.3** Let $G$ be a 5-connected graph and $x \in V(G)$. Suppose for any $T \subseteq G$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, $G/T$ is not 5-connected. Then $G$ contains $K_4^−$, or for any $(T, S_T, A, B) \in Q_x$ with $|V(A)|$ minimum, $T \cong K_3$.

**Proof.** Let $(T, S_T, A, B) \in Q_x$ with $|V(A)|$ minimum, and assume $T \cong K_2$. Then $|S_T| = 5$. Let $a \in N(x) \cap V(A)$. By Lemma 2.6.2, there exists $(T', S_{T'}, C, D) \in Q_x$ such that $\{a, x\} \subseteq V(T')$. Note that $T' \cong K_2$ and $|S_{T'}| = 5$, or $T' \cong K_3$ and $|S_{T'}| \in \{5, 6\}$. We may assume $|V(A)| \geq 1$; for, if not, then $G$ contains $K_4^−$ by Lemma 2.6.1.

We may assume that if $A \cap C \neq \emptyset$ then $|(S_T \cup S_T) - V(B \cup D)| \geq |S_T| + 1$. For, suppose $A \cap C \neq \emptyset$ and $|(S_T \cup S_T) - V(B \cup D)| \leq |S_T|$. If $|V(A \cap C)| \geq 2$ or $T' \cong K_3$ then $(T', (S_T \cup S_T) - V(B \cup D), A \cap C, B \cup D) \in Q_x$ and $|V(A \cap C)| \leq |V(A) - \{a\}| < |V(A)|$, contradicting the choice of $(T, S_T, A, B)$ that $|V(A)|$ is minimum. So assume $|V(A \cap C)| = 1$ and $T' \cong K_2$. Then $|(S_T \cup S_T) - V(B \cup D)| = |S_T| = 5$ and $|V(C)| \geq 2 \leq |V(D)|$.

Assume for the moment $A \cap D = \emptyset$. By Lemma 2.6.1, we may assume $|S_T \cap V(A)| = 4$ (as $|S_T| = 5$ and $|V(A)| \geq 5$); so $|S_T \cap V(B)| = 0$, $|S_T \cap V(C)| = 0$, and $|S_T \cap S_T| = 1$. Since $|V(C)| \geq 2$, $B \cap C \neq \emptyset$. So $S_T \cap S_T$ is a 1-cut in $G$, contradicting the assumption that $G$ is 5-connected. Hence, $A \cap D \neq \emptyset$. We may assume $|V(A \cap D)| \geq 2$; as otherwise, since $G$ is 5-connected, $G[(A \cap C) \cup (A \cap D) \cup \{a, x\}] \cong K_4^−$. Then $|(S_T \cup S_T) −$
\[ V(B \cup C) \geq |S_T| + 1; \text{ otherwise, } (T', (S_{T'} \cup S_T) - V(B \cup C), A \cap D, B \cup C) \in Q_x \]
and \( 2 \leq |V(A \cap D)| < |V(A)| \), contradicting the choice of \((T, S_T, A, B)\) that \(|V(A)|\) is minimum. Hence, \(|(S_{T'} \cup S_T) - V(A \cup D)| = |S_T| + |S_{T'}| - |(S_{T'} \cup S_T) - V(B \cup C)| \leq 4. \]

Since \( G \) is 5-connected, \( B \cap C = \emptyset \). Since \(|(S_{T'} \cup S_T) - V(B \cup D)| = 5, |S_T \cap V(C)| \leq 3. \]
Therefore, \(|V(C)| \leq 4 < |V(A)|\), a contradiction.

Similarly, we may assume that if \( A \cap D \neq \emptyset \) then \(|(S_{T'} \cup S_T) - V(B \cup C)| \geq |S_{T'}| + 1. \]

Suppose \( A \cap C = A \cap D = \emptyset \). Then, since \(|V(A)| \geq 5\) and \(|S_T| \leq 5\), \(|S_T \cap V(A)| = |V(A)| = 5, |S_T \cap S_{T'}| = 1\), and \(|S_{T'} \cap V(B)| = 0. \) Since \(|S_T| = 5\) and \( G \) is 5-connected, we see that \( B \cap C = \emptyset \) or \( B \cap D = \emptyset \). However, this implies \(|V(C)| \leq 4\) or \(|V(D)| \leq 4\), contradicting the choice of \((T, S_T, A, B)\) that \(|V(A)|\) is minimum.

We may thus assume \( A \cap C \neq \emptyset \). Then \(|(S_{T'} \cup S_T) - V(B \cup D)| \geq |S_{T'}| + 1. \) So \(|(S_{T'} \cup S_T) - V(A \cup C)| = |S_T| + |S_{T'}| - |(S_{T'} \cup S_T) - V(B \cup D)| \leq 4. \) Since \( G \) is 5-connected, \( B \cap D = \emptyset. \) In addition, \( A \cap D \neq \emptyset \); as otherwise, \(|V(D)| \leq 4 < |V(A)|\), contradicting the choice of \((T, S_T, A, B)\) that \(|V(A)|\) is minimum. Therefore, \(|(S_{T'} \cup S_T) - V(B \cup C)| \geq |S_{T'}| + 1. \) Hence, \(|(S_{T'} \cup S_T) - V(A \cup D)| = |S_T| + |S_{T'}| - |(S_{T'} \cup S_T) - V(B \cup C)| \leq 4. \)

Since \( G \) is 5-connected, \( B \cap C = \emptyset. \) Thus, \(|V(B)| \leq |S_{T'} - V(T')| = 4, \) contradicting the fact \(|V(A)| \geq 5\) and \(|V(A)|\) is minimum.

The next lemma will allow us to assume that if \((T, S_T, A, B) \in Q_x\) with \(|V(A)|\) minimum and \((T', S_{T'}, C, D) \in Q_x\) with \( T' \cap A \neq \emptyset \) then \( T \cong K_3 \) and \( T' \cong K_3. \)

**Lemma 2.6.4** Let \( G \) be a 5-connected graph and \( x \in V(G) \). Suppose for any \( T \subseteq G \) with \( x \in V(T) \) and \( T \cong K_2 \) or \( T \cong K_3 \), \( G/T \) is not 5-connected. Let \((T, S_T, A, B) \in Q_x\) with \(|V(A)|\) minimum and \((T', S_{T'}, C, D) \in Q_x\) with \( T' \cap A \neq \emptyset \). Suppose \( T' \cong K_2. \) Then one of the following holds:

(i) \( G \) contains a TK5 in which \( x \) is not a branch vertex.

(ii) \( G \) contains \( K_4^- \).
(iii) There exist distinct \(x_1, x_2, x_3 \in N(x)\) such that for any distinct \(y_1, y_2 \in N(x) - \{x_1, x_2, x_3\}\), \(G' := G - \{xv : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}\) contains \(TK_5\).

Proof. By Lemma 2.6.3, we may assume \(T \cong K_3\). By Lemma 2.4.6, we may further assume \(|S_T| = 6\). Note the symmetry between \(C\) and \(D\), and assume that \(V(T) \subseteq S_T - V(D)\). Since \(|V(T')| = 2\), \(|S_{T'}| = 5\).

Suppose \(A \cap C \neq \emptyset\). Then \(|(S_{T'} \cup S_T) - V(B \cup D)| \geq 7\); otherwise, \((T, (S_{T'} \cup S_T) - V(B \cup D), A \cap C, B \cup D) \in Q_x\) and \(0 < |V(A \cap C)| < |V(A)|\), contradicting the choice of \((T, S_T, A, B)\) that \(|V(A)|\) is minimum. Hence, \(|(S_{T'} \cup S_T) - V(A \cup C)| = |S_T| + |S_{T'}| - |(S_{T'} \cup S_T) - V(B \cup D)| \leq 4\). Since \(G\) is 5-connected, \(B \cap D = \emptyset\). We may assume \(A \cap D \neq \emptyset\); otherwise, \(|V(D)| \leq 4\) and, by Lemma 2.6.1, (ii) holds. We may also assume \(|V(D)| > |V(A)|\); otherwise, \((T', S_{T'}, D, C) \in Q_x\) and, by Lemma 2.6.3, \(G\) contains \(K_4^-\). Hence, \(|V(D) \cap S_T| > |V(A \cap C)| + |V(A) \cap S_T| \geq |V(A) \cap S_T| + 1\). Then, since \(|S_T| = 6\) and \(V(T) \subseteq S_T - V(D)\), \(|V(D) \cap S_T| = 3\) and \(|V(A) \cap S_T| = 1\). Hence, \(|(S_{T'} \cup S_T) - V(B \cup D)| \leq 4\), a contradiction as \(G\) is 5-connected.

Now assume \(A \cap C = \emptyset\). Then, since \(|S_{T'} \cap V(A)| \leq 4\), we may assume \(A \cap D \neq \emptyset\) by Lemma 2.6.1.

Suppose \(|(S_{T'} \cup S_T) - V(B \cup C)| = 5\). Then, since \(|V(A \cap D)| < |V(A)|\), \(|V(A \cap D)| = 1\); otherwise, \((T', (S_{T'} \cup S_T) - V(B \cup C), A \cap D, B \cup C)\) contradicts the choice of \((T, S_T, A, B)\) that \(|V(A)|\) is minimum. Hence by Lemma 2.6.1, we may assume \(|V(A) \cap S_{T'}| = 4\); so \(V(B) \cap S_{T'} = V(D) \cap S_T = \emptyset\). Since \(G\) is 5-connected, \(B \cap D = \emptyset\). So \(|V(D)| = 1\), a contradiction.

Hence, we may assume \(|(S_{T'} \cup S_T) - V(B \cup C)| \geq 6\). Then \(S_T \cap V(D) \neq \emptyset\) because \(|S_T| = 5\). By Lemma 2.6.1, we may assume \(B \cap C \neq \emptyset\) (otherwise \(|V(C)| \leq 4\)). Hence, since \(G\) is 5-connected, \(|(S_{T'} \cup S_T) - V(A \cup D)| \geq 5\). Since \(|(S_{T'} \cup S_T) - V(A \cup D)| + |(S_{T'} \cup S_T) - V(B \cup C)| = |S_T| + |S_{T'}| = 11\), \(|(S_{T'} \cup S_T) - V(A \cup D)| = 5\). If \(|V(B \cap C)| = 1\) then, since \(G\) is 5-connected, \(G[T \cup (B \cap C)] \cong K_5^-\). If \(|V(B \cap C)| \geq 2\) then, since \(V(T) \subseteq (S_{T'} \cup S_T) - V(A \cup D)\), the assertion follows from Lemma 2.4.6.  

27
The proofs of the remaining two results in this section use Lemmas 2.5.1, 2.5.2 and 2.5.3. The following result will allow us to assume that if $(T, S_T, A, B) \in Q_x$ is chosen to minimize $|V(A)|$ then $N(x) \cap V(A) \neq \emptyset$, which in turn will allow us to choose another quadruple at $x$.

**Lemma 2.6.5** Let $G$ be a 5-connected nonplanar graph and $x \in V(G)$. Suppose for any $H \subseteq G$ with $x \in V(H)$ and $H \cong K_2$ or $H \cong K_3$, $G/H$ is not 5-connected. Let $(T, S_T, A, B) \in Q_x$ minimizing $|V(A)|$. Then $N(x) \cap V(A) \neq \emptyset$, or one of the following holds:

(i) $G$ contains a $TK_5$ in which $x$ is not a branch vertex.

(ii) $G$ contains $K_4^−$.

(iii) There exist distinct $x_1, x_2, x_3 \in N(x)$ such that for any distinct $u_1, u_2 \in N(x) - \{x_1, x_2, x_3\}$, $G' := G - \{xv : v \notin \{x_1, x_2, x_3, u_1, u_2\}\}$ contains $TK_5$.

**Proof.** Suppose $N(x) \cap V(A) = \emptyset$. Then, since $G$ is 5-connected, $|S_T| = 6$ and $T \cong K_3$. Let $V(T) = \{x, x_1, x_2\}$ and $S_T = \{x, x_1, x_2, v_1, v_2, v_3\}$. By Lemma 2.4.8, we may assume $N(x_1) \subseteq V(A) \cup \{x, x_2\}$, and any three independent paths in $G_A := G[A + (S_T - \{x\})] - E(S_T)$ from $\{x_1, x_2\}$ to $v_1, v_2, v_3$, respectively, with two from $x_1$ and one from $x_2$, must include a path from $x_2$ to $v_1$.

We wish to apply Lemma 2.5.1. Let $G_A'$ be obtained from $G_A$ by adding a new vertex $x'_1$ and joining $x'_1$ to each vertex in $N(x_1) \cap V(G_A)$ with an edge. Thus, in $G_A'$, $x_1$ and $x'_1$ have the same set of neighbors. Note that $\{x_1, x'_1, x_2\}$ and $\{v_1, v_2, v_3\}$ are independent sets in $G_A'$.

**Claim 1.** There is no separation $(A_1, A_2)$ in $G_A'$ such that $|V(A_1 \cap A_2)| \leq 3$, $\{x_1, x'_1, x_2\} \subseteq V(A_1)$ and $\{v_1, v_2, v_3\} \subseteq V(A_2)$.

For, suppose such $(A_1, A_2)$ does exist. Then $\{x_1, x'_1\} \not\subseteq V(A_1 \cap A_2)$; for, otherwise, $A_1 - \{x_1, x'_1, x_2\} \neq \emptyset$ (as $\{x_1, x'_1, x_2\}$ is independent in $G_A'$ and $x_2$ has a neighbor in $V(A)$) and, hence, $(V(A_1 \cap A_2) - \{x'_1\}) \cup \{x, x_2\}$ is a cut in $G$ of size at most 4, a contradiction.
Thus, we may assume by symmetry that \( x_1 \notin V(A_1 \cap A_2) \). Then \((A_1, A_2)\) may be chosen so that \( x'_1 \notin V(A_1 \cap A_2) \) (as \( x'_1 \) has the same set of neighbors as \( x_1 \) in \( G'_A \)). Moreover, \( V(A_1) - V(A_2) \subseteq \{x_1, x'_1, x_2\} \); otherwise \( S'_T := V(A_1 \cap A_2) \cup V(T) \) is a cut in \( G \) with \( |S'_T| \leq 6 \), and \( G - S'_T \) has a component strictly contained in \( A \), contradicting the choice of \((T, S_T, A, B)\) that \( |V(A)| \) is minimum.

Since \( G \) is 5-connected and \( N(x_1) \subseteq V(A) \cup \{x, x_2\} \), \( V(A_1 \cap A_2) \cup \{x, x_2\} \) is not a 4-cut in \( G \). So \( x_2 \in V(A_1) - V(A_2) \) and \( |V(A_1 \cap A_2)| = 3 \). Since \( G \) is 5-connected and \( V(A_1) - V(A_2) \subseteq \{x_1, x'_1, x_2\} \), \( N(x_1) = \{x, x_2\} \cup V(A_1 \cap A_2) \). Since \( N(x_2) \cap V(A_1) \neq \emptyset \), there exists \( v \in V(A_1 \cap A_2) \) such that \( vx_2 \in E(G) \). Now \( G[\{v, x, x_1, x_2\}] \cong K_4^- \) and \((ii)\) holds. This completes the proof of Claim 1.

Since any three disjoint paths in \( G'_A \) from \( \{x_1, x_2, x'_1\} \) to \( \{v_1, v_2, v_3\} \) contains a path from \( x_2 \) to \( v_1 \), it follows from Claim 1 and Lemma 2.5.1 that \( G'_A \) has a separation \((J, L)\) such that \( V(J \cap L) = \{w_0, \ldots, w_n\} \), \((J, w_0, \ldots, w_n)\) is 3-planar, \((L, (x_1, x_2, x'_1), (v_2, v_1, v_3))\) is a ladder along some sequence \( b_0 \ldots b_m \), where \( b_0 = x_2 \), \( b_m = v_1 \), and \( w_0 \ldots w_n \) is the reduced sequence of \( b_0 \ldots b_m \). (Note that if \((ii)\) of Lemma 2.5.1 holds then, by Claim 1, \((G'_A, (x_1, x_2, x'_1), (v_2, v_1, v_3))\) is a rung, and we let \( L = G'_A \) and \( J \) consist of \( v_1 \) and \( x_2 \).)

Since \( L \) is a ladder, \( L \) contains three disjoint paths \( P_1, P_2, P_3 \) from \( x_1, x_2, x'_1 \), respectively, to \( \{v_1, v_2, v_3\} \), with \( v_1 \in V(P_2) \). Without loss of generality, we may further assume that \( v_2 \in V(P_1) \) and \( v_3 \in V(P_3) \). Let \((R_i, (a_{i-1}, b_{i-1}, c_{i-1}), (a_i, b_i, c_i))\), \( i \in [m] \), be the rungs in \( L \), with \( a_i \in V(P_1) \), \( b_i \in V(P_2) \) and \( c_i \in V(P_3) \) for \( i = 0, \ldots, m \). Since \( G \) is 5-connected, \((J, w_0, \ldots, w_n)\) is planar and, by Lemmas 2.5.2 and 2.5.3, we may assume that the rungs in \( L \) have the simple structures as in Lemma 2.5.3.

**Claim 2.** There exist \( t \in V(A) \) and independent paths \( Q_1, Q_2, Q_3, Q_4, Q_5 \) in \( G_A \) such that \( Q_1, Q_2, Q_3, Q_4 \) are from \( t \) to \( x_1, x_2, v_1, v_2 \), respectively, and \( Q_5 \) is from \( x_1 \) to \( v_3 \); and there exist \( t \in V(A) \) and independent paths \( Q'_1, Q'_2, Q'_3, Q'_4, Q'_5 \) in \( G_A \) such that \( Q'_1, Q'_2, Q'_3, Q'_4 \) are from \( t \) to \( x_1, x_2, v_1, v_3 \), respectively, and \( Q'_5 \) is from \( x_1 \) to \( v_2 \).

We may assume that for \( i \in [m] \), \((R_i, (a_{i-1}, b_{i-1}, c_{i-1}), (a_i, b_i, c_i))\) is not of type \((iv)\)
as in Lemma 2.5.3. For, suppose \((R_i, (a_{i-1}, b_{i-1}, c_{i-1}), (a_i, b_i, c_i))\) is of type \((iv)\) for some \(i \in [m]\), and let \(v \in V(R_i) = (\{a_{i-1}, b_{i-1}, c_{i-1}\} \cup \{a_i, b_i, c_i\})\). Then Claim 2 holds with \(v, va_{i-1} \cup a_{i-1}P_1 x_1, vb_{i-1} \cup b_{i-1}P_2 x_2, vb_i \cup b_iP_2 v_1, va_i \cup a_iP_1 v_2, P_3\) as \(t, Q_1, Q_2, Q_3, Q_4, Q_5\), respectively, and with \(v, vc_{i-1} \cup c_{i-1}P_3 x_1, vb_{i-1} \cup b_{i-1}P_2 x_2, vb_i \cup b_iP_2 v_1, vc_i \cup c_iP_3 v_3, P_1\) as \(t, Q'_1, Q'_2, Q'_3, Q'_4, Q'_5\), respectively.

We claim that there exists \(q \in [m]\), such that \(x_1b_q \in E(G)\). Let \(q \geq 1\) be the smallest integer such that \((R_q, (a_{q-1}, b_{q-1}, c_{q-1}), (a_q, b_q, c_q))\) is not of type \((ii)\) as in Lemma 2.5.3, which must exist as \(x_1 \notin \{v_1, v_2, v_3\}\). Then \(a_{q-1} = x_1\) and \(c_{q-1} = x'_1\). Since \(G\) is 5-connected, \((R_q, (a_{q-1}, b_{q-1}, c_{q-1}), (a_q, b_q, c_q))\) cannot be of type \((iii)\) (thus, must be of type \((i)\) as in Lemma 2.5.3. Since \(x_1\) and \(x'_1\) have the same set of neighbors in \(G'_{A}, a_q \neq x_1\) and \(c_q \neq x'_1\). Since \(G\) is 5-connected, \(V(R_q) = \{x_1, x'_1, a_q, b_q, c_q\}\). Since \(N(x_1) \subseteq V(A) \cup \{x, x_2\}\) and \(G\) is 5-connected, \(x_1b_q \in E(G)\).

We choose such \(q\) to be maximum. Note that \(q \neq 0\) as \(x_1b_0 \notin E(G(A))\). We now show the existence of \(t\) and \(Q_i, i \in [5]\); the proof of the existence of \(t\) and \(Q'_i, i \in [5]\), is symmetric (by switching the roles of \(v_2, P_1\) and \(v_3, P_3\)).

We may assume that for any choice of \(P_1, P_3\) there does not exist \(r, q < r \leq m\), such that \(L\) has disjoint paths \(S, S'\) from \(b_r, x_1\) to \(v_2, v_3\), respectively, and internally disjoint from \(J \cup P_2\). For, suppose for some choice of \(P_1, P_3\) such \(r, S, S'\) exist. By Claim 1, \(J \cup P_2\) is 2-connected. So let \(P'_2\) denote the path between \(x_2\) and \(v_1\) in \(J \cup P_2\) such that the cycle \(P'_2 \cup P_2\) bounds the infinite face of \(J \cup P_2\). Let \(t \in V(P'_2)\) such that \(x_2t \in E(P'_2)\). If there exist independent paths \(L_1, L_2\) in \(J \cup P_2\) from \(t\) to \(b_q, b_r\), respectively, and internally disjoint from \(P'_2\), then \(L_1 \cup b_q x_1, L_2 \cup S, tx_2, tP'_2 v_1, S'\) give the desired \(Q_1, Q_2, Q_3, Q_4, Q_5\), respectively. Thus we may assume that such \(L_1, L_2\) do not exist. So \(J \cup P_2\) has a separation \((J_1, J_2)\) such that \(|V(J_1 \cap J_2)| \leq 3, t \in V(J_1) - V(J_2),\) and \(\{b_q, b_r, v_1, x_2\} \subseteq V(J_2)\). By planarity of \(J \cup P_2\), \(V(J_1 \cap J_2)\) contains \(x_2\) and a vertex \(t' \in V(tP'_2 v_1)\). Since \(V(J_1 \cap J_2)\) cannot be a cut in \(G\), we must have \(|V(J_1 \cap J_2)| = 3, t' = v_1,\) and \(V(J_1 \cap J_2) - \{t', x_2\} \subseteq V(b_rP_2 v_1)\). Let \(b_s \in V(J_1 \cap J_2) - \{t', x_2\}\). Then \(V(T) \cup \{a_s, b_s, c_s\}\) is a cut in \(G\) separating
\( \bigcup_{i=1}^{n} R_i \) from \( B + t \), contradicting the choice of \((T, S_T, A, B)\) that \( |V(A)| \) is minimum.

Hence, for any \( j > q \), \((R_j, (a_{j-1}, b_{j-1}, c_{j-1}), (a_j, b_j, c_j))\) must be of type (i) or (ii) as in Lemma 2.5.3 and there is no edge in \( G'_A \) from \( P_2 \) to \( P_1 - x_1 \). Also notice that, for \( j \leq q \) with \( b_{j-1} \neq b_q \), because of edges \( x_1 b_q, x'_1 b_q \) in \( G'_A \), \((R_j, (a_{j-1}, b_{j-1}, c_{j-1}), (a_j, b_j, c_j))\) must be of type (ii) as in Lemma 2.5.3. For \( j \leq q \) with \( b_{j-1} = b_q \), we see that \( V(R_j) = \{x_1, x'_1, a_j, b_q, c_j\} \) as \( G \) is 5-connected, and we may assume that \( b_q a_j \notin E(G) \) (otherwise, \( b_q, b_q x_1, b_q P_2 x_2, b_q P_q v_1, b_q a_q \cup a_q P_1 v_2, P_3 \) give the desired \( t, Q_1, Q_2, Q_3, Q_4, Q_5 \).

Thus, we may assume that for some \( j > q \), \( \{a_{j-1}, c_{j-1}\} \cap \{a_j, c_j\} = \emptyset \). For, otherwise, \((G_A, x_1, x_2, v_1, v_2, v_3)\) is planar, and the assertion follows from Lemma 2.4.5.

If \( R_j - a_{j-1} \) contains disjoint paths \( S_1, S_2 \) from \( b_j, c_{j-1} \) to \( a_j, c_j \), respectively, then \( b_j \) and the paths \( S_1 \cup a_j P_1 v_2, x_1 P_3 c_{j-1} \cup S_2 \cup c_j P_3 v_3 \) contradict the nonexistence of \( b_r, S, S' \). So assume \( S_1, S_2 \) do not exist. Then by Lemma 2.4.10, \((R_j - a_{j-1}, a_j, c_j, b_j, c_{j-1})\) is planar.

By Lemma 2.4.5, we may assume \( |V(R_j - a_{j-1})| \leq 5 \).

If \( |V(R_j - a_{j-1})| = 5 \) then there exists \( v \in V(R_j) - \{a_{j-1}, a_j, b_j, c_{j-1}, c_j\} \) such that \( v \) is adjacent to all of \( \{a_{j-1}, a_j, b_j, c_{j-1}, c_j\} \); so \( b_j \) and the paths \( b_j v a_j \cup a_j P_1 v_2, P_3 \) contradict the nonexistence of \( b_r, S, S' \).

Hence, we may assume \( |V(R_j - a_{j-1})| = 4 \). Then, since \( R_j \) has no cut of size at most 3 separating \( \{a_{j-1}, b_{j-1}, c_{j-1}\} \) from \( \{a_j, b_j, c_j\} \), we must have \( a_{j-1} c_j, a_j c_{j-1} \in E(G) \).

Note that there exists \( t > q \) such that \( L \) has a path \( Z \) from \( b_t \) to \( z \in V(x_1 P_1 a_{j-1} - x_1) \cup V(x'_1 P_3 c_{j-1} - x'_1) \) and internally disjoint from \( J \cup P_1 \cup P_2 \cup P_3 \); for otherwise, \( \{a_j, b_j, c_j, x_1\} \) would be a cut in \( G \). If \( z \in V(x_1 P_1 a_{j-1} - x_1) \) then \( b_t \) and the paths \( Z \cup z P_1 v_2, P_3 \) contradict the nonexistence of \( b_r, S, S' \). So assume \( z \in V(x_1 P_3 c_{j-1} - x_1) \). Then \( b_t \) and the paths \( Z \cup z P_3 c_{j-1} \cup c_{j-1} a_j \cup a_j P_1 v_2, x_1 P_1 a_{j-1} \cup a_{j-1} c_j \cup c_j P_3 v_3 \) contradict the nonexistence of \( b_r, S, S' \), with \( x'_1 P_3 c_{j-1} \cup c_{j-1} a_j \cup a_j P_1 v_2, x_1 P_1 a_{j-1} \cup a_{j-1} c_j \cup c_j P_3 v_3 \) as \( P_1, P_3 \), respectively.

This completes the proof of Claim 2.

Now that we have the paths in Claim 2, we turn to \( G_B := G[B + S_T - x_1] \). Choose \( x_3 \in N(x) \cap V(B) \), let \( u_1 := x_3 \) and let \( u_2 \in N(x) - \{x_1, x_2, x_3\} \) be arbitrary. Note that
$u_2 \in S_T \cup V(B)$. We wish to prove $(iii)$ by attempting to find a $TK_5$ in $G' := G - \{xv : v \notin \{u_1, u_2, x_1, x_2\}\}$. Since $G$ is 5-connected and $N(x_1) \cap V(B) = \emptyset$, $G_B$ has four independent paths $B_1, B_2, B_3, B_4$ from $u_1$ to $v_1, v_2, v_3, x_2$, respectively, and we may assume that these paths are induced.

**Claim 3.** We may assume $u_2 \notin S_T$.

For, suppose $u_2 \in S_T$. If $u_2 = v_1$ then $T \cup Q_1 \cup Q_2 \cup (Q_3 \cup v_1 x) \cup u_1 x \cup B_4 \cup (B_2 \cup Q_4) \cup (B_3 \cup Q_5)$ is a $TK_5$ in $G'$ with branch vertices $t, u_1, x, x_1, x_2$. If $u_2 = v_2$ then $T \cup Q_1 \cup Q_2 \cup (Q_4 \cup v_2 x) \cup u_1 x \cup B_1 \cup (B_1 \cup Q_3) \cup (B_3 \cup Q_5)$ is a $TK_5$ in $G'$ with branch vertices $t, u_1, x, x_1, x_2$. Now assume $u_2 = v_3$. Then $T \cup Q'_1 \cup Q'_2 \cup (Q'_4 \cup v_3 x) \cup u_1 x \cup B_4 \cup (B_1 \cup Q'_3) \cup (B_2 \cup Q'_5)$ is a $TK_5$ in $G'$ with branch vertices $t, u_1, x, x_1, x_2$. This completes the proof of Claim 3.

Let $P$ be a path in $G_B$ from $u_2$ to some $w_2 \in V(B_1 \cup B_2 \cup B_3 \cup B_4) - \{u_1\}$ and internally disjoint from $B_1 \cup B_2 \cup B_3 \cup B_4$.

**Claim 4.** We may assume that for any choice of $P$, $w_2 \in V(B_4)$.

For, if $w_2 \in V(B_1)$ then $T \cup Q_1 \cup Q_2 \cup (Q_3 \cup v_1 B_1 w_2 \cup P \cup u_2 x) \cup u_1 x \cup B_4 \cup (B_2 \cup Q_4) \cup (B_3 \cup Q_5)$ is a $TK_5$ in $G'$ with branch vertices $t, u_1, x, x_1, x_2$. If $w_2 \in V(B_2)$ then $T \cup Q_1 \cup Q_2 \cup (Q_4 \cup v_2 B_2 w_2 \cup P \cup u_2 x) \cup u_1 x \cup B_4 \cup (B_1 \cup Q_3) \cup (B_3 \cup Q_5)$ is a $TK_5$ in $G'$ with branch vertices $t, u_1, x, x_1, x_2$. If $w_2 \in V(B_3)$ then $T \cup Q'_1 \cup Q'_2 \cup (Q'_4 \cup v_3 B_3 w_2 \cup P \cup u_2 x) \cup u_1 x \cup B_4 \cup (B_1 \cup Q'_3) \cup (B_2 \cup Q'_5)$ is a $TK_5$ in $G'$ with branch vertices $t, u_1, x, x_1, x_2$. This completes the proof of Claim 4.

Let $U_2$ denote the $(B_1 \cup B_2 \cup B_3)$-bridge of $G_B$ containing $B_4 + u_2$. That is, $U_2$ is the subgraph of $G_B$ induced by the edges in the component of $G_B - (B_1 \cup B_2 \cup B_3)$ containing $B_4 + u_2$ and the edges from that component to $B_1 \cup B_2 \cup B_3$.

**Claim 5.** We may assume that $V(U_2) \cap V(B_2 \cup B_3) = \{u_1\}$.

For, suppose there exists $w \in V(U_2) \cap V(B_2 \cup B_3)$ such that $w \neq u_1$. By symmetry, we may assume $w \in V(B_2 - u_1)$ and choose $w$ so that $wB_2v_2$ is minimal.
Then $U_2$ has a path $X$ between $x_2$ to $w$ and internally disjoint from $B_1 \cup B_2 \cup B_3$, and a path from $u_2$ to some $u'_2 \in V(X)$ and internally disjoint from $X \cup B_1 \cup B_2 \cup B_3$. Since $G$ is 5-connected, $U_2$ has four independent paths from $u'_2$ to four distinct vertices in $V(U_2) \cap V(B_1 \cup B_2 \cup B_3)$ and internally disjoint from $B_1 \cup B_2 \cup B_3$. Thus, by Lemma 2.4.11, $U_2$ contains independent paths $L_1, L_2, L_3, L_4$ from $u'_2$ to $u_2, x_2, w, w'$, respectively, and internally disjoint from $B_1 \cup B_2 \cup B_3$, where $w' \in V(B_1 \cup B_2 \cup B_3)$.

If $w' \in V(wB_2u_1 - w)$ then $T \cup (L_1 \cup u_2x) \cup L_2 \cup (L_3 \cup wB_2v_2 \cup P_1) \cup (u_1B_2w' \cup L_4) \cup u_1x \cup (B_1 \cup P_2) \cup (B_3 \cup P_3)$ is a $TK_5$ in $G'$ with branch vertices $u_1, u'_2, x, x_1, x_2$. (Note we identify $x'_1$ with $x_1$ when we use $P_3$.)

If $w' \in V(B_1 - u_1)$ then $T \cup Q'_1 \cup Q'_2 \cup (Q'_4 \cup B_3 \cup u_1x) \cup (L_1 \cup u_2x) \cup L_2 \cup (L_3 \cup wB_2v_2 \cup Q'_4) \cup (L_4 \cup w'B_1v_1 \cup Q'_5)$ is a $TK_5$ in $G'$ with branch vertices $t, u'_2, x, x_1, x_2$.

If $w' \in V(B_3 - u_1)$ then $T \cup Q_1 \cup Q_2 \cup (Q_4 \cup B_1 \cup u_1x) \cup (L_1 \cup u_2x) \cup L_2 \cup (L_3 \cup wB_2v_2 \cup Q_4) \cup (L_4 \cup w'B_3v_3 \cup Q_5)$ is a $TK_5$ in $G'$ with branch vertices $t, u'_2, x, x_1, x_2$.

This completes the proof of Claim 5.

Now let $z \in V(B_1 \cap U_2)$ such that $zB_1v_1$ is minimal. Since $G$ is 5-connected, there exists a path $Y$ in $G_B - x$ from some $y \in V(zB_1u_1) - \{u_1, z\}$ to some $y' \in V(B_2 \cup B_3) - \{u_1\}$ and internally disjoint from $U_2 \cup B_1 \cup B_2 \cup B_3$.

Claim 6. We may assume that $G[U_2 - B_1 + z]$ has no independent paths from $u_2$ to $x_2, z$, respectively.

For, suppose $G[U_2 - B_1 + z]$ (and hence $G[U_2 \cup zB_1u_1]$) has independent paths from $u_2$ to $x_2, z$, respectively. Then by Lemma 2.4.11, $G[U_2 \cup zB_1u_1]$ has independent paths $L_1, L_2, L_3, L_4$ from $u_2$ to distinct vertices $x_2, z, z_1, z_2$, respectively, and internally disjoint from $B_1$, where $u_1, z_2, z_1, z$ occur on $B_1$ in the order listed. Possibly, $u_1 = z_2$.

If $y' \in V(B_2 - u_1)$ then $T \cup Q'_1 \cup Q'_2 \cup (Q'_4 \cup B_3 \cup u_1x) \cup u_2x \cup L_1 \cup (L_2 \cup zB_1v_1 \cup Q'_5) \cup (L_3 \cup zB_1y \cup Y \cup y'B_2v_2 \cup Q'_5)$ is a $TK_5$ in $G'$ with branch vertices $t, u_2, x, x_1, x_2$.

If $y' \in V(B_3 - u_1)$ then $T \cup Q_1 \cup Q_2 \cup (Q_4 \cup B_2 \cup u_1x) \cup u_2x \cup L_1 \cup (L_2 \cup zB_1v_1 \cup Q_5) \cup (L_3 \cup zB_1y \cup Y \cup y'B_3v_3 \cup Q_5)$ is a $TK_5$ in $G'$ with branch vertices $t, u_2, x, x_1, x_2$.  


By Claim 6, $G[U_2 - B_1 + z]$ has a 1-separation $(U_{21}, U_{22})$ such that $u_2 \in V(U_{21}) - V(U_{22})$ and $\{x_2, z\} \subseteq V(U_{22})$. We choose this separation so that $U_{22}$ is minimal. Let $u'_2$ denote the unique vertex in $V(U_{21} \cap U_{22})$. By the minimality of $U_{22}$, we see that $U_{22}$ has independent paths $L_1, L_2$ from $u'_2$ to $x_2, z$, respectively.

Claim 7. We may assume that $u'_2$ has exactly two neighbors in $U_{22}$.

For, otherwise, by the minimality of $U_{22}$, $G[U_{22} \cup zB_1u_1] - u_1$ has three independent paths from $u'_2$ to three distinct vertices in $V(zB_1u_1 - u_1) \cup \{x_2\}$. So by Lemma 2.4.11, $G[U_{22} \cup zB_1u_1] - u_1$ has independent paths $L'_1, L'_2, L'_3$ from $u'_2$ to $x_2, z, z_1$, respectively, and internally disjoint from $B_1$, where $z, z_1, u_1$ occur on $B_1$ in order. Let $L$ be a path in $U_{21}$ from $u_2$ to $u'_2$.

If $y' \in V(B_2 - u_1)$ then $T \cup Q'_1 \cup Q'_2 \cup (Q'_4 \cup B_3 \cup u_1x) \cup (L \cup u_2x) \cup L'_1 \cup (L'_2 \cup zB_2v_1 \cup Q'_3) \cup (L'_2 \cup z_1B_1y \cup Y \cup y'B_2v_2 \cup Q'_5)$ is a $TK_5$ in $G'$ with branch vertices $t, u'_2, x, x_1, x_2$.

If $y' \in V(B_3 - u_1)$ then $T \cup Q_1 \cup Q_2 \cup (Q_4 \cup B_2 \cup u_1x) \cup (L \cup u_2x) \cup L'_1 \cup (L'_2 \cup zB_2v_1 \cup Q_3) \cup (L'_3 \cup z_1B_1y \cup Y \cup y'B_3v_3 \cup Q_5)$ is a $TK_5$ in $G'$ with branch vertices $t, u'_2, x, x_1, x_2$.

This completes the proof of Claim 7.

Since $G$ is 5-connected, it follows from Claim 7 that $u'_2$ has at least two neighbors in $U_{21}$. Since all paths from $u_2$ to $B_1 \cup B_2 \cup B_3 \cup B_4$ must end on $B_4$, $G[U_{21} \cup zB_1u_1] - \{z, u_1\}$ has independent paths $L_3, L_4$ from $u'_2$ to $z_1, u_2$, respectively, and internally disjoint from $B_1$, where $z_1 \in V(zB_1u_1) - \{z, u_1\}$.

If $y' \in V(B_2 - u_1)$ then $T \cup Q'_1 \cup Q'_2 \cup (Q'_4 \cup B_3 \cup u_1x) \cup (L_4 \cup u_2x) \cup L_1 \cup (L_2 \cup zB_2v_1 \cup Q'_3) \cup (L_3 \cup z_1B_1y \cup Y \cup y'B_2v_2 \cup Q'_5)$ is a $TK_5$ in $G'$ with branch vertices $t, u'_2, x, x_1, x_2$.

If $y' \in V(B_3 - u_1)$ then $T \cup Q_1 \cup Q_2 \cup (Q_4 \cup B_2 \cup u_1x) \cup (L_4 \cup u_2x) \cup L_1 \cup (L_2 \cup zB_2v_1 \cup Q_3) \cup (L_3 \cup z_1B_1y \cup Y \cup y'B_3v_3 \cup Q_5)$ is a $TK_5$ in $G'$ with branch vertices $t, u'_2, x, x_1, x_2$.

We conclude this section with another technical lemma, which deals with a special case that occurs in the proof of Lemma 2.7.5. It is included in this section because its proof also makes use of Lemmas 2.5.1, 2.5.2 and 2.5.3.
Lemma 2.6.6 Let $G$ be a 5-connected nonplanar graph and $x \in V(G)$. Let $(T, S_T, A, B) \in \mathcal{Q}_x$ such that $|V(A)|$ is minimum, and suppose there exists $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ such that $T' \cong K_3$, $T' \cap A \neq \emptyset$, $V(A \cap C) = S_T \cap V(C) = V(B \cap D) = V(B) \cap S_{T'} = \emptyset$, $|V(A) \cap S_{T'}| = |V(D) \cap S_T| = |V(D \cap T)| = 1$, and $|S_T \cap S_{T'}| = 5$. Suppose for any $H \subseteq G$ with $x \in V(H)$ and $H \cong K_2$ or $H \cong K_3$, we have $G/H$ is not 5-connected, $|V(H \cap A)| \leq 1$, and $H \cong K_3$ when $H \cap A \neq \emptyset$. Then one of the following holds:

(i) $G$ has a $TK_5$ in which $x$ is not a branch vertex.

(ii) $G$ contains $K_4$.

(iii) There exist $x_1, x_2, x_3 \in N(x)$ such that, for any distinct $y_1, y_2 \in N(x) - \{x_1, x_2, x_3\}$, $G' := G - \{vx : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}$ contains $TK_5$.

Proof. Note that $|S_T| = |S_T \cap S_{T'}| + |V(D \cap T)| = 6$. Let $V(T) = \{x, w, x_1\}$ and $T' = \{x, a, b\}$ such that $V(A) \cap S_{T'} = \{a\}$ and $V(D) \cap S_T = \{w\}$, and let $S_T \cap S_{T'} = \{x, x_1, b, z_1, z_2\}$. Then $|V(D)| = |V(A)| = |V(A \cap D)| + 1$. Moreover,

1. $|N(s) \cap V(A)| \geq 2$ for $s \in \{b, z_1, z_2\}$, for, otherwise, $(T, (S_T - \{s\}) \cup (N(s) \cap V(A)), A - N(s), G[B + s]) \in \mathcal{Q}_x$, contradicting the choice of $(T, S_T, A, B)$ that $|V(A)|$ is minimum. We may assume that

2. $G$ has no edge from $T - x$ to $T' - x$,

as otherwise $G[T \cup T']$ contains $K_4$ and (ii) holds. We may also assume

3. $N(x_1) \cap V(D) \neq \{w\}$ and $N(w) \cap V(A) \neq \emptyset$,

for, otherwise, let $S := S_T \setminus \{x_1\}$ and $B' = G[B + x_1]$ if $N(x_1) \cap V(D) = \{w\}$, and let $S := S_T \setminus \{w\}$ and $B' = G[B + w]$ if $N(w) \cap V(A) = \emptyset$; then $(xw, S, A, B') \in \mathcal{Q}_x$, and (ii) follows from Lemma 2.6.3. We may further assume that

4. for any $x' \in N(x) \cap V(A \cap D)$, $xx'z_1x$ or $xx'z_2x$ is a triangle.
For, let \( x' \in N(x) \cap V(A \cap D) \). By Lemma 2.6.2, we may assume that there exists \( H \subseteq G \) with \( x, x' \in V(H) \) and \( H \cong K_2 \) or \( H \cong K_3 \). By the assumption of this lemma, \( H \cong K_3 \) and \( V(H) \cap S_T \neq \{x\} \). If \( V(H) \cap \{b, x_1\} \neq \emptyset \) then \( H \cup T \) or \( H \cup T' \) contains \( K^4_- \). So we may assume \( V(H) \cap \{z_1, z_2\} \neq \emptyset \) and, hence, \( xx'z_1x \) or \( xx'z_2x \) is a triangle.

We may assume that

\[
(5) \quad |N(x) \cap V(A \cap D)| \leq 2.
\]

For, otherwise, by (4), there exist \( i \in [2] \) and distinct \( x', x'' \in N(x) \cap V(A \cap D) \cap N(z_i) \). So \( G[x', x'', x, z_i] \) contains \( K^4_- \), and \((ii)\) holds.

We now distinguish two cases.

**Case 1.** \( z_i \notin N(x) \) for \( i \in [2] \).

Then by (4), \( N(x) \cap V(A \cap D) = \emptyset \). We prove that \((iii)\) holds with \( x_2 = w \) and \( x_3 = b \). Let \( y_1, y_2 \in N(x) - \{x_1, x_2, x_3\} \). Since \( G \) is 5-connected and \( z_1, z_2 \notin N(x) \), we may assume \( y_1 \in V(B \cap C) \). Then \( G_B := G[B + \{b, x_1, z_2\}] \) has independent paths \( Y_1, Y_2, Y_3, Y_4 \) from \( y_1 \) to \( z_1, z_2, x_1, b \), respectively.

We may assume that \( wz_i \notin E(G) \) for \( i \in [2] \). For, suppose \( wz_1 \in E(G) \). If \( G[A + \{b, w, x_1\}] \) has independent paths \( Q_1, Q_2 \) from \( b \) to \( x_1, w \), respectively, then \( T \cup bx \cup Q_1 \cup Q_2 \cup y_1x \cup (Y_1 \cup z_1w) \cup Y_3 \cup Y_4 \) is a TK_5 in \( G' \) with branch vertices \( b, w, x, x_1, y_1 \). So we may assume that such \( Q_1, Q_2 \) do not exist. Then \( G[A + \{b, w, x_1\}] \) has a cut vertex \( v \) separating \( b \) from \( \{w, x_1\} \). Let \( D \) denote the component of \( G[A + \{b, w, x_1\}] - v \) containing \( b \). Since \( |N(b) \cap V(A)| \geq 2 \) (by (1)), \( |V(D)| \geq 2 \). Now \( \{b, v, x, z_1, z_2\} \) is a cut in \( G \), and \( G \) has a separation \( (G_1, G_2) \) such that \( V(G_1 \cap G_2) = \{b, v, x, z_1, z_2\} \), \( |V(G_1)| \geq 6 \) and \( \{a, b\} \subseteq V(G_1) \), and \( B + \{w, x_1\} \subseteq G_2 \). By the choice of \((T, S_T, A, B)\) with \( |V(A)| \) minimum, \( |V(G_1)| = 6 \). Let \( u \in V(G_1) - V(G_2) \). If \( u = a \) then, \( V(G_1 \cap G_2) \subseteq N(a) \) (since \( G \) is 5-connected) and \( bv \in E(G) \) (since \( |N(b) \cap V(A)| \geq 2 \); so \( G[\{a, b, v, x\}] \cong K^4_- \), and \((ii)\) holds. So assume \( u \neq a \). Then \( v = a \) and \( G[\{b, u, v, x\}] \) contains \( K^4_- \); so \((ii)\) holds.
We may assume that \( G_A := G[A+\{b, w, x_1, z_1, z_2\}] \) does not contain three independent paths, with one from \( x_1 \) to \( b \), one from \( b \) to \( w \), and one from \( w \) to \( z_i \) for some \( i \in [2] \). For, otherwise, such three paths and \( T \cup bx \cup y_1x \cup Y_1 \cup Y_3 \cup Y_4 \) form a \( TK_5 \) in \( G' \) with branch vertices \( b, w, x, x_1, y_1 \).

We wish to apply Lemma 2.5.1. Let \( G'_A \) be the graph obtained from \( G_A \) by identifying \( z_1 \) and \( z_2 \) as \( z' \), and duplicating \( w, b \) with \( w', b' \), respectively (adding edges from \( w' \) to all vertices in \( N(w) \), and from \( b' \) to all vertices in \( N(b) \)). Then any three disjoint paths in \( G'_A \) from \( \{w, x_1, w'\} \) to \( \{b, z', b'\} \), if exist, must contain a path from \( x_1 \) to \( z' \).

Suppose \( G'_A \) has a separation \( (A_1, A_2) \) such that \( |V(A_1 \cap A_2)| \leq 2 \), \( \{w, x_1, w'\} \subseteq V(A_1) \), and \( \{b, z', b'\} \subseteq V(A_2) \). Since \( w \) and \( w' \) have the same set of neighbors in \( G'_A \), we may assume \( \{w, w'\} \subseteq V(A_1 \cap A_2) \) or \( \{w, w'\} \cap V(A_1 \cap A_2) = \emptyset \). If \( \{w, w'\} \subseteq V(A_1 \cap A_2) \) then \( V(A_1) = \{x_1\} \cup V(A_1 \cap A_2) \) as \( \{x, x_1, w\} \) cannot be a cut in \( G \); hence, \( N(x_1) \cap V(D) = \{w\} \), contradicting (3). So \( \{w, w'\} \cap V(A_1 \cap A_2) = \emptyset \). Suppose \( \{b, b', z'\} \cap V(A_1 \cap A_2) = \emptyset \). Then, since \( wz_i \notin E(G) \) for \( i \in [2] \), \( V(A_1 \cap A_2) \cup \{x_1, x\} \) is a cut in \( G \) separating \( w \) from \( B + \{b, z_1, z_2\} \), contradicting the fact that \( G \) is 5-connected. So \( \{b, b', z'\} \cap V(A_1 \cap A_2) \neq \emptyset \). Note that \( \{b, b'\} \not\subseteq V(A_1 \cap A_2) \); as otherwise \( \{b, x, x_1\} \) would be a cut in \( G \). Thus, we may assume that \( b, b' \not\subseteq V(A_1 \cap A_2) \) as \( b \) and \( b' \) have the same set of neighbors in \( G'_A \). Hence, \( z' \in V(A_1 \cap A_2) \). Now \( S := \{x, x_1, z_1, z_2\} \cup (V(A_1 \cap A_2) - \{z'\}) \) is a cut in \( G \) separating \( w \) from \( B + b \). Since \( G \) is 5-connected, \( x_1 \notin V(A_1 \cap A_2) \). If \( |V(A_1 - x_1 - A_2)| \geq 2 \) then \( (xx_1, S, A_1 - x_1 - A_2, G - S - A_1) \in Q_x \) which contradicts the choice of \( (T, S_T, A, B) \) with \( |V(A)| \) minimum. So \( V(A_1 - x_1 - A_2) = \{w\} \). Since \( G \) is 5-connected, \( wz_i \in E(G) \) for \( i \in [2] \), a contradiction.

Hence, by Lemma 2.5.1, \( G'_A \) has a separation \( (J, L) \) such that \( V(J \cap L) = \{w_0, \ldots, w_n\} \), \( (J, w_0, \ldots, w_n) \) is planar (since \( G \) is 5-connected), \( (L, (w, x_1, w'), (b, z', b')) \) is a ladder along a sequence \( b_0 \ldots b_m \), where \( b_0 = x_1 \), \( b_m = z' \), and \( w_0 \ldots w_n \) is the reduced sequence of \( b_0 \ldots b_m \). Moreover, we may assume that \( L \) has disjoint induced paths \( P_1, P_2, P_3 \) from \( w, x_1, w' \) to \( b, z', b' \), respectively, and \( J \) is a connected plane graph with \( P_2 \) as part of the
Let \( J \), we let \( \text{outer walk of } J \) and \( w_0, \ldots, w_n \) occurring on \( P_2 \) in order. (When \((ii)\) of Lemma 2.5.1 holds, we let \( J = P_2 \).) Note that by Lemmas 2.5.2 and 2.5.3, each rung of \((L,(w,x_1,w'),(b,z',b'))\) is of type \((i)-(iv)\) as in Lemma 2.5.3, with possible exceptions of those rungs containing \( z' \).

Let \((R_j,(a_{j-1},b_{j-1},c_{j-1}),(a_j,b_j,c_j)), j \in [m]\), be the rungs in \((L,(w,x_1,w'),(b,z',b'))\) such that \( a_j \in V(P_1) \) and \( c_j \in V(P_3) \) for \( j = 0, 1, \ldots, m \).

We now show that there exists \( t \in N(w) \) such that \( t \in V(P_2) - \{x_1, z'\} \). For, suppose such \( t \) does not exist. Choose the largest \( j \) such that \( \{w,w'\} \subseteq V(R_j) \) and \((R_j,(a_{j-1},b_{j-1},c_{j-1}),(a_j,b_j,c_j))\) is not of type \((ii)\) in Lemma 2.5.3, which is well defined as \( w \neq b \). Since \( G \) is 5-connected and \( w \) and \( w' \) have the same set of neighbors in \( G_A' \), \((R_j,(a_{j-1},b_{j-1},c_{j-1}),(a_j,b_j,c_j))\) cannot be of type \((iii)\) as in Lemma 2.5.3. Moreover, \((R_j,(a_{j-1},b_{j-1},c_{j-1}),(a_j,b_j,c_j))\) is not of type \((iv)\) as in Lemma 2.5.3, as otherwise \( G \) contains \( K_4^- \) (obtained from \( R_j - \{b_{j-1},b_j\} \) after identifying \( w \) with \( w' \)).

So \((R_j,(a_{j-1},b_{j-1},c_{j-1}),(a_j,b_j,c_j))\) is of type \((i)\) as in Lemma 2.5.3. Now \( V(R_j) = \{a_j,b_j,c_j,w,w'\} \), as otherwise \( \{a_j,b_j,c_j,w\} \) would be a cut in \( G \). Then \( wb_j \in E(G) \); for otherwise, \( N(w) \subseteq \{a_j,c_j,x,x_1\} \), a contradiction. Hence \( t := b_j \) is as desired.

Without loss of generality, we may assume that the edge of \( P_2 \) incident with \( z' \) corresponds to the edge of \( G \) incident with \( z_1 \). We view \( P_3 \) as a path in \( G_A \) from \( b \) to \( w \). Then \( G_A - V(P_1 \cup P_3) - z_2 \) has independent paths from \( t \) to \( x_1, z_1 \), respectively. Hence, by Lemma 2.4.11, \( G_A \) has five independent paths \( Q_1, Q_2, Q_3, Q_4, Q_5 \) from \( t \) to \( x_1, w, z_1, (V(P_1 \cup P_3) - \{w\}) \cup \{z_2\} \), respectively, with only \( t \) in common, and internally disjoint from \( P_1 \cup P_3 \). Without loss of generality, we may assume that \( Q_4 \) ends at \( t' \in V(P_3) \).

If \( G_B - x \) contains disjoint paths \( S_1, S_2 \) from \( z_1, b \) to \( y_1, x_1 \), respectively, then \( T \cup bx \cup P_1 \cup S_2 \cup Q_1 \cup Q_2 \cup (Q_3 \cup S_1 \cup y_1 x) \cup (Q_4 \cup t'P_3b) \) is \( T'K_5 \) in \( G' \) with branch vertices \( b, t, w, x, x_1 \). Hence, we may assume such \( S_1, S_2 \) do not exist. Then by Lemma 2.4.10, there exists a collection \( D \) of subsets of \((G_B - x) - \{z_1, b, y_1, x_1\}\) such that \((G_B - x, D, z_1, b, y_1, x_1)\) is 3-planar.
If \((G_B - x, \{b, x_1, z_1, z_2\})\) is planar then the assertion of the lemma follows from Lemma 2.4.5, with the cut \(\{b, x, x_1, z_1, z_2\}\) giving the required 5-separation for Lemma 2.4.5.

So we may assume that either \(D = \emptyset\) and \(z_2\) does not belong to the facial walk of \(G_B - x\) containing \(\{b, x_1, y_1, z_1\}\), or \(D = \{D\}\) for some \(D \subseteq V(G_B - x) - \{b, x_1, y_1, z_1\}\) and \(z_2 \in D\). Thus, since \(G\) is 5-connected and \((G_B - x, \{z_1, b, y_1, x_1\})\) is 3-planar, \(G_B - x\) has disjoint paths \(S'_1, S'_2\) from \(z_2, b\) to \(y_1, x_1\), respectively. Moreover, if \(b\) has degree at least two in \(G_B - x\) then \(G_B - x\) has independent paths \(Y, Y'_2, Y'_3, Y'_4\), with \(Y\) from \(b\) to \(x_1\) and \(Y'_2, Y'_3, Y'_4\) from \(y_1\) to \(z_2, x_1, b\), respectively.

We may assume that \(G'_A - J\) contains a path \(Z\) from \(z_2\) to some \(z'_2 \in V(P_1 \cup P_3) - \{b, b'\}\) and internally disjoint from \(P_1 \cup P_3\). For, suppose not. Then, since \(|N(z_2) \cap V(A)| \geq 2\) (by (1)), \(z_2\) has at least two neighbors in \(J - z'\). Then \(G'_A - V(P_1 \cup P_3) - z_1\) has independent paths from \(t\) to \(x_1, z_2\), respectively; for otherwise, \(G'_A - V(P_1 \cup P_3) - z_1\) has a cut vertex \(v \in V(tP_2x_2)\) separating \(t\) from \(\{x_1, z_2\}\) and, hence, \(V(T) \cup \{v, z_1, z_2\}\) is a cut in \(G\), contradicting the choice of \(S_T\) with \(|V(A)|\) minimum. Hence, by Lemma 2.4.11, \(G_A\) has five independent paths \(Q'_1, Q'_2, Q'_3, Q'_4, Q'_5\) from \(t\) to \(x_1, w, z_2, (V(P_1 \cup P_3) - \{w, b'\}) \cup \{z_1\}\), respectively, with only \(t\) in common, and internally disjoint from \(P_1 \cup (P_3 - \{b', w'\})\).

Without loss of generality, we may assume that \(Q'_4\) ends at \(t'' \in V(P_3)\). Then \(T \cup bx \cup P_1 \cup S'_2 \cup Q'_1 \cup Q'_2 \cup (Q'_3 \cup S'_1 \cup y_1x) \cup (Q'_4 \cup t''P_3b)\) is \(TK_5\) in \(G'\) with branch vertices \(b, t, w, x, x_1\).

Without loss of generality, we may assume that \(z'_2 \in V(P_3)\). We may further assume that \(b\) has only one neighbor in \(G_B - x\); for, otherwise, \(T \cup bx \cup P_1 \cup Y \cup y_1x \cup (Y'_2 \cup Z \cup z'_2P_3w) \cup Y'_3 \cup Y'_4\) is a \(TK_5\) in \(G'\) with branch vertices \(b, w, x, x_1, y_1\).

Thus, since \(G\) is 5-connected and \(bw \notin E(G)\) (by (2)), \(b\) has a neighbor \(u \in V(A) - V(P_1 \cup P_3)\). We choose \(u\) and the rung \((R_j, (a_j-1, b_j-1, c_j-1)), (a_j, b_j, c_j)\) such that \(b, b', u \in V(R_j)\). Since \(b\) and \(b'\) have the same set of neighbors in \(G'_A\), \(a_{j-1} = b\) if, and only if, \(c_{j-1} = b'\). Moreover, we must have \(b_j = z'\) because of the path \(Z\).

First, suppose \(b_{j-1} = z'\). Then \(a_{j-1} \neq b\) and \(c_{j-1} \neq b'\). If \(z_2\) has no neighbor in \(V(G'_A - \(39\))
$J - R_j$ then $V(T) \cup \{a_{j-1}, c_{j-1}, z_1\}$ is a cut in $G$ separating $a_{j-1}P_1w \cup c_{j-1}P_3w \cup (J - z')$ from $B \cup (R_j - \{b', z'\})$, contradicting the choice of $(T, S_T, A, B)$ with $|V(A)|$ minimum. Thus, $z_2$ has a neighbor in $V(G' \setminus J - R_j)$; so the above path $Z$ may be chosen to be disjoint from $R_j$. Let $S$ be a path in $R_j - \{a_{j-1}, c_{j-1}\}$ from $b$ to $z_1$ (which must exist as otherwise $\{a_{j-1}, c_{j-1}, z_2\} \cup V(T)$ is a cut in $G$ contradicting the choice of $(T, S_T, A, B)$ that $|V(A)|$ is minimum). So $T \cup bx \cup P_1 \cup (S \cup z_1P_2x_1) \cup y_1x \cup (Y_2 \cup Z \cup z_2P_3w) \cup Y_3 \cup Y_4$ is $TK_5$ in $G'$ with branch vertices $b, w, x, x_1, y_1$.

Now assume $b_{j-1} \neq z' = b_j$. Since $b_{j-1} \neq b_j$ and since $b$ and $b'$ have the same set of neighbors in $G'_{A}$, we must have $a_{j-1} = b$ and $c_{j-1} = b'$. If $u \in \{b_{j-1}, b_j\}$ then, since $bz' \notin E(G'_{A})$, $u = b_{j-1}$; and let $S = bb_{j-1}$. Now suppose $u \notin \{b_{j-1}, b_j\}$. Then $\{b, b_{j-1}, x, z_1, z_2\}$ is a cut in $G$ separating $u$ from $(J - z') \cup B$. By the choice of $(T, S_T, A, B)$ that $|V(A)|$ is minimum, $\{u\} = V(R_j) - \{b,b',b_{j-1}, z'\}$. Since $G$ is 5-connected, $N(u) = \{b, b_{i-1}, x, z_1, z_2\}$. Let $S = bb_{j-1}$. Since $|N(z_2) \cap V(A)| \geq 2$ (by (1)), the path $Z$ may be chosen to be disjoint from $R_j$. So $T \cup bx \cup P_1 \cup (S \cup b_{j-1}P_2x_1) \cup y_1x \cup (Y_2 \cup Z \cup z_2P_3w) \cup Y_3 \cup Y_4$ is $TK_5$ in $G'$ with branch vertices $b, w, x, x_1, y_1$.

**Case 2.** $N(x) \cap \{z_1, z_2\} \neq \emptyset$.

Without loss of generality, we may assume $xz_1 \in E(G)$. We may further assume $z_1$ is not adjacent to any of $\{a, b, w, x_1\}$; for otherwise, $G[T + z_1]$ or $G[T + z_1]$ contains $K_4^-$, and (ii) holds. We wish to prove (iii), with $x_2 = b$ and $x_3 = z_1$. Let $y_1, y_2 \in N(x) - \{b, x_1, z_1\}$ be distinct.

**Subcase 2.1.** There exists some $i \in [2]$ such that $y_i \in V(B) \cup \{z_2\}$.

Without loss of generality, assume $y_1 \in V(B) \cup \{z_2\}$ and, whenever possible, let $y_1 \in V(B)$. Let $G_B := G[B + \{b, x_1, z_1, z_2\}]$. When $y_1 \in V(B)$ let $t = y_1$ and let $Y_1, Y_2, Y_3, Y_4, Y_5$ be independent paths in $G[B]$ from $t$ to $z_1, y_1, b, x_1, z_2$, respectively. When $y_1 = z_2$ let $t = y_1$ and let $Y_1, Y_2, Y_3, Y_4$ be independent paths in $G[B]$ from $t$ to $z_1, y_1, b, x_1, z_2$, respectively. Let $G_A = G[A + \{b, w, x_1, z_1\}]$.

We may assume that there is no cycle in $G_A$ containing $\{b, x_1, z_1\}$. For, such a cycle and
\[ xb \cup xx_1 \cup xz_1 \cup Y_1 \cup (Y_2 \cup y_1, x) \cup Y_3 \cup Y_4 \text{ is a TK}_5 \text{ in } G' \text{ with branch vertices } b, t, x, x_1, z_1. \]

We may also assume that \( G_A \) is 2-connected. To see this, we first assume \( N(x_1) \cap N(w) = \{x\} \); for otherwise, letting \( u \in (N(x_1) \cap N(w)) - \{x\} \) we see that \( G[T + u] \) contains \( K_4^- \) and (ii) holds. Therefore, since \( N(w) \cap V(A) \neq \emptyset \neq N(x_1) \cap V(A) \) (by (3)), it suffices to show that \( G[A + \{b, z_1\}] \) is 2-connected. So assume for a contradiction that there exists a separation \((A_1, A_2)\) in \( G[A + \{b, z_1\}] \) such that \( |V(A_1 \cap A_2)| \leq 1 \). Without loss of generality, let \( |\{b, z_1\} \cap V(A_1)| \leq 1 \). Then \( V(A_1) \not\subseteq V(A_2) \cup \{b, z_1\} \) as \( |N(s) \cap V(A)| \geq 2 \) for \( s \in \{b, z_1\} \) (by (1)). Hence, \( V(T) \cup (\{b, z_1\} \cap A_1) \cup V(A_1 \cap A_2) \cup \{z_2\} \) is a cut in \( G \) of size at most 6 which separates \( A_1 \) from the rest of \( G \), contradicting the choice of \( (T, S_T, A, B) \) that \( |V(A)| \) is minimum.

Then, since \( G_A \) has no cycle containing \( \{b, x_1, z_1\} \), (i), or (ii), or (iii) of Lemma 2.4.12 holds for \( G_A \) and \( \{b, x_1, z_1\} \). So for each \( u \in \{b, x_1, z_1\} \), \( G_A \) has a 2-cut \( S_u \) separating \( u \) from \( \{b, x_1, z_1\} - \{u\} \), and let \( D_u \) denote a union of components of \( G_A - S_u \) such that \( u \in V(D_u) \) for \( u \in \{b, x_1, z_1\} \) and \( D_b, D_{x_1}, D_{z_1} \) are pairwise disjoint. We choose \( S_u \) and \( D_u \), \( u \in \{b, x_1, z_1\} \), to maximize \( D_b \cup D_{x_1} \cup D_{z_1} \). Note that, since \( wx_1 \in E(G), w \notin V(D_b \cup D_{z_1}) \).

We claim that for \( u \in \{b, x_1, z_1\} \), \( V(D_u) = \{u\} \). For, otherwise, \( S := S_u \cup \{u, x, z_2\} \) is a cut in \( G \) separating \( D_u - u \) from the rest of \( G \). If \( |V(D_u)| \geq 3 \) then \( (ux, S, D_u, G - S - D_u) \in Q_2 \) contradicts the choice of \( (T, S_T, A, B) \) with \( |V(A)| \) minimum. So let \( V(D_u) = \{u, u'\} \) and let \( S_u = \{s_u, t_u\} \). Since \( G \) is 5-connected, \( N(u') = \{s_u, t_u, u, x, z_2\} \). Since \( |N(u) \cap V(A + w)| \geq 2 \) (by (1) and (3)), we may assume that \( us_u \in E(G) \). Then \( G[\{s_u, u', u, x, z\}] \) contains \( K_4^- \), and (ii) holds.

For \( u \in \{b, x_1, z_1\} \), let \( S_u = \{s_u, t_u\} \). Since \( G_A \) is 2-connected, \( \{us_u, ut_u\} \subseteq E(G) \). Note \( a \in \{s_b, t_b\} \); so we may assume \( s_bt_b \notin E(G) \) because otherwise \( G[\{x, b, s_b, t_b\}] \) contains \( K_4^- \), and (ii) holds. Similarly, \( w \in \{s_{x_1}, t_{x_1}\} \) and we may assume \( s_{x_1}t_{x_1} \notin E(G) \). If (i) of Lemma 2.4.12 occurs then \( ax_1 \in E(G) \), contradicting (2). If (iii) of Lemma 2.4.12 occurs then let \( R_1, R_2 \) be the components of \( G_A - V(D_b \cup D_{x_1} \cup D_{z_1}) \) and assume without
loss of generality that \( s_u \in V(R_1) \) and \( t_u \in V(R_2) \) for \( u \in \{b, x_1, z_1\} \). By symmetry, assume \( w \notin V(R_1) \). Hence, \((xb, \{x, b, x_1, s_{z_1}, z_2\}, R_1 - s_{z_1}, G - R_1 - \{x, b, x_1, z_2\}) \in Q_x \) with \( 2 \leq |V(R_1 - s_{z_1})| < |V(A)| \), contradicting the choice of \((T, S_T, A, B)\).

So we may assume that \((ii)\) of Lemma 2.4.12 holds. Without loss of generality let \( R_1, R_2 \) be the components of \( G - V(D_b \cup D_{x_1} \cup D_{z_1}) \) containing \( z = s_b = s_{x_1} = s_{z_1}, \{t_b, t_{x_1}, t_{z_1}\} \), respectively. By (2), \( z \neq a \) and \( z \neq w \). So \( a = t_b \) and \( w = t_{x_1} \). Thus, we may assume \( xz \notin E(G) \) as, otherwise, \( G[T + z] \) contains \( K_t \) and \((ii)\) holds. Hence, \( R_1 = R_2 \) (otherwise \( z \) would have degree at most 4 in \( G \)). By (1) and by the maximality of \( D_b \cup D_{x_1} \cup D_{z_1}, G[R_2 + z_2] \) is 2-connected (since \( G \) is 5-connected).

We claim that there exist distinct \( t_1, t_2 \in \{a, w, t_{z_1}\} \) such that \( G[R_2 + z_2] \) contains disjoint paths \( P_1, P_2 \) from \( z, t_1 \) to \( z_2, t_2 \), respectively. For, suppose \( \{a, w\} \) cannot serve as \( \{t_1, t_2\} \). Then, by Lemma 2.4.10, \((G[R_2 + z_2], a, z_2, w, z)\) is 3-planar. Hence, \((G[R_2 + z_2]\) has disjoint paths from \( z, a \) to \( z_2, t_{z_1} \), respectively, or disjoint paths from \( z, w \) to \( z_2, t_{z_1} \), respectively.

Suppose \( z_2 \neq y_1 \). Recall the definition of \( t \) and the paths \( Y_1, Y_2, Y_3, Y_4, Y_5 \). If \( \{t_1, t_2\} = \{a, w\} \) then \( bxx_1zb \cup xz_1z_2 \cup (x_1w \cup P_2 \cup ab) \cup (y_2 \cup y_1x) \cup (y_5 \cup P_1) \cup Y_3 \cup Y_4 \) is a \( TK_5 \) in \( G' \) with branch vertices \( b, t, x, x_1, z \). If \( \{t_1, t_2\} = \{a, t_{z_1}\} \) then \( bzx_1zb \cup xx_1z_2 \cup (z_{t_{z_1}} \cup P_2 \cup ab) \cup Y_1 \cup (y_2 \cup y_1x) \cup Y_3 \cup (y_5 \cup P_1) \) is a \( TK_5 \) in \( G' \) with branch vertices \( b, t, x, z, z_1 \). If \( \{t_1, t_2\} = \{w, t_{z_1}\} \) then \( x_1xz_1xz_2 \cup xbz \cup (x_1w \cup P_2 \cup t_{z_1}, z_1) \cup Y_1 \cup (y_2 \cup y_1x) \cup Y_4 \cup (y_5 \cup P_1) \) is a \( TK_5 \) in \( G' \) with branch vertices \( t, x, x_1, z, z_1 \).

So assume \( z_2 = y_1 \). Then \( y_2 \neq z_2 \); and hence, by the choice of \( y_1 \), we have \( y_2 \in V(A) \cup \{w\} \). If \( R_2 - z \) has independent paths \( S_1, S_2, S_3 \) from \( y_2 \) to \( a, w, t_{z_1} \), respectively, then \( xbz_1x \cup y_2x \cup (S_1 \cup ab) \cup (S_2 \cup wx_1) \cup Y_3 \cup Y_4 \cup (y_1 \cup z_{t_{z_1}} \cup S_3) \cup (y_2 \cup z_2x) \) is a \( TK_5 \) in \( G' \) with branch vertices \( b, t, x, x_1, y_2 \). So assume such \( S_1, S_2, S_3 \) do not exist.

Then \( R_2 \) has a separation \((A_1, A_2)\) such that \( z \in V(A_1 \cap A_2) \), \(|V(A_1 \cap A_2)| \leq 3, y_2 \in V(A_1 - A_2) \) and \( \{a, w, t_{z_1}\} \subseteq V(A_2) \). Thus \( S := \{x, z_2\} \cup V(A_1 \cap A_2) \) is a 5-cut in \( G \) separating \( y_2 \) from \( B \cup A_2 \cup \{b, x_1, z_1, z\} \). Hence, by the choice of \((T, S_T, A, B)\) (with
|V(A)| minimum), V(A_1 - A_2) = \{y_2\}. Therefore, since G is 5-connected, N(y_2) = S.

By the maximality of D_b \cup D_{x_1} \cup D_{z_1}, R_2 - \{y_2, z\} has a path Q from a to w. Then bxx_1zb \cup (ba \cup Q \cup wx_1) \cup zy_2x \cup (Y_1 \cup z_1z) \cup (Y_2 \cup z_2x) \cup Y_3 \cup Y_4 is a TK_5 in G' with branch vertices b, t, x, x_1, z.

**Subcase 2.2.** y_1, y_2 \in V(A) \cup \{w\}.

First, we show that we may assume y_1 = w. For, suppose y_1, y_2 \in V(A). Then by Lemma 2.6.2, for each i \in [2] there exists (T_i, S_{T_i}, A_i, B_i) \in Q_x such that x, y_i \in V(T_i) and T_i \cong K_2 or T_i \not\cong K_3. By the assumption of this lemma, we have T_i \cong K_3 and V(A) \cap S_{T_i} = \{y_i\}. Hence, \{b, w, x_1, z_1, z_2\} \cap V(T_i) \neq \emptyset for i \in [2]. Without loss of generality, we may assume that y_1 \neq a. By the symmetry between z_1 and z_2, we may also assume z_1 \in V(T_1); for, otherwise, G[T + y_1] or G[T' + y_1] contains K^-_4 and (ii) holds. Therefore, we may choose S_{T_1} = V(T_1) \cup \{b, x_1, z_2\}. Note the symmetry between T_1, S_{T_1} and T, S_T, and we may choose T_1, S_{T_1} as T, S_T, respectively. So we may assume y_1 = w (as y_1 now plays the role of w).

Let t \in V(B), and let L_1, L_2, L_3, L_4 be independent paths in G_B = G[B + \{b, x_1, z_1, z_2\}] from t to z_1, z_2, b, x_1, respectively. Let G_A := G[A + \{b, w, x_1, z_2\}]. Note that, by the same argument as in Subcase 2.1 (with z_2 in place of z_1), we may assume that G_A is 2-connected.

We may assume that G_A does not contain independent paths from z_2, w, b to w, b, x_1, respectively; for otherwise, these paths and T \cup bx \cup (L_1 \cup z_1x) \cup L_2 \cup L_3 \cup L_4 form a TK_5 in G with branch vertices b, t, w, x, x_1.

Hence, since G_A is 2-connected, wz_2 \notin E(G). We may assume that wz_1 \notin E(G); else G[T + z_1] contains K^-_4 and (ii) holds. Therefore, since G is 5-connected, it follows from (2) that

\[|N(w) \cap V(A \cap D)| \geq 3.\]

Let G'_A be the graph obtained from G_A by duplicating w, b with w', b', respectively, and adding all edges from w' to N(w), and from b' to N(b). Then any three disjoint paths in
For, otherwise, \(G'\) has no cut of size at most 2 separating \(\{x_1, w, w'\}\) from \(\{b, b', z_2\}\). For, otherwise, \(G'\) has a separation \((A_1, A_2)\) such that \(|V(A_1 \cap A_2)| \leq 2\), \(\{x_1, w, w'\} \subseteq V(A_1)\) and \(\{b, b', z_2\} \subseteq V(A_2)\). Note that \(V(A_1 \cap A_2) \neq \{w, w'\}\) as otherwise, \(w\) would be a cut vertex in \(G_A\). Further, \(\{w, w'\} \cap V(A_1 \cap A_2) = \emptyset\); for, otherwise, since \(w\) and \(w'\) have the same set of neighbors in \(G'\), it follows from (3) that \(V(A_1 \cap A_2) - \{w, w'\}\) would be a cut in \(G_A\) of size at most one. On the other hand, \(V(A_1 - A_2) \subseteq \{x_1, w\}\); otherwise \((T, V(T) \cup \{z_1\} \cup V(A_1 \cap A_2), (A_1 - A_2) - w', G - (T \cup A_1)) \in \mathcal{Q}_x\) with 
\[1 \leq |(A_1 - A_2) - w'| < |A|,\]
contradicting the choice of \((T, S_T, A, B)\). However, this implies \(|N(w) \cap V(A \cap D)| \leq |V(A_1 \cap A_2)| \leq 2\), a contradiction.

Hence by Lemma 2.5.1, \(G'_A\) has a separation \((J, L)\) such that \(V(J \cap L) = \{w_0, \ldots, w_n\}\), \((J, w_0, \ldots, w_n)\) is 3-planar, \(L, (w, x_1, w'), (b, z_2, b')\) is a ladder along some sequence \(b_0 \ldots b_m\), where \(b_0 = z_2, b_m = x_1\), and \(w_0 \ldots w_n\) is the reduced sequence of \(b_0 \ldots b_m\).

Let \(P_1, P_2, P_3\) be three disjoint paths in \(L\) from \(w, x_1, w'\) to \(b, z_2, b'\), respectively, and assume that they are induced in \(G'_A\). (Let \(L = G'_A\) and \(J = P_2\) if \((ii)\) of Lemma 2.5.1 holds.) Let \((R_i, (a_{i-1}, b_{i-1}, c_{i-1}), (a_i, b_i, c_i)), i \in [m]\), be the rungs in \(L\) with \(a_i \in V(P_1)\) and \(c_i \in V(P_3)\) for \(i = 0, 1, \ldots, m\).

Since \(|N(w) \cap V(A \cap D)| \geq 3\) and \(P_1, P_3\) are induced paths in \(G'_A\), there exists \(w^* \in (N(w) \cap V(A)) - V(P_1 \cup P_3)\). We show that there exists \(u \in V(P_2)\) such that \(G'[G_A + \{x, z_1\}]\) has five independent paths \(Q_1, Q_2, Q_3, Q_4, Q_5\) from \(u\) to distinct vertices \(x_1, w, z_2, u_1, u_2\), respectively, with \(u_1, u_2 \in V(P_1 - w) \cup V(P_3 - \{b', w'\}) \cup \{x, z_1\}\), and internally disjoint from \(P_1 \cup (P_3 - \{b', w'\})\). If \(w^* \in V(P_2)\) then let \(u = w^*\) and we see that there exist independent paths in \(G_A - (V(P_1 - w) \cup V(P_3 - \{b', w'\}))\) from \(u\) to \(x_1, w, z_2\), respectively; then the paths \(Q_1, \ldots, Q_5\) exist by Lemma 2.4.11. Now suppose \(w^* \notin V(P_2)\).

Let \((R_i, (a_{i-1}, b_{i-1}, c_{i-1}), (w, b_i, w'))\) be the rung in \(L\) containing \(\{w, w', w^*\}\). Since \(w\) and \(w'\) have the same set of neighbors in \(G'_A\), \(w = a_{i-1}\) iff \(w' = c_{i-1}\). If \(w = a_{i-1}\) and \(w' = c_{i-1}\)
then $S^*_T := V(T) \cup \{b_{i-1}, b_i, z_1\}$ is a cut in $G$ of size at most 6, and $G - S^*_T$ has a component of size smaller than $|V(A)|$, contradicting the choice of $(T, S_T, A, B)$. So $w \neq a_{i-1}$ and $w' \neq c_{i-1}$. Suppose $R_i - x_1$ has a separation $(R', R'')$ such that $|V(R' \cap R'')| \leq 2$, $w \in V(R' - R'')$, and $\{a_{i-1}, c_{i-1}, b_{i-1}, b_i\} - \{x_1\} \subseteq V(R'')$. Then we may assume $w' \in V(R' - R'')$ as $w$ and $w'$ have the same set of neighbors in $G'_{A_i}$. Therefore, since $|N(w) \cap V(A \cap D)| \geq 3$, $S^*_T := V(T) \cup V(R' \cap R'') \cup \{z_1\}$ is a cut in $G$ of size at most 6, and $G - S^*_T$ has a component of size smaller than $|V(A)|$, contradicting the choice of $(T, S_T, A, B)$. Thus we may assume, by Lemma 2.4.11, $R_i - x_1$ contains three independent paths from $w$ to $a_{i-1}, c_{i-1}, \{b_{i-1}, b_i\} - \{x_1\}$, respectively, and internally disjoint from $\{b_{i-1}, b_i\}$. Again since $w$ and $w'$ have the same set of neighbors in $G'_{A_i}$, the parts of $P_1, P_3$ inside $R$ can be modified so that the three paths in $R_i$ correspond to $wP_1a_{i-1}, w'P_3c_{i-1}$ and a path from $w$ to some $u \in \{b_{i-1}, b_i\} - \{x_1\}$ and internally disjoint from $P_1 \cup P_2 \cup P_3$. Thus, there exist independent paths in $G_A - (V(P_1 - w) \cup V(P_3 - \{b', w'\}))$ from $u$ to $x_1, w, z_2$, respectively. Now the paths $Q_1, \ldots, Q_5$ exist by Lemma 2.4.11.

We may assume $u_1 = z_1$ and $u_2 = x$. For, otherwise, we may assume by symmetry that $u_1 \in V(P_1)$. If $G_B - x$ has disjoint paths $B_1, B_2$ from $z_1, b$ to $z_2, x_1$, respectively, then $T \cup bx \cup P_3 \cup B_2 \cup Q_1 \cup Q_2 \cup (Q_3 \cup B_1 \cup z_1x) \cup (Q_4 \cup u_1P_1b)$ is a $TK_5$ in $G$ with branch vertices $b, u, w, x, x_1$. (Here we view $P_3$ as a path in $G$ by identifying $b', w'$ with $b, w$, respectively.) So we may assume that such $B_1, B_2$ do not exist. Then by Lemma 2.4.10, $(G_B - x, z_1, b, z_2, x_1)$ is planar; so the assertion of the lemma follows from Lemma 2.4.5.

We may also assume $|N(b) \cap V(B)| \leq 1$. For, suppose $|N(b) \cap V(B)| \geq 2$. Then, since $G$ is 5-connected, $G[B + \{b, x_1, z_2\}]$ contains independent paths $B_1, B_2$ from $b$ to $x_1, z_2$, respectively. Hence, $T \cup bx \cup P_3 \cup B_1 \cup Q_1 \cup Q_2 \cup (Q_3 \cup B_2) \cup (Q_4 \cup z_1x)$ is a $TK_5$ in $G$ with branch vertices $b, u, w, x, x_1$, where we view $P_3$ as a path in $G'$ by identifying $b', w'$ with $b, w$, respectively.

Then we may assume $|N(b) \cap V(A + z_2)| \geq 3$ as otherwise, $bz_1 \in E(G)$ by (2); so $G[T' + z_1]$ contains $K_4^-$ and (ii) holds. Let $b^* \in (N(b) \cap V(A + z_2)) - V(P_1 \cup P_3)$.  

45
If $b^* \in V(P_2)$ let $z = b^*$ and let $P = bz$ which is internally disjoint from $P_1 \cup P_2 \cup P_3$. Now suppose $b^* \notin V(P_2)$. Let $(R_j, (b, b_{j-1}, b'), (a_j, b_j, c_j))$ be the rung in $L$ containing $\{b, b', b^*\}$. Since $b$ and $b'$ have the same set of neighbors in $G'_A$, $b = a_j$ iff $b' = c_j$. If $b = a_j$ and $b' = c_j$ then, since $az_1 \notin E(G)$, $S_i^+ := V(T') \cup \{b_{j-1}, b_j, z_1\}$ is a cut in $G$ of size $6$ and $G - S_i^+$ has a component of size smaller than $|V(A)|$, contradicting the choice of $(T, S_T, A, B)$. So $b \neq a_j$ and $b' \neq c_j$. We claim that $P_1 \cap R_j$ and $P_3 \cap R_j$ may be modified so that $G_A$ contains a path $P$ from $b$ to some $z \in V(P_2)$ and internally disjoint from $P_1 \cup P_2 \cup (P_3 - \{b', w'\})$. If $R_j$ contains three independent paths from $b$ to $a_j, c_j, \{b_{j-1}, b_j\}$, respectively, and internally disjoint from $\{a_j, c_j, b_{j-1}, b_j\}$, then $P_1 \cap R_j, P_3 \cap R_j$ can be modified so that the three paths in $R_j$ correspond to $bP_1a_j, b'P_3c_j$ and a path $P$ from $b$ to $z \in \{b_{j-1}, b_j\}$ and internally disjoint from $P_1 \cup P_2 \cup (P_3 - \{b', w'\})$. So assume that such three paths in $R_j$ do not exist. Then by the existence of $bP_1a_j$ and $b'P_3c_j$ and by Lemma 2.4.11, $R_j$ has no three independent paths from $b$ to $\{a_j, c_j, b_{j-1}, b_j\}$ and internally disjoint from $\{a_j, c_j, b_{j-1}, b_j\}$. Thus $R_j$ has a separation $(A_1, A_2)$ with $|V(A_1 \cap A_2)| \leq 2$, $V(A_1 \cap A_2) \subseteq V(P_1 \cup P_3)$, $b, b^* \in V(A_1 - A_2)$ and $\{a_j, c_j, b_{j-1}, b_j\} \subseteq V(A_2)$. Since $b'$ is a copy of $b$, we may assume $b' \in V(A_1 - A_2)$. Now, since $az_1 \notin E(G)$, $V(A_1 \cap A_2) \cup \{x, b, z_1\}$ is a cut in $G$; so $V(A_1) = V(A_1 \cap A_2) \cup \{b, b', b^*\}$ by the choice of $(T, S_T, A, B)$ that $|V(A)|$ is minimum. Then $b^*x, b^*z_1 \in E(G)$ (as $G$ is 5-connected); so $G[\{x, b^*, b, z_1\}]$ contains $K_{4,1}$, and $(ii)$ holds.

Suppose $R_i \neq R_j$. Since $G$ is 5-connected, $G[B + \{b, x_1\}]$ has a path $B_1$ from $b$ to $x_1$. Since $Q_3$ is internally disjoint from $P_1 \cup P_3$, we may assume that $z \in V(Q_3)$ and $P$ is also internally disjoint from $Q_3$. Hence, $T \cup bx \cup P_3 \cup B_1 \cup Q_1 \cup Q_2 \cup (uQ_3z \cup P) \cup (Q_4 \cup z_1x)$ is a $TK_5$ in $G'$ with branch vertices $b, u, w, x, x_1$, where we view $P_3$ as a path in $G$ by identifying $b', w'$ with $b, w$, respectively.

So $R_i = R_j$. Then $a_{i-1} = b$ and $c_{i-1} = b'$. Recall $bw \notin E(G)$ (by (2)). Since $w$ and $w'$ (respectively, $b$ and $b'$) have the same set of neighbors in $G'_A$, it follows from Lemma 2.5.3 that $b_{i-1} = b_i$. Then $\{b, b_i, w, x, z_1\}$ is a cut in $G$ separating $P_1 \cup (P_3 - \{b', w'\})$ from
In this section, we explore the structure of $G$. Suppose $T$ below allows us to assume that if $T \cap C = \emptyset$ then $A \cap C = \emptyset$. The lemma below allows us to assume that if $T \cap C = \emptyset$ then $A \cap C = \emptyset$.

**Lemma 2.7.1** Let $G$ be a 5-connected nonplanar graph and $x \in V(G)$. Suppose for any $H \subseteq G$ with $x \in V(H)$ and $H \cong K_2$ or $H \cong K_3$, $G/H$ is not 5-connected. Let $(T, S_T, A, B) \in Q_x$ with $|V(A)|$ minimum and $(T', S_{T'}, C, D) \in Q_x$ with $T' \cap A \neq \emptyset$. Suppose $T \cap C = \emptyset$. Then $A \cap C = \emptyset$, or one of the following holds:

(i) $G$ contains a $TK_5$ in which $x$ is not a branch vertex.

(ii) $G$ contains $K_4^-$.

(iii) There exist $x_1, x_2, x_3 \in N(x)$ such that for any $y_1, y_2 \in N(x) - \{x_1, x_2, x_3\}$, $G - \{xy : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}$ contains $TK_5$.

**Proof.** We may assume $T \cong K_3$ (by Lemma 2.6.3) and $T' \cong K_3$ (by Lemma 2.6.4). Suppose $A \cap C \neq \emptyset$.

Then $|(S_T \cup S_{T'}) - V(B \cup D)| \geq 7$; otherwise $(T', (S_{T'} \cup S_T) - V(B \cup D), A \cap C, B \cup D) \in Q_x$ and $1 \leq |V(A \cap C)| \leq |V(A - a)| < |V(A)|$, contradicting the choice of $(T, S_T, A, B)$ that $|V(A)|$ is minimum. Hence $|(S_T \cup S_{T'}) - V(A \cup C)| = 5$, as $|S_T| = |S_{T'}| = 6$. Since $T \cap C = \emptyset$, $V(T) \subseteq (S_T \cup S_{T'}) - V(A \cup C)$.

Suppose $|V(B \cap D)| \geq 2$. Then $G$ has a separation $(G_1, G_2)$ such that $V(G_1 \cap G_2) = (S_T \cup S_{T'}) - V(A \cup C)$ and $|V(G_i)| \geq 7$. So the assertion of this lemma follows from Lemma 2.4.6.
Lemma 2.7.2 Let $|V(B \cap D)| \leq 1$. Therefore, by the minimality of $|V(A)|$, $|S_T \cap V(D)| \geq |S_T \cap V(A)|$. But this implies that $|S_T| \geq |(S_T \cup S_T') - V(B \cup D)| \geq 7$, a contradiction.

We need a lemma for finding paths to deal with a special case when $A \cap C = \emptyset$ for quadruples $(T, S_T, A, B), (T', S_{T'}, C, D) \in Q_x$.

**Lemma 2.7.2** Let $G$ be a 5-connected nonplanar graph and $x \in V(G)$, and suppose for any $H \subseteq G$ with $x \in V(H)$ and $H \cong K_2$ or $H \cong K_3$, $G/H$ is not 5-connected. Let $(T, S_T, A, B) \in Q_x$ with $|V(A)|$ minimum and $(T', S_{T'}, C, D) \in Q_x$ with $T' \cap A \neq \emptyset$.

Let $V(T) = \{x, x_1, x_2\}$ and $V(T') = \{x, a, b\}$ with $a \in V(A)$. Suppose $A \cap C = \emptyset$, $|S_T| = 6 = |S_{T'}|$, $V(T) \subseteq V(C)$, $|(S_T \cup S_{T'}) - V(B \cup C)| = 7$, and $(S_T \cup S_{T'}) - V(B \cup C \cup T \cup T') = \{x_3, x_4\}$. Then $G$ contains $K_4^-$, or the following statements hold:

1. $N(b) \cap V(A - a) \neq \emptyset$ and if $t \in N(b) \cap V(A - a)$ then $G[(A - a) + \{b, x_1, x_2, x_3, x_4\}]$ has independent paths from $t$ to $b, x_1, x_2, x_3, x_4$, respectively, and

2. if $b \in S_T$ then $G[A + \{b, x_1, x_2\}]$ has independent paths from $b$ to $x_1, x_2$, respectively.

**Proof.** First, we note that $N(b) \cap V(A - a) \neq \emptyset$. For, otherwise, $(T, (S_T \cup S_{T'}) - V(B \cup C) - \{b\}, A - a, G[B \cup C + b]) \in Q_x$. By the choice of $(T, S_T, A, B)$ that $|V(A)|$ is minimum, we must have $V(A - a) = \emptyset$. So $G$ contains $K_4^-$ by Lemma 2.6.1.

To complete the proof of (i), let $t \in N(b) \cap V(A - a)$. If $G[(A - a) + \{x_1, x_2, x_3, x_4\}]$ has four independent paths from $t$ to $x_1, x_2, x_3, x_4$, respectively, then these four paths and $tb$ give the desired five paths. So we may assume that such four paths do not exist. Then $G[(A - a) + \{x_1, x_2, x_3, x_4\}]$ has a separation $(G_1, G_2)$ such that $|V(G_1 \cap G_2)| \leq 3$, $t \in V(G_1 - G_2)$ and $\{x_1, x_2, x_3, x_4\} \subseteq V(G_2)$. Hence, $(T', V(T') \cup V(G_1 \cap G_2), G_1 - G_2, G - T' - G_1) \in Q_x$ and $1 \leq |V(G_1 - G_2)| \leq |V(A - a)| < |V(A)|$, contradicting the choice of $(T, S_T, A, B)$.

To prove (ii), let $b \in S_T$ and assume that the two paths in (ii) do not exist. Note that if $b \in V(T)$ then $T \cup T'$ contains $K_4^-$. So we may assume $b \notin V(T)$. Then, $G[A + \{b, x_1, x_2\}]$
has a separation \((G_1, G_2)\) such that \(|V(G_1) \cap V(G_2)| \leq 1\), \(b \in V(G_1) - V(G_2)\) and \({x_1, x_2}\) \(\subseteq V(G_2)\). Since \(N(b) \cap V(A - a) \neq \emptyset\) and \(|V(G_1) \cap V(G_2)| \leq 1\), \(|V(G_1 - G_2)| \geq 2\). Let \(S_{bx} = (S_T - \{x_1, x_2\}) \cup V(G_1 \cap G_2)\), and let \(F = G_1 - S_{bx}\). Then \(|V(F)| \geq 1\) as \(|V(G_1 - G_2)| \geq 2\). If \(|V(F)| \geq 2\) then \((bx, S_{bx}, F, G - S_{bx} - F) \in Q_x\) with \(2 \leq |V(F)| < |V(A)|\), contradicting the choice of \((T, S_T, A, B)\) that \(|V(A)|\) is minimum. So assume \(|V(F)| = 1\) and let \(v \in V(F)\). Since \(G\) is 5-connected, \(v\) is adjacent to all vertices in \(S_{bx}\). If \(v \neq a\) then \(V(G_1 \cap G_2) = \{a\}\); so \(G[\{a, b, v, x\}]\) contains \(K_4^-\). Now assume \(v = a\). Let \(w \in V(G_1 \cap G_2)\). Since \(N(b) \cap V(A - a) \neq \emptyset\), \(bw \in E(G)\). So \(G[\{a, b, w, x\}]\) contains \(K_4^-\).

In the next two lemmas, we consider the case when quadruples \((T, S_T, A, B)\) and \((T', S_{T'}, C, D)\) may be chosen so that \(|V(T' \cap A)| = 2\).

**Lemma 2.7.3** Let \(G\) be a 5-connected nonplanar graph and \(x \in V(G)\). Suppose for any \(H \subseteq G\) with \(x \in V(H)\) and \(H \cong K_2\) or \(H \cong K_3\), \(G/H\) is not 5-connected. Let \((T, S_T, A, B) \in Q_x\) with \(|V(A)|\) minimum. Suppose there exists \((T', S_{T'}, C, D) \in Q_x\) such that \(T' \cong K_3\) and \(|V(T' \cap A)| = 2\). Then one of the following holds:

(i) \(G\) contains a TK5 in which \(x\) is not a branch vertex.

(ii) \(G\) contains \(K_4^-\).

(iii) There exist \(x_1, x_2, x_3 \in N(x)\) such that for any \(y_1, y_2 \in N(x) - \{x_1, x_2, x_3\}\), \(G - \{xv : v \notin \{x_1, x_2, x_3, y_1, y_2\}\}\) contains TK5.

(iv) \(|S_T \cap S_T'| = 1\), \(|S_T' \cap V(B)| = 2\), and either \(|S_T \cap V(C)| = 2\) and \(T \cap C = \emptyset\) or \(|S_T \cap V(D)| = 2\) and \(T \cap D = \emptyset\).

**Proof.** We may assume \(T \cong K_3\) (by Lemma 2.6.3). We may also assume that \(|S_T| = |S_T'| = 6\); for, otherwise, (i) or (ii) or (iii) follows from Lemma 2.4.6. We may further assume \(|V(A)| \geq 5\); as otherwise, by Lemma 2.6.1, \(G\) contains \(K_4^-\) and (ii) holds.
Let \( T' = \{a, b, x\} \) with \( a, b \in V(A) \). By symmetry, assume \( T \cap C = \emptyset \). Then, by Lemma 2.7.1, we may assume \( A \cap C = \emptyset \). Now \( B \cap C \neq \emptyset \); for, otherwise, \(|V(C)| = |S_T \cap V(C)| \leq 3\), contradicting the choice of \((T, S_T, A, B)\) that \(|V(A)|\) is minimum. Hence, \( S_T \cap V(C) \neq \emptyset \) as \( S_T' - \{a, b\} \) is not a cut in \( G \). Moreover, \( A \cap D \neq \emptyset \); for otherwise, \(|V(A) \cap S_T'| = 5\) and, hence, \(|S_T' \cap S_T| = 1\) and \(|S_T \cap V(B)| = 0\); so \((S_T \cup S_T') - V(A \cup D)\) is a cut in \( G \) of size at most 4 and separating \( B \cap C \) from \( A \cup D \), a contradiction.

We claim that \(|(S_T \cup S_T') - V(B \cup C)| = 7\) and \(|(S_T \cup S_T') - V(A \cup D)| = 5\). First, note that \(|(S_T \cup S_T') - V(B \cup C)| \geq 7\); otherwise, \((T', (S_T \cup S_T') - V(B \cup C), A \cap D, B \cup C) \in \mathcal{Q}_x\) and \(1 \leq |V(A \cap D)| \leq |V(A - a)| < |V(A)|\), contradicting the choice of \((T, S_T, A, B)\) that \(|V(A)|\) is minimum. Also note that \(|(S_T \cup S_T') - V(A \cup D)| \geq 5\) since \(B \cap C \neq \emptyset\) and \(G\) is 5-connected. Thus the claim follows from the fact that \(|(S_T \cup S_T') - V(B \cup C)| + |(S_T \cup S_T') - V(A \cup D)| = |S_T| + |S_T'| = 12\).

We may assume that \(|S_T \cap V(C)| \neq 1\) or \(|S_T \cap V(A)| \neq 2\). For, suppose \(S_T \cap V(C) = \{c\}\) and \(S_T' \cap V(A) = \{a, b\}\). If \(a, b \in N(c)\) then \(G[T' + c]\) contains \(K^+_4\) and (ii) holds. So by the symmetry between \(a\) and \(b\), we may assume that \(ca \notin E(G)\). Then \((T, (S_T - c) \cup \{b\}, A - b, G[B + c]) \in \mathcal{Q}_x\), contradicting the choice of \((T, S_T, A, B)\) that \(|V(A)|\) is minimum.

We may also assume \(T \cap D \neq \emptyset\); for, otherwise, since \(A \cap D \neq \emptyset\), (i) or (ii) or (iii) follows from Lemma 2.7.1. Therefore, \(S_T \cap V(D) \neq \emptyset\). Note that \(1 \leq |S_T \cap S_T'| \leq 4\), and we distinguish four cases according to \(|S_T \cap S_T'|\).

Suppose \(|S_T \cap S_T'| = 4\). Then \(S_T' \cap V(B) = \emptyset\) and \(|S_T \cap V(C)| = |S_T \cap V(D)| = 1\). Therefore, by the minimality of \(|V(A)|\), \(B \cap D \neq \emptyset\). Hence, \(S_T - V(C)\) is a 5-cut in \(G\) and \(V(T) \subseteq S_T - V(C)\). By the choice of \((T, S_T, A, B)\) that \(|V(A)|\) is minimum, \(|V(B \cap D)| \geq 5\). Now (i) or (ii) or (iii) follows from Lemma 2.4.6.

Consider \(|S_T \cap S_T'| = 3\). Suppose for the moment \(S_T \cap V(B) = \emptyset\). Then \(|S_T \cap V(C)| = 2\) as \(|(S_T \cup S_T') - V(A \cup D)| = 5\). So \(B \cap D = \emptyset\) as otherwise \(S_T - V(C)\) would be a 4-cut in \(G\). However, this implies \(|V(D)| < |V(A)|\), contradicting the choice of
\((T, S_T, A, B)\) that \(|V(A)|\) is minimum. So \(S_{T'} \cap V(B) \neq \emptyset\). Therefore, since \(|S_{T'}| = 6\), we have \(|S_{T'} \cap V(B)| = 1\) and \(S_{T'} \cap V(A) = \{a, b\}\). Since \(|(S_T \cup S_{T'}) - V(A \cup D)| = 5\), \(|S_T \cap V(C)| = 1\). This is a contradiction, as we have \(|S_T \cap V(C)| \neq 1\) or \(|S_{T'} \cap V(A)| \neq 2\).

Now let \(|S_T \cap S_{T'}| = 2\). First, assume \(|S_T \cap V(C)| = 1\). Then \(|S_{T'} \cap V(B)| = 2\) (as \(|(S_T \cup S_{T'}) - V(A \cup D)| = 5\)) and, hence, \(|S_{T'} \cap V(A)| = 2\) (as \(|S_{T'}| = 6\)), a contradiction. So we may assume that \(|S_T \cap V(C)| \geq 2\), which implies \(|S_{T'} \cap V(B)| \leq 1\) as \(|(S_T \cup S_{T'}) - V(A \cup D)| = 5\). Hence, since \(|S_T| = |S_{T'}| = 6\), \(|S_{T'} \cap V(A)| \geq 3\) and \(|S_T \cap V(D)| \leq 2\). Therefore, by the minimality of \(|V(A)|\), \(B \cap D \neq \emptyset\). Thus \((S_T \cap S_{T'}) - V(A \cup C)\) is a 5-cut in \(G\) and contains \(V(T)\). So \(|V(B \cap D)| \geq 5\) by the minimality of \(|V(A)|\). Now (i) or (ii) or (iii) follows from Lemma 2.4.6.

Finally, assume \(|S_T \cap S_{T'}| = 1\). If \(|S_{T'} \cap V(B)| = 2\) then \(|S_T \cap V(C)| = 2\) (as \(|(S_T \cup S_{T'}) - V(A \cup D)| = 5\)); so (iv) holds. If \(|S_{T'} \cap V(B)| = 3\) then \(|S_T \cap V(C)| = 1\) (since \(|(S_T \cup S_{T'}) - V(A \cup D)| = 5\)) and \(S_{T'} \cap V(A) = \{a, b\}\) (as \(|S_{T'}| = 6\)), a contradiction. Hence, we may assume \(|S_{T'} \cap V(B)| \leq 1\). Then \(|S_T \cap V(C)| \geq 3\) (since \(|(S_T \cup S_{T'}) - V(A \cup D)| = 5\), \(|S_{T'} \cap V(A)| \geq 4\), and \(|(S_T \cup S_{T'}) - V(A \cup C)| \leq 4\). Hence, since \(G\) is 5-connected, \(B \cap D = \emptyset\); so \(|V(D)| < |V(A)|\). However, this shows that \((T', S_{T'}, D, C)\) contradicts the choice of \((T, S_T, A, B)\).

Next, we take care of the case when (iv) of Lemma 2.7.3 holds.

**Lemma 2.7.4** Let \(G\) be a 5-connected nonplanar graph and \(x \in V(G)\), and suppose for any \(H \subseteq G\) with \(x \in V(H)\) and \(H \cong K_2\) or \(H \cong K_3\), \(G/H\) is not 5-connected. Let \((T, S_T, A, B) \in \mathcal{Q}_x\) with \(|V(A)|\) minimum and \((T', S_{T'}, C, D) \in \mathcal{Q}_x\) with \(T' \cap A \neq \emptyset\). Suppose \(T \cap C = \emptyset\), \(S_T \cap S_{T'} = \{x\}\) and \(|S_T \cap V(C)| = |S_{T'} \cap V(B)| = 2\). Then one of the following holds:

(i) \(G\) contains a \(TK_5\) in which \(x\) is not a branch vertex.

(ii) \(G\) contains \(K_4^-\).
(iii) There exist \( x_1, x_2, x_3 \in N(x) \) such that, for any \( y_1, y_2 \in N(x) - \{x_1, x_2, x_3\}, G' := G - \{xv : v \notin \{x_1, x_2, x_3, y_1, y_2\}\} \) contains \( TK_5 \).

**Proof.** We may assume \( T \cong K_3 \) (by Lemma 2.6.3) and \( T' \cong K_3 \) (by Lemma 2.6.4). By Lemma 2.6.1, we may assume \( |V(A)| \geq 5 \). We may further assume that \(|S_T| = |S_T'| = 6\); for, otherwise, the assertion follows from Lemma 2.4.6.

Let \( V(T) = \{x, x_1, x_2\}, V(T') = \{x, a, b\}, S_T \cap V(C) = \{p_1, p_2\}, S_T' \cap V(B) = \{c_1, c_2\}, S_T \cap V(A) = \{a, b, q\}, \) and \( S_T \cap V(D) = \{x_1, x_2, w\} \). Since \( T \cap C = \emptyset \), we may assume by Lemma 2.7.1 that \( A \cap C = \emptyset \). Then \( B \cap C \neq \emptyset \) by the minimality of \(|V(A)|\).

We may assume \( N(p_1) \cap V(A) = \{a, q\} \) and \( N(p_2) \cap V(A) = \{b, q\} \). To see this, for \( i \in [2] \), let \( S_i := (S_T - \{p_i\}) \cup (N(p_i) \cap \{a, b, q\}) \) which is a cut in \( G \) and containing \( V(T) \). If \( N(p_i) \cap \{a, b, q\} = \emptyset \) then \( |S_i| = 5 \) and the assertion of this lemma follows from Lemma 2.4.6. If \( |N(p_i)\cap\{a,b,q\}| = 1 \) then \( (T, S_i, A - (N(p_i)\cap\{a,b,q\}), S_i, G[B + p_i]) \in Q_x \), contradicting the choice of \( (T, S_T, A, B) \) that \(|V(A)|\) is minimum. Hence, we may assume that \( |N(p_i)\cap\{a,b,q\}| \geq 2 \) for \( i \in [2] \). We may assume \( \{a, b\} \not\subset N(p_i) \) for \( i \in [2] \); as otherwise, \( G[T' + p_i] \) contains \( K_4^- \) and (ii) holds. Moreover, \( N(p_1)\cap\{a,b,q\} \neq N(p_2)\cap\{a,b,q\} \), as otherwise, \( S := (S_T - \{p_1, p_2\}) \cup (N(p_1) \cap \{a, b, q\}) \) is a cut in \( G \) containing \( V(T) \); so \( (T, S, A - (N(p_1)\cap\{a,b,q\}), G[B + \{p_1, p_2\}] \in Q_x \), contradicting the choice of \( (T, S_T, A, B) \) with \(|V(A)|\) minimum. Hence, we may assume \( N(p_1) \cap V(A) = \{a, q\} \) and \( N(p_2) \cap V(A) = \{b, q\} \).

Note that \( N(x_i) \cap V(B) \neq \emptyset \) for \( i \in [2] \); for, otherwise, \( S := V(T') \cup \{q, x_{3-i}, w\} \) is a cut in \( G \), and \( (T', S, G[(A \cap D) + x_i], G[B + \{p_1, p_2\}] \in Q_x \), contradicting the choice of \( (T, S_T, A, B) \) that \(|V(A)|\) is minimum. Moreover, we may assume \( N(w) \cap V(B) \neq \emptyset \); as otherwise, \( S_T - \{w\} \) is a 5-cut in \( G \) and \( V(T) \subseteq S_T - \{w\} \), and the assertion of this lemma follows from Lemma 2.4.6.

We wish to prove (iii) with \( x_3 = b \). Let \( y_1, y_2 \in N(x) - \{x_1, x_2, x_3\} \) be distinct. Choose \( v \in \{y_1, y_2\} \setminus \{a\} \). We may assume \( v \notin \{p_1, p_2\} \); as otherwise \( G[T' + v] \) contains \( K_4^- \) and (ii) holds. By Lemma 2.7.2, we may choose \( t \in N(b) \cap V(A - a) \)
such that $G[(A-a) + \{b, q, x_1, x_2, w\}]$ has independent paths $P_1, P_2, P_3, P_4, P_5$ from $t$ to $b, x_1, x_2, w, q$ respectively. We distinguish four cases according to the location of $v$.

Case 1. $v \in V(B)$.

Let $W$ be the component of $B$ containing $v$. First, suppose $N(x_i) \cap W \neq \emptyset$ for $i \in \{2\}$. Then there exists $v^* \in V(W)$ such that $G[W + \{x_1, x_2\}]$ has three independent paths from $v^*$ to $v, x_1, x_2$, respectively. Hence by Lemma 2.4.11, $G[W + (S_T - \{x\})]$ has independent paths $Q_1, Q_2, Q_3, Q_4$ from $v^*$ to $v, x_1, x_2, u$, respectively, and internally disjoint from $S_T$, where $u \in S_T - \{x, x_1, x_2\}$. If $u = w$ then $T \cup (P_1 \cup bx) \cup P_2 \cup P_3 \cup (Q_1 \cup v) \cup Q_2 \cup Q_3 \cup (Q_4 \cup P_4)$ is a $TK_5$ in $G'$ with branch vertices $t, v^*, x, x_1, x_2$. If $u = p_i$ for some $i \in \{2\}$ then $T \cup (P_1 \cup bx) \cup P_2 \cup P_3 \cup (Q_1 \cup v) \cup Q_2 \cup Q_3 \cup (Q_4 \cup p_iq \cup P_5)$ is a $TK_5$ in $G'$ with branch vertices $t, v^*, x, x_1, x_2$.

Thus, we may assume that $N(x_1) \cap W = \emptyset$. Since $G$ is 5-connected, $N(x_2) \cap W \neq \emptyset$. So $G[W + (S_T - \{x_1\})]$ has independent paths $Q_1, Q_2, Q_3, Q_4, Q_5$ from $v$ to $x, x_2, w, p_1, p_2$, respectively. Clearly, we may assume that $Q_1 = vx$. Since $N(x_1) \cap V(B) \neq \emptyset$, let $W'$ be a component of $B$ with $N(x_1) \cap V(W') \neq \emptyset$. Since $G$ is 5-connected, there exists $i \in \{2\}$ such that $N(p_i) \cap V(W') \neq \emptyset$. Hence, $G[W' + \{x_1, p_i\}]$ has a path $R$ from $x_1$ to $p_i$, and, by symmetry, assume $R$ is from $x_1$ to $p_1$. Now $T \cup (P_1 \cup bx) \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup (Q_3 \cup P_4) \cup (Q_4 \cup R)$ is a $TK_5$ in $G'$ with branch vertices $t, v, x, x_1, x_2$.

Case 2. $v \in V(A \cap D)$.

First, we show that $G[(A \cap D) + \{q, w, x_1, x_2\}]$ has independent paths $P_1', P_2', P_3', P_4', P_5'$ from $v$ to $q, x, x_1, x_2, w$, respectively (and we may assume that $P_2' = vx$). This is clear if $G[(A \cap D) + \{q, w, x_1, x_2\}]$ has independent paths from $v$ to $q, x_1, x_2, w$, respectively. So we may assume that $G[(A \cap D) + \{q, w, x_1, x_2\}]$ has a separation $(G_1, G_2)$ such that $|V(G_1 \cap G_2)| \leq 3$, $v \in V(G_1 - G_2)$ and $\{q, w, x_1, x_2\} \subseteq V(G_2)$. Then $S := V(T') \cup V(G_1 \cap G_2)$ is a cut in $G$, and $(T', S, G_1 - G_2, G - S - G_1) \in Q_x$, contradicting the choice of $(T, S_T, A, B)$ that $|V(A)|$ is minimum.
Suppose $B$ has a component $W$ such that $N(x_i) \cap W \neq \emptyset$ for $i \in [2]$. Then there exists $z \in V(W)$ such that $G[W + \{x_1, x_2\}]$ has independent paths from $z$ to $x_1, x_2$, respectively. Hence by Lemma 2.4.11, $G[W + (S_T - \{x\})]$ has four independent paths $Q_1, Q_2, Q_3, Q_4$ from $x_1, x_2, u_1, u_2$, respectively, and internally disjoint from $S_T$, where $u_1, u_2 \in \{w, p_1, p_2\}$ are distinct. If $\{u_1, u_2\} = \{w, p_1\}$ then we may assume $u_1 = w$ and $u_2 = p_1$; now $T \cup P_2' \cup P_3' \cup P_4' \cup Q_1 \cup Q_2 \cup (Q_3 \cup P_5') \cup (Q_4 \cup p_1 abx)$ is a $TK_5$ in $G'$ with branch vertices $v, x, x_1, x_2, z$. If $\{u_1, u_2\} = \{w, p_2\}$ then we may assume $u_1 = w$ and $u_2 = p_2$; now $T \cup P_2' \cup P_3' \cup P_4' \cup Q_1 \cup Q_2 \cup (Q_3 \cup P_5') \cup (Q_4 \cup p_2 abx)$ is a $TK_5$ in $G'$ with branch vertices $v, x, x_1, x_2, z$. So assume $\{u_1, u_2\} = \{p_1, p_2\}$. We may further assume $u_i = p_i$ for $i \in [2]$. Then $T \cup P_2' \cup P_3' \cup P_4' \cup Q_1 \cup Q_2 \cup (Q_3 \cup p_1 q \cup P_5') \cup (Q_4 \cup p_2 abx)$ is a $TK_5$ in $G'$ with branch vertices $v, x, x_1, x_2, z$.

Hence, we may assume that no component of $B$ contains neighbors of both $x_1$ and $x_2$. Since $G$ is 5-connected, we may assume by symmetry that $Z$ is a component of $B$ such that $N(x_1) \cap V(Z) = \emptyset$ and $N(x_2) \cap V(Z) \neq \emptyset$. Again, since $G$ is 5-connected, $G[Z + (S_T - \{x_1\})]$ has five independent paths $Q_1, Q_2, Q_3, Q_4, Q_5$ from some $z \in V(Z)$ to $x_2, w, p_1, p_2, x$, respectively. Since $N(x_1) \cap V(B) \neq \emptyset$, let $Z'$ be a component of $B$ with $N(x_1) \cap Z' \neq \emptyset$. Then $N(x_2) \cap V(Z') = \emptyset$. So $G[Z' + \{x_1, p_1\}]$ contains a path $R$ from $x_1$ to $p_1$. Now $T \cup P_2' \cup P_3' \cup P_4' \cup (Q_1 \cup Q_2 \cup (Q_3 \cup R) \cup (Q_2 \cup P_5') \cup (Q_4 \cup p_2 abx)$ is a $TK_5$ in $G'$ with branch vertices $v, x, x_1, x_2, z$.

Case 3. $v = q$.

Suppose $B$ has a component $Z$ such that $\{w, x_1, x_2\} \subseteq N(Z)$. Then there exists $z \in V(Z)$ such that $G[Z + \{w, x_1, x_2\}]$ has independent paths from $z$ to $w, x_1, x_2$, respectively. By Lemma 2.4.11, $G[Z + (S_T - \{x\})]$ has independent paths $Q_1, Q_2, Q_3, Q_4$ from $z$ to $x_1, x_2, w, u$, respectively, and internally disjoint from $S_T$, where $u \in \{p_1, p_2\}$. Let $S = Q_4 \cup p_1 abx$ if $u = p_1$ and $S = Q_4 \cup p_2 abx$ if $u = p_2$. Then $T \cup Q_1 \cup Q_2 \cup S \cup (P_1 \cup Q_3) \cup P_2 \cup P_3 \cup (P_5 \cup qx)$ is a $TK_5$ in $G'$ with branch vertices $t, x, x_1, x_2, z$. 

54
So we may assume that no component of $B$ is adjacent to all of $x_1, x_2$ and $w$. Since $N(w) \cap V(B) \neq \emptyset$, there exists a component $Z$ of $B$ such that $N(w) \cap V(Z) \neq \emptyset$. Since $G$ is 5-connected, we may assume by symmetry that $N(x_2) \cap V(Z) \neq \emptyset$. Then $N(x_1) \cap V(Z) = \emptyset$. Since $G$ is 5-connected, $G[Z + (S_T - \{x_1\})]$ has independent paths $Q_1, Q_2, Q_3, Q_4, Q_5$ from some $z \in V(Z)$ to $x_2, w, p_1, p_2, x$, respectively. Since $N(x_1) \cap V(B) \neq \emptyset$, there exists some component $Z'$ of $B$ with $N(x_1) \cap V(Z') \neq \emptyset$. Hence, $N(x_2) \cap V(Z') = \emptyset$ or $N(w) \cap V(Z') = \emptyset$; so $G[Z' + \{x_1, p_1\}]$ contains a path $R$ from $x_1$ to $p_1$. Now $T \cup Q_1 \cup (Q_3 \cup R) \cup (Q_4 \cup p_2 bx) \cup (P_4 \cup Q_2) \cup P_2 \cup P_3 \cup (P_5 \cup qx)$ is a $TK_5$ in $G'$ with branch vertices $t, x, x_1, x_2, z$.

Case 4. $v = w$.

Suppose $B$ has a component $Z$ such that $\{w, x_1, x_2\} \subseteq N(Z)$. Then there exists $z \in V(Z)$ such that $G[Z + \{w, x_1, x_2\}]$ has three independent paths from $z$ to $w, x_1, x_2$, respectively. Hence, by Lemma 2.4.11, $G[Z + (S_T - \{x\})]$ has independent paths $Q_1, Q_2, Q_3, Q_4$ from $z$ to $x_1, x_2, w, u$, respectively, and internally disjoint from $S_T$, where $u = p_i$ for some $i \in [2]$. Then $T \cup Q_1 \cup Q_2 \cup (Q_3 \cup wx) \cup (P_1 \cup bx) \cup P_2 \cup P_3 \cup (P_5 \cup qp_i \cup Q_4)$ is a $TK_5$ in $G'$ with branch vertices $t, x, x_1, x_2, z$.

Hence, we may assume that no component of $B$ is adjacent to all of $w, x_1, x_2$. Since $N(w) \cap V(B) \neq \emptyset$, $B$ has a component $Z$ such that $N(w) \cap V(Z) \neq \emptyset$. Since $G$ is 5-connected, we may assume by symmetry that $N(x_2) \cap V(Z) \neq \emptyset$. Then $N(x_1) \cap V(Z) = \emptyset$. Since $G$ is 5-connected, $G[Z + (S_T - \{x_1\})]$ has five independent paths $Q_1, Q_2, Q_3, Q_4, Q_5$ from $z$ to $x_2, w, p_1, p_2, x$, respectively. Since $N(x_1) \cap V(B) \neq \emptyset$, $B$ has a component $Z'$ such that $N(x_1) \cap V(Z') \neq \emptyset$. Then $N(x_2) \cap V(Z') = \emptyset$ or $N(w) \cap V(Z') = \emptyset$; so $G[Z' + \{x_1, p_1\}]$ contains a path $R$ from $x_1$ to $p_1$. Now $T \cup Q_1 \cup (Q_2 \cup wx) \cup (Q_3 \cup R) \cup (P_1 \cup bx) \cup P_2 \cup P_3 \cup (P_5 \cup qx)$ is a $TK_5$ in $G'$ with branch vertices $t, x, x_1, x_2, z$. 

We end this section with the following lemma which deals with another special case when $(T, S_T, A, B) \in \mathcal{Q}_x$ with $|V(A)|$ minimum, $(T', S_{T'}, C, D) \in \mathcal{Q}_x$ with $T' \cap A \neq \emptyset$, and $A \cap C = \emptyset$.  

55
Lemma 2.7.5 Let $G$ be a 5-connected nonplanar graph and $x \in V(G)$ such that for any $H \subseteq G$ with $x \in V(H)$ and $H \cong K_2$ or $H \cong K_3$, $G/H$ is not 5-connected. Let $(T, S_T, A, B) \in Q_x$ with $|V(A)|$ minimum, and $(T', S_{T'}, C, D) \in Q_x$ with $T' \cap A \neq \emptyset$. Suppose $A \cap C = \emptyset$, $|S_T| = 6$, $|S_{T'}| = 6$, $V(T') \cap S_T = \{x, b\}$, $V(T' \cap A) = S_{T'} \cap V(A) = \{a\}$ and $V(C) \cap S_T = \emptyset$. Then, one of the following holds:

(i) $G$ contains a $TK_5$ in which $x$ is not a branch vertex.

(ii) $G$ contains $K_4^-$.

(iii) There exist distinct $x_1, x_2 \in N(x)$ such that for any distinct $y_1, y_2 \in N(x) - \{b, x_1, x_2\}$, $G' := G - \{xv : v \notin \{x_1, x_2, b, y_1, y_2\}\}$ contains $TK_5$.

Proof. By assumption, $V(T') = \{a, b, x\}$ with $a \in V(A)$ and $b, x \in S_T \cap S_{T'}$. Let $V(T) = \{x, x_1, x_2\}$ and $S_T = \{b, x, x_1, x_2, x_3, x_4\}$. We wish to prove (iii) with $x_3 = b$; so let $y_1, y_2 \in N(x) - \{b, x_1, x_2\}$ be distinct. Let $v \in \{y_1, y_2\} - \{a\}$.

Note that $B \cap C \neq \emptyset$ as $S_{T'}$ is a cut. So $|(S_T \cup S_{T'}) - V(A \cup D)| \geq 5$. Moreover, we may assume $A \cap D \neq \emptyset$ by Lemma 2.6.1. So $|(S_T \cup S_{T'}) - V(B \cup C)| \geq 7$ by the minimality of $|V(A)|$. Since $|S_T| = |S_{T'}| = 6$,

$$|(S_T \cup S_{T'}) - V(A \cup D)| = 5 \text{ and } |(S_T \cup S_{T'}) - V(B \cup C)| = 7.$$ 

We may assume that $N(x_i) \cap V(B) \neq \emptyset$ for $i \in [2]$. For, suppose this is not true and by symmetry assume $N(x_1) \cap V(B) = \emptyset$. Let $S = (S_T - \{x_1\}) \cup \{a\}$, $C' = B$, and $D' = G[(A - a) + x_1]$. Then $(T', S, C', D') \in Q_x$. We now apply Lemma 2.6.6 to $(T, S_T, A, B)$ and $(T', S, C', D')$. Note that $|S \cap S_T| = 5$, $V(A \cap C') = S_T \cap V(C') = S \cap V(B) = V(B \cap D') = \emptyset$, and $|S \cap V(A)| = |S_T \cap V(D')| = |V(T \cap D')| = 1$. To verify the other condition in Lemma 2.6.6, let $(H, S_H, C_H, D_H) \in Q_x$ such that $H \cong K_2$ or $H \cong K_3$. Then we may assume that $H \cong K_3$ when $H \cap A \neq \emptyset$ (by Lemma 2.6.4) and that $|V(H \cap A)| \leq 1$ (by Lemmas 2.7.3 and 2.7.4). Therefore, the assertion of this lemma.
follows from Lemma 2.6.6. Hence, we may assume \( N(x_i) \cap B \neq \emptyset \) for \( i \in [2] \).

We may assume that for any component \( W \) of \( B \), \( N(b) \cap W \neq \emptyset \); for, otherwise, \( S_T - \{ b \} \) is a 5-cut in \( G \), and the assertion of this lemmas follows from Lemma 2.4.6. We consider three cases according to the location of \( v \).

**Case 1.** \( v \in V(B) \).

Let \( B_v \) be the component of \( B \) containing \( v \). First, suppose \( N(x_i) \cap V(B_v) \neq \emptyset \) for \( i \in [2] \). Then \( G[B_v + \{ x_1, x_2 \}] \) has independent paths from some \( v^* \in V(B_v) \) to \( v, x_1, x_2 \), respectively. Thus, by Lemma 2.4.11, \( G[B_v + S_T - x] \) has independent paths \( P_1, P_2, P_3, P_4 \) from \( v^* \) to \( v, x_1, x_2, u \), respectively, and internally disjoint from \( S_T \), where \( u \in \{ b, x_3, x_4 \} \). Suppose \( u = b \). By Lemma 2.7.2, we may assume that \( G[A + \{ b, x_1, x_2 \}] \) contains independent paths \( R_1, R_2 \) from \( b \) to \( x_1, x_2 \), respectively. Then \( T \cup R_1 \cup R_2 \cup bx \cup (P_1 \cup vx) \cup P_2 \cup P_3 \cup P_4 \) is a \( TK_5 \) in \( G' \) with branch vertices \( b, v^*, x, x_1, x_2 \). So we may assume by symmetry that \( u = x_3 \). By Lemma 2.7.2 again, we may choose \( t \in N(b) \cap V(A-a) \) and let \( Q_1, Q_2, Q_3, Q_4, Q_5 \) be independent paths in \( G[(A-a)+\{ b, x_1, x_2, x_3, x_4 \}] \) from \( t \) to \( b, x_1, x_2, x_3, x_4 \), respectively. Then, \( T \cup (Q_1 \cup bx) \cup Q_2 \cup Q_3 \cup (P_1 \cup vx) \cup P_2 \cup P_3 \cup (P_4 \cup Q_4) \) is a \( TK_5 \) in \( G' \) with branch vertices \( t, v^*, x, x_1, x_2 \).

Therefore, we may assume by symmetry that \( N(x_1) \cap V(B_v) = \emptyset \). Since \( G \) is 5-connected, \( G[B_v + S_T - x_1] \) has independent paths \( P_1, P_2, P_3, P_4, P_5 \) from \( v \) to \( b, x_2, x_3, x_4 \), respectively, and we may assume that \( P_1 = vx \). Since \( N(x_1) \cap V(B) \neq \emptyset \), \( B \) has a component \( B_{x_1} \) such that \( N(x_1) \cap V(B_{x_1}) \neq \emptyset \). Again, since \( G \) is 5-connected, \( N(x_j) \cap V(B_{x_j}) \neq \emptyset \) for some \( j \in \{ 3, 4 \} \), and we may assume \( j = 3 \). Then \( G[B_{x_1} + \{ x_1, x_3 \}] \) contains a path \( Q \) from \( x_1 \) to \( x_3 \). Let \( t \in N(b) \cap V(A-a) \). By Lemma 2.7.2, we may assume that \( G[(A-a)+\{ b, x_1, x_2, x_3, x_4 \}] \) has independent paths \( Q_1, Q_2, Q_3, Q_4, Q_5 \) from \( t \) to \( b, x_1, x_2, x_3, x_4 \), respectively. Then \( T \cup (Q_1 \cup bx) \cup Q_2 \cup Q_3 \cup (P_5 \cup Q_5) \cup (P_4 \cup Q) \cup P_1 \cup P_3 \) is a \( TK_5 \) in \( G' \) with branch vertices \( t, v, x, x_1, x_2 \).

**Case 2.** \( v \in V(A \cap D) \).
We claim that $G[\{A - a\} + \{x, x_1, x_2, x_3, x_4\}]$ has independent paths $P_1, P_2, P_3, P_4, P_5$ from $v$ to $x, x_1, x_2, x_3, x_4$, respectively (and we may assume $P_1 = vx$). This is clear if $G[(A - a) + \{x_1, x_2, x_3, x_4\}]$ has independent paths from $v$ to $x_1, x_2, x_3, x_4$, respectively; so we may assume such paths do not exist. Then there exists a separation $(G_1, G_2)$ in $G[(A - a) + \{x_1, x_2, x_3, x_4\}]$ such that $|V(G_1 \cap G_2)| \leq 3, v \in V(G_1 - G_2)$, and $\{x_1, x_2, x_3, x_4\} \subseteq V(G_2)$. Let $S := V(G_1 \cap G_2) \cup V(T')$, which is a cut in $G$ of size at most 6. Since $G$ is 5-connected, $|V(G_1 \cap G_2)| \geq 2$. Then, $(T', S, G_1 - G_2, (G - S) - G_1) \in \mathcal{Q}_x$ and $1 \leq |V(G_1 - G_2)| \leq |V(A - a)| < |V(A)|$, contradicting the choice of $(T, S_T, A, B)$ that $|V(A)|$ is minimum.

Suppose that $B$ has a component $W$ such that $N(x_i) \cap V(W) \neq \emptyset$ for $i \in [2]$. Then there exists $w \in V(W)$ such that $G[W + b]$ has independent paths from $w$ to $x_1, x_2, b$, respectively. By Lemma 2.4.11, $G[B + S_T]$ has independent paths $Q_1, Q_2, Q_3, Q_4, Q_5$ from $w$ to $x_1, x_2, b, u_1, u_2$, respectively, and internally disjoint from $S_T$, where $u_1, u_2 \in \{x, x_3, x_4\}$ are distinct. By symmetry, we may assume $u_1 = x_3$. Then $T \cup P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup (Q_3 \cup bx) \cup (Q_4 \cup P_4)$ is a $TK_5$ in $G'$ with branch vertices $v, w, x, x_1, x_2$.

Hence, we may assume that no component of $B$ is adjacent to both $x_1$ and $x_2$. Let $W$ be a component of $B$ such that $N(x_2) \cap V(W) \neq \emptyset$. Then $N(x_1) \cap V(W) = \emptyset$. Since $G$ is 5-connected, $G[W + S_T - x_1]$ has independent paths $Q_1, Q_2, Q_3, Q_4, Q_5$ from some $w \in V(W)$ to $b, x_2, x_3, x_4, x$, respectively. Since $N(x_1) \cap V(B) \neq \emptyset, B$ has a component $B_x$ such that $N(x_1) \cap V(B_x) \neq \emptyset$. Then $N(x_2) \cap V(B_x) = \emptyset$. Again, since $G$ is 5-connected, $G[B_x + \{x_1, x_3\}]$ contains a path $R$ from $x_1$ to $x_3$. Now $T \cup P_1 \cup P_2 \cup P_3 \cup (Q_1 \cup bx) \cup Q_2 \cup (Q_3 \cup R) \cup (Q_4 \cup P_5)$ is a $TK_5$ in $G'$ with branch vertices $v, w, x, x_1, x_2$.

**Case 3.** $v \in S_T$.

We may assume that $v = x_3$. By Lemma 2.7.2, we may assume $t \in N(b) \cap V(A - a)$ and $G[(A - a) + \{b, x_1, x_2, x_3, x_4\}]$ has independent paths $P_1, P_2, P_3, P_4, P_5$ from $t$ to $b, x_1, x_2, x_3, x_4$, respectively, with $P_1 = tb$. Also by Lemma 2.7.2, we may assume that $G[A + \{b, x_1, x_2\}]$ has independent paths $Q_1, Q_2$ from $b$ to $x_1, x_2$, respectively.

58
Suppose $B$ has a component $W$ such that $\{x_1, x_2\} \subseteq N(W)$. Then there exists $w \in V(W)$ such that $G[W + \{b, x_1, x_2\}]$ has independent paths from $w$ to $b, x_1, x_2$, respectively. So by Lemma 2.4.11, $G[B + S_T]$ has independent paths $R_1, R_2, R_3, R_4, R_5$ from $w$ to $x_1, x_2, b, u_1, u_2$, respectively, and internally disjoint from $S_T$, where $u_1, u_2 \in \{x, x_3, x_4\}$ are distinct. Assume by symmetry that $u_2 \in \{x_3, x_4\}$. If $u_1 = x_3$, then $T \cup bx \cup Q_1 \cup Q_2 \cup R_1 \cup R_2 \cup R_3 \cup (R_4 \cup x_3)$ is a TK$_5$ in $G'$ with branch vertices $b, w, x, x_1, x_2$. If $u_1 = x_4$, then $T \cup (P_4 \cup x_3) \cup P_2 \cup P_3 \cup R_1 \cup R_2 \cup (R_3 \cup bx) \cup (R_4 \cup P_5)$ is a TK$_5$ in $G'$ with branch vertices $t, w, x, x_1, x_2$.

Thus, we may assume that no component of $B$ is adjacent to both $x_1$ and $x_2$. Since $G$ is 5-connected, we may assume by symmetry that $W$ is a component of $B$ such that $N(x_2) \cap V(W) \neq \emptyset$ and $N(x_1) \cap V(W) = \emptyset$. Let $w \in V(W)$. Since $G$ is 5-connected, $G[W + S_T - x_1]$ has independent paths $R_1, R_2, R_3, R_4, R_5$ from $w$ to $x, x_2, x_3, x_4, b$, respectively. Since $N(x_1) \cap B \neq \emptyset$, $B$ has a component $B_x$ such that $N(x_1) \cap V(B_x) \neq \emptyset$. Then $N(x_2) \cap V(B_x) = \emptyset$. Since $G$ is 5-connected, $G[B_x + \{x_1, x_4\}]$ contains a path $R$ from $x_1$ to $x_4$. Now $T \cup bx \cup Q_1 \cup Q_2 \cup R_2 \cup (R_3 \cup x_3) \cup R_5 \cup (R_4 \cup R)$ is a TK$_5$ in $G'$ with branch vertices $b, w, x, x_1, x_2$.

2.8 Proof of Theorem 1.2.1

In this section, we complete the proof of Theorem 1.2.1, using the lemmas we have proved so far. Let $G$ be a 5-connected nonplanar graph. We proceed to find a TK$_5$ in $G$. By Lemma 2.4.1, we may assume that

(1) $G$ contains no $K_4^-$.

Let $M$ denote a maximal connected subgraph of $G$ such that

$$H := G/M$$

is 5-connected and nonplanar, and contains no $K_4^-$. 

59
Note that $|V(M)| = 1$ (i.e., $H = G$) is possible. Let $x$ denote the vertex of $H$ resulting from the contraction of $M$. Then, for any $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, one of the following holds:

$$H/T \text{ contains } K_4^-, \text{ or } H/T \text{ is planar, or } H/T \text{ is not } 5\text{-connected}.$$  

For convenience, we will use $x_T$ to denote the vertex of $H/T$ resulting from the contraction of $T$. We may assume that

(2) for any $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, if $F$ is a $TK_5$ in $H/T$ then $x_T$ is a branch vertex of $F$.

For, suppose that $F$ is a $TK_5$ in $H/T$ in which $x_T$ is not a branch vertex. If $x_T \notin V(F)$ then $F$ is also $TK_5$ in $G$. So assume $x_T \in V(T)$. Let $u, v \in V(F)$ such that $x_T u, x_T v \in E(F)$. Since $M$ is connected, $G[M + \{u, v\}]$ has a path $P$ from $u$ to $v$. Thus, $(F - x) \cup P$ is a $TK_5$ in $G$. So we may assume (2).

Suppose there exists $T \subseteq V(H)$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, such that $H/T$ is 5-connected and planar. Then by Lemma 2.4.9, $H - T$ contains $K_4^-$, contradicting (1). So

(3) for any $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, if $H/T$ is 5-connected then $H/T$ is nonplanar.

We now show that

(4) if $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$ and if $x_1, x_2, x_3 \in N_{H/T}(x_T)$ such that $H/T - \{x_Tv : v \notin \{u_1, u_2, x_1, x_2, x_3\}\}$ contains $TK_5$ for every choice of distinct $u_1, u_2 \in N_{H/T}(x_T) - \{x_1, x_2, x_3\}$, then $G$ contains $TK_5$.

To prove (4), let $A = N_G(M \cup T) = N_{H/T}(x_T)$. Consider the subgraph $G[M \cup T + A]$. Since $M \cup T$ is connected, there is a vertex $v \in V(M \cup T)$ such that $G[M \cup T + \{x_1, x_2, x_3\}]$
has independent paths from $v$ to $x_1, x_2, x_3$, respectively. Since $G$ is 5-connected, $G[M \cup T + A]$ has five independent paths from $v$ to $A$ with only $v$ in common and internally disjoint from $A$. Hence, by Lemma 2.4.11, there exist distinct $u_1, u_2 \in A - \{x_1, x_2, x_3\}$ such that $G[M \cup T + A]$ has five independent paths $P_1, P_2, P_3, P_4, P_5$ from $v$ to $x_1, x_2, x_3, u_1, u_2$, respectively, and internally disjoint from $A$. Now suppose $F$ is a $TK_5$ in $H/T - \{x_T v : v \notin \{x_1, x_2, x_3, u_1, u_2\}\}$. Then $F - x_T$ and the four paths among $P_1, P_2, P_3, P_4, P_5$ corresponding to the four edges at $x_T$ in $F$ form a $TK_5$ in $G$. Hence, we may assume (4).

By (3), we have two cases: for some $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, $H/T$ is 5-connected and nonplanar but contains $K_4^{-}$; or for every $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, $H/T$ is not 5-connected.

Case 1. There exists $T \subseteq H$ with $x \in V(T)$ and $T \cong K_2$ or $T \cong K_3$ such that $H/T$ is 5-connected and nonplanar, and $H/T$ contains $K_4^{-}$.

Let $K \subseteq H/T$ such that $K \cong K_4^{-}$, and let $V(K) = \{x_1, x_2, y_1, y_2\}$ with $y_1y_2 \notin E(H)$.

By (1), $x_T \in V(K)$.

Subcase 1.1. $x_T$ has degree 2 in $K$.

Then we may assume that the notation is chosen so that $x_T = y_2$. By Lemma 2.4.2, one of the following holds:

(i) $H/T$ contains a $TK_5$ in which $x_T$ is not a branch vertex.

(ii) $H/T - x_T$ contains $K_4^{-}$.

(iii) $H/T$ has a 5-separation $(G_1, G_2)$ such that $V(G_1 \cap G_2) = \{x_T, a_1, a_2, a_3, a_4\}$, and $G_2$ is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding $x_T$ and the edges $x_Tb_i$ for $i \in [4]$.

(iv) For $w_1, w_2, w_3 \in N_{H/T}(x_T) - \{x_1, x_2\}$, $H/T - \{x_T v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$ contains $TK_5$.

Note that (i) does not occur because of (2), and (ii) does not occur because of (1).
Now suppose \((iii)\) occurs. First, assume \(|V(G_1)| \geq 7\). Then by Lemma 2.4.3, for any \(u_1, u_2 \in N(x_T) - \{b_1, b_2, b_3\}\), \(H/T - \{x_Tv : v \notin \{b_1, b_2, b_3, u_1, u_2\}\}\) contains \(TK_5\). Hence, by (4) (with \(x_i\) as \(b_i\) for \(i \in [3]\)), \(G\) contains \(TK_5\). So we may assume that \(|V(G_1)| = 6\), and let \(v \in V(G_1 - G_2)\). By (1), \(a_ia_{i+1} \notin E(G)\) for \(i \in [4]\), where \(a_5 = a_1\). Hence, since \(G\) is 5-connected, \(a_1a_3, a_2a_4 \in E(G)\). Now \((H - x_T) - \{a_1v, a_1b_4, a_4v, a_4b_4\}\) is a \(TK_5\) with branch vertices \(a_2, a_3, b_1, b_2, b_3\), contradicting (2).

Finally, suppose \((iv)\) holds. Then, by (4) (with \(w_1, w_2, w_3\) as \(x_3, u_1, u_2\), respectively), we see that \(G\) contains \(TK_5\).

Subcase 1.2. \(x_T\) has degree 3 in \(K\).

Then we may assume that the notation is chosen so that \(x_T = x_1\). By Lemma 2.4.4, one of the following holds:

(i) \(H/T\) contains a \(TK_5\) in which \(x_T\) is not a branch vertex.

(ii) \(H/T - x_T\) contains \(K_4^-\), or \(H/T\) contains a \(K_4^-\) in which \(x_T\) is of degree 2.

(iii) \(x_2, y_1, y_2\) may be chosen so that for any distinct \(z_0, z_1 \in N_{H/T}(x_T) - \{x_2, y_1, y_2\}\), \(H/T - \{x_Tv : v \notin \{z_0, z_1, x_2, y_1, y_2\}\}\) contains \(TK_5\).

By (2), (i) does not occur. If (ii) holds then, by (1), \(H/T\) contains \(K_4^-\) in which \(x_T\) is of degree 2; and we are back in Subcase 1.1. If (iii) holds then \(G\) contains \(TK_5\) by (4).

Case 2. \(H/T\) is not 5-connected for each \(T \subseteq H\) with \(x \in V(T)\) and \(T \cong K_2\) or \(T \cong K_3\).

Let \(Q_x\) denote the set of all quadruples \((T, S_T, A, B)\), such that

- \(T \subseteq V(H), x \in V(T),\) and \(T \cong K_2\) or \(T \cong K_3\),

- \(S_T\) is a cut in \(H\) with \(V(T) \subseteq S_T\), \(A\) is a nonempty union of components of \(H - S_T\), and \(B = H - S_T - A \neq \emptyset\),

- if \(T \cong K_3\) then \(5 \leq |S_T| \leq 6\), and
• if $T \cong K_2$ then $|S_T| = 5$, $|V(A)| \geq 2$, and $|V(B)| \geq 2$.

Among all the quadruples in $Q_x$, we select $(T, S_T, A, B)$ such that $|V(A)|$ is minimum.

Since $K_4^- \not\subset H$, $T \cong K_3$ (by Lemma 2.6.3) and there exists $a \in V(A)$ such that $ax \in E(H)$ (by Lemma 2.6.5 and by (2) and (4)). By Lemma 2.6.2, there exists $T' \subseteq H$ such that $x \in V(T')$ and $T' \cong K_2$ or $T' \cong K_3$, and there exists $(T', S_{T'}, C, D) \in Q_x$. Again since $K_4^- \not\subset H$, $T' \cong K_3$ by Lemma 2.6.4 and by (2) and (4).

We may assume, without loss of generality, that $T \cap C = \emptyset$. Hence, by Lemma 2.7.1 and by (2) and (4), $A \cap C = \emptyset$ (since $K_4^- \not\subset H$). We may assume $B \cap C \neq \emptyset$; for otherwise, $|V(A)| \leq |V(C)| = |V(C) \cap S_T| \leq 3$ and, by Lemma 2.6.1, $H$ contains $K_4^-$, a contradiction.

We may assume that $|V(T') \cap S_T| = 2$ for any choice of $(T', S_{T'}, C, D) \in Q_x$ with $T' \cap A \neq \emptyset$; otherwise, by Lemmas 2.7.3 and 2.7.4, we derive a contradiction to (2), or (4), or the fact $K_4^- \not\subset H$. Hence, since $K_4^- \not\subset H$, we have $A \cap D \neq \emptyset$ by Lemma 2.6.1.

Note that $|S_T| = |S_{T'}| = 6$; for otherwise, by Lemma 2.4.6, we derive a contradiction to (2), or (4), or the fact $K_4^- \not\subset H$. We claim that $|(S_T \cup S_{T'}) - V(B \cup C)| = 7$ and $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$. First, note that $|(S_T \cup S_{T'}) - V(B \cup C)| \geq 7$; otherwise, $(T', (S_T \cup S_{T'}) - V(B \cup C), A \cap D, G[B \cup C]) \in Q_x$ and $1 \leq |V(A \cap D)| < |V(A)|$, contradicting the choice of $(T, S_T, A, B)$ with $|V(A)|$ minimum. Since $H$ is 5-connected and $B \cap C \neq \emptyset$, $|(S_T \cup S_{T'}) - V(A \cup D)| \geq 5$. So the claim follows from the fact that $|(S_T \cup S_{T'}) - V(B \cup C)| + |(S_T \cup S_{T'}) - V(A \cup D)| = |S_T| + |S_{T'}| = 12$.

If $S_T \cap V(C) = \emptyset$ for some choice $(T', S_{T'}, C, D)$ then $|S_T \cap V(A)| = 1$ as $|S_{T'}| = 6$ and $|(S_T \cup S_{T'}) - V(A \cup D)| = 5$; so by Lemma 2.7.5, we derive a contradiction to (2), or (4), or the fact $K_4^- \not\subset H$.

Hence, we may assume that

$$S_T \cap V(C) \neq \emptyset$$
for any choice of \((T', S_{T'}, C, D) \in Q_x\) with \(T' \cap A \neq \emptyset\). Then \(2 \leq |S_T \cap S_{T'}| \leq 4\) as 
\(|(S_T \cup S_{T'}) - V(A \cup D)| = 5\).

Suppose \(|S_T \cap S_{T'}| = 4\). Then \(|S_T \cap V(B)| = 0\) and \(|S_T \cap V(C)| = 1\), as \(|(S_T \cup S_{T'}) - V(A \cup D)| = 5\). Since \(|S_T| = |S_{T'}| = 6\), \(|S_T \cap V(D)| = 1\) and \(|S_{T'} \cap V(A)| = 2\). Hence, \(B \cap D \neq \emptyset\) (since \(|V(D)| \geq |V(A)|\)). So \(S_T - V(C)\) is a 5-cut in \(H\) and \(V(T') \subseteq S_T - V(C)\). Note \(|V(B \cap D)| \geq 2\); for otherwise, since \(H\) is 5-connected, \(H[T \cup (B \cap D)]\) contains \(K^5_4\), a contradiction. Hence, by Lemma 2.4.6, we derive a contradiction to (2), or (4), or the fact \(K^5_4 \notin H\).

Now assume \(|S_T \cap S_{T'}| = 3\). Then, \(|S_T \cap V(B)| \leq 1\) as \(|(S_T \cup S_{T'}) - V(A \cup D)| = 5\) and \(|S_T \cap V(C)| > 0\). Suppose \(|S_T \cap V(B)| = 0\). Then \(|S_T \cap V(A)| = 3\) as \(|S_{T'}| = 6\). So \(|S_T \cap V(D)| = 1\) since \(|(S_T \cup S_{T'}) - V(B \cup C)| = 7\). Thus, since \(H\) is 5-connected, \(B \cap D = \emptyset\). However, this implies that \(|V(D)| < |V(A)|\), a contradiction. So \(|S_T \cap V(B)| = 1\).

Then \(|S_T \cap V(A)| = 2\) as \(|S_{T'}| = 6\), and \(|S_T \cap V(C)| = 1\) as \(|(S_T \cup S_{T'}) - V(A \cup D)| = 5\). Let \(q \in S_T \cap V(A - T')\), \(S' := (S_T - \{q\}) \cup (S_T \cap V(C))\), \(C' := B \cap C\), and \(D' = G[D + q]\).

Then \((T', S', C', D') \in Q_x\) with \(T' \cap A \neq \emptyset\) and \(T \cap C' = \emptyset\). However, \(S_T \cap V(C') = \emptyset\), a contradiction. Finally, assume \(|S_T \cap S_{T'}| = 2\). Suppose \(|S_T \cap V(C)| \geq 2\). Then \(|S_T \cap V(B)| \leq 1\) (as \(|(S_T \cup S_{T'}) - V(A \cup D)| = 5\)), and \(|S_T \cap V(A)| \geq 3\) (as \(|S_{T'}| = 6\)). So \(B \cap D \neq \emptyset\) as \(|V(D)| \geq |V(A)|\). Hence, \((S_T \cup S_{T'}) - V(A \cup C)\) is a 5-cut in \(H\) and contains \(V(T)\). If \(|V(B \cap D)| = 1\) then, since \(H\) is 5-connected, \(H[T \cup (B \cap D)]\) contains \(K^5_4\), a contradiction. So \(|V(B \cap D)| \geq 2\). Then, by Lemma 2.4.6, we derive a contradiction to (2), or (4), or the fact \(K^5_4 \notin H\). Therefore, we may assume \(|S_T \cap V(C)| = 1\). Hence, \(|S_T \cap V(D)| = 3\) (as \(|S_T| = 6\), \(|S_T \cap V(B)| = 2\) (as \(|(S_T \cup S_{T'}) - V(A \cup D)| = 5\)), and \(|S_T \cap V(A)| = 2\) (as \(|S_{T'}| = 6\)). Let \(q \in S_T \cap V(A - T')\), \(S' := (S_T - \{q\}) \cup (S_T \cap V(C))\), \(C' := B \cap C\), and \(D' = G[D + q]\). Then \((T', S', C', D') \in Q_x\) with \(T' \cap A \neq \emptyset\) and \(T \cap C' = \emptyset\). However, \(S_T \cap V(C') = \emptyset\), a contradiction.
REFERENCES


VITA

Yan Wang is born in Shanghai, China in 1990. Between 2006 and 2010, he studied Information Security at Shanghai Jiao Tong University. He received a Diplôme de l’Ecole polytechnique from Ecole Polytechnique in France in 2013. He is a Ph.D. candidate in Algorithms, Combinatorics and Optimization at Georgia Institute of Technology.