Title:
Scattering and Excitation of Acoustic Waves by Turbulent Flames

Project Participants

Senior Personnel
Name: Lieuwen, Tim
Worked for more than 160 Hours: Yes
Contribution to Project:

Post-doc
Name: Wu, Lei
Worked for more than 160 Hours: Yes
Contribution to Project:
Performed theoretical analyses of flame-acoustic wave interactions.

Graduate Student
Name: Cho, Ju
Worked for more than 160 Hours: Yes
Contribution to Project:
Mr. Cho developed the theory for wave scattering from turbulent flames

Undergraduate Student

Technician, Programmer

Other Participant

Research Experience for Undergraduates

Organizational Partners

Other Collaborators or Contacts
We have interacted with various companies that will ultimately be end-users of models developed as a result of this work. These companies include General Electric, Pratt&Whitney, Gas Research Institute, and Solar Turbines.

Activities and Findings

Research and Education Activities:
We have performed theoretical analyses of acoustic wave/flame interactions. The key accomplishments have been developing transfer functions that describe the response of flames to equivalence ratio oscillations, a model that is now widely used by industry. In addition, we have developed an analytical approach and obtained the first results for a first-principles (as opposed to phenomenological) based description of sound wave interactions with turbulent flames. This work has been performed in close coordination with another NSF project that is experimentally investigating the generation of sound by flames
Findings: (See PDF version submitted by PI at the end of the report)
The major research findings are detailed in the attached document.

Training and Development:
This project has enabled the participants to learn new research skills, by requiring the understanding of complex theory developed in the past for understanding radar and acoustic scattering from rough surfaces. It has also provided teaching opportunities for Lei Wu, as he has had several opportunities at Georgia Tech to present his work to other graduate students in seminars. Finally, it has supported the PhD training of one graduate student.

Outreach Activities:
We have visited a local partner high school and described what engineers do and how mathematics, physics, and science are needed to solve many very interesting problems. We have also worked with local high school teachers and supported their employment in our lab for the summer. Finally, we have hosted one high school student to work in our lab.

Journal Publications


Books or Other One-time Publications
Editor(s): T. Lieuwen, V. Yang

Editor(s): T. Lieuwen and V. Yang
Collection: Combustion Instabilities in Gas Turbine Engines

Web/Internet Site

Other Specific Products

Contributions within Discipline:
This work has enabled the development of analytical techniques to model acoustic-wave flame interactions that fully incorporates modern turbulent combustion theory. As such, it enables the analysis of this and/or related problems from first principles, as opposed to highly phenomenological approaches. Significantly, the models developed under this program have been widely incorporated by industry into internal predictive tools.
Contributions to Other Disciplines:
This work has extended techniques for analysis of acoustic scattering from rough surfaces that are not passive.

Contributions to Human Resource Development:
This project has financially supported the research of a post-doctoral fellow and graduate research assistant.

Contributions to Resources for Research and Education:
Developed best practices for utilization of high school teachers and students in our combustion lab.

Contributions Beyond Science and Engineering:
We have outreached to a local high school to educate them about the exciting opportunities in the engineering field. We have also mentored high school teachers and students who have worked with us in our lab.

Categories for which nothing is reported:

Organizational Partners
Any Web/Internet Site
Any Product
# TABLE OF CONTENTS

NOMENCLATURE ...............................................................................................................1

CHAPTER 1  BACKGROUND ANALYSIS ....................................................................4
   1.1  PROBLEM STATEMENT AND BASIC ASSUMPTIONS ..............................................4
   1.2  LINEARIZED WAVE EQUATIONS .........................................................................5
   1.3  EVALUATION OF SCATTERING AMPLITUDES.....................................................7

CHAPTER 2  STOCHASTIC ANALYSIS OF SCATTERED FIELDS .....................15
   2.1  FORMULATION OF AVERAGED ACOUSTIC ENERGY FLUX ...............................16
   2.2  EVALUATION OF ACOUSTIC ENERGY FLUX OF SCATTERED FIELDS ...............18
   2.3  BUDGET OF NET ENERGY FLUXES .................................................................22
   2.4  CALCULATION OF NET ENERGY FLUX USING GAUSSIAN STATISTICS ...........25
   2.5  RESULTS AND DISCUSSION ............................................................................32

REFERENCES .....................................................................................................................40
NOMENCLATURE

\( c \) speed of sound

\( D \) transmission coefficient

\( \hat{e}_x, \hat{e}_y, \hat{e}_z \) unit vectors in Cartesian coordinates

\( E \) normalized acoustic energy flux

\( \Delta E \) net acoustic energy flux

\( f \) frequency

\( f_c \) frequency of flame surface oscillation (:= \( 1/T_c \))

\( \tilde{f}_0 \) dimensionless frequency of an incident wave (:= \( \Omega_0 t_c / (2\pi) \))

\( h \) flame front position

\( I \) acoustic intensity

\( k \) horizontal components of wave number vector (:= \( (k_x, k_y, k_z) = k_x \hat{e}_x + k_y \hat{e}_y + k_z \hat{e}_z \))

\( \vec{K} \) wave number vector (:= \( (k_x, k_y, k_z) = k_x \hat{e}_x + k_y \hat{e}_y + k_z \hat{e}_z \))

\( l_c \) correlation length of flame front

\( \tilde{l}_c \) dimensionless correlation length of flame front (:= \( K_0 l_c \))

\( M_S \) mean flame speed mach number (:= \( \bar{S}_L^{(1)} / c_i \))

\( \bar{n} \) vector normal to the flame surface

\( p \) acoustic pressure

\( P_i \) incident acoustic pressure

\( q_k \) vertical component of wave number vector
\( r \)  horizontal coordinates \( (= (x, y)) \)

\( R \)  reflection coefficient

\( \vec{R} \)  3 dimensional coordinates \( (= (x, y, z)) \)

\( S_{N_1, N_2} \)  scattering amplitude (from medium \( N_2 \) into medium \( N_1 \))

\( S_L \)  laminar flame speed for unburned gas

\( t \)  time

\( t_c \)  correlation time of flame front

\( \bar{v} \)  acoustic velocity

\( W \)  power spectral density

\( \bar{W} \)  correlation function

\( x, y, z \)  Cartesian coordinates

\( < > \)  ensemble average

**Greeks**

\( \beta \)  jump factor in acoustic velocity

\( \delta( ) \)  dirac delta function

\( \delta( ) \)  kronecker delta function

\( \varepsilon \)  dimensionless magnitude of linearized incident pressure \( (=|P_i|/(\rho_i c_i^2)) \)

\( \phi \)  polar angle

\( \Lambda \)  temperature ratio \( (=T_2/T_1) \)

\( \Delta \)  laplacian \( (=\partial_{xx} + \partial_{yy} + \partial_{zz}) \)

\( \nabla \)  horizontal gradient \( (= (\partial_x, \partial_y) = \vec{e}_x \partial_x + \vec{e}_y \partial_y) \)
\( \nabla \) three-dimensional gradient \( (= (\partial_x, \partial_y, \partial_z)) = \vec{e}_x \partial_x + \vec{e}_y \partial_y + \vec{e}_z \partial_z) \)

\( \theta \) dimensionless activation energy

\( \rho \) density

\( \sigma \) rms height or scattering cross section

\( \bar{\sigma} \) dimensionless rms height of flame front \( (= K_0 \sigma) \)

\( \tau \) the ratio of reference time to correlation time \( (= t_r / t_c) \)

\( \omega \) angular frequency

\( \psi \) velocity potential

**Superscripts**

\( (\cdot)' \) fluctuating value or integral variable

\( (\cdot) \) normalized value

\( (\cdot) \) three-dimensional vector

(1),(2) medium (1) and (2), respectively

**Subscripts**

0 incident wave or zero-th order term

1,2 medium (1) and (2) or 1st and 2nd order perturbation terms

av average value

c complex value

co coherent energy flux

I incident wave or medium containing it
inc  incoherent energy flux
J  velocity jump effect
n n-th order perturbation term
R  reflection
T  transmission
total total acoustic energy flux
ω  wrinkling effect

CHAPTER 1  BACKGROUND ANALYSIS

1.1  Problem statement and basic assumptions

Figure 1 illustrates the schematics of flame surface and acoustic fields where plane incident waves impinge obliquely upon a corrugated flame surface and, subsequently, scattered waves are reflected and transmitted by the interaction with the flame surface.
Several assumptions are made to render the theoretical approach tractable.

1) The heights and slopes of a corrugated flame surface are small

2) Mean flame extends without boundary. This assumption is justified since the present analysis does not deal with any edge effects such as diffraction from the flame edges.

3) Multiple scattering is not considered, i.e., all scattering is assumed to be single scattering.

4) The flame front is temperature discontinuity that separates the unburned reactants and burned products.

1.2 Linearized wave equations

The time-dependent acoustic wave fields are described by linearized wave equation in time domain.

\[
\left( \Delta - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) p(\mathbf{R},t) = -\rho_0 \frac{\partial Q(\mathbf{R},t)}{\partial t} \quad (1.1)
\]

where \( \Delta = \partial_{xx} + \partial_{yy} + \partial_{zz} \). We define \( \psi \) as the acoustic velocity potential:

\[
\mathbf{v} = \nabla \psi(\mathbf{R},t) \quad (1.2)
\]

where \( \nabla = (\partial_x, \partial_y, \partial_z) \). Then it follows from linearized Euler’s equation,

\[
\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p, \quad \text{that the acoustic pressure is also expressed in terms of } \psi:\n\]

\[
p = -\rho_0 \frac{\partial \psi(\mathbf{R},t)}{\partial t} \quad (1.3)
\]
Hereafter we use $\psi$ instead of $p$ to describe acoustic wave fields. Then Eq. (1.1) has the form in terms of $\psi$:

$$
\left( \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi(\tilde{R}, t) = Q(\tilde{R}, t)
$$

(1.4)

$\psi$ in Eq. (1.4) represents acoustic wave field in free space caused by arbitrary distribution of sound sources, $Q$. Now consider media (1) and (2) that are separated by a (flame front) boundary as shown in Figure 1. Let the incident wave be a plane wave that is propagating in medium (1) with a horizontal component of wave vector $k_0$ and frequency $\omega_0$ and impinging upward upon the (flame front) boundary. Then the total acoustic fields in each media can be described by the following forms considering reflected and transmitted waves:

In medium (1),

$$
\psi^{(1)}(\tilde{R}, t) = \text{Re}[\psi_e^{(1)}]
$$

(1.5)

where

$$
\frac{\psi_e^{(1)}}{A_1} = \left( \rho_1 q_0^{(1)} \right)^{-1/2} e^{i(k_0 r + q_0^{(1)} z - \omega_0 t)} + \int \int \int S^{11}(k, k_0, \omega, \omega_0) (\rho_1 q_k^{(1)})^{-1/2} e^{i(k r + q_k^{(1)} z - \omega t)} dkd\omega
$$

(1.6)

$$
A_1 = \left| \frac{\rho_1}{\omega_0} \right|^{1/2}, \quad q_0^{(1)} = [(\omega_0 / c_1)^2 - k_0^2]^{1/2}, \quad q_k^{(1)} = [(\omega / c_1)^2 - k^2]^{1/2}, \quad k = |k|
$$

In medium (2),

$$
\psi^{(2)}(\tilde{R}, t) = \text{Re}[\psi_e^{(2)}]
$$

(1.7)

where

$$
\frac{\psi_e^{(2)}}{A_2} = \int \int \int S^{21}(k, k_0, \omega, \omega_0) (\rho_2 q_k^{(2)})^{-1/2} e^{i(k r + q_k^{(2)} z - \omega t)} dkd\omega
$$

(1.8)

$$
q_k^{(2)} = [(\omega / c_2)^2 - k^2]^{1/2}
$$
In Eq. (1.6), $P_I$ is a complex amplitude of an incident wave pressure and $A_0$ is a scaling factor needed to match with the unit of pressure $q_0^{(1)}$, $q_k^{(1)}$ and $q_k^{(2)}$ denote the vertical components of wave number of incident and scattered waves in each medium, and these can be a complex value when $\omega/c < k$, representing exponentially decaying waves. (Note that $S_{N_1N_2}^{N_1N_2}$ denotes scattering amplitude of waves that are scattered into medium $N_1$ by incident waves from medium $N_2$. Therefore, $S_1^{11}$ denotes reflected waves and $S_2^{21}$ denotes transmitted waves.)

### 1.3 Evaluation of scattering amplitudes

The boundary conditions at the flame front, $z = h(r,t)$, are the continuity of acoustic pressure and acoustic velocity jump condition, as is derived by Lieuwen\cite{1} of the accuracy of $O(M_S^2)$.

Pressure continuity: \[ p^{(1)} = p^{(2)} \] \[ (3.1) \]

Normal velocity jump: \[ \frac{\nu_{N_1}^{(2)}(t)}{c^{(1)}} - \frac{\nu_{N_1}^{(1)}(t)}{c^{(1)}} = (\Lambda - 1)M_S \left( \frac{m_1(t)}{m_0} - \frac{p_1(t)}{p_0} \right) + O(M_S^2) \] \[ (3.2) \]

where $m_1(t)/m_0$ can be calculated by linearizing the expression for the burning rate reported by Peters and Ludford \cite{2}:

\[ \frac{m_1(t)}{m_0} = \frac{\gamma - 1}{\gamma p_0} \left( \frac{\alpha}{2} p_1(t) + \theta_r \frac{\partial p_1(t)}{\partial t} \right) \] \[ (3.3) \]

Now these boundary conditions can be expressed in terms of velocity potential from Eqs. (1.2) and (1.3).

Pressure continuity:

\[ \rho_1 \left. \frac{\partial \psi_{e_r}^{(1)}(\tilde{R},t)}{\partial t} \right|_{z=h(r,t)} = \rho_2 \left. \frac{\partial \psi_{e_r}^{(2)}(\tilde{R},t)}{\partial t} \right|_{z=h(r,t)} \] \[ (3.4) \]
Normal velocity jump:

\[
\left( \frac{\partial \psi^{(2)}_c(\hat{\mathbf{R}},t)}{\partial n} - \frac{\partial \psi^{(1)}_c(\hat{\mathbf{R}},t)}{\partial n} + \beta_\chi \frac{\partial \psi^{(2)}_c(\hat{\mathbf{R}},t)}{\partial t} + \beta_\gamma \frac{\partial^2 \psi^{(2)}_c(\hat{\mathbf{R}},t)}{\partial t^2} \right)_{z=h(r,t)} = 0
\]  

(3.5)

where \( \beta_\chi = \frac{(\Lambda-1) \rho_2 {\overline{S}}^{(1)}_L}{p_0} \left( \frac{\alpha (\gamma-1)}{2} - 1 \right), \quad \beta_\gamma = \frac{(\Lambda-1)(\gamma-1) \rho_2 t, \theta {\overline{S}}^{(1)}_L}{p_0} \)

Substituting Eqs. (1.6) and (1.8) into Eqs. (3.4) and (3.5) yields the expression which relates scattering amplitude in medium (1) to one in medium (2) in integral forms.

\[
\rho_1 \left[ \omega_0 (\rho_1 q^{(1)}_0)^{-1/2} e^{i(k_r r + q^{(1)}_0 h(r,t) - q^{(1)}_0)} + \int_\omega \int_k \omega S^{(1)}(\mathbf{k},k_r,\omega,\omega_0)(\rho_1 q^{(1)}_k)^{-1/2} e^{i(k_r r + q^{(1)}_k h(r,t) - q^{(1)}_0)} dk d\omega \right]
\]

\[
= \rho_2 \int_\omega \int_k \omega S^{(2)}(\mathbf{k},k_r,\omega,\omega_0)(\rho_2 q^{(2)}_k)^{-1/2} e^{i(k_r r + q^{(2)}_k h(r,t) - q^{(2)}_0)} dk d\omega \]  

(3.6)

\[
\left(1+|\nabla h|^2\right)^{-1/2} \left( \rho_1 q^{(1)}_0 \right)^{-1/2} \left( q^{(1)}_0 - \nabla h(r,t) \cdot \mathbf{k}_0 \right) e^{i(k_r r + q^{(1)}_0 h(r,t) - q^{(1)}_0)}
\]

\[
- \left(1+|\nabla h|^2\right)^{-1/2} \int_\omega \int_k \omega S^{(1)}(\mathbf{k},k_r,\omega,\omega_0)(\rho_1 q^{(1)}_k)^{-1/2} \left( q^{(1)}_k + \nabla h(r,t) \cdot \mathbf{k} \right) e^{i(k_r r + q^{(1)}_k h(r,t) - q^{(1)}_0)} dk d\omega
\]

\[
= \int_\omega \int_k \omega S^{(2)}(\mathbf{k},k_r,\omega,\omega_0)(\rho_2 q^{(2)}_k)^{-1/2} \left[ \left(1+|\nabla h|^2\right)^{-1/2} \left( q^{(2)}_k - \nabla h(r,t) \cdot \mathbf{k} \right) + \beta_\chi \omega - i \beta_\gamma \omega^2 \right] e^{i(k_r r + q^{(2)}_k h(r,t) - q^{(2)}_0)} dk d\omega
\]

An approximate solution for scattering amplitude \( S^{11} \) and \( S^{21} \) in Eqs. (3.6) and (3.7) can be obtained by using perturbation method, i.e., expanding the scattering amplitudes in powers of \( h \).

\[
S^{N_i N_j}_{0} = S^{N_i N_j}_{1} + S^{N_i N_j}_{2} + \cdots
\]  

(3.8)

where \( S^{N_i N_j}_{n} \sim O(h^n) \). Substituting Eq. (3.8) into Eqs. (3.6) and (3.7) and expanding

\[
\left(1+|\nabla h|^2\right)^{-1/2} \text{ and } \exp(igh(r,t)) \text{ in a power series of } h \text{ yield}
\]
\[ \rho_1 \left[ \omega_0 (\rho_0 q_0^{(1)})^{-1/2} e^{i(k_0 r - \omega_0 t)} \left\{ 1 + i q_0^{(1)} h(r,t) - \frac{1}{2} (q_0^{(1)} h(r,t))^2 + \cdots \right\} + \int \int \omega (S_{0}^{11} + S_{1}^{11} + S_{2}^{11} + \cdots) (\rho_1 q_k^{(1)})^{-1/2} e^{i(k r - \omega t)} \left\{ 1 - i q_k^{(1)} h(r,t) - \frac{1}{2} (q_k^{(1)} h(r,t))^2 + \cdots \right\} dk d\omega \right] \]

\[= \rho_2 \int \int \omega (S_{0}^{21} + S_{1}^{21} + S_{2}^{21} + \cdots) (\rho_2 q_k^{(2)})^{-1/2} e^{i(k r - \omega t)} \left\{ 1 + i q_k^{(2)} h(r,t) - \frac{1}{2} (q_k^{(2)} h(r,t))^2 + \cdots \right\} dk d\omega \]

(3.9)

\[\left(1 - \frac{1}{2} \left| \nabla h \right|^2 + O(h^4) \right) \left( \rho_0 q_0^{(1)} \right)^{-1/2} \left( q_0^{(1)} - \nabla h(r,t) \cdot k_0 \right) \left\{ 1 + i q_0^{(1)} h(r,t) - \frac{1}{2} (q_0^{(1)} h(r,t))^2 + O(h^3) \right\} e^{i(k_0 r - \omega_0 t)} \]

\[- \int \int \omega \left(1 - \frac{1}{2} \left| \nabla h \right|^2 + O(h^4) \right) \left( S_{0}^{11} + S_{1}^{11} + S_{2}^{11} + O(h^3) \right) \left( \rho_1 q_k^{(1)} \right)^{-1/2} \left( q_k^{(1)} + \nabla h(r,t) \cdot k \right) \]

\[\times \left\{ 1 - i q_k^{(1)} h(r,t) - \frac{1}{2} (q_k^{(1)} h(r,t))^2 + O(h^3) \right\} e^{i(k r - \omega t)} dk d\omega \]

\[= \int \int \omega \left( S_{0}^{21} + S_{1}^{21} + S_{2}^{21} + O(h^3) \right) \left( \rho_2 q_k^{(2)} \right)^{-1/2} \left[ \left( 1 - \frac{1}{2} \left| \nabla h \right|^2 + O(h^4) \right) \left( q_k^{(2)} - \nabla h(r,t) \cdot k \right) + \beta(\omega) \right) \]

\[\times \left\{ 1 + i q_k^{(2)} h(r,t) - \frac{1}{2} (q_k^{(2)} h(r,t))^2 + O(h^3) \right\} e^{i(k r - \omega t)} dk d\omega \]

(3.10)

Collecting the terms of \( O(h^0) \) in Eqs. (3.9) and (3.10) leads to the solution of zeroth order of scattering amplitude.

\[ S_{0}^{11}(k,k_0,\omega,\omega_0) = R_j (k, \omega) \delta(k - k_0) \delta(\omega - \omega_0) \]

(3.11)

\[ S_{0}^{21}(k,k_0,\omega,\omega_0) = D_j (k, \omega) \delta(k - k_0) \delta(\omega - \omega_0) \]

(3.12)

where

\[ R_j (k, \omega) = \frac{\rho_2 q_k^{(1)} - \rho_1 \left( q_k^{(2)} + \beta(\omega) \right)}{\rho_2 q_k^{(1)} + \rho_1 \left( q_k^{(2)} + \beta(\omega) \right)} , \]

\[ D_j (k, \omega) = \frac{2 \left( \rho_1 \rho_2 q_k^{(1)} q_k^{(2)} \right)^{1/2}}{\rho_2 q_k^{(1)} + \rho_1 \left( q_k^{(2)} + \beta(\omega) \right)} ; \beta(\omega) = \beta_0 + i \beta_0 \omega^2 \]
Reflection / transmission coefficients, $R$ and $D$, in the above equations represent specular reflection / transmission from flat mean flame surface. Collecting the terms of $O(h^1)$ in Eqs. (3.9) and (3.10) leads to the solution of first order of scattering amplitude.

$$S_{11}^{(k, k_0, \omega, \omega_0)} = A_j(k, k_0, \omega, \omega_0) h(k - k_0, \omega - \omega_0)$$

$$S_{21}^{(k, k_0, \omega, \omega_0)} = B_j(k, k_0, \omega, \omega_0) h(k - k_0, \omega - \omega_0)$$

where

$$A_j(k, k_0, \omega, \omega_0) = \frac{2i(q_0^{(1)} q_k^{(1)})^{1/2}}{\left[\rho_2 q_0^{(1)} + \rho_1 \left(q_0^{(2)} + \beta(\omega_0)\right)\right] \left[\rho_2 q_k^{(1)} + \rho_1 \left(q_k^{(2)} + \beta(\omega)\right)\right]} \times$$

$$\left[\frac{\omega_0}{\omega} \rho_1 \left(q_k^{(2)} + \beta(\omega)\right) \left\{(\rho_2 - \rho_1)q_0^{(2)} - \rho_1 \beta(\omega_0)\right\} + \rho_2 \alpha_j(k, k_0, \omega_0)\right]$$

$$B_j(k, k_0, \omega, \omega_0) = \frac{2i(\rho_1 \rho_2 q_0^{(1)} q_k^{(2)})^{1/2}}{\left[\rho_2 q_0^{(1)} + \rho_1 \left(q_0^{(2)} + \beta(\omega_0)\right)\right] \left[\rho_2 q_k^{(1)} + \rho_1 \left(q_k^{(2)} + \beta(\omega)\right)\right]} \times$$

$$\left[-\frac{\omega_0}{\omega} q_k^{(1)} \left\{(\rho_2 - \rho_1)q_0^{(2)} - \rho_1 \beta(\omega_0)\right\} + \alpha_j(k, k_0, \omega_0)\right]$$

$$\alpha_j(k, k_0, \omega_0) = \rho_2 \left\{\left(\frac{\omega_0}{c_1}\right)^2 - k \cdot k_0\right\} - \rho_1 \left\{\left(\frac{\omega_0}{c_2}\right)^2 - k \cdot k_0 + q_0^{(2)} \beta(\omega_0)\right\}$$

Second order approximation of scattering amplitude can be obtained by collecting the terms of $O(h^2)$ in Eqs. (3.9) and (3.10). Note that terms including $|\nabla h|^2$ in Eq. (3.10) should be expressed in terms of $h^2$ as the following procedure shows:
\[
\frac{1}{(2\pi)^3} \iiint \nabla h(r, t) e^{-i(k' - k_0) \cdot r - (\omega - \omega_0) t} \, dt \, dr
\]

\[
= \frac{1}{(2\pi)^3} \iiint \left( \int \int \int h(k_1, \omega_1) k_1 e^{i(k_1 \cdot r - \omega_1 t)} \, d\omega_1 d\mathbf{k}_1 \right) \left( \int \int \int h(k_2, \omega_2) k_2 e^{i(k_2 \cdot r - \omega_2 t)} \, d\omega_2 d\mathbf{k}_2 \right) \times e^{-i(k' - k_0) \cdot r - (\omega - \omega_0) t} \, dt \, dr
\]

\[
= -\iiint \int k_1 \cdot (k' - k_0 - k_1) h(k' - k_0, \omega', \omega_0, \omega) \delta(\omega + \omega' - \omega_0 - \omega_0) \, d\omega \, d\mathbf{k}_1
\]

\[
(k = k_0 + k_1, \omega = \omega_0 + \omega_1 \text{ yields})
\]

\[
= -\iiint (k' - k) \cdot (k' - k_0) h(k' - k, \omega' - \omega_0) \, d\omega \, d\mathbf{k}
\]

(3.18)

Then the 2nd order scattering amplitudes has the final form

\[
\left\{ \begin{align*}
S_2^{(1)}(k, k_0, \omega, \omega_0) \\
S_2^{(2)}(k, k_0, \omega, \omega_0)
\end{align*} \right\} = \frac{\rho_2 \rho_2 q_2^{(1)} q_2^{(2)}}{\rho_2 q_2^{(1)} + \rho_1 (q_2^{(2)} + \beta(\omega))} \left\{ \begin{align*}
\iiint \int \int F_j(k', k_0, k, \omega', \omega_0, \omega) h(k - k', \omega - \omega') h(k' - k_0, \omega' - \omega_0) \, dk' \, d\omega' \\
\iiint \int \int G_j(k', k_0, k, \omega', \omega_0, \omega) h(k' - k_0, \omega' - \omega_0) \, dk' \, d\omega'
\end{align*} \right\}
\]

(3.19)

where

\[
F_j(k', k_0, k, \omega', \omega_0, \omega) = \frac{(\rho_2 \rho_2 q_0^{(1)} q_2^{(2)})^{1/2}}{\rho_2 q_0^{(1)} + \rho_1 (q_0^{(2)} + \beta(\omega_0))} \times
\]

\[
\left\{ \begin{align*}
&\left( \frac{\omega_0}{c_1} \right)^2 - \left( \frac{\omega}{c_2} \right)^2 \left[ \frac{\omega_0}{\omega} (q_2^{(2)} + \beta(\omega)) - (q_0^{(2)} + \beta(\omega_0)) \right] + \beta(\omega_0) \left[ k' \cdot (k' - k_0) + k_0 \cdot (k' - k_0) \right] \\
&+ \frac{i}{(q_2^{(2)})^{1/2}} \left( \frac{\omega}{q_2^{(2)}} \rho_2 q_0^{(1)} \right)^{1/2} \left( q_2^{(2)} + \beta(\omega) \right) + \left( \frac{\rho_2}{\rho_1 q_0^{(1)}} \right)^{1/2} \left( \frac{\omega}{c_1} \right)^2 \left[ k' \cdot k \right] A_j(k', k_0, \omega', \omega_0) \\
&+ \frac{i}{(q_2^{(2)})^{1/2}} \left( \frac{\omega}{q_2^{(2)}} \rho_2 q_0^{(1)} \right)^{1/2} \left( q_2^{(2)} + \beta(\omega) \right) - (q_2^{(2)})^{-1/2} \left[ \left( \frac{\omega}{c_2} \right)^2 - k' \cdot k + \beta(\omega') q_2^{(2)} \right] B_j(k', k_0, \omega', \omega_0)
\end{align*} \right\}
\]

(3.20)
\[ G_j(k',k_0,k,\omega',\omega_0,\omega) = \frac{-(q_0^{(1)}/q_k^{(1)})^{1/2}}{\rho_2q_0^{(1)} + \rho_1(q_0^{(2)} + \beta(\omega_0))} \times \]
\[
\left\{ \left( \frac{\omega}{c_1} \right)^2 - \left( \frac{\omega}{c_2} \right)^2 \right\} \left[ \frac{\omega}{\omega'} \rho_2q_k^{(1)} + \rho_1(q_k^{(2)} + \beta(\omega_0)) \right] - \rho_1\beta(\omega_0) \left[ k' \cdot (k-k') + k_0 \cdot (k'-k_0) \right] \right\} + i \left\{ -\omega \left( q_k^{(1)}/q_k^{(1)'} \right)^{1/2} + \left( q_k^{(1)}/q_k^{(1)'} \right)^{1/2} \left[ \frac{\omega'}{c_1} \right] - k' \cdot k \right\} A_j(k',k_0,\omega',\omega_0) \]
\[-i \left\{ \frac{\omega'}{\omega} \left( \frac{\rho_2}{\rho_1} q_k^{(1)}/q_k^{(2)'} \right)^{1/2} + \left( \frac{\rho_2}{\rho_1} q_k^{(1)}/q_k^{(2)'} \right)^{1/2} \left[ \frac{\omega'}{c_2} \right] - k' \cdot k + \beta(\omega')q_k^{(2)'} \right\} B_j(k',k_0,\omega',\omega_0) \]

(3.21)

Scattering amplitude up to 2nd order can now be of the following form by combining

Eqs. (3.11), (3.12), (3.13), (3.14), and (3.19).

\[ S_{11}^{n}(k,k_0,\omega,\omega_0) \]
\[
= \sum_{n=0}^{2} S_{11}^{n}(k,k_0,\omega,\omega_0) + O(h^3) \]
\[
= R_j(k,\omega) \delta(k-k_0) \delta(\omega-\omega_0) + A_j(k,k_0,\omega,\omega_0) h(k-k_0,\omega-\omega_0) \]
\[ + \frac{1}{2} D_j(k,\omega) \int \int F_j(k',k_0,k,\omega',\omega_0) h(k-k',\omega-\omega') h(k'-k_0,\omega'-\omega_0) dk' d\omega' \]

(3.22)

\[ S_{21}^{n}(k,k_0,\omega,\omega_0) \]
\[
= \sum_{n=0}^{2} S_{21}^{n}(k,k_0,\omega,\omega_0) + O(h^3) \]
\[
= D_j(k,\omega) \delta(k-k_0) \delta(\omega-\omega_0) + B_j(k,k_0,\omega,\omega_0) h(k-k_0,\omega-\omega_0) \]
\[ + \frac{1}{2} D_j(k,\omega) \int \int G_j(k',k_0,k,\omega',\omega_0) h(k-k',\omega-\omega') h(k'-k_0,\omega'-\omega_0) dk' d\omega' \]

(3.23)

The velocity potential in medium (1) and (2) from Eqs. (1.6) and (1.8) can be expressed with the second order accuracy with respect to flame front height as follows.
\[
\psi_{c(1)}(\bar{R}, \omega) = \frac{(\rho_{1}q_{0(1)}^{(1)})^{-1/2}}{A_{1}} e^{i(k_{\omega} r + q_{1}^{(1)} z - \omega_{1} t)} + (\rho_{1}q_{0}^{(1)})^{-1/2} R_{j}(k_{\omega}, \omega_{0}) e^{i(k_{\omega} r - q_{1}^{(1)} z - \omega_{1} t)} \\
+ \int \int \int (\rho_{1}q_{k}^{(1)})^{-1/2} A_{j}(k_{\omega}, k_{\omega}, \omega, \omega_{0}) h(k - k_{\omega}, \omega - \omega_{0}) e^{i(k_{\omega} r + q_{1}^{(1)} z - \omega_{1} t)} dkd \omega \tag{3.24}
\]
\[
+ \frac{1}{2} \int \int \int (\rho_{1}q_{k}^{(1)})^{-1/2} D_{j}(k, \omega) e^{i(k_{\omega} r + q_{1}^{(1)} z - \omega_{1} t)} \times \left[ \int \int F_{j}(k_{\omega}, k_{\omega}, k_{\omega}, \omega, \omega, \omega) h(k - k_{\omega}, \omega - \omega_{0}) h(k' - k_{\omega}, \omega' - \omega_{0}) dk'd \omega' \right] dkd \omega
\]
\[
\psi_{c(2)}(\bar{R}, \omega) = \frac{(\rho_{2}q_{0}^{(2)})^{-1/2}}{A_{1}} D_{j}(k_{\omega}, \omega_{0}) e^{i(k_{\omega} r + q_{2}^{(2)} z - \omega_{1} t)} + \int \int \int (\rho_{2}q_{k}^{(2)})^{-1/2} B_{j}(k_{\omega}, k_{\omega}, \omega, \omega_{0}) h(k - k_{\omega}, \omega - \omega_{0}) e^{i(k_{\omega} r + q_{2}^{(2)} z - \omega_{1} t)} dkd \omega \tag{3.25}
\]
\[
+ \frac{1}{2} \int \int \int (\rho_{2}q_{k}^{(2)})^{-1/2} D_{j}(k, \omega) e^{i(k_{\omega} r + q_{2}^{(2)} z - \omega_{1} t)} \times \left[ \int \int G_{j}(k_{\omega}, k_{\omega}, k_{\omega}, \omega, \omega, \omega) h(k - k_{\omega}, \omega - \omega_{0}) h(k' - k_{\omega}, \omega' - \omega_{0}) dk'd \omega' \right] dkd \omega
\]

where the forms of \( R_{j}, D_{j}, A_{j}, B_{j}, F_{j}, \) and \( G_{j} \) are found in Eqs. (3.11), (3.12), (3.15), (3.16), (3.20), and (3.21). Dimensionless form of acoustic pressure and velocity fields then follows from substituting the above forms of velocity potential into Eq. (1.3).

In medium (1),
\[
\tilde{p}_{e}^{(1)}(\bar{R}, t) \equiv \frac{p_{e}^{(1)}}{|P_{e}|} = -\rho_{e} \text{Re} \left[ \frac{\partial \psi_{c(1)}(\bar{R}, t)}{\partial t} \right] = \tilde{p}_{t}^{(1)}(\bar{R}, t) + \tilde{p}_{R}^{(1)}(\bar{R}, t) \tag{3.26}
\]

where
\[
\tilde{p}_{t}^{(1)}(\bar{R}, t) = \text{Re} \left[ i e^{i(k_{\omega} r + q_{1}^{(1)} z - \omega_{1} t)} \right]
\]

13
\[ \tilde{\nu}^{(1)}(\tilde{R},t) = \frac{1}{c_1} \text{Re} \left[ \int \int \omega \left( \frac{q_0(1)}{q_k(1)} \right)^{1/2} S^{11}(k,k_0,\omega,\omega_0) e^{i(k - q^{(1)}_k z - \omega t)} dk d\omega \right] \]

\[ = \text{Re} \left\{ iR_j(k_0,\omega_0) e^{i(k - q^{(1)}_k z - \omega \omega')} \right\} \]

\[ + i \int \int \frac{\omega}{\omega_k} \left( \frac{q_0(1)}{q_k(1)} \right)^{1/2} A_j(k,k_0,\omega,\omega_0) h(k - k_0,\omega - \omega_0) e^{i(k - q^{(1)}_k z - \omega \omega')} dk d\omega \]

\[ + \frac{i}{2} \int \int \frac{\omega}{\omega_k} \left( \frac{q_0(1)}{q_k(1)} \right)^{1/2} D_j(k,\omega) e^{i(k - q^{(1)}_k z - \omega \omega')} \times \]

\[ \left\{ \int \int F_j(k',k_0,k,\omega',\omega_0) h(k' - k_0,\omega' - \omega_0) dk' d\omega' \right\} dk d\omega \]

\[ \tilde{v}^{(1)}(\tilde{R},t) = \frac{\tilde{v}^{(1)}(\tilde{R},t)}{c_1} = \frac{\psi}{c_1} \text{Re} \left[ \int \int \omega \left( \frac{q_0(1)}{q_k(1)} \right)^{1/2} S^{11}(k,k_0,\omega,\omega_0) \tilde{K}^{(1)}(k) e^{i(k - q^{(1)}_k z - \omega \omega')} dk d\omega \right] \]

\[ = \frac{\psi}{c_1} \text{Re} \left\{ i\tilde{K}_0(k_0,\omega_0) e^{i(k - q^{(1)}_k z - \omega \omega')} \right\} \]

\[ + i \int \int \frac{\omega}{\omega_k} \left( \frac{q_0(1)}{q_k(1)} \right)^{1/2} A_j(k,k_0,\omega,\omega_0) h(k - k_0,\omega - \omega_0) \tilde{K}_0(k) e^{i(k - q^{(1)}_k z - \omega \omega')} dk d\omega \]

\[ + \frac{i}{2} \int \int \frac{\omega}{\omega_k} \left( \frac{q_0(1)}{q_k(1)} \right)^{1/2} D_j(k,\omega) \tilde{K}_0(k) e^{i(k - q^{(1)}_k z - \omega \omega')} \times \]

\[ \left\{ \int \int F_j(k',k_0,k,\omega',\omega_0) h(k' - k_0,\omega' - \omega_0) dk' d\omega' \right\} dk d\omega \]

In medium (2),

\[ \tilde{p}^{(2)}(\tilde{R},t) \equiv \frac{p^{(2)}}{p^{(1)}} = - \frac{\rho_2}{\rho_1} \text{Re} \left[ \frac{\partial \psi^{(2)}(\tilde{R},t)}{\partial t} \right] \]

\[ = \text{Re} \left\{ i \int \int \omega \left( \frac{\rho_2 q_0(1)}{\rho_1 q_k(1)} \right)^{1/2} S^{21}(k,k_0,\omega,\omega_0) e^{i(k - q^{(2)}_k z - \omega \omega')} dk d\omega \right\} \]
\[
\begin{align*}
&= \Re \left\{ i \left( \frac{\rho_2 q_0^{(1)}}{\rho_1 q_0^{(2)}} \right)^{1/2} D_j(k_0, \omega_0) e^{i(k_0 r + q_0^{(2)} z - \omega_0 t)} \right\} \\
&+ i \int \int \left. \frac{\omega_k}{\omega_0} \left( \frac{\rho_2 q_0^{(1)}}{\rho_1 q_0^{(2)}} \right)^{1/2} B_j(k, k_0, \omega, \omega_0) h(k - k_0, \omega - \omega_0) e^{i(k_0 r + q_0^{(2)} z - \omega_0 t)} \right| dkd\omega \\
&+ \frac{i}{2} \int \int \left. \frac{\omega_k}{\omega_0} \left( \frac{\rho_2 q_0^{(1)}}{\rho_1 q_0^{(2)}} \right)^{1/2} D_j(k, \omega) \times \right. \\
&\left. \left\{ \int \int G_j(k, k_0, k, \omega, \omega_0) h(k - k_0, \omega - \omega_0) e^{i(k_0 r + q_0^{(2)} z - \omega_0 t)} \right| dkd\omega \right\}
\end{align*}
\]

\[
\tilde{\psi}_e^{(2)}(\tilde{R}, t) \equiv \tilde{\psi}_e^{(2)} = \frac{1}{c_1} \Re \left[ \nabla \psi_e^{(2)}(\tilde{R}, t) \right] \\
= \frac{\varepsilon c_1}{\omega_0} \Re \left\{ i \int \int \left( \frac{\rho_2 q_0^{(1)}}{\rho_1 q_0^{(2)}} \right)^{1/2} S_{21}^{(2)}(k, k_0, \omega, \omega_0) \tilde{K}_e^{(2)} e^{i(k_0 r + q_0^{(2)} z - \omega_0 t)} \right| dkd\omega \right\}
\]

\[
= \frac{\varepsilon c_1}{\omega_0} \left( \frac{\rho_2 q_0^{(1)}}{\rho_2 q_0^{(2)}} \right)^{1/2} \Re \left\{ i D_j(k_0, \omega_0) \tilde{K}_e^{(2)} e^{i(k_0 r + q_0^{(2)} z - \omega_0 t)} \right\} \\
+ i \int \int (q_0^{(2)} / q_0^{(2)})^{1/2} B_j(k, k_0, \omega, \omega_0) h(k - k_0, \omega - \omega_0) \tilde{K}_e^{(2)} e^{i(k_0 r + q_0^{(2)} z - \omega_0 t)} \left| dkd\omega \right\} \\
+ \frac{i}{2} \int \int (q_0^{(2)} / q_0^{(2)})^{1/2} D_j(k, \omega) \tilde{K}_e^{(2)} e^{i(k_0 r + q_0^{(2)} z - \omega_0 t)} \times \\
\left\{ \int \int G_j(k, k_0, k, \omega, \omega_0) h(k - k_0, \omega - \omega_0) h(k - k_0, \omega - \omega_0) \right| dkd\omega \right\}
\]

where \( \tilde{K}_{e \pm}^{(m)} = k + q_0^{(m)} \tilde{e}_z \), \( \tilde{K}_e^{(m)} = k + q_0^{(m)} \tilde{e}_z \) \( (m = \{1, 2\}) \), \( \varepsilon = |P_1|/(\rho_1 c_1^2) \)

\[\text{CHAPTER 2} \quad \text{STOCHASTIC ANALYSIS OF SCATTERED FIELDS}\]

This section deals with evaluation of acoustic energy flux before and after scattering to see how the acoustic energy is balanced or amplified / damped through scattering process when the incident waves are scattered by randomly moving turbulent flames. Statistical analysis will
be incorporated to describe random motion of turbulent flame and resultant scattered acoustic fields. Statistics of spatial homogeneity and temporal stationarity will be assumed in the following analysis.

2.1 Formulation of averaged acoustic energy flux

Recalling time-averaged intensity first, the mean acoustic energy flux is defined as one being averaged over T which is either an acoustic time period or interminably long time interval ([3], p. 25)

\[
I_{av\_time} = \frac{1}{T} \int_{-T/2}^{T/2} p(t)\bar{v}(t)dt = \frac{1}{2} \text{Re}(P\bar{V}^*)
\]

which is valid for a single frequency wave, i.e., \( p(t) = \text{Re}(Pe^{-i\omega t}) \) and \( \bar{v}(t) = \text{Re}(\bar{V}e^{-i\omega t}) \). The wave fields of interest in this paper, however, are those of multi-frequencies which are produced by scattering from randomly moving turbulent flames due to Doppler frequency shift. Note that the scattered waves are of multi-frequencies even if an incident wave is of single frequency. The following analysis derives formulation of evaluating a time-averaged intensity of multi frequency wave fields has the form for \( p(t) = \text{Re}\left[ \int_{\omega} P(\omega)e^{-i\omega t}d\omega \right] \) and

\[
\bar{v}(t) = \text{Re}\left[ \int_{\omega} \bar{V}(\omega)e^{-i\omega t}d\omega \right]:
\]

\[
I_{av\_time} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} p(t)\bar{v}(t)dt
\]

\[
= \frac{1}{4} \int_{\omega} \int_{\omega'} P(\omega) \left( \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} e^{-i(\omega+\omega')t}dt \right) + \bar{V}^*(\omega') \left( \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} e^{-i(\omega-\omega')t}dt \right) d\omega' d\omega + C.C.
\]

\[
= \frac{1}{2} \text{Re}\left\{ \int_{\omega} P(\omega) \left( \int_{\omega'} \left[ \bar{V}(-\omega') + \bar{V}^*(\omega') \right] d\omega' \right) d\omega \right\}
\]
Note that (4.2) reduces to $\text{Re}(P\hat{V}^*)/2$ for a single-frequency wave, i.e.,

$$P(\omega) = P\delta(\omega - \omega_0)$$

and $\hat{V}(\omega) = \hat{V}\delta(\omega - \omega_0)$ since, for nonzero $\omega_0$,

$$I_{\text{av, time}} = \frac{1}{2}\text{Re}\left\{ \int_{\omega} P\delta(\omega - \omega_0)(\hat{V}\delta_{\omega(\omega_0)} + \hat{V}^*\delta_{\omega(0)})d\omega \right\}$$

$$= \frac{1}{2}\text{Re}\left\{ P(\hat{V}\delta_{\omega(\omega_0)} + \hat{V}^*\delta_{\omega_0}) \right\} = \frac{1}{2}\text{Re}(P\hat{V}^*)$$

(4.3)

where $\delta_{\omega(\omega_0)}$ is Kronecker’s delta function and

$$\int_{\omega' \to \omega} \delta(\omega' - \omega_0)d\omega' = \delta_{\omega(\omega_0)} = \begin{cases} 1 & (\omega = \omega_0) \\ 0 & (\omega \neq \omega_0) \end{cases}$$

are used. Similarly, space-averaged intensity can be utilized in order to obtain a time and space-averaged intensity for multi-frequency and multi-directional waves of the form

$$p(\mathbf{R}, t) = \text{Re}\left[ \iint_{\omega k} P(k, \omega, z)e^{i(kr - \omega t)}dkd\omega \right]$$

(4.4)

$$\tilde{v}(\mathbf{R}, t) = \text{Re}\left[ \iint_{\omega k} \tilde{V}(k, \omega, z)e^{i(kr - \omega t)}dkd\omega \right]$$

which yields a time and space-averaged intensity in the form

$$\tilde{I}_{\text{av}} = \lim_{\Delta T \to \infty} \frac{1}{\Delta T} \int_{|t|,|y| \leq \sqrt{T}/2} \int_{-T/2}^{T/2} p(\mathbf{R}, t)\tilde{v}(\mathbf{R}, t)dt dr$$

$$= \frac{1}{2}\text{Re}\left\{ \iint_{\omega k} P(k, \omega, z)\left[ \iint_{\omega' \to \omega' \to \omega' \to \omega} (\tilde{V}(-k', -\omega', z) + \tilde{V}^*(k', \omega', z))dk'd\omega' \right] dkd\omega \right\}$$

(4.5)

Ensemble average for the above equation can be taken to describe statistical characteristics of acoustic fields that obey stochastic properties.

$$\langle \tilde{I}_{\text{av}} \rangle = \frac{1}{2}\text{Re}\left\{ \iint_{\omega k} \left[ \iint_{\omega' \to \omega} \left( P(k, \omega, z)\tilde{V}(-k', -\omega', z) \right) dk'd\omega' \right] dkd\omega \right\}$$

(4.6)
2.2 Evaluation of acoustic energy flux of scattered fields

To account for statistical characteristics of scattered acoustic fields, statistical analysis of scattering amplitude should be conducted first since scattered acoustic fields are described by scattering amplitude. Taking ensemble average of scattering amplitude in Eq. (3.22).

\[
\left\langle S^{11}(k, k_0, \omega, \omega_0) \right\rangle = R_j(k, \omega)\delta(k - k_0)\delta(\omega - \omega_0) + A_j(k, k_0, \omega, \omega_0)\left\langle h(k - k_0, \omega - \omega_0) \right\rangle \\
+ \frac{1}{2} D_j(k, \omega) \int \int \int F_j(k', k_0, k, \omega', \omega_0, \omega) \left\langle h(k - k', \omega - \omega')h(k' - k_0, \omega' - \omega_0) \right\rangle dk'd\omega'
\]

where ensemble average of \( h \) vanishes.

\[
\left\langle h(k - k_0, \omega - \omega_0) \right\rangle = \frac{1}{(2\pi)^3} \iiint_{r} \left\langle h(r, t) \right\rangle e^{-i(k - k_0) \cdot r - (\omega - \omega_0) t} dr dt = 0
\]

Ensemble average of \( h^2 \) term is related to the power spectral density as follows. ([4], p. 80)

\[
\left\langle h(k_1, \omega_1)h(k_2, \omega_2) \right\rangle = W(k_1, \omega_1)\delta(k_1 + k_2)\delta(\omega_1 + \omega_2)
\]

\( W(k_1, \omega_1) \) is the power spectral density of flame front height, which is defined as Fourier transform of correlation function of flame front height ([5], pp. 490)

\[
W(k_1, \omega_1) = \frac{1}{(2\pi)^3} \iiint_{\xi} \tilde{W}(\xi, \eta)e^{-i(k_1 \cdot \xi - \omega_1 \eta)} d\eta d\xi
\]

Substituting Eqs. (4.8) and (4.9) into Eq. (4.7) yields the form

\[
\left\langle S^{11}(k, k_0, \omega, \omega_0) \right\rangle = \left\langle V_j(k, \omega) \right\rangle \delta(k - k_0)\delta(\omega - \omega_0)
\]

where

\[
\left\langle V_j(k, \omega) \right\rangle = R_j(k, \omega) + \frac{1}{2} D_j(k, \omega) \int \int \int F_j(k', k_0, k, \omega', \omega_0, \omega)W(k - k', \omega - \omega') dk'd\omega'
\]
which implies that mean (ensemble-averaged) scattering amplitude consists only of the waves that propagate in a specular direction and that 1st order term has no contribution to mean scattering amplitude while zeroth and 2nd order terms do have contribution. \( \langle V_z(k, \omega) \rangle \) in the above equation is referred to as the mean reflection coefficient. The above analysis enables one to evaluate ensemble average of acoustic fields to yield net energy flux. Comparing Eq. (4.4) with reflected acoustic pressure / velocity fields in Eqs. (3.26) and (3.27) yields the forms in terms of scattering amplitude.

\[
P(k, \omega, z) = i \frac{\omega}{\omega_0} \left( \frac{q_{0(1)}^{(1)}}{q_{k}^{(1)}} \right)^{1/2} S^{11}(k, k_0, \omega, \omega_0) e^{-iq_0^{(1)}z} \tag{4.12}
\]

\[
\tilde{V}(k, \omega, z) = i \frac{\varepsilon_{01}}{\omega_0} \left( \frac{q_{0}^{(1)}/q_{k}^{(1)}}{q_{k}^{(1)}} \right)^{1/2} S^{11}(k, k_0, \omega, \omega_0) \tilde{K}_+^{(1)} e^{-iq_0^{(1)}z} \tag{4.12}
\]

The terms in the integral of Eq. (4.6) are then expressed in terms of the second moments of scattering amplitude.

\[
\left\langle P(k, \omega, z) \tilde{V}(\mathbf{k}', -\mathbf{\omega}', z) \right\rangle
= \frac{\varepsilon_{01} \omega}{\omega_0^2} \left( \frac{q_{0}^{(1)}}{q_{k}^{(1)}} \right)^{1/2} \left( \frac{q_{0}^{(1)}}{q_{k}^{(1)}} \right)^{1/2} S^{11}(k, k_0, \omega, \omega_0) S^{11}(\mathbf{k}', \mathbf{k}_0, -\mathbf{\omega}', -\mathbf{\omega}_0) \tilde{K}_+^{(1)} e^{-i(q_0^{(1)}+q_{k}^{(1)})z} \tag{4.13}
\]

\[
\left\langle P(k, \omega, z) \tilde{V}^*(\mathbf{k}', \mathbf{\omega}', z) \right\rangle
= \frac{\varepsilon_{01} \omega}{\omega_0^2} \left( \frac{q_{0}^{(1)}}{q_{k}^{(1)}} \right)^{1/2} \left( \frac{q_{0}^{(1)}}{q_{k}^{(1)}} \right)^{1/2} S^{11}(k, k_0, \omega, \omega_0) S^{11*}(\mathbf{k}', \mathbf{k}_0, \mathbf{\omega}', \mathbf{\omega}_0) \tilde{K}_+^{(1)*} e^{-i(q_0^{(1)}-q_{k}^{(1)})z} \tag{4.14}
\]

\[
\tilde{K}_+^{(m)} = k' \pm q_{k}^{(m)} \hat{e}_z
\]

where the second moments of scattering amplitude are further evaluated as
The above equations lead finally to the acoustic energy flux reflected from unit area of mean flame surface \((z = 0)\) whose normal vector \(\vec{n}\) is equal to \(\vec{e}_z\).

\[
\mathcal{T}_{\text{av}} \cdot \vec{n} =
\left[ \frac{\varepsilon c_I q_0^{(1)}}{2 \omega_0} \right] - \frac{1}{2\omega_0} \left[ \left| V_j(k_0, \omega_0) \right|^2 + \int \int \frac{\omega}{\omega_0} |A_j(k, \omega, \omega_0)|^2 W(k - k_0, \omega - \omega_0) dk d\omega \right] + O(h^3) \tag{4.17}
\]

The first term in Eq. (4.17) represents coherent energy flux of the waves reflected from mean flame surface. The second term is related to incoherent energy flux. The reflected energy flux normalized by incident energy flux can be obtained by dividing Eqs. (4.17) by

\[
\mathcal{T}_{\text{av}} \cdot \vec{n} = \frac{\varepsilon c_I q_0^{(1)}}{2 \omega_0}, \text{ where terms of } O(h^3) \text{ and higher are neglected.}
\]
Transmitted acoustic energy flux can be evaluated in a similar way to the preceding analysis of reflection energy flux to have the form

\[ E_T = \left\{ \vec{i}_{av} \right\}_T \cdot \vec{n} / \left\| \vec{i}_{av,1} \cdot \vec{n} \right\| \]

\[ = \Re \left[ \frac{q_0^{(2)}}{q_0^{(2)}} \right] \left\langle T_j(k_0, \omega_0) \right\rangle^2 + \int \int \frac{\omega}{\omega_0} |B_j(k, k_0, \omega, \omega_0)|^2 W(k - k_0, \omega - \omega_0)dkd\omega \]

\[ = \Re \left[ \frac{q_0^{(2)}}{q_0^{(2)}} \right] \left| D_j(k_0, \omega_0) \right|^2 \left( 1 + \int \int \Re \left[ H_j(k, k_0, k_0, \omega, \omega_0, \omega_0) \right] W(k - k_0, \omega - \omega_0)dkd\omega \right) \]

\[ + \int \int \frac{\omega}{\omega_0} |B_j(k, k_0, \omega, \omega_0)|^2 W(k - k_0, \omega - \omega_0)dkd\omega \]

where \( \left\langle T_j(k, \omega) \right\rangle = D_j(k, \omega) \left[ 1 + \frac{1}{2} \int \int G_j(k', k_0, k, \omega', \omega_0, \omega') W(k - k', \omega - \omega')dk'd\omega' \right] \)

Combining Eqs. (4.18) and (4.19) yields total scattering energy flux.

\[ E_{total} = E_R + E_T \]

\[ = |R_j(k_0, \omega_0)|^2 + \Re \left[ \frac{q_0^{(2)}}{q_0^{(2)}} \right] |D_j(k_0, \omega_0)|^2 + \int \int \Re \left[ H_j(k, k_0, k_0, \omega, \omega_0, \omega_0) \right] W(k - k_0, \omega_0 - \omega)dkd\omega \]

\[ + \int \int \frac{\omega}{\omega_0} \left[ \Re \left[ \frac{q_k^{(1)}}{q_k^{(1)}} \right] |A_j(k, k_0, \omega, \omega_0)|^2 + \Re \left[ \frac{q_k^{(2)}}{q_k^{(2)}} \right] |B_j(k, k_0, \omega, \omega_0)|^2 \right] W(k - k_0, \omega - \omega_0)dkd\omega \]

\[ + O(h^3) \]

(4.20)
where

\[ H_j(k, k_0, k_0, \omega, \omega_0, \omega_0) = R_j^*(k_0, \omega_0)D_j(k_0, \omega_0)F_j(k, k_0, k_0, \omega, \omega_0, \omega_0) + \text{Re} \left( \frac{q_0^{(2)}}{|q_0^{(2)}|} \right) |D_j(k_0, \omega_0)|^2 G_j(k, k_0, k_0, \omega, \omega_0, \omega_0) \]

The first three terms, which include \( R \), \( D \), and \( H \), on the right-hand side of Eq. (4.20) represent coherent energy flux of scattering acoustic fields. The last two terms, which include \( A \) and \( B \), represent incoherent energy flux of reflection and transmission scattering fields, respectively. The term including \( H \) is responsible for reduction of coherent field which describes coherent energy transfer to incoherent field due to corrugated flame surface. It can be shown that Eq. (4.20) yields unity in case of no velocity jump (\( \beta = 0 \)) and no unsteady motion (\( \omega = \omega_0 \)), which manifests acoustic energy balance stating that incoherent energy produced by wrinkled flame originates from coherent energy and total energy is conserved if flame’s unsteady effects, i.e., unsteady heat release and unsteady motion, are not considered. Acoustic energy balance can also be expressed as sum of coherent and incoherent fluxes.

\[
\left| \langle V(k_0, \omega_0) \rangle \right|^2 + \text{Re} \left( \frac{q_0^{(2)}}{|q_0^{(2)}|} \right) \left| \langle T(k_0, \omega_0) \rangle \right|^2 + \iint_{|k|<\frac{\omega}{c_1}} \sigma_R(k, k_0)dk + \iint_{|k|<\frac{\omega}{c_2}} \sigma_T(k, k_0)dk = 1 + O(h^3) \tag{4.21}
\]

with \( \sigma_R(k, k_0) \equiv \left| A(k, k_0, \omega_0, \omega_0) \right|^2 W(k - k_0) \), \( \sigma_T(k, k_0) \equiv \left| B(k, k_0, \omega_0, \omega_0) \right|^2 W(k - k_0) \)

where the last two terms, i.e., scattering cross sections \( \sigma_R \) and \( \sigma_T \), represent incoherent (diffuse) energy flux for reflected and transmitted wave fields, respectively.

### 2.3 Budget of net energy fluxes

Note that, if either velocity jump due to unsteady mass burning rate or Doppler frequency shift due to flame’s unsteady motion is considered, then energy balance in Eq. (4.21) does not hold.
because the acoustic energy is amplified or damped. In another words, net acoustic energy flux is attributed to two factors: acoustic velocity jump due to unsteady heat release and unsteady motion of flame front. Net energy flux can further be separated into coherent and incoherent flux. Total acoustic energy in Eq. (4.20) can be rewritten in the form

\[
E_{total} = \left| \langle V_{j} (k_0, \omega_0) \rangle \right|^2 + \text{Re} \left( \frac{q_0^{(2)}}{q_0^{(2)}} \right) \left| \langle T_{j} (k_0, \omega_0) \rangle \right|^2
\]

\[+
\int \int \sigma_{R,j} (k, k_0, \omega, \omega_0) dk d\omega + \int \int \sigma_{T,j} (k, k_0, \omega, \omega_0) dk d\omega \quad (4.22)
\]

where the scattering cross sections are

\[
\sigma_{R,j} (k, k_0, \omega, \omega_0) = \frac{\omega}{\omega_0} \left| A_{j} (k, k_0, \omega, \omega_0) \right|^2 W (k - k_0, \omega - \omega_0)
\]

\[
\sigma_{T,j} (k, k_0, \omega, \omega_0) = \frac{\omega}{\omega_0} \left| B_{j} (k, k_0, \omega, \omega_0) \right|^2 W (k - k_0, \omega - \omega_0)
\]

Note that energy balance equation in Eq. (4.21) is a special case of Eq. (4.22) with no jump (\( \beta = 0 \)) and constant frequency (\( \omega = \omega_0 \)). Then total net energy flux is evaluated by subtracting Eq. (4.21) from Eq. (4.22) to yield

\[
\Delta E = E_{total} - 1 = \Delta E_{R,co} + \Delta E_{T,co} + \Delta E_{R,inc} + \Delta E_{T,inc} \quad (4.23)
\]

where

\[
\Delta E_{R,co} = \left| \langle V_{j} (k_0, \omega_0) \rangle \right|^2 - \left| R (k_0, \omega_0) \right|^2
\]

\[
\Delta E_{T,co} = \text{Re} \left( \frac{q_0^{(2)}}{q_0^{(2)}} \right) \left| \langle T_{j} (k_0, \omega_0) \rangle \right|^2 - \left| D (k_0, \omega_0) \right|^2
\]

\[
\Delta E_{R,inc} = \int \int \sigma_{R,j} (k, k_0, \omega, \omega_0) dk d\omega
\]

\[
\Delta E_{T,inc} = \int \int \sigma_{T,j} (k, k_0, \omega, \omega_0) dk d\omega
\]
The first term, $\Delta E_{R,co}$, is net coherent energy flux of reflected waves. $\Delta E_{T,co}$ is net coherent energy flux of transmitted waves, which vanishes when the incident angle is beyond critical angle because no energy is transmitted ($\text{Re}(q_0^{(2)}) = 0$). $\Delta E_{R,inc}$ and $\Delta E_{T,inc}$ are net incoherent energy flux of reflected and transmitted waves, respectively, which results from surface wrinkling and unsteady heat release. As mentioned previously, net coherent energy flux, $\Delta E$, is attributed to two factors: One is due to acoustic velocity jump due to unsteady heat release across the flame, $\Delta_j(E)$. The other is due to flame’s wrinkling, $\Delta_w(E)$, which accounts for both temporal (unsteady motion) and spatial wrinkling of flame fronts. This has the form for coherent flux

$$\Delta E_{co} = \Delta_j(E_{co}) + \Delta_w(E_{co})$$

(4.24)

where

$$\Delta_j(E_{co}) = \|V_j(k_0, \omega_0)\|^2 - \|V(k_0, \omega_0)\|^2 + \text{Re}\left(\frac{q_0^{(2)}}{|q_0^{(2)}|}\right)\left(\|T_j(k_0, \omega_0)\|^2 - \|T(k_0, \omega_0)\|^2\right)$$

$$\Delta_w(E_{co}) = \|V(k_0, \omega_0)\|^2 + \text{Re}\left(\frac{q_0^{(2)}}{|q_0^{(2)}|}\right)\|T(k_0, \omega_0)\|^2 - 1$$

Note that $\Delta_w(E_{co}) < 0$ for constant frequency, which implies that coherent energy is only damped into incoherent field due to spatial wrinkling. Similarly net incoherent energy flux is attributed to velocity jump and surface wrinkling effects.

$$\Delta E_{inc} = \Delta_j(E_{inc}) + \Delta_w(E_{inc})$$

(4.25)

where

$$\Delta_j(E_{inc}) = \int \int_{\omega, |k| < c_1} [\sigma_{R,j}(k, k_0, \omega, \omega_0) - \sigma_R(k, k_0, \omega, \omega_0)] dk d\omega$$

$$+ \int \int_{\omega, |k| > c_2} [\sigma_{T,j}(k, k_0, \omega, \omega_0) - \sigma_T(k, k_0, \omega, \omega_0)] dk d\omega$$
\[ \Delta_\omega(E_{\text{inc}}) = \int \int \int \sigma_R(k, k_\theta, \omega, \omega_0) dk d\omega + \int \int \sigma_I(k, k_\theta, \omega, \omega_0) dk d\omega \]

Note that \( \Delta_\omega(E_{\text{inc}}) > 0 \) for constant frequency, \( \omega = \omega_0 \), which implies that incoherent energy is only produced due to spatial wrinkling.

### 2.4 Calculation of net energy flux using Gaussian statistics

Since evaluation of net energy flux requires integrations of terms including \( F_J \), \( G_J \), \( A_J \), and \( B_J \) over \( k, \omega \)-space as shown in Eqs. (4.18), (4.19), and (4.23), these integrations should be conducted first. \( F_J \) in Eq. (3.20) can be rewritten in terms of \( k \) and \( k_\theta \) by substituting \( A_J \) and \( B_J \) into \( F_J \) to yield the form after some manipulations

\[
F_J(k, k_\theta, \omega, \omega_0, \omega_0) = D_J(k_\theta, \omega_0) \left[ -\frac{\beta(\omega_0) (k - k_\theta)^2}{2} + C_{10}(k, \omega) + C_{11}(k, \omega) k \cdot k_\theta + C_{12}(k, \omega)(k \cdot k_\theta)^2 \right] \quad (4.26)
\]

Statistics of flame surface height is assumed to observe the Gaussian correlation function in time and space.

\[
\bar{W}(\xi, \eta) \equiv \langle h(r_1, t_1)h(r_2, t_2) \rangle = \langle h^2(r, t) \rangle e^{-|r_1 - r_2|^2 / l_c^2 - (t_1 - t_2)^2 / t_c^2} = \sigma^2 e^{-\xi^2 / l_c^2 - \eta^2 / t_c^2} \quad (4.27)
\]

where \( l_c \) and \( t_c \) are correlation length and time, respectively. Then the power spectrum of flame height is obtained from Eq. (4.10).

\[
W(k, \omega) = \frac{\sigma^2}{(2\pi)^3} \int \int e^{\xi^2 / l_c^2 - \eta^2 / t_c^2} e^{-i(k \cdot x - \omega t)} d\eta d\xi
\]

\[
= \frac{\sigma^2 t_c l_c^2}{8\pi^{3/2}} \exp\left\{-\left[(\omega t_c)^2 + (|k| l_c)^2 \right] / 4 \right\} \quad (4.28)
\]
which implies that a Gaussian correlation function yields a Gaussian power spectrum. Note that, if a surface is stationary, correlation time goes to infinity \( t_c \to \infty \) and the power spectrum of a stationary surface is approximated by direct delta function using one of choices of direct delta function, i.e., \( \frac{t_c}{\sqrt{\pi}} \exp[-(\omega t_c)^2] \to \delta(\omega) \) as \( t_c \to \infty \). Therefore,

\[
W(k, \omega) \equiv W(k)\delta(\omega) \quad \text{with} \quad W(k) = \frac{(\sigma l_c)^2}{4\pi} \exp\left[\frac{-|k l_c|^2}{4}\right] \quad (4.29)
\]

Integration of \( F_I W \) in Eq. (4.18) is over \( k_x, k_y, \) and \( \omega \), which is a triple integration. Utilizing a polar coordinate, however, can reduce an order of integration to yield a double integration.

\[
k = k_x \hat{e}_x + k_y \hat{e}_y, \quad k_x = k \cos \theta, \quad k_y = k \sin \theta
\]

\[
k_0 = k_{x0} \hat{e}_x + k_{y0} \hat{e}_y, \quad k_{x0} = k_0 \cos \theta_0, \quad k_{y0} = k_0 \sin \theta_0
\]

which lead to

\[
k \cdot k_0 = kk_0 \cos(\theta - \theta_0)
\]

\[
q_k^{(m)} = \left((\omega/c_m)^2 - k^2\right)^{1/2} \quad m = \{1, 2\}
\]

\[
W(k_0 - k, \omega_0 - \omega) = \frac{\sigma^2 l_c^2}{8\pi^{3/2}} \exp\left\{-\left[\left((\omega_0 - \omega) t_c\right)^2 + \left(k_0 - k l_c\right)^2\right]/4\right\}
\]

\[
= \frac{\sigma^2 l_c^2}{8\pi^{3/2}} \exp\left[-\left((\omega - \omega_0)^2 + (k^2 + k_0^2 - 2kk_0 \cos(\theta - \theta_0))l_c^2\right)/4\right] \quad (4.30)
\]

The following integral formulas will also be used.

\[
\int_0^{2\pi} \exp\left(k \cdot k_0 l_c^2 / 2\right) d\theta = 2\pi l_0 (kk_0 l_c^2 / 2) \quad (4.31)
\]

\[
\int_0^{2\pi} k \cdot k_0 \exp\left(k \cdot k_0 l_c^2 / 2\right) d\theta = 2\pi kk_0 I_1(kk_0 l_c^2 / 2) \quad (4.32)
\]
\[\int_0^{2\pi} (k \cdot k_0)^2 \exp \left( k \cdot k_0 l_c^2 / 2 \right) d\theta = \pi (kk_0)^2 \left[ I_0(kk_0l_c^2 / 2) + I_2(kk_0l_c^2 / 2) \right] \quad (4.33)\]

where \(J( )\) is a Bessel function and \(I( )\) is a modified Bessel function. Then integration of \(F_j\) \(W\) can now be performed.

\[
\begin{align*}
\int_k F_j (k,k_0,k_0,\omega,\omega_0,\omega_0)W (k_0 - k,\omega_0 - \omega) dk &= \frac{D_j(k_0,\omega_0)}{q_0^{(2)}} \frac{\sigma^2 t_f l_c^2}{8\pi^{3/2}} \exp \left[ -((\omega_0 - \omega)t_f)^2 / 4 \right] \times \\
&\quad \left\{ -\frac{\beta(\omega_0)}{2} \int k \exp \left[ -(k^2 + k_0^2)l_c^2 / 4 \right] dk \right. \\
&\quad \left. + \int_0^\infty \sum_{K=0}^\infty C_{ij}(k,\omega)(k \cdot k_0) \exp \left[ -(k^2 + k_0^2 - 2k \cdot k_0)l_c^2 / 4 \right] dk \right\} k d\theta dk \\
&= \frac{D_j(k_0,\omega_0)}{q_0^{(2)}} \frac{\sigma^2 t_f l_c^2}{8\pi^{3/2}} \exp \left[ -((\omega_0 - \omega)t_f)^2 / 4 \right] \times \\
&\quad \left\{ -\frac{\beta(\omega_0)}{2} \int k^{2} \exp \left[ -(k^2 + k_0^2)l_c^2 / 4 \right] dk \right. \\
&\quad \left. + \int_0^\infty \sum_{K=0}^\infty C_{ij}(k,\omega) \int_0^{2\pi} \exp(k \cdot k_0l_c^2 / 2) d\theta \right\} \int_0^\infty \sum_{K=0}^\infty C_{ij}(k,\omega) \\
&= \frac{D_j(k_0,\omega_0)}{q_0^{(2)}} \frac{\sigma^2 t_f l_c^2}{\pi^{1/2}} \exp \left[ -((\omega_0 - \omega)^2 t_f^2 / 4 \right] \times \\
&\quad \left\{ -\frac{\beta(\omega_0)}{l_c^2} \int \frac{e^{-k^2l_c^2 / 4}}{8} \int_0^{2\pi} \exp \left( 2C_{i0}(k,\omega)I_0(kk_0l_c^2 / 2) + 2kk_0C_{11}(k,\omega)I_1(kk_0l_c^2 / 2) \right) \right. \\
&\quad \left. \left[ I_0(kk_0l_c^2 / 2) + I_2(kk_0l_c^2 / 2) \right] k e^{-k^2l_c^2 / 4} dk \right\} \quad (4.34)
\end{align*}
\]

using \(\int_0^\infty \int k^{2} \exp \left[ -(k^2 + k_0^2)l_c^2 / 4 \right] dk \) \(= \int_0^\infty \int k^{2} \exp \left[ -(k^2l_c^2) / 4 \right] k d\theta dk = 16\pi l_c^4\).

Eq. (4.34) can be rearranged in terms of dimensionless parameters of rms height of flame front, \(\tilde{\sigma} := K_0\sigma \) \((K_0 = \omega_0 / c_1)\), correlation length, \(\tilde{l}_c := K_0l_c\), incident wave frequency,
\( \tilde{\omega}_0 := \omega_0 t_c \) (alternatively, \( \tilde{f}_0 := \omega_0 t_c / (2\pi) \)), the ratio of reference time to correlation time, \( \tau := t_c / t \), and polar angle of incidence, \( \tilde{k}_0 := k_0 / K_0 = \sin \phi_0^{(1)} \) to yield

\[
\int k \left( F_j(k,k_0,k_0,\omega,\omega_0,\omega_0) W(k_0 - k, \omega_0 - \omega) \right) dk
\]

\[
= \frac{2 \left( (\rho^2 / \rho_0) \tilde{q}_0^{(1)} / \tilde{q}_0^{(2)} \right)^{1/2}}{(\rho^2 / \rho_0) \tilde{q}_0^{(1)} + \tilde{q}_0^{(2)} + \beta(\tilde{\omega}_0) \pi^{1/2}} \tilde{\sigma}_c^2 \tilde{t}_c e^{-(\tilde{\omega}_0 - \tilde{\omega})^2 / 4} \chi
\]

\[
= \left[ \frac{\tilde{B}(\tilde{\omega}_0)}{L^2} + \frac{\tilde{I}_c^2}{8} e^{-(\tilde{\omega}_0 - \tilde{\omega})^2 / 4} \chi \right]
\]

\[
\int_{k=0}^\infty \left( 2 \tilde{C}_{\omega}(\tilde{k}, \tilde{\omega}) I_0(\tilde{k}_{0c}^2 / 2) + 2 \tilde{\kappa}_{k0} \tilde{C}_{\omega1}(\tilde{k}, \tilde{\omega}) I_1(\tilde{k}_{0c}^2 / 2) \right) \tilde{k} e^{-(\tilde{k}_{0c})^2 / 4} d\tilde{k}
\]

where

\[
\tilde{C}_{\omega m}(\tilde{k}, \tilde{\omega}) := C_{\omega m}(\tilde{k}K_0, \tilde{\omega} / t_c) / (K_0)^3 \quad (m = \{0, 1, 2\})
\]

\[
\tilde{q}_k^{(i)} := \tilde{q}_0^{(i)} / K_0 = \left[ \left( \frac{\tilde{\omega}}{\tilde{\omega}_0} \right)^2 - \tilde{k}^2 \right]^{1/2}, \quad \tilde{q}_k^{(2)} := \tilde{q}_0^{(2)} / K_0 = \left[ \left( \frac{\tilde{\omega}}{\tilde{\omega}_0} \right)^2 - \tilde{k}_0^2 \right]^{1/2}
\]

\[
\tilde{q}_0^{(1)} := \tilde{q}_0 / K_0 = \left( 1 - (\tilde{k}_0)^2 \right)^{1/2}, \quad \tilde{q}_0^{(2)} := \tilde{q}_0^{(2)} / K_0 = \left[ \left( \frac{c_1}{c_2} \right)^2 - (\tilde{k}_0)^2 \right]^{1/2}
\]

\[
\tilde{B}(\tilde{\omega}) := \frac{\beta(\tilde{\omega})}{K_0} = \frac{\beta_x \omega - i \beta_y \omega^2}{K_0} = \frac{c_1 \tilde{\omega}}{c_2 \tilde{\omega}_0} \left[ \frac{\alpha}{2} \right] \frac{(\gamma - 1) - \gamma - i(\gamma - 1)\theta \tilde{\omega}}{\Lambda^{1/2}}
\]

Note that the right hand side of Eq. (4.35) contains \( \tilde{q}_0^{(2)} \) in the denominator, which implies that the integration in Eq. (4.35) diverges for a critical angle of incidence because \( \tilde{q}_0^{(2)} \mid_{\tilde{q}_0^{(2)} = \tilde{q}_0^{(2)}} = \left[ (c_1 / c_2)^2 - (\sin \phi_0^{(1)})^2 \right]^{1/2} = 0 \) by Snell’s law. This problem can be settled by multiplying Eq. (4.35) by \( D_j(k_0, \omega_0) = \frac{2 \left( \Lambda \tilde{q}_0^{(1)} \tilde{q}_0^{(2)} \right)^{1/2}}{\Lambda \tilde{q}_0^{(1)} + \tilde{q}_0^{(2)} + \tilde{B}(\tilde{\omega}_0) \chi} \) (using \( \rho^2 / \rho_0 = \Lambda^{-1} \)).
\[ D_j(k_0, \omega_0) \int \int \int \int F_j(k, k_0, k_0, \omega, \omega_0, \omega_0) W(k_0 - k, \omega_0 - \omega) dk \]

\[ = \frac{4q_0^{(1)} \sigma^2}{\Lambda \left[ \Lambda^{-1}q_0^{(1)} + q_0^{(2)} + \tilde{p}(\tilde{\omega}_0) \right]^2 \pi^{1/2} e^{-(\tilde{\omega}_0 - \tilde{\omega})^2 / 4}} \times \]

\[ \left[ -\frac{\tilde{p}(\tilde{\omega}_0)}{\tilde{l}_c^2} \frac{\tilde{l}_c^2}{8} e^{-(\tilde{k}_0^2)^2 / 4} \times \right] \]

\[ \int_{\tilde{l}_c}^{\infty} \left\{ 2\tilde{C}_{10}(\tilde{k}, \tilde{\omega}) I_0(\tilde{k}\tilde{c}_0^2 / 2) + 2k\tilde{k}_0 \tilde{C}_{11}(\tilde{k}, \tilde{\omega}) I_1(\tilde{k}\tilde{c}_0^2 / 2) \right\} k e^{-(\tilde{k}_0^2)^2 / 4} \right] \]

which is further integrated over \( \omega \) to yield the form by using \( \rho_2 / \rho_1 = \Lambda^{-1} \) and \( c_2 / c_1 = \Lambda^{1/2} \)

\[ D_j(k_0, \omega_0) \int \int \int \int \int F_j(k, k_0, k_0, \omega, \omega_0, \omega_0) W(k_0 - k, \omega_0 - \omega) dk \]

\[ = \frac{2q_0^{(1)} \sigma^2}{\Lambda \left[ \Lambda^{-1}q_0^{(1)} + q_0^{(2)} + \tilde{p}(\tilde{\omega}_0) \right]^2 \pi^{1/2} e^{-(\tilde{\omega}_0 - \tilde{\omega})^2 / 4}} \times \]

\[ \left[ -\frac{4\tilde{p}(\tilde{\omega}_0)}{\tilde{l}_c^2} \frac{\tilde{l}_c^2}{8} e^{-(\tilde{k}_0^2)^2 / 4} \times \right] \]

\[ \int_{\tilde{l}_c}^{\infty} \left\{ 2\tilde{C}_{10}(k, \omega) I_0(k \tilde{c}_0^2 / 2) + 2k\tilde{k}_0 \tilde{C}_{11}(k, \omega) I_1(k \tilde{c}_0^2 / 2) \right\} k e^{-(\tilde{k}_0^2)^2 / 4} \right] \]

where

\[ \tilde{C}_{10}(k, \omega) = \frac{1}{\Lambda^{-1}q_0^{(1)} + q_0^{(2)} + \tilde{p}(\omega)} \times \]

\[ \left\{ \begin{array}{l}
\left[ 1 - \Lambda^{-1}q_0^{(1)} + \tilde{p}(\omega) \right] \left( \frac{q_0^{(1)} (q_0^{(2)} + \tilde{p}(\tilde{\omega}_0)) + \Lambda \tilde{p}(\omega)}{\Lambda^{-1}(\omega - \omega_0)^2 + \tilde{p}(\omega)q_0^{(2)}} \right) \\
+ \frac{\omega}{\omega_0} \left( \frac{q_0^{(1)} + \tilde{p}(\omega)}{\omega} \right)^2 + q_0^{(1)} \left[ \Lambda^{-1} \frac{\omega}{\omega_0} + \tilde{p}(\omega)q_0^{(2)} \right]
\end{array} \right\} \]

\[ -\frac{\omega}{\omega_0} \left( \frac{q_0^{(1)} + \tilde{p}(\omega)}{\omega} \right)^2 + q_0^{(1)} \tilde{p}(\omega) \right\} \]

\[ -\frac{\omega}{\omega_0} \left( \frac{q_0^{(1)} + \tilde{p}(\omega)}{\omega} \right)^2 + q_0^{(1)} \tilde{p}(\omega) \right\} \]
\[\tilde{C}_{11}(k, \omega) = \frac{1}{\Lambda^{-1} \tilde{q}^{(1)}_k + \tilde{q}^{(2)}_k + \tilde{B}(\omega)} \times \]
\[
\left(1 - \Lambda^{-1}\right) \left[\tilde{q}^{(2)}_0 \tilde{B}(\omega_0) + \tilde{q}^{(2)}_k \tilde{B}(\omega) - \frac{\omega}{\omega_0} \left(\tilde{q}^{(1)}_k + \tilde{q}^{(2)}_k\right) \left(\tilde{q}^{(2)}_0 + \tilde{B}(\omega_0)\right)\right]
\]
\[-\frac{\tilde{\omega}_b}{\omega} \left(1 - \Lambda^{-1}\right) \tilde{q}^{(2)}_0 + \tilde{B}(\omega_0) \left(\tilde{q}^{(1)}_k + \tilde{q}^{(2)}_k + \tilde{B}(\omega)\right)\]
\[
\tilde{C}_{12}(k, \omega) = \frac{-(1 - \Lambda^{-1})^2}{\Lambda^{-1} \tilde{q}^{(1)}_k + \tilde{q}^{(2)}_k + \tilde{B}(\omega)}
\]
\[
\tilde{q}^{(1)}_0 = \left(1 - k_0^2\right)^{1/2}, \quad \tilde{q}^{(2)}_0 = \left(\Lambda^{-1} - k_0^2\right)^{1/2}, \quad \tilde{q}^{(1)}_k = \left[\left(\frac{\omega}{\omega_0}\right)^2 - k^2\right]^{1/2}, \quad \tilde{q}^{(2)}_k = \left[\Lambda^{-1} \left(\frac{\omega}{\omega_0}\right)^2 - k^2\right]^{1/2}
\]
\[
\tilde{B}(\omega) = \frac{\omega}{\omega_0} \left[\frac{\alpha}{2} (\gamma - 1) - \gamma - i(\gamma - 1)\theta \tau \omega\right] (1 - \Lambda^{-1})M_S
\]

Similarly integration of \(G_j\) has a form
\[
\int \int \int_{\omega, k} G_j(k, k_0, k_0, \omega, \omega_0, \omega_0) W(k_0 - k, \omega_0 - \omega) dk d\omega
\]
\[
= \tilde{\sigma}^2 \left(\Lambda^{-1} - 1\right) + \tilde{\sigma}^2 \left(\frac{\rho_2}{\rho_1}\right) \tilde{q}^{(1)}_0 + \tilde{q}^{(2)}_0 + \tilde{B}(\omega_0) \times \left[\frac{4 \tilde{\beta}(\omega_0)}{\tilde{I}_c^4} + \tilde{I}_c^2 e^{-\tilde{I}_c^2/4} \frac{4 \pi^{1/2}}{4 \pi^{1/2}} \times \int_{\omega = -\infty}^{\infty} \int_{k = 0}^{\infty} \left(2 \tilde{C}_{30}(k, \omega) L_0(k_0, \omega_0) / 2 + 2k_0 \tilde{C}_{31}(k, \omega) L_1(k_0, \omega_0) / 2\right) \tilde{C}_{32}(k, \omega) \left[I_0(k_0, \omega_0) L_0(k_0, \omega_0) / 2 + I_1(k_0, \omega_0) L_1(k_0, \omega_0) / 2\right] \right) e^{-\left(k_0^2 + (\omega_0 - \omega)^2\right)/4} dk d\omega\]

where
\[
\tilde{C}_{30}(k, \omega) = \frac{1}{\Lambda^{-1}\tilde{q}^{(1)}_k + \tilde{q}^{(2)}_k + \tilde{\beta}(\omega)} \times \\
\left[ (1 - \Lambda^{-1})\tilde{q}^{(2)}_0 + \tilde{\beta}(\tilde{\omega}_0) \right] \left[ \Lambda^{-1}\tilde{q}^{(1)}_k \left[ \Lambda^{-1} - 1 \right] \tilde{q}^{(2)}_k - \tilde{\beta}(\omega) \right] \\
+ \frac{\tilde{\omega}_0}{\omega} \left( \tilde{q}^{(2)}_0 + \tilde{\beta}(\omega) \right) \left( \frac{\omega}{\tilde{\omega}_0} \right)^2 + \tilde{q}^{(1)}_k \left[ \Lambda^{-1} \left( \frac{\omega}{\tilde{\omega}_0} \right)^2 + \tilde{\beta}(\omega)\tilde{q}^{(2)}_k \right] \\
- \left( \Lambda^{-1}\tilde{q}^{(1)}_0, \tilde{q}^{(2)}_0 + \tilde{\beta}(\tilde{\omega}_0) \right) \left( \tilde{q}^{(1)}_k + \tilde{q}^{(2)}_k \right) + \tilde{\beta}(\omega) \\
\left[ \Lambda^{-1}\tilde{q}^{(1)}_0, \tilde{q}^{(2)}_0 + \tilde{\beta}(\tilde{\omega}_0) \right] \left( \tilde{q}^{(1)}_k + \tilde{q}^{(2)}_k \right) + \tilde{\beta}(\omega) \\
\right) \\
\]
\[ \int \int \sigma_{T,j}(k, k_0, \omega, \omega_0) \, dk \, d\omega = \int \int \frac{\partial}{\partial \omega_0} \left[ B_j(k, k_0, \omega, \omega_0) \right] \, W(k - k_0, \omega - \omega_0) \, dk \, d\omega \]

\[ = \frac{\Lambda^{-1} \tilde{\sigma}_j^2 \tilde{q}_0^{(1)}}{2\pi^{1/2} \tilde{\alpha}_0} \left( \frac{\Lambda^{-1} \tilde{q}_0^{(1)} + \tilde{q}_0^{(2)} + \tilde{\beta}(\tilde{\alpha}_0)}{2} \right)^2 \]

\[ \times \int_{\omega = -\infty}^{\infty} \left( 2 \tilde{C}_{40}(k, \omega) I_0(k \tilde{k}_0 \tilde{j}_c^2 / 2) + 2 k \tilde{k}_0 \tilde{C}_{41}(k, \omega) I_1(k \tilde{k}_0 \tilde{j}_c^2 / 2) \right) \kappa \omega e^{-\left( k \tilde{j}_c^2 + (\tilde{\alpha}_0 - \omega)^2 \right)/4} \, dk \, d\omega \]

where \( \tilde{C}_{40}(k, \omega) = \frac{\tilde{q}_k^{(2)} \tilde{C}_{43}(k, \omega)^2}{\Lambda^{-1} \tilde{q}_k^{(1)} + \tilde{q}_k^{(2)} + \tilde{\beta}(\omega)} \), \( \tilde{C}_{41}(k, \omega) = \frac{2(1 - \Lambda^{-1}) \tilde{q}_k^{(2)} \text{Re}[\tilde{C}_{43}(k, \omega)]}{\Lambda^{-1} \tilde{q}_k^{(1)} + \tilde{q}_k^{(2)} + \tilde{\beta}(\omega)} \)

\[ \tilde{C}_{42}(k, \omega) = \frac{(1 - \Lambda^{-1})^2 \tilde{q}_k^{(2)}}{\Lambda^{-1} \tilde{q}_k^{(1)} + \tilde{q}_k^{(2)} + \tilde{\beta}(\omega)} \], \( \tilde{C}_{43}(k, \omega) = \frac{\tilde{q}_k^{(1)} (\Lambda^{-1} - 1) \tilde{q}_k^{(2)} - \tilde{\beta}(\tilde{\alpha}_0)}{\Lambda^{-1} \tilde{q}_k^{(1)} + \tilde{q}_k^{(2)} + \tilde{\beta}(\omega)} \)

### 2.5 Results and Discussion

Figure 2 shows variation of net energy flux with \( \tilde{f}_0 \) and \( \tau \) for incidence polar angle of 10 deg. Total energy flux, as shown in (a), is damped most by -10 % for \( \tilde{f}_0 = 0.1 \). Such damping results from coherent flux damping by -12 % and incoherent flux production by 2 %, as shown in (b) and (c). These coherent flux damping and incoherent flux production are due mostly to wrinkling effect, not much to jump effect, as shown in (d) and (e). This follows from the fact that a smaller value of \( \tilde{f}_0 \) yields higher frequency of flame surface oscillation \( (f_c > f_0) \). Note that \( \tilde{f}_0 = f_0 / f_c \) which enhances unsteady wrinkling effect. A smaller value of \( \tilde{f}_0 \) also yields a smaller value of \( \tilde{\beta}(\tilde{\alpha}_0) \) leading to smaller jump effect. Note that smaller jump effect at smaller \( \tilde{f}_0 \) explains little dependence of net energy flux upon \( \tau \). (\( \tau \) only appears in \( \tilde{\beta} \)). For \( \tilde{f}_0 \) of O(1), total net energy flux is very small, shown in (a), because
coherent damping occurs almost as much as incoherent production occurs, shown in (b) and (c). Note in (d) and (e) that jump effect becomes significant for $\tilde{f}_o > 1$ even though coherent and incoherent fluxes due to jump effect are nearly canceled with each other for $\tilde{f}_o < 4$. For larger value of $\tilde{f}_o (> 4)$, however, net coherent flux due to jump effect starts to increase significantly, which results in amplification in total net energy flux for larger value of $\tau (> 4)$, as shown in (a). Local minimization of net coherent flux along a curve of $\tilde{f}_o \tau \approx 40$, shown in (b), is attributed mainly to jump effect, as shown in (d).
(b)
Figure 2  Dependence of net energy flux upon $\tilde{f}_0$ and $\tau$ ($\bar{c}=0.3$, $\tilde{l}_c=1.5$, $\phi_0=10$ deg)

Figure 3 shows the dependence of net energy flux upon $\tilde{l}_c$ and $\phi_0$ for $\tilde{f}_0 = \tau = 0.1$. Total energy flux, as shown in (a), is damped except for near-critical angle. It also shows that total net energy flux has little dependence upon $\tilde{l}_c$, as opposed to large dependence upon
incidence angle. Coherent flux, as shown in (b), has a similar trend to total energy flux except that coherent flux is damped for entire range of $\tilde{I}_c$ and incidence angle, and most coherent damping occurs at supercritical angle at about 35 °. Incoherent flux, as shown in (c), is always produced for entire range of $\tilde{I}_c$ and incidence angle. The amount of incoherent production is nearly constant for sub- and supercritical angle, respectively, with a sudden jump across critical angle. As such more incoherent flux is produced for supercritical angle than for subcritical angle.
Figure 3  Dependence of net energy flux upon $\tilde{I}_c$ and $\phi_0$ ($\tilde{\sigma} = 0.3$, $\tilde{f}_0 = \tau = 0.1$)
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