

Final Report for Period: 07/2008 - 06/2009

Submitted on: 07/03/2009

Principal Investigator: Ghomi, Mohammad .

Award ID: 0332333

Organization: GA Tech Res Corp - GIT

Submitted By:

Ghomi, Mohammad - Principal Investigator

Title:

CAREER: Classical Problems in Differential Geometry, Topology, and Convexity

Project Participants

Senior Personnel

Name: Ghomi, Mohammad

Worked for more than 160 Hours: Yes

Contribution to Project:

Post-doc

Name: Stojanovic, Gordana

Worked for more than 160 Hours: Yes

Contribution to Project:

Graduate Student

Name: Tyurina, Yulia

Worked for more than 160 Hours: Yes

Contribution to Project:

Yulia wrote her PhD thesis on 'Existence and Nonexistence of Skew Branes' and received support in the Summer of 2004.

Name: van Erp, Erik

Worked for more than 160 Hours: Yes

Contribution to Project:

Erik helped PI teach an undergraduate class the PI taught in the Spring on 2004.

Name: Krysiak, James

Worked for more than 160 Hours: Yes

Contribution to Project:

James Krysiak is working on the Alexandrov's conjecture on the intrinsic diameter and area of convex surfaces.

Undergraduate Student

Name: McCoy, Zachary

Worked for more than 160 Hours: Yes

Contribution to Project:

Zachary, a student at Yale University, participated both in Penn State's REU in the Summer of 2004, and Penn State's MASS program the following Fall. During this period he worked on Alexandrov's conjecture on intrinsic diameter of convex surfaces and received a fellowship at the end of the Fall semester.

Name: Krysiak, James

Worked for more than 160 Hours: Yes

Contribution to Project:

James, a student at Alfred University, participated both in Penn State's REU in the Summer of 2004, and Penn State's MASS program the following Fall. During this period he worked on Alexandrov's conjecture on intrinsic diameter of convex surfaces and received a

fellowship at the end of the Fall semester.

Name: Friend, Arthur

Worked for more than 160 Hours: Yes

Contribution to Project:

AJ Friend worked on the unfolding problem for convex polytopes, and wrote a program which unfolds a given convex polytope into the plane.

Technician, Programmer

Other Participant

Research Experience for Undergraduates

Name: Bush, Casey

Worked for more than 160 Hours: Yes

Contribution to Project:

Casey recieved Summer support from the PI in order to participate in Penn State's REU program in the Summer of 2004. The PI supervised Casey's research during this period on the rigidity of closed surfaces. In the following two semesters Casey took undergraduate and graduate classes which the PI taught on differential geometry.

Years of schooling completed: Freshman

Home Institution: Same as Research Site

Home Institution if Other:

Home Institution Highest Degree Granted(in fields supported by NSF): Associate's Degree

Fiscal year(s) REU Participant supported:

REU Funding: No Info

Name: Culter, Christopher

Worked for more than 160 Hours: Yes

Contribution to Project:

Christopher, a student at University of California at Berkeley, recieved support from the PI in order to participate in Penn State's REU in the Summer of 2004. He did his research on outer billiards.

Years of schooling completed: Freshman

Home Institution: Same as Research Site

Home Institution if Other:

Home Institution Highest Degree Granted(in fields supported by NSF): Associate's Degree

Fiscal year(s) REU Participant supported:

REU Funding: No Info

Name: Nakamura, Brian

Worked for more than 160 Hours: Yes

Contribution to Project:

Brian Nakamura worked on the Alexandrov conjecture on the intrinsic diameter and area of convex surfaces.

Years of schooling completed: Junior

Home Institution: Same as Research Site

Home Institution if Other:

Home Institution Highest Degree Granted(in fields supported by NSF): Doctoral Degree

Fiscal year(s) REU Participant supported: 2006

REU Funding: REU supplement

Name: DeMarco, Bobby

Worked for more than 160 Hours: Yes

Contribution to Project:

Boby Demarco worked on the converse of the four vertes theorem on surfaces with constant curvature.

Years of schooling completed: Sophomore

Home Institution: Other than Research Site

Home Institution if Other: Universit of Delaware

Home Institution Highest Degree Granted(in fields supported by NSF): Doctoral Degree

Fiscal year(s) REU Participant supported: 2006

REU Funding: REU supplement

Organizational Partners

PENNSYLVANIA STATE UNIVERSITY

Other Collaborators or Contacts

Kossowski, Marek (University of South Carolina)

Worked for more than 160 Hours: Yes

Contribution to Project: Coauthor of the paper

'Prescribing principal curvatures and directions of hypersurfaces with boundary.'

Tabachnikov, Serge (Penn State University)

Worked for more than 160 Hours: Yes

Contribution to Project: Coauthor of the paper

'Totally skew embedding of manifolds.'

Choe, Jaigyoung (Seol National University)

Worked for more than 160 Hours: Yes

Contribution to Project: Coauthor of the paper

'Total curvature of hypersurfaces with convex boundary', and

'Relative isoperimetric inequality outside convex domains in R^n '

Ritore, Manuel (University of Granada)

Worked for more than 160 Hours: Yes

Contribution to Project: Coauthor of the paper

'Total curvature of hypersurfaces with convex boundary', and

'Relative isoperimetric inequality outside convex domains in R^n '

Connell, Christopher (Indiana University, Bloomington)

Worked for more than 160 Hours: Yes

Contribution to Project: Coauthor of the paper

'Topology of negatively curved real algebraic surfaces in R^3 '

Alexander, Stephanie (UIUC)

Worked for more than 160 Hours: Yes

Contribution to Project: Coauthor of the paper

'Topology of Submanifolds with prescribed boundary'

Wong, Jeremy (Toronto)

Worked for more than 160 Hours: Yes

Contribution to Project: Coauthor of the paper

'Topology of Submanifolds with prescribed boundary'

Greene, Robert (UCLA)

Worked for more than 160 Hours: Yes
 Contribution to Project: Coauthor of the paper
 'Relative isometric embeddings of Riemannian manifolds'

Activities and Findings

Research and Education Activities:

TALKS:

U. of Alabama, Birmingham, Southeast Geom. Seminar III, May '03
 U. of Georgia, Georgia Topology Conference, Jun '03
 Conf. on Monge-Ampere Equations, Banff, Canada, Aug. '03
 Penn State, Dynamical Systems and Geometry Seminar, Sep. '03
 Penn State, MASS Colloquium (for undergraduates), Oct. '03
 Georgia Tech, Geometry Seminar, Nov. '03
 Penn State, Topology Seminar, Nov. '03
 University of Pennsylvania, Geometry Seminar, Mar '04
 The XIII School of Diff. Geometry, U. of Sao Paulo, Plenary Talk, Jul. '04.
 IMPA, Rio de Janeiro, Geometry Seminar, Jun '04.
 Penn State, MASSfest Conf. for REU students, Aug '04.
 Penn State, Geometry Seminar, Sep. '04.
 Georgia Tech, Geometry Seminar, Dec. '04.
 Columbia U., Geometry and Analysis Seminar, Mar. '05.
 U. Alabama Birmingham, Colloquium, Oct. '05.
 Indiana University, Bloomington Geometry Workshop, Apr '06.
 Canadian Math Soc, Calgary, Session on Discrete and Convex Geometry, Jun '06.
 University of Georgia, Geometry Seminar, Sep '06.
 Emory University, Differential Geometry/Analysis Seminar, Oct '06.
 Johns Hopkins U., Conf. on Geometric Analysis and Elliptic PDE's, Oct '06.
 Korean Institute for Advanced Study, Seoul, Conf on Geometric Analysis, Sept '07.
 Duke University, Geometry Seminar, Oct '07.
 University of Notre Dame, Felix Klein Seminar, Feb '08.
 University of Georgia, Geometry Seminar, Mar '08.
 George Washington University, Geometry Seminar, Mar '08.
 XV Brazilian School of Differential Geometry, Fortaleza, Plenary Talk, July 2008
 Penn State U., MASS Colloquium, Sep. 2008
 Penn State U., Geometry Seminar, Sep. 2008
 AMS Western, Vancouver, Session on Convex Geometry, Oct. 2008
 Georgia Tech, Colloquium, Oct. 2008
 Rutgers U., Newark, Conference in honor of Herman Gluck, Nov. 2008
 University of Calgary, Colloquium and Fejes Toth Lecture, Mar. 2009
 University of Calgary, Discrete Geometry Seminar, Mar. 2009
 AMS Central, Urbana, Session on Differential Geometry, Mar. 2009
 College of Charleston, Colloquium, Apr. 2009
 University of Minnesota, Differential Geometry Seminar, May 2009

GRADUATE CLASSES TAUGHT:

Introduction to the h-principle, Fall '03, PSU.

Geometry and Topology II, Spring '05, PSU.
 Differential Topology, Fall '05 Georgia Tech.
 Differential Geometry I, Fall '06, Georgia Tech.
 Differential Geometry II, Fall '06, Georgia Tech.

UNDERGRADUATE CLASSES TAUGHT:

Introduction to Modern Geometry, Spring '04.
 Differential Geometry and Topology of Curves and Surfaces, Fall '04.
 Honors Calculus III, Spring '06.
 Differential Geometry, Fall '07.
 Calculus II, Spring '08.

SERVED AS REU ADVISOR in the Summer of 2004, Penn State U, and in the Summer of 2006 at Georgia Tech.

Findings:

The paper 'Totally skew embeddings of manifolds' with Serge Tabachnikov is concerned with obtaining bounds on the least dimension of an affine space that can contain an n -dimensional submanifold without any pairs of parallel or intersecting tangent lines at distinct points. This problem is closely related to the generalized vector field problem, non-singular bilinear maps, and the immersion problem for real projective spaces.

The paper on 'h-Principles for hypersurfaces with prescribed principal curvatures and directions', uses Gromov's h-principle, specifically the holonomic approximation theorems, to prove that any compact orientable hypersurface with boundary immersed (resp. embedded) in Euclidean space is regularly homotopic (resp. isotopic) to a hypersurface with principal directions which may have any prescribed homotopy type, and principal curvatures each of which may be prescribed to within an arbitrary small error of any constant. Further we construct regular homotopies (resp. isotopies) which control the principal curvatures and directions of hypersurfaces in a variety of ways. These results generalize theorems of Gluck and Pan on embedding and knotting of positively curved surfaces in 3-space.

In the paper 'Total positive curvature of hypersurfaces with convex boundary', with J. Choe and M. Ritore, it is proved that if the boundary of a compact hypersurface in Euclidean n -space lies on the boundary of a convex body and meets that convex body orthogonally from the outside, then the total positive curvature of the hypersurface is bigger than or equal to half the area of the $(n-1)$ -sphere. Also we obtain necessary and sufficient conditions for the equality to hold.

In the paper 'Relative isoperimetric inequality outside convex domains in R^n ' which is also joint work with J. Choe and M. Ritore, it is proved that the area of a hypersurface which traps a given volume outside of a convex body in Euclidean n -space must be greater than or equal to the area of a hemisphere trapping the given volume on one side of a hyperplane.

In the paper 'Topology of surfaces with connected shades', the PI proves that any closed orientable two dimensional manifold may be smoothly embedded in Euclidean 3-space so as to have connected shades (a.k.a. shadows) with respect to all directions of illumination. This generalizes some earlier findings of the PI when he solved 'Wente's Shadow Problem'.

In the paper 'h-Principles of curves and knots of constant curvature', the PI proves that smooth curves of constant curvature satisfy, in the sense of Gromov, the relative C^1 -dense h-principle in the space of immersed curves. In particular, in the isotopy class of any given C^1 knot f in Euclidean space R^n there exists a smooth knot g of constant curvature which is C^1 -close to f . Further we show that if f is C^2 , then the curvature of g may be set equal to any constant c which is equal to or bigger than the maximum curvature of f . Furthermore, we may require that g be tangent to along any finite set of prescribed points, and coincide with f over any compact set with an open neighborhood where f has constant curvature c . The proof involves some basic convexity theory and a sharp estimate for the position of the average value of a parameterized curve within its convex hull.

In the paper 'Topology of negatively curved real algebraic surfaces in R^3 ', which is joint work with C. Connell, we find quartic examples of smooth embedded negatively curved surfaces in R^3 homeomorphic to singly and doubly punctured tori, and a triply punctured sphere. These constitute explicit solutions to Hadamard's problem on constructing complete surfaces with negative curvature and Euler characteristic in R^3 . Further we show that our solutions have the optimal degree of algebraic complexity via a topological classification for smooth cubic surfaces with a negatively curved component in R^3 : any such component must either be topologically a plane or an annulus. In particular, there exists no cubic solutions to Hadamard's problem.

In the paper on 'Topology of Riemannian submanifolds with prescribed boundary, which is joint work with Stephanie Alexander and Jeremy Wong, We prove that a smooth compact immersed submanifold of codimension 2 in R^n , $n > 2$, bounds at most finitely many topologically distinct compact nonnegatively curved hypersurfaces. This settles a question of Guan and Spruck related to a problem of Yau. Analogous results for complete fillings of arbitrary Riemannian submanifolds are obtained as well. On the other hand, we show that these finiteness theorems may not hold if the codimension is too high, or the prescribed boundary is not sufficiently regular.

In the paper on 'Relative isometric embedding of Riemannian manifolds, which is joint work with Rober Greene, we prove the existence of C^1 isometric embeddings, and smooth approximate isometric embeddings, of Riemannian manifolds into Euclidean space with prescribed values in a neighborhood of a point.

In the paper 'A Riemannian four vertex theorem for surfaces with boundary' PI proves that every metric of constant curvature on a compact surface M with boundary bdM induces at least four vertices, i.e., local extrema of geodesic curvature, on a connected component of bdM , if, and only if, M is simply connected. Indeed, when M is not simply connected, we construct hyperbolic, parabolic, and elliptic metrics of constant curvature on M which induce only two critical points of geodesic curvature on each component of bdM . With few exceptions, these metrics are obtained by removing the singularities and a perturbation of flat structures on closed surfaces.

Another paper on four vertex theorems is 'Vertices of closed curves in Riemannian surfaces', where the PI uncovers some connections between the topology of a complete Riemannian surface M and the minimum number of vertices, i.e., critical points of geodesic curvature, of closed curves in M . In particular we show that the space forms with finite fundamental group are the only surfaces in which every simple closed curve has more than two vertices. Further we characterize the simply connected space forms as the only surfaces in which every closed curve bounding a compact immersed surface has more than two vertices.

Training and Development:

PhD Student:

Ioulia Tiourina completed her PhD thesis in the Summer of 2005 on skew surfaces in R^4 .

Jim Krysiak is currently working on Alexandrov's conjecture.

Postdoc:

Gordanna Stoyanovic, 2007-2009 at Georgia Tech, who worked on the skew submanifolds and embeddings.

Teaching Assistants:

Erik Van Erp has gained more experience in discussing differential geometry problems with undergraduate students. He assisted the PI in teaching an undergraduate class on differential geometry.

Robert Yuncken assisted the PI in teaching an undergraduate differential geometry class, as a part of PSU's MASS program, in the Fall of '04. He attended all lectures and wrote solutions to homework problems.

REU and Undergraduate Students:

During Penn State's REU program in the Summer of '04 the PI helped supervise or provide support for the research of several students:

Casey Bush (PSU) and Toan Phan (Bucknell) worked jointly on the problem of rigidity for closed surfaces.

Zachary McCoy (Yale) and James Krysiak (Alfred University) worked on Alexandrov's conjecture on the diameter of convex surfaces.

Dmitry Vaintrob (University of Oregon) worked on the Bicycle problem.

Christopher Cutler (Berkeley) worked on outer billiards.

In the Fall of '04, the PI taught the class on Differential Geometry during the Penn State's MASS program and supervised the following research projects:

Jenna Hammang (Valparaiso U.), Distortion of Curves.

Rob King (Clark U.), Geometry of minimal surfaces bounded by convex planar curves.

Sarah Mall (St. Edwards's U.), the DNA inequality.

Casey Bush (PSU), Schwartz-Child solution in General Relativity.

James Krysiak (Alfred U.), Alexandrov's conjecture on degenerate surfaces and surfaces of revolution.

Zachary McCoy (Yale), Alexandrov's conjecture for convex polytopes with few sides.

Van Cyr (U. Buffalo), the bicycle problem.

At Georgia Tech, the PI supervised the research of AJ Friend on the unfolding problem for convex polytopes.

Further, he supervised two REU students in the Summer of 2006: Brian Nakamura and Bobby Demarco.

Currently, the PI is supervising a PhD student, Jim Krysiak, at Georgia Tech.

Outreach Activities:

The PI presented a talk on 'Solution to the shadow problem' in the MASS colloquium at Penn State University in the Fall of 2003. MASS is a semester long program which brings talented undergraduates to Penn State each year for intensive study of Mathematics.

In the Summer of 2004, the PI served as an advisor to several REU students, including one from high school, and also gave a talk at Penn States MASSfest conference for REU students.

In the Fall of 2004, the PI taught a class on differential geometry as a part of Penn State's MASS program.

Since arriving at Georgia Tech, in the Fall of 2005, the PI has been organizing a series of mathematical talks aimed at a general audience, especially the undergraduate students, the first three speakers in this series have been Tom Banchoff and John Sullivan, and John Morgan. Each event attracted about 100 audience members from the entire campus.

Journal Publications

A. Abrams, J. Cantarella, J. Fu, M. Ghomi, and R. Howard, "Circles minimize most knot energies", *Topology*, p. 381, vol. 42, (2003). Published,

S. Alexander and M. Ghomi, "The convex hull property and topology of hypersurfaces with nonnegative curvature", *Advances in Mathematics*, p. 324, vol. 180, (2003). Published,

M. Ghomi, "A smooth convex loop with vanishing projections", *Topology*, p. 245, vol. 43, (2004). Published,

M. Ghomi, "Optimal smoothing for convex polytopes", *Bulletin of London Mathematical Society*, p. 483, vol. 36, (2004). Published,

M. Ghomi, "Shortest periodic billiard trajectories", *Geometric and Functional Analysis (GAFA)*, p. 295, vol. 14, (2004). Published,

S. Alexander and M. Ghomi, "The convex hull property of noncompact hypersurfaces with positive curvature", *American Journal of Mathematics*, p. 891-897, vol. 126, (2004). Published,

M. Ghomi, "Tangent bundle embeddings of manifolds in Euclidean space", *Comment. Math. Helv.*, p. 259, vol. 81, (2006). Published,

M. Ghomi and S. Tabachnikov, "h-Principles for hypersurfaces with Prescribed principal curvatures and directions", Tran. Amer. Math. Soc., p. 129, vol. 358, (2006). Published,

M. Ghomi and S. Tabachnikov, "Totally skew embedding of manifolds", Math. Z., p. , vol. 258, (2008). Published,

M. Ghomi, "Nonexistence of skew loops on ellipsoids", Proc. Amer. Math. Soc, p. 3687, vol. 133, (2005). Published,

J. Choe, M. Ghomi, and M. Ritore, "Total positive curvature of hypersurfaces with convex boundary", J. Differential Geom., p. 129, vol. 72, (2006). Published,

J. Choe, M. Ghomi, M. Ritore, "Relative isoperimetric inequality outside convex domains in R^n ", Calc. Var. Partial Differential Equations, p. 42, vol. 29, (2007). Published,

M. Ghomi, "Topology of Surfaces with Connected Shades", Asian J. Math, p. , vol. , (). Accepted,

M. Ghomi, "h-Principles for curves and knots of constant curvature", Geom. Dedicata, p. , vol. , (2008). Accepted,

C. Connell, and M. Ghomi, "Topology of negatively curved real algebraic surfaces in R^3 ", J. Reine Angew. Math., p. 1, vol. 624, (2008). Published,

S. Alexander; M. Ghomi; J. Wong, "Topology of Riemannian submanifolds with prescribed boundary", Preprint, p. , vol. , (2008). Preprint,

M. Ghomi; R. Greene, "Relative isometric embeddings of Riemannian manifolds", To appear in Transactions of American Mathematical Society, p. , vol. , (2008). Accepted,

Ghomi, Mohammad, "A Riemannian four vertex theorem for surfaces with boundary", Submitted to Journal of Differential Geometry, p. , vol. , (2009). Submitted,

Ghomi, Mohammad, "Vertices of closed curves in Riemannian surfaces", Submitted to Journal of Topology, p. , vol. , (2009). Published,

Books or Other One-time Publications

Web/Internet Site

URL(s):

www.math.gatech.edu/~ghomi

Description:

This site contains the preprints for the PI's papers as well as links to websites which the PI set up for his graduate and undergraduate classes. These websites contain lecture notes and Mathematica software packages written by the PI.

Other Specific Products

Product Type:

Software (or netware)

Product Description:

Mathematica software packages for exploring differential geometry of curves and surfaces. There are three main ones on planar curves, space curves, and surfaces. The one on planar curves is close to its final form, and the rest will be refined further. Each notebook contains dozens of programs for computing various geometric quantities or generating graphs.

Sharing Information:

These software packages are currently available freely on line at:

www.math.gatech.edu/~ghomi

Contributions

Contributions within Discipline:

The PI's work on 'Prescribing principal curvatures and directions of hypersurfaces with boundary' uses the h-principle of Gromov -- a relatively recent development in differential geometry and topology -- to generalize some theorems of Gluck and Pan on the structure of positively curved surfaces with boundary in Euclidean space. It is hoped that this result will not only encourage further study of the h-principle, but also prompt more people to search for applications of recent techniques to classical problems in differential geometry.

The work on 'Skew submanifolds' is concerned with a Whitney type topological problem (finding the least codimension for an embedding of a manifold in Euclidean space) in the setting of affine or projective geometry (via the requirement that the embedding have no pairs of parallel or intersecting tangent lines). This is a new area of research initiated by the PI and his collaborators, Bruce Solomon and Serge Tabachnikov. This field contains many interesting open problems which provide opportunity for interaction between topology, differential geometry, and algebraic geometry.

The two papers which have been written in collaboration with J. Choe and M. Ritoré are concerned with a generalization of the classical isoperimetric inequality, which states that the sphere is the surface of least area bounding a given volume. Here we obtain an analogous result for surfaces which trap a given volume outside a convex body. Study of isoperimetric inequalities has been a very active area of research in geometric analysis ever since this field has existed, and has been directly or indirectly responsible for a development of many techniques which have found application elsewhere.

PI's work on shades on illuminated surfaces answers a fundamental question in this area: when does a surface have connected shades? This field is of interest in computer vision where 'shape from shading' problems are studied, and also has applications in geometric variational problems; specifically, the stability of constant mean curvature surfaces, or soap films.

Another work on h-principle in differential geometry is on knots of constant curvature where the PI proves that any curve in Euclidean space may be C^1 -approximated by a curve of constant curvature. Thus one could say that there exists no 'visual difference' between an arbitrary smooth curve in \mathbb{R}^3 and one whose curvature is constant. More importantly, the PI obtains a sharp estimate for the curvature of the approximating curve.

In the joint work with Chris Connell, the PI explores the relation between differential geometry and classical algebraic geometry by studying Hadamard's problem on constructing complete surfaces of negative curvature in \mathbb{R}^3 with nontrivial topology. In particular we construct explicit examples of quartic surfaces with negative curvature and Euler characteristic and show that there exist no such cubic surfaces.

In joint work with Stephanie Alexander and Jeremy Wong we use methods from the theorem of Alexandrov spaces with curvature bounded below to answer affirmatively a question of Guan and Spruck on finiteness of topological types of positively curved hypersurfaces in Euclidean space bounded by a given boundary. This question is related to a problem of Yau: when does a curve in Euclidean 3-space bound a surface of positive curvature. Further, to prove our main results here, we develop a relative version of Nash's isometric embedding theorem in joint work with Robert Greene, which proved the existence of isometric embeddings with prescribed values in a neighborhood of a point. This result is used to show that the finiteness theorem mentioned above does not hold when the codimension is too high.

Contributions to Other Disciplines:

The PI's work on Shades or Shadows on illuminated surfaces are of interest in computer vision, where the 'shape from shading' problems are studied.

Contributions to Human Resource Development:

As described elsewhere in this report, the PI has directed the research of graduate, undergraduate, and high school students.

Contributions to Resources for Research and Education:

The PI has developed several Mathematica software packages for studying curves and surfaces. These are freely available on the PI's website.

Contributions Beyond Science and Engineering:

Conference Proceedings

Categories for which nothing is reported:

Any Book

Contributions: To Any Beyond Science and Engineering

Any Conference