6-CONNECTED GRAPHS ARE TWO-THREE LINKED

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Shijie Xie

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Approved by:

Dr. Xingxing Yu, Advisor
School of Mathematics
Georgia Institute of Technology

Dr. Robin Thomas
School of Mathematics
Georgia Institute of Technology

Dr. Prasad Tetali
School of Mathematics
Georgia Institute of Technology

Dr. Richard Peng
School of Computer Science
Georgia Institute of Technology

Dr. Lutz Warnke
School of Mathematics
Georgia Institute of Technology

Date Approved: October 24, 2019
No, emptiness is not nothingness. Emptiness is a type of existence. You must use this existential emptiness to fill yourself.

*Liu Cixin, The Three-Body Problem*
To my parents and my wife.
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SUMMARY

Let $G$ be a graph and $a_0, a_1, a_2, b_1, b_2$ be distinct vertices of $G$. Motivated by their work on Four Color Theorem, Hadwiger’s conjecture for $K_6$, and Jørgensen’s conjecture, Robertson and Seymour asked when does $G$ contain disjoint connected subgraphs $G_1, G_2$, such that \( \{a_0, a_1, a_2\} \subseteq V(G_1) \) and \( \{b_1, b_2\} \subseteq V(G_2) \). We prove that if $G$ is 6-connected then such $G_1, G_2$ exist. Joint work with Robin Thomas and Xingxing Yu.
CHAPTER 1
INTRODUCTION AND BACKGROUND

1.1 Introduction to Hadwiger’s conjecture and 2-3 linked graphs

The Four Color Theorem [1, 2, 3] asserts that every loopless planar graph admits a vertex 4-coloring. The related problem was first put forward by Francis Guthrie in 1852, who asked whether it is true that any planar map can be colored with four colors such that adjacent regions receive different colors. In 1976, Appel and Haken [1] claimed a proof of the Four Color Theorem with the help of a computer. However, some computer-free parts of their proof are complicated and tedious to verify. In 1997, Robertson, Sanders, Seymour, and Thomas [2, 3] gave a much simpler proof for the Four Color Theorem.

According to Kuratowski’s theorem [4], a graph is planar if and only if it contains no $K_5$-subdivision or $K_{3,3}$-subdivision. Moreover, it is well known that any 3-connected nonplanar graph other than $K_5$ contains a $K_{3,3}$-subdivision. Hence, as an extension of the Four Color Theorem, it is natural to ask whether every graph without $K_5$-subdivision is also 4-colorable. More generally, Hajós [5] conjectured that for any positive integer $k$, any graph containing no $K_{k+1}$-subdivision is $k$-colorable. This conjecture is true for $k \leq 3$, but Catlin [5] found counterexamples to this conjecture for each $k \geq 6$. However, the cases for $k = 4$ and $k = 5$ are still open. Efforts have been made to resolve Hajós’ conjecture for $k = 4$. Yu and Zickfeld [6] proved that a minimum counterexample to Hajós’ conjecture when $k = 4$ must be 4-connected. Moreover, Sun and Yu [7] showed that if $G$ is a minimum counterexample to Hajós’ conjecture and $S$ is a 4-cut in $G$ then $G - S$ has exactly two components. In fact, if one can show a minimum counterexample to Hajós’ conjecture for $k = 4$ is 5-connected, then Hajós’ conjecture for $k = 4$ will immediately follow from the Kelmans-Seymour conjecture [8, 9]: Every 5-connected nonplanar graph
contains $K_5$-subdivision. This Kelmans-Seymour conjecture was recently proved by He, Wang, and Yu [10, 11, 12, 13].

While Hajós’ conjecture concerns the chromatic number of graphs with no $K_{k+1}$-subdivision, Hadwiger [14], in 1943, conjectured a far-reaching generalization of the Four Color Theorem in terms of $K_{k+1}$-minor: For any positive integer $k$, if a graph contains no $K_{k+1}$-minor then it is $k$-colorable.

It is easy to prove that Hadwiger’s conjecture holds for $k \leq 2$. Hadwiger [14] and Dirac [15] proved the case for $k = 3$. For $k = 4$, Hadwiger’s conjecture is equivalent to the Four Color Theorem by the result of Wagner [16], which characterized graphs containing no $K_5$-minor and showed that Four Color Theorem implies that graphs containing no $K_5$-minor are 4-colorable. The case $k = 5$ can also be reduced to the Four Color Theorem, as shown by Robertson, Seymour, and Thomas [17]. However, this conjecture remains open for $k \geq 6$.

In fact, there are also many other interesting results related to Hadwiger’s conjecture. Suppose Hadwiger’s conjecture is false for some $k$, and let $G$ be a minor minimal counterexample. Dirac [15] showed that $G$ is 5-connected when $k \geq 5$, and Mader [18] showed that $G$ is 6-connected when $k \geq 5$, and 7-connected when $k \geq 6$. Kawarabayashi and G. Yu [19] proved that $G$ is $(2k/27)$-connected, improving upon an earlier bound in [20].

Let the stability number $\alpha(G)$ of a graph $G$ denote the size of the largest stable set in $G$. Then every $n$-vertex graph $G$ has chromatic number at least $\lceil n/\alpha(G) \rceil$, and should contain a clique minor of this size if Hadwiger’s conjecture is true. In 1982, Duchet and Meyniel [21] proved that every $n$-vertex graph $G$ has a $K_k$-minor where $k \geq n/(2\alpha(G) - 1)$. Moreover, there has been a subsequent improvement by Fox [22]. And then Balogh and Kostochka [23] further improved the result, and showed that every $n$-vertex graph $G$ has a $K_k$-minor where $k \geq 0.51338n/\alpha(G)$. Later, in 2007, Kawarabayashi and Song [24] proved that every $n$-vertex graph $G$ with $\alpha(G) \geq 3$ has a $K_k$-minor where $k \geq n/(2\alpha(G) - 2)$. 2
For an \( n \)-vertex graph \( G \) with \( \alpha(G) = 2 \), the Duchet-Meyniel theorem implies that there is a \( K_k \)-minor with \( k \geq n/3 \), which was strengthened by Böhme, Kostochka and Thomason [25] in 2011. They proved that every \( n \)-vertex graph with chromatic number \( t \) has a \( K_k \)-minor where \( k \geq (4t - n)/3 \).

A graph is \textit{claw-free} if no vertex has three pairwise nonadjacent neighbours. So graphs with stability number two are claw-free. Fradkin [26] showed that every \( n \)-vertex connected claw-free graph \( G \) with \( \alpha(G) \geq 3 \) has a \( K_k \)-minor where \( k \geq n/\alpha(G) \). Furthermore, in 2010, Chudnovsky and Fradkin [27] proved that every claw-free graph \( G \) with no \( K_{k+1} \)-minor is \( \lfloor 3k/2 \rfloor \)-colorable.

Since line graphs are claw-free, these results about claw-free graphs are related to a theorem of Reed and Seymour. They showed [28] that Hadwiger’s conjecture is true for line graphs (of multigraphs).

We say that \( H \) is an \textit{odd minor} of \( G \) if \( H \) can be obtained from a subgraph \( G' \) of \( G \) by contracting a set of edges that is a cut of \( G' \). Clearly, a graph contains \( K_3 \) as an odd minor if and only if it is not 2-colorable. In 1979, Catlin [5] showed that if \( G \) has no \( K_4 \) odd minor then \( G \) is 3-colorable. A \textit{fully odd} \( K_4 \) in \( G \) is a subgraph of \( G \) which is obtained from \( K_4 \) by replacing each edge of \( K_4 \) by a path of odd length in such a way that the interiors of these six paths are disjoint. Zang [29] in 1998 and, independently, Thomassen [30] in 2001 proved the conjecture of Toft [31] that if \( G \) contains no fully odd \( K_4 \) then \( G \) is 3-colorable.

In 1995, Gerards and Seymour conjectured a strengthening of Hadwiger's conjecture (see [32]) that for every \( k \geq 0 \), if \( G \) has no \( K_{k+1} \) odd minor, then \( G \) is \( k \)-colorable, which is known to be true for \( k \leq 3 \). More interesting results and open problems about Hadwiger’s conjecture and its variations can be found in [33], which was written by Seymour in 2016.

Now, we come back and spend a bit more space on the \( k = 5 \) case of the Hadwiger conjecture. As we mentioned, Mader [18] proved that any minor minimal counterexample to the Hadwiger conjecture for \( k = 5 \) is 6-connected. Jørgensen [34] conjectured that every 6-connected graph contains a \( K_6 \)-minor or has a vertex whose removal results in a planar
graph. Therefore, if Jørgensen’s conjecture holds, then Hadwiger’s conjecture for \( k = 5 \) easily reduces to the Four Color Theorem. In 2017, Kawarabayashi, Norine, Thomas, and Wollan [35] showed that Jørgensen’s conjecture holds for sufficiently large graphs.

In their work [17], Robertson, Seymour, and Thomas proved that Jørgensen’s conjecture holds for each 6-connected graph in which some edge is contained in four triangles. (However, they were not able to resolve the Jørgensen conjecture. Instead, they explored different structures of a minimum counterexample to the Hadwiger conjecture.) It is natural and useful to extend this result to graphs in which some edge is contained in three triangles: Given a 6-connected graph \( G \) and triangles \( a_ib_1b_2a_i \) for \( i = 0, 1, 2 \) in \( G \), can we prove that \( G \) contains \( K_6 \)-minor or has a vertex whose removal results in a planar graph?

A first step is to prove that 6-connected graphs are two-three linked: If \( G \) is a 6-connected graph and \( a_0, a_1, a_2, b_1, b_2 \) are distinct vertices of \( G \), then \( G \) contains disjoint connected subgraphs \( G_1, G_2 \) such that \( \{a_0, a_1, a_2\} \subseteq V(G_1) \) and \( \{b_1, b_2\} \subseteq V(G_2) \). In fact, Robertson and Seymour asked for a characterization of two-three linked graphs. We believe that we have such a characterization except that it is quite complicated (even to state) and its proof is long.

1.2 A main theorem about 2-3 linked graphs

For convenience, we use \( (G, a_0, a_1, a_2, b_1, b_2) \) to denote a graph \( G \) and distinct vertices \( a_0, a_1, a_2, b_1, b_2 \) of \( G \), and call it a rooted graph. A cluster in a graph \( G \) is a set \( \mathcal{X} \) of disjoint subsets of \( V(G) \) such that each member of \( \mathcal{X} \) induces a connected subgraph of \( G \). We say that a rooted graph \( (G, a_0, a_1, a_2, b_1, b_2) \) is feasible if there exists a cluster \( \{X_1, X_2\} \) in \( G \) such that \( \{a_0, a_1, a_2\} \subseteq X_1 \) and \( \{b_1, b_2\} \subseteq X_2 \). We can now state our result as follows.

**Theorem 1.2.1** Let \( (G, a_0, a_1, a_2, b_1, b_2) \) be a rooted graph, and assume \( G + b_1b_2 + \{a_ib_j : i = 0, 1, 2 \text{ and } j = 1, 2\} \) is 6-connected. Then \( (G, a_0, a_1, a_2, b_1, b_2) \) is feasible.

We may view the problem of characterizing feasible rooted graphs as a generalization
of the following problem of characterizing 2-linked graphs: Given a graph $G$ and four distinct vertices $a_1, a_2, b_1, b_2$ of $G$, when does $G$ contain disjoint paths from $a_1, a_2$ to $b_1, b_2$, respectively? Several characterizations of 2-linked graphs are known in [36, 37, 38, 39] and have been used extensively in the literature for proving important structural results on graphs (e.g., in the graph minors project of Robertson and Seymour).

Suppose $\gamma := (G, a_0, a_1, a_2, b_1, b_2)$ is an infeasible rooted graph such that $b_1b_2 \notin E(G)$, $a_ib_j \notin E(G)$ for $i = 0, 1, 2$ and $j = 1, 2$, and $G^* := G + b_1b_2 + \{a_ib_j : i = 0, 1, 2$ and $j = 1, 2\}$ is 6-connected. A $B$-bridge of $G$ is a subgraph of $G$ induced by all edges in a component of $G - V(B)$ and all edges from that component to $B$.

In Chapter 2, we will present the proof of our main theorem, and in Chapter 3, some future works will be introduced.

In fact, in section 2.1, we show that for some $i \in \{0, 1, 2\}$, $G$ has an $a_i$-frame $A, B$ in $(G, a_0, a_1, a_2, b_1, b_2)$, that is $G - a_i$ has disjoint paths $A$ from $a_{i-1}$ to $a_{i+1}$ and $B$ from $b_1$ to $b_2$ (with $a_{-1} = a_2, a_3 = a_0$). Moreover, given an $a_i$-frame $A, B$ for some $i \in \{0, 1, 2\}$, we will prove some useful properties. For example, we prove that the $B$-bridge of $G$ containing $a_i$ can be drawn in a disk in which no two edges cross, and $b_1, b_2, a_i$ occur on the boundary of the disk.

In section 2.2, we further show that $\gamma$ has a good frame and an ideal frame. For an ideal $a_i$-frame $A, B$ in $\gamma$, roughly speaking, we group the $(A \cup B)$-bridges of $G$ not containing $a_i$ into slim connectors and fat connectors.

In sections 2.3 and 2.4, we deal with the case when there exists at least one fat connector in $A, B$. In section 2.5, we solve the case when there does not exist any fat connector. In this case, $G - A$ can be drawn in a disk in which no two edges cross, $b_1, b_2, a_i$ occur on the boundary of the disk, and any $A-B$ path in $G$ is induced by a single edge. So the structure of $G$ is quite simple in some sense. However, in both cases, we will try to find a configuration consisting of paths with special properties, and use them to force a small cut in $G$ or show that $(G, a_0, a_1, a_2, b_1, b_2)$ is feasible.
For readers’ convenience, we also draw Figure 1.1 containing the illustration of structures of some important special graphs, which shows a sketch of our proof idea.

Finally, we end this chapter with some notation and terminology. Let $G_1, G_2$ be two graphs. We use $G_1 \cup G_2$ (respectively, $G_1 \cap G_2$) to denote the graph with vertex set $V(G_1) \cup V(G_2)$ (respectively, $V(G_1) \cap V(G_2)$) and edge set $E(G_1) \cup E(G_2)$ (respectively, $E(G_1) \cap E(G_2)$). Let $G$ be a graph, a separation in $G$ is a pair $(G_1, G_2)$ of edge-disjoint subgraphs $G_1, G_2$ of $G$ such that $G = G_1 \cup G_2$. And $|V(G_1) \cap V(G_2)|$ is the order of the separation $(G_1, G_2)$. Let $P$ be a path, and let $u, v \in V(P)$. Then we write $P[u, v] := P[u, v] - v, P(u, v) := P[u, v] - u$, and $P(u, v) := P[u, v] - \{u, v\}$. For any positive integer $m$, we let $[m] := \{1, \ldots, m\}$. 
\[(G, a_0, a_1, a_2, b_1, b_2)\]

\[\rightarrow\]

An \(a_0\)-frame

\[\rightarrow\]

A good frame

\[\rightarrow\]

Two different ideal frames

- An ideal frame with a fat connector
- An ideal frame with only slim connectors

Figure 1.1: A flow chart of proof
CHAPTER 2
THE PROOF OF MAIN THEOREM

2.1 Frames

In the first section of this chapter, we state some known results and prove some lemmas that we will use. In particular, we show that an infeasible rooted graph must contain a “frame” which consists of two disjoint paths.

A result we use often is Seymour’s characterization of 2-linked graphs [37] (with equivalent versions in [36, 38, 39]). To state this result we introduce several concepts. A disk representation of a graph $G$ is a drawing of $G$ in a disk in which no two edges cross. A 3-planar graph $(G, A)$ consists of a graph $G$ and a set $A = \{A_1, ..., A_k\}$ of pairwise disjoint subsets of $V(G)$ (possibly $A = \emptyset$) such that

(i) for $i \neq j$, $N_G(A_i) \cap A_j = \emptyset$,

(ii) for $1 \leq i \leq k$, $|N_G(A_i)| \leq 3$, and

(iii) if $p(G, A)$ denotes the graph obtained from $G$ by (for each $i$) deleting $A_i$ and adding edges joining every pair of distinct vertices in $N_G(A_i)$, then $p(G, A)$ can be drawn in the plane without crossing edges.

If, in addition, $b_0, b_1, ..., b_n$ are vertices in $G$ such that $b_i \notin A$ for $0 \leq i \leq n$ and $A \in \mathcal{A}$, $p(G, A)$ can be drawn in a closed disk with no edge crossings, and $b_0, b_1, ..., b_n$ occur on the boundary of the disk in this cyclic order, then we say that $(G, A, b_0, b_1, ..., b_n)$ is 3-planar. If there is no need to specify $\mathcal{A}$, we may simply say that $(G, b_0, b_1, ..., b_n)$ is 3-planar. If $\mathcal{A} = \emptyset$, we say that $(G, b_0, b_1, ..., b_n)$ is planar. Moreover, we say that a face of (the disk representation of) $G$ is finite, if the face is inside the disk.
Lemma 2.1.1 (Seymour, 1980) Let $G$ be a graph with distinct vertices $x_1, x_2, x_3, x_4$. Then either $(G, x_1, x_2, x_3, x_4)$ is 3-planar, or $G$ has a cluster $\{X_1, X_2\}$ such that $\{x_1, x_3\} \subseteq X_1$ and $\{x_2, x_4\} \subseteq X_2$.

We say that a sequence $(\alpha_1, \ldots, \alpha_n)$ is larger than $(\beta_1, \ldots, \beta_m)$ with respect to the lexicographic ordering if either

(i) $m < n$ and $\alpha_i = \beta_i$ for $i = 1, \ldots, m$, or

(ii) there exists $j \in [\min(m, n)]$ with $\alpha_j > \beta_j$ and $\alpha_i = \beta_i$ for all $i < j$.

We will also use the following lemma to modify paths.

Lemma 2.1.2 Let $G$ be a connected graph and $P$ be a path between vertices $u_1$ and $u_2$ of $G$, and let $C$ denote a component of $G - P$. Then one of the following holds:

- $G$ has a separation $(G_1, G_2)$ such that $|V(G_1 \cap G_2)| \leq 2$, $V(C) \cup \{u_1, u_2\} \subseteq V(G_1)$, and $|V(G_2 - G_1)| \geq 1$, or

- $G$ has an induced path $Q$ from $u_1$ to $u_2$ such that $G - Q$ is connected with $C \subseteq (G - Q)$.

Proof. We choose a path $Q$ in $G$ from $u_1$ to $u_2$ and label the components of $G - Q$ as $C_1, \ldots, C_n$ such that $C \subseteq C_1$ and $|V(C_2)| \geq \cdots \geq |V(C_n)|$, and, subject to this, $s(Q) := (|V(C_1)|, |V(C_2)|, \cdots, |V(C_n)|)$ is maximum under the lexicographical ordering. Note that $Q$ is well defined because of $P$.

Then $Q$ is an induced path in $G$. For, otherwise, let $Q'$ be the induced path in $G[Q]$ from $u_1$ to $u_2$ then $s(Q') > s(Q)$, a contradiction. If $n = 1$ then the assertion of the lemma holds. So assume $n \geq 2$.

Let $l_n, r_n \in N_G(C_n) \cap V(Q)$ such that $Q[l_n, r_n]$ is maximal. We may assume there exists $C_j$ with $j < n$ such that $N_G(C_j) \cap Q(l_n, r_n) \neq \emptyset$; otherwise, $G$ has a separation $(G_1, G_2)$ such that $V(G_1 \cap G_2) = \{l_n, r_n\}$, $V(C) \cup \{u_1, u_2\} \subseteq V(G_1)$, and $V(C_n) \subseteq V(G_2 - G_1)$, a contradiction.
Now let \( Q' \) be an induced path between \( u_1 \) and \( u_2 \) in \( G[Q \cup C_n] \) such that \( Q' \cap Q(l_n, r_n) = \emptyset \). Clearly, \( s(Q') > s(Q) \) under the lexicographical ordering, a contradiction. □

In the remainder of this paper, we will always assume that

- \( \gamma := (G, a_0, a_1, a_2, b_1, b_2) \) is a given rooted graph such that \( b_1b_2 \notin E(G), a_ib_j \notin E(G) \) for \( i = 0, 1, 2 \) and \( j = 1, 2 \), and
- \( G^* := G + b_1b_2 + \{a_ib_j : i = 0, 1, 2 \text{ and } j = 1, 2 \} \) is 6-connected.

When we write \( a_{i+j} \), we understand that the subscript \( i+j \) is taken modulo 3. In the next two lemmas, we show that \( G \) does not admit certain separations.

**Lemma 2.1.3** \( G \) has no separation \((G_1, G_2)\) such that \( V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4, c_5, c_6\} \), \( |V(G_2 - G_1)| \geq 2 \), \( \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1) \), and \((G_2, c_1, c_2, c_3, c_4, c_5, c_6)\) is planar.

**Proof.** For, otherwise, let \( G'_2 := G_2 + \{c_1c_2, c_2c_3, c_3c_4, c_4c_5, c_5c_6, c_6c_1, c_1c_3, c_3c_5, c_5c_1\} \), which is planar as \((G_2, c_1, c_2, c_3, c_4, c_5, c_6)\) is planar.

Since \( G^* \) is 6-connected, \( G_2 \) has at least one edge from each \( c_i \) to \( V(G_2 - G_1) \) and, hence, the number of edges in \( G_2 \) with at least one end in \( V(G_2 - G_1) \) is at least \((6|V(G_2 - G_1)| + 6)/2 = 3|V(G_2 - G_1)| + 3 = 3|V(G_2)| - 15\). Thus, \( G'_2 \) has at least \( 3|V(G_2)| - 15 + 9 = 3|V(G_2)| - 6 \) edges.

Thus, \( G'_2 \) is a planar graph with exactly \( 3|V(G'_2)| - 6 \) edges and each \( c_i \) has a unique neighbor in \( G_2 - G_1 \). Note that \( G'_2 \) must be a planar triangulation. Therefore, the neighbors of \( c_1, \ldots, c_6 \) in \( G_2 - G_1 \) are the same. Hence, since \( G^* \) is 6-connected, \( |V(G_2 - G_1)| = 1 \), a contradiction. □

**Lemma 2.1.4** \( G \) has no separation \((G_1, G_2)\) such that \( |V(G_1 \cap G_2)| = 4 \) and for some permutation \( \pi \) of \( \{0, 1, 2\} \), \( a_{\pi(0)}, a_{\pi(1)}, b_j \in V(G_2 - G_1), |V(G_2 - G_1)| \geq 4 \), \( a_{\pi(2)}, b_{3-j} \in V(G_1) \), and \((G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, V(G_1 \cap G_2))\) is planar.

**Proof.** Suppose to the contrary that such a separation \((G_1, G_2)\) exists in \( G \) and let \( V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\} \) such that \((G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_4, c_3, c_2, c_1)\) is planar. Let \( X := \)
V(G_2 - G_1) - \{a_{\pi(0)}, a_{\pi(1)}, b_j\}. Since G^* is 6-connected, we see that G_2 has at least two edges from b_j to X and at least three edges from a_{\pi(i)} to X for i \in \{0, 1\}.

Further, for any i \in [4], c_i has a neighbor in X. For, otherwise, suppose, for some i \in [4], c_i has no neighbor in X. Then by applying Lemma 2.1.3 to the separation (G[V(G_1) \cup \{c_i\}], G_2 - c_i) in G, we see that |X| = 1. It then follows from planarity that b_j has at most one neighbor in X, a contradiction.

Hence, the number of edges in G_2 with at least one end in X is at least (6|X| + 1 + 1 + 1 + 1 + 3 + 3 + 2)/2 = 3|X| + 6. So G'_2 := G_2 + \{c_1c_2, c_2c_3, c_3c_4, c_4a_{\pi(1)}, a_{\pi(1)}b_j, b_ja_{\pi(0)}, a_{\pi(0)}c_1, c_2a_{\pi(0)}, c_2b_j, c_2c_4, c_4b_j\} has edges at least 3|X| + 6 + 11 = 3(|X| + 7) - 4. On the other hand, since G'_2 is planar (as (G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_4, c_3, c_2, c_1) is planar), G'_2 has at most 3(|X| + 7) - 6 edges, a contradiction. \[\Box\]

For i \in \{0, 1, 2\}, an a_i-frame in \gamma consists of disjoint paths A from a_{i-1} to a_{i+1} and B from b_1 to b_2 in G - a_i, such that A is induced in G, G - A is connected, and the B-bridge of G containing a_i does not contain A. The next lemma says that if \gamma is infeasible then it has a frame.

**Lemma 2.1.5** If \gamma is infeasible then there exists i \in \{0, 1, 2\} such that \gamma has an a_i-frame.

**Proof.** Since G^* is 6-connected, G - \{a_0, a_1, a_2\} contains an induced path P from b_1 to b_2 such that G - \{a_0, a_1, a_2\} - P \neq \emptyset. By Lemma 2.1.2, G - \{a_0, a_1, a_2\} has an induced path Q from b_1 to b_2 such that C := G - \{a_0, a_1, a_2\} - Q is connected and C \neq \emptyset.

Note that there exists a permutation i, j, k of \{0, 1, 2\} such that N_G(a_j) \cap V(C) \neq \emptyset and N_G(a_k) \cap V(C) \neq \emptyset, or N_G(a_j) \cap V(C) = \emptyset and N_G(a_k) \cap V(C) = \emptyset. In the former case, G - a_i contains disjoint paths from b_1, a_j to b_2, a_k, respectively. In the latter case, N_G(a_j) \cap V(Q(b_1, b_2)) \neq \emptyset and N_G(a_k) \cap V(Q(b_1, b_2)) \neq \emptyset; so we have a path in G[Q(b_1, b_2) + \{a_j, a_k\}] from a_j to a_k and a path from b_1 to b_2 in G - \{a_0, a_1, a_2\} - Q(b_1, b_2).

Hence, there exists i \in \{0, 1, 2\} such that G - a_i has disjoint paths A* and B from a_{i-1}, b_1 to a_{i+1}, b_2, respectively. Since \gamma is infeasible, a_i and A* are contained in different
components of $G - B$. Hence, $a_i$ and $B$ are contained in a component of $G - A^*$. So by Lemma 2.1.2, $G$ has an induced path $A$ between $a_{i-1}$ and $a_{i+1}$ such that $G - A$ is connected and $B + a_i \subseteq G - A$. Since $\gamma$ is infeasible, the $B$-bridge of $G$ containing $a_i$ does not contain $A$. Hence, $A, B$ is an $a_i$-frame in $\gamma$. 

In the next two lemmas, we derive useful information about frames in $\gamma$, seen at Figure 2.1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{a0-frame.png}
\caption{An $a_0$-frame}
\end{figure}

\textbf{Lemma 2.1.6} \textit{Suppose $\gamma$ is infeasible and $A, B$ is an $a_i$-frame in $\gamma$. Let $A_i(B)$ denote the $B$-bridge of $G$ containing $a_i$, and let $V(A_i(B) \cap B) = \{d_1, \ldots, d_t\}$ such that $b_1, d_1, \ldots, d_t, b_2$ occur on $B$ in this order. Then $(A_i(B) \cup B, a_i, b_1, d_1, \ldots, d_t, b_2)$ is planar.}

\textit{Proof.} Let $G' = G/A$, and let $a'$ denote the vertex representing the contraction of $A$. Since $\gamma$ is infeasible, $G'$ has no disjoint paths from $a', b_1$ to $a_i, b_2$, respectively. So by Lemma 2.1.1, there exists a set $S$ of pairwise disjoint subsets of $V(G')$, such that $(G', S, a', b_1, a_i, b_2)$ is 3-planar.

Note that for any $S \in S$, $a' \in N_{G'}(S)$. For, otherwise, $N_G(S)$ is a cut in $G^*$ separating $S$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction as $G^*$ is 6-connected.
Thus, for any $S \in \mathcal{S}$, we have $|N_{G'}(S) \cap V(B)| \leq 2$. Hence, $S \cap A_i(B) = \emptyset$. For otherwise, since $a' \in N_{G'}(S)$, there exists $u \in V(A_i(B) \cap B)$, such that $u \in S$. But then $G - A$ contains three internally disjoint paths from $u$ to $b_1, b_2, a_i$, respectively, a contradiction to the existence of cut $N_{G'}(S)$. Therefore, $A_i(B) \subseteq G' - \bigcup_{S \in \mathcal{S}} S$, and $G' - \bigcup_{S \in \mathcal{S}} S$ has a disk representation with $b_1, b_2, a_i$ on the boundary of the disk. Thus, $A_i(B) \cup B$ inherits a disk representation with $b_1, b_2, a_i$ occurring on the boundary of the disk. Since $A_i(B) \cup B - B$ has only one component, $(A_i(B) \cup B, a_i, b_1, d_1, \ldots, d_t, b_2)$ is planar. □

Suppose $A, B$ is an $a_i$-frame in $\gamma$. Let $A_i(B)$ denote the $B$-bridge of $G$ containing $a_i$. By a double cross in $A, B$ we mean a pair of disjoint connected subgraphs $A', B'$ (in this order) of $G - (A_i(B) - B)$ for which there exist $a_1', a_2' \in V(A)$ and $b_1', b_2' \in V(B)$, such that $V(A')$ includes $a_1', a_2'$ and at least one vertex of $B(b_1', b_2')$ and is otherwise disjoint from $A \cup B[b_1, b_1'] \cup B[b_2', b_2]$, and $V(B')$ includes $b_1', b_2'$ and at least one vertex of $A(a_1', a_2')$ and is otherwise disjoint from $B \cup A[a_1, a_2] \cup A[a_2, a_2']$. The vertices $a_1', a_2', b_2', b_1'$ (in this order) are called the terminals of the double cross.

**Lemma 2.1.7** If $\gamma$ is infeasible then there is no double cross in $\gamma$.

**Proof.** Without loss of generality, assume $A, B$ is an $a_0$-frame in $\gamma$. Suppose $A', B'$ is a double cross in $A, B$ with terminals $a_1', a_2', b_2', b_1'$. Let $H = A(a_1', a_2') \cup B(b_1', b_2') \cup (A' - \{a_1', a_2'\}) \cup (B' - \{b_1', b_2'\})$. Consider the graph $G'$ obtained from $G$ by contracting $H$ to a single vertex $h$.

Since $G^*$ is 6-connected, then, combined with the existence of four disjoint paths $A[a_1, a_1'], A[a_2, a_2'], B[b_1, b_1'], B[b_2', b_2]$ and Menger’s theorem, $G'$ contains five vertex disjoint paths between $\{a_1', a_2', b_1', b_2', h\}$ and $\{a_0, a_1, a_2, b_1, b_2\}$. So $G$ contains five disjoint paths $P_i, i = 1, \ldots, 5$, (also internally disjoint from $H$) joining $a_1', a_2', b_1', b_2'$ and $H$ to $\{a_0, a_1, a_2, b_1, b_2\}$. Without loss of generality, assume that $a_1 \in V(P_1), a_2 \in V(P_2), b_1 \in V(P_3), b_2 \in V(P_4)$, and $a_0 \in V(P_5)$.  

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Let \( S_1 = (V(P_1 \cup P_2 \cup P_3)) \cap (\{a_1', a_2', b_1', b_2'\} \cup V(H)) \), and \( S_2 = (V(P_3 \cup P_4)) \cap (\{a_1', a_2', b_1', b_2'\} \cup V(H)) \). Using the properties of a double cross, we can show that \( H \) contains a cluster \( \{H_1, H_2\} \) such that \( S_i \subseteq V(H_i), i = 1, 2 \). Let \( X_1 := H_1 \cup V(P_1 \cup P_2 \cup P_3) \) and \( X_2 := V(P_3 \cup P_4) \cup H_2 \). Then \( \{X_1, X_2\} \) is a cluster in \( G \), a contradiction. \( \square \)

We conclude this section by considering intersections of special cuts in a planar graph, and investigating when they force another cut or interesting structures of the graph.

**Lemma 2.1.8** Let \( \gamma \) be infeasible with an \( a_0 \)-frame \( A, B \), and let \( G_0 \) be obtained from \( G^* \) by deleting the component of \( G^* - B \) containing \( A \). Suppose \((G_0, a_0, b_1, B, b_2)\) is planar, and \( G_0 \) has 3-cuts \( \{a_0', b_1', b_2'\} \) and \( \{a_0'', b_1'', b_2''\} \) separating \( \{a_0, b_1, b_2\} \) from \( B[b_1', b_2'] \) and \( B[b_1'', b_2''] \), respectively, such that \( b_1, b_1', b_1'', b_2', b_2'', b_2 \) occur on \( B \) in order, \( b_1' \neq b_2'' \), and \( G_0 \) contains a path from \( B(b_1', b_2') \) to \( a_0 \) and internally disjoint from \( B \). Then one of the following holds:

(i) \( \{b_1', b_2'\} \) is contained in a 3-cut of \( G_0 \) separating \( \{a_0, b_1, b_2\} \) from \( B[b_1', b_2'] \).

(ii) \( \{b_1', b_2'\} = \{b_1, b_2\} \), and \( a_0' = a_0'' = a_0 \).

(iii) \( \{a_0', b_1', b_2'\} = \{a_0, b_1, b_2\} \), \( b_2' \) is a cut vertex of \( G_0 \) separating \( b_2 \) from \( \{a_0, b_1\} \), and \( a_0', a_0'', b_2', b_2'' \) are incident with some finite face of \( G_0 \).

(iv) \( \{a_0', b_1', b_2'\} = \{a_0, b_1, b_2\} \), \( b_1' \) is a cut vertex of \( G_0 \) separating \( b_1 \) from \( \{a_0, b_2\} \), and \( a_0', a_0'', b_1', b_1'' \) are incident with some finite face of \( G_0 \).

**Proof.** We may assume \( a_0' \neq a_0'' \). For, otherwise, since \((G_0, a_0, b_1, B, b_2)\) is planar, either \( \{a_0', b_1', b_2'\} \) is a 3-cut in \( G_0 \) separating \( \{a_0, b_1, b_2\} \) from \( B[b_1', b_2'] \) and (i) holds, or \( \{a_0', b_1', b_2'\} = \{a_0, b_1, b_2\} \) and (ii) holds.

For \( i \in [2] \), let \( F_i' \) be a finite face of \( G_0 \) incident with both \( b_i' \) and \( a_0' \) and let \( F_i'' \) be a finite face of \( G_0 \) incident with both \( b_i'' \) and \( a_0'' \). Since \( a_0' \neq a_0'', b_1', b_1'', b_2', b_2'' \) occur on \( B \) in order, and \( G_0 \) contains a path from \( B(b_1', b_2') \) to \( a_0 \) and internally disjoint from \( B \), we have \( F_i' = F_i'' \) for some \( i \in [2] \).
By symmetry, we may assume $F_1' = F_2''$. Then $a_0', a_0'', b_0', b_0''$ are incident with some finite face of $G_0$. Thus, either \(\{a_0', b_0', b_0''\}\) is a 3-cut of $G_0$ separating \(\{a_0, b_1, b_2\}\) from $B[b_0'', b_0']$, or \(\{a_0', b_0', b_0''\}\) = \(\{a_0, b_1, b_2\}\) and $b_0'$ is a cut vertex of $G_0$ separating $b_1$ from \(\{a_0, b_2\}\). So (i) or (iv) holds, a contradiction.

□

Lemma 2.1.9 Let $\gamma$ be infeasible and $A, B$ be an $a_0$-frame in $\gamma$, and let $G_0$ be obtained from $G^*$ by deleting the component of $G^* - B$ containing $A$. Suppose $(G_0, a_0, b_1, B, b_2)$ is planar, and $G_0$ has four distinct vertices $b_1'', b_1', b_2'', b_2'$ with $b_1, b_1', b_1'', b_2', b_2'$ on $B$ in order, and $b_1', b_2'$ are incident with some finite face of $G_0$.

(i) If \(\{b_1', b_2'\}\) is a 2-cut in $G_0$ separating $B[b_1', b_2']$ from \(\{a_0, b_1, b_2\}\), then $b_1', b_1'', b_2', b_2''$ are incident with some finite face of $G_0$, and \(\{a_0', b_0', b_0''\}\) is a 2-cut in $G_0$ separating $B[b_0', b_0'']$ from \(\{a_0, b_1, b_2\}\).

(ii) If there exists a vertex $a_0'$ in $G_0$, such that \(\{a_0', b_1', b_2'\}\) is a 3-cut in $G_0$ separating $B[b_1', b_2']$ from \(\{a_0, b_1, b_2\}\), then one of the following occurs:

(a) $a_0', b_0', b_1', b_2''$ are incident with some finite face of $G_0$, and \(\{a_0', b_0', b_0''\}\) is a 3-cut in $G_0$ separating $B[b_0'', b_0']$ from \(\{a_0, b_1, b_2\}\) or \(\{a_0', b_0', b_0''\}\) = \(\{a_0, b_1, b_2\}\);

(b) $a_0', b_0', b_2', b_2'$ are incident with some finite face of $G_0$, and \(\{b_1', b_1''\}\) is a 2-cut in $G_0$ separating $B[b_1', b_1'']$ from \(\{a_0, b_1, b_2\}\).

Proof. Let $F''$ be a finite face of $G_0$ incident with $b_1', b_2'$. To prove (i), we let $F'$ be a finite face of $G_0$ incident with $b_1', b_2'$. Since $b_1, b_1'', b_1', b_2', b_2'$ occur on $B$ in order, $F' = F''$, and so (i) holds.

Next, we prove (ii). For each $i \in [2]$, we let $F_i'$ be a finite face of $G_0$ incident with both $b_i'$ and $a_0'$. Since $b_1, b_1'', b_1', b_2', b_2'$ occur on $B$ in order, then $F_1' = F''$ or $F_2' = F''$. Now, if $F_1' = F''$, then (a) of (ii) holds; if $F_2' = F''$, then (b) of (ii) holds. □
2.2 Good frames and ideal frames

In this section, we fix $\gamma = (G, a_0, a_1, a_2, b_1, b_2)$ and $G^* = G + b_1b_2 + \{a_ib_j : i = 0, 1, 2 \text{ and } j = 1, 2\}$, assume that $\gamma$ is infeasible, and then show that $\gamma$ has a special frame with good properties. For an $a_i$-frame $A, B$ in $\gamma$, we fix the following notation:

- $\alpha(A, B) = |\{b_i : N_G(b_i) \cap V(A_i(B) - a_i - B) \neq \emptyset\}|$, and
- $c(A, B) = |\{v \in V(A_i(B) \cap B) - \{b_1, b_2\} : \{v, a_i\} \text{ separates } b_1 \text{ from } b_2 \text{ in } A_i(B) \cup B\}|$.

We say that an $a_i$-frame $A, B$ in $\gamma$ is good (seen at Figure 2.4), if among all the frames in $\gamma$,

(i) $\alpha(A, B)$ is maximum,
(ii) subject to (i), $c(A, B)$ is minimum,

(iii) subject to (ii), $A_i(B)$ is maximal.

Figure 2.4: A good frame and its connectors

**Lemma 2.2.1** Suppose $A, B$ is a good frame in $\gamma$. Let $i \in \{0, 1, 2\}$ and $A', B'$ be disjoint paths in $G - a_i$ from $a_{i-1}, b_1$ to $a_{i+1}, b_2$, respectively.

(i) If, for some $j \in [2]$, $G$ has a path $B_0$ from $a_i$ to $b_j$ that is internally disjoint from $A', B'$, then $\alpha(A, B) \geq 1$.

(ii) If $\{a_i, b_1, b_2\}$ is contained in a component of $G - (A' \cup (B' - \{b_1, b_2\}))$, then $\alpha(A, B) = 2$.

(iii) If $G$ has a path $B''$ from $b_1$ to $b_2$ that is internally disjoint from $A', B'$, then $\alpha(A, B) = 2$ and $c(A, B) = 0$.

**Proof.** We first prove (i). We see that $B', B_0$ are contained in some component of $G - A'$. By Lemma 2.1.2 and the existence of $A'$, there exists an induced path $A^*$ from $a_{i-1}$ to $a_{i+1}$, such that $G - A^*$ is connected, and $B', B_0 \subseteq G - A^*$. Since $\gamma$ is infeasible, $A^*$ and $a_i$
are in different components of $G - B'$. So $A^*, B'$ is a frame. By the existence of $B_0$, $\alpha(A^*, B') \geq 1$, and so $\alpha(A, B) \geq 1$.

Similarly, for (ii), let $C$ be the component of $G - (A' \cup (B' - \{b_1, b_2\}))$ containing $b_1, b_2, a_i$, we may assume there exists an induced path $A^*$ from $a_{i-1}$ to $a_{i+1}$, such that $G - A^*$ is connected, and $B', C \subseteq G - A^*$. So $A^*, B'$ is a frame. By the existence of $C$, $\alpha(A^*, B') = 2$, and so $\alpha(A, B) = 2$.

For (iii), since $\gamma$ is infeasible, $B' \cup B'' + a_i$ must be contained in a component of $G - A'$. Hence, we may assume that $B'' + a_i$ is contained in a component of $G - (A' \cup (B' - \{b_1, b_2\}))$. So by (ii), $\alpha(A, B) = 2$. Now by Lemma 2.1.2 and the existence of $A'$, there exists an induced path $A^*$ from $a_{i-1}$ to $a_{i+1}$, such that $G - A^*$ is connected, and $B' \cup B'' + a_i \subseteq G - A^*$. So $A^*, B'$ is a frame. Since $B'' + a_i$ is contained in a component of $G - (A' \cup (B' - \{b_1, b_2\}))$, we see that $c(A, B) = 0$. 

For a frame $A, B$ in $\gamma$, an $A$-$B$ bridge is an $(A \cup B)$-bridge of $G$ that intersects both $A$ and $B$. Let $M$ be an $A$-$B$ bridge, $l, r \in V(A \cap M)$, and $l', r' \in V(B \cap M)$, such that $A[l, r]$ and $B[l', r']$ are maximal. Then we say that $l, r$ are the extreme hands of $M$, and that $l', r'$ are the feet of $M$. We say that $M$ lies on $B[b'_1, b'_2]$ for some $b'_1, b'_2 \in V(B)$, if $B[l', r'] \subseteq B[b'_1, b'_2]$. We say that $M$ is fat if $|V(M \cap B)| \geq 2$ and non-fat if $|V(M \cap B)| = 1$.

**Lemma 2.2.2** Suppose $A, B$ is a good $a_0$-frame in $\gamma$. Let $\{d_1, \cdots, d_t\} = V(B \cap A_0(B))$ such that $b_1, d_1, \cdots, d_t, b_2$ occur on $B$ in order, and let $d_0 = b_1, d_{t+1} = b_2$. Then the following conclusions hold:

(i) For any $i \in [t]$, $G - (A_0(B) - (B - d_i))$ does not contain disjoint paths from $a_1, b_1$ to $a_2, b_2$, respectively.

(ii) For any $A$-$B$ bridge $M$, $M \cap B \subseteq B[d_{i-1}, d_i]$ for some $i \in [t + 1]$.

(iii) Let $N$ be a $B$-bridge of $G$ not containing $A$ or $a_0$, then $|V(N \cap B)| \geq 4$, and $N \cap B \subseteq B[d_{i-1}, d_i]$ for some $i \in [t + 1]$. 

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Proof. First, we note that (ii) and (iii) follow immediately from (i). So we prove (i). Suppose (i) fails, and let $A^*, B'$ be disjoint paths in $G - (A_0(B) - (B - d_i))$ from $a_1, b_1$ to $a_2, b_2$, respectively.

Then $A_0(B) \cup B'$ is contained in a component of $G - A^*$. By Lemma 2.1.2 and the existence of $A^*$, there exists an induced path $A'$ from $a_1$ to $a_2$, such that $G - A'$ is connected, and $A_0(B) \cup B' \subseteq G - A'$. So $A', B'$ is a frame in $\gamma$. Now, due to the existence of $d_i$, the $B$-bridge of $G$ containing $a_0$ is properly contained in the $B'$-bridge of $G$ containing $a_0$, a contradiction.

An $a_i$-frame $A, B$ in $\gamma$ is ideal if $A, B$ is a good frame such that

(i) the union of $B$-bridges of $G$ not containing $A$ or $a_i$ is maximal,

(ii) subject to (i), the union of fat $A$-$B$ bridges is maximal,

(iii) subject to (ii), the number of non-fat $A$-$B$ bridges is minimum.

**Lemma 2.2.3** Suppose $A, B$ is an ideal $a_0$-frame in $\gamma$. Then all $A$-$B$ bridges are fat.

Proof. Let $M$ be a non-fat $A$-$B$ bridge with extreme hands $l, r$ and foot $u$. Then $V(M \cap A(l, r)) \neq \emptyset$, to avoid the cut $\{l, r, u\}$ in $G^*$. Note that $M - u - A(l, r)$ has a path from $l$ to $r$. Hence, by Lemma 2.1.2, $M \cup A[l, r] - u$ contains an induced path $P$ from $l$ to $r$, such that $M \cup A[l, r] - u - P$ is connected with $A(l, r) \subseteq M \cup A[l, r] - u - P$. Let $A' := A[a_1, l] \cup P \cup A[r, a_2]$. We show that $A', B$ contradicts the choice of $A, B$.

Clearly, $A', B$ is a good frame, and the union of those $B$-bridges of $G$ not containing $A$ or $a_0$ is equal to the union of those $B$-bridges of $G$ not containing $A'$ or $a_0$. Moreover, $A(l, r)$ is contained in a non-fat $A'$-$B$ bridge; otherwise, the union of those fat $A'$-$B$ bridges properly contains the union of those fat $A$-$B$ bridges, a contradiction.

Let $M_1, \ldots, M_k$ be the $A$-$B$ bridges such that for each $i \in [k]$, $M_i \cap A(l, r) \neq \emptyset$, $M_i \neq M$. Then $k \neq 0$; otherwise, $G$ has at least two disjoint edges from $A(l, r)$ to $B$ (as $G^*$ is 6-connected), which contradicts that $A(l, r)$ is contained in a non-fat $A'$-$B$ bridge.
Since \( M_i \cap A(l, r) \neq \emptyset \) for \( i \in [k] \), \( \bigcup_{i \in [k]} M_i \) and \( A(l, r) \) are contained in a same non-fat \( A'-B \) bridge; so \( M_1, \ldots, M_k \) are non-fat \( A-B \) bridges. Now, since \( M \cup A[l, r] - u - P \) is connected with \( A(l, r) \subseteq M \cup A[l, r] - u - P \), then \( \bigcup_{i \in [k]} M_i \) and \( M \cup A[l, r] - u - P \) are contained in one single \( A'-B \) bridge. Hence, the number of non-fat \( A'-B \) bridges is strictly smaller than the number of non-fat \( A-B \) bridges, a contradiction.

Let \( A, B \) be a good \( a_i \)-frame in \( \gamma \), let \( \{ d_1, \ldots, d_t \} = V(B \cap A_i(B)) \) with \( b_1, d_1, \ldots, d_t, b_2 \) on \( B \) in order, and let \( d_0 = b_1 \) and \( d_{t+1} = b_2 \). For any \( i \in [t+1] \), we let \( J_i^* \) be the union of \( B[d_{i-1}, d_i] \), all the edges between \( A \) and \( B[d_{i-1}, d_i] \), all those \( A-B \) bridges \( M \) with \( M \cap B \subseteq B[d_{i-1}, d_i] \), and all those \( B \)-bridges \( N \) of \( G \) with \( (A + a_i) \cap N = \emptyset \) and \( N \cap B \subseteq B[d_{i-1}, d_i] \). Let \( u_1, u_2 \in V(A \cap J_i^*) \), such that \( a_1, u_1, u_2, a_2 \) occur on \( A \) in order with \( A[u_1, u_2] \) maximal. Then we say \( J_i = G[V(J_i^* \cup A[u_1, u_2])] \) is an \( A-B \) connector, and \( u_1, u_2 \) are the extreme hands of \( J_i \). We say that \( d_{i-1}, d_i \) are the feet of \( J_i \). Note that our definition does not require \( J_i \cap J_j = \emptyset \) for \( i \neq j \).

An \( A-B \) connector \( J \) (with feet \( v_1, v_2 \) and extreme hands \( u_1, u_2 \)) is slim if \( (J - A[u_1, u_2], B[v_1, v_2]) \) is planar, and each edge of \( J \) with exactly one end in \( A[u_1, u_2] \) has its other end in \( B[v_1, v_2] \) (seen at Figure 2.5). Thus, no slim \( A-B \) connector contains an \( A-B \) bridge. If \( J \) is not a slim connector, we say that \( J \) is a fat \( A-B \) connector (seen at Figure 2.6).

![Figure 2.5: A slim connector](image-url)
Lemma 2.2.4 Let $A, B$ be an ideal $a_0$-frame in $\gamma$, and $J$ be an $A$-$B$ connector with feet $v_1, v_2$ and extreme hands $u_1, u_2$, such that $V(J) \setminus \{u_1, u_2, v_1, v_2\} \neq \emptyset$. Then

(i) $u_1 \neq u_2$, there exists a unique $j \in [2]$ such that $G$ has an $A$-$B$ path from $B[b_j, v_j]$ to $A(u_1, u_2)$, and $(J - v_j, A[u_1, u_2], v_{3-j})$ is planar, and

(ii) if $J$ is fat then $N_G(v_j) \cap V(J - v_j - A) \nsubseteq L_p$ for $p \in [2]$, where $L_p$ denotes the subpath of the outer walk of $(J - v_j, A[u_1, u_2], v_{3-j})$ from $u_p$ to $v_{3-j}$ without going through $u_{3-p}$.

Proof. Since $V(J) \setminus \{u_1, u_2, v_1, v_2\} \neq \emptyset$ and $G^*$ is 6-connected, then $u_1 \neq u_2$ and $G$ has an $A$-$B$ path from $B - B[b_1, b_2]$ to $A(u_1, u_2)$. By Lemma 2.1.7, there exists a unique $j \in [2]$ such that $G$ has an $A$-$B$ path from $B[b_j, v_j]$ to $A(u_1, u_2)$.

To prove $(J - v_j, A[u_1, u_2], v_{3-j})$ is planar, let $T$ be an $A$-$B$ path from $t' \in B[b_j, v_j]$ to $t \in A(u_1, u_2)$. If $J - v_j$ contains disjoint paths $A^*, B^*$ from $u_1, t$ to $u_2, v_{3-j}$, respectively, then $A' := A[a_1, u_1] \cup A^* \cup A[u_2, a_2]$ and $B' := B[b_j, t'] \cup T \cup B^* \cup B[v_{3-j}, b_{3-j}]$ are disjoint paths in $G - v_j - (A_0(B) - B)$ from $a_1, b_1$ to $a_2, b_2$, respectively; which contradicts (i) of Lemma 2.2.2. So assume that such $A^*, B^*$ do not exist. Then by Theorem 2.1.1, there exist $m \geq 0$ and a set $\mathcal{D} = \{D_1, \ldots, D_m\}$ of pairwise disjoint nonempty subsets of $V(J - v_j) - \{u_1, u_2, t, v_{3-j}\}$ such that $(J - v_j, D, u_1, t, u_2, v_{3-j})$ is 3-planar. We choose $D_1, \ldots, D_m$ such that $\bigcup_{i \in [m]} D_i$ is minimal. Then for all $p \in [m]$, $G[D_p \cup N_{J-v_j}(D_p)]$ does not have a disk representation with $N_{J-v_j}(D_p)$ occurring on the boundary of the disk (or else, $D_p$ could be chosen to be empty). Obviously, $|D_p| \geq 2$.
Note that \( J - v_j - A[u_1, u_2] \) is connected. For, otherwise, let \( C \) be a component of \( J - v_j - A[u_1, u_2] \) disjoint from \( B(v_j, v_{3-j}) \). Then \( N_G(C) \subseteq V(A[u_1, u_2]) \cup \{v_j\} \). Since \( G - A \) is connected, \( v_j \in N_G(C) \); hence, \( G[V(C) \cup N_G(C)] - E(A) \) is a non-fat \( A-B \) bridge, contradicting Lemma 2.2.3.

If \( m = 0 \) then \( D = \emptyset \), and \( (J - v_j, u_1, t, u_2, v_{3-j}) \) is planar; so \( (J - v_j, A[u_1, u_2], v_{3-j}) \) is planar as \( J - v_j - A[u_1, u_2] \) is connected. Hence, \( m \geq 1 \). Since \( G^* \) is 6-connected, for all \( p \in [m] \), \( N_{J-v_j}(D_p) \cup \{v_j\} \) is not a cut of \( G \) separating \( D_p \) from other vertices. So \( D_p \cap V(A) \neq \emptyset \). Since \( D_p \cap \{u_1, u_2, t, v_{3-j}\} = \emptyset \), \( |N_{J-v_j}(D_p) \cap A| \geq 2 \). Moreover, since \( A \) is an induced path and \( G[D_p \cup N_{J-v_j}(D_p)] \) does not have a disk representation with \( N_{J-v_j}(D_p) \) occurring on the boundary of the disk, \( D_p \not\subseteq V(A) \). Thus, \( N_{J-v_j}(D_p) \not\subseteq V(A) \) as \( J - v_j - A[u_1, u_2] \) is connected. So \( |N_{J-v_j}(D_p)| = 3 \) and \( |N_{J-v_j}(D_p) \cap A| = 2 \). Moreover, if we let \( \{s_1, s_2, s\} = N_{J-v_j}(D_p) \) such that \( s \notin V(A) \) and \( u_1, s_1, s_2, u_2 \) occur on \( A \) in order, then \( J - v_j \) has a path \( D \) from \( s \) to \( v_{3-j} \) disjoint from \( A \); or else, there exists a non-fat \( A-B \) bridge with foot \( v_j \), or \( G - A \) is not connected. Moreover, since \( G^* \) is 6-connected, \( G \) has an \( A-B \) path \( R \) from \( r' \in V(B - B[v_1, v_2]) \) to \( v \in V(A(s_1, s_2)) \). By Lemma 2.1.7, \( r' \in B[b_j, v_j] \).

Let \( H := G[D_p \cup N_{J-v_j}(D_p)] \). If \( H \) contains disjoint paths \( X', R_1 \) from \( s_1, r \) to \( s_2, s \), respectively, then the paths \( A' := A[a_1, s_1] \cup X' \cup A[s_2, a_2] \) and \( B' := B[b_j, r'] \cup R \cup R_1 \cup D \cup B[v_{3-j}, b_{3-j}] \) in \( G - (A_0(B) - B) - v_j \) from \( a_1, b_1 \) to \( a_2, b_2 \), respectively, contradict Lemma 2.2.2. So such \( X' \) and \( R_1 \) do not exist. By Lemma 2.1.1, there exist \( n \geq 0 \) and a set \( V = \{V_1, \ldots, V_n\} \) of pairwise disjoint subsets of \( D_p \) such that \( (H, V, s_1, r, s_2, s) \) is 3-planar. However, we see that \( \{D_1, \ldots, D_m\} \setminus \{D_p\} \cup \{V_1, \ldots, V_n\} \) contradicts our choice of \( \{D_1, \ldots, D_m\} \). This completes the proof of (i).

Next, we prove (ii). Since \( J \) contains disjoint paths \( A[u_1, u_2] \) and \( B[v_1, v_2] \), \( N_G(v_j) \cap V(J - v_j - A) \neq \emptyset \). Suppose \( N_G(v_j) \cap V(J - v_j - A) \subseteq L_p \) for some \( p \in [2] \). Let \( u \in N_G(v_j) \cap V(L_p) \), such that \( u \neq u_p \), and \( L_p[u_p, u] \) is minimal. Since \( (J - v_j, A[u_1, u_2], v_{3-j}) \) is planar, \( J - v_j - A[u_1, u_2] \) is also planar. Let \( P' \) denote the subpath of the outer walk of
Let $B' = B[b_j, v_j] \cup \{v_ju\} \cup P' \cup B[v_{3-j}, b_{3-j}]$. Then $A, B'$ is a good frame. The union of those $B$-bridges of $G$ not containing $A$ and $a_0$ is contained in the union of those $B'$-bridges of $G$ not containing $A$ and $a_0$, which forces $B = B'$ by the choice of $A, B$. Moreover, by Lemma 2.2.3 and the planarity of $J - v_j$, each edge of $J$ with exactly one end in $A[u_1, u_2]$ has its other end in $B[v_1, v_2]$; so $J$ is a slim connector, a contradiction. 

\[ \square \]

### 2.3 Core frames

In this section, we consider the situation when there is a fat connector for some ideal frame in $\gamma$ (seen at Figure 2.7). The first two lemmas study the structure inside fat connectors, and show that each fat connector has a core in which we can find various disjoint paths.

**Lemma 2.3.1** Suppose $A, B$ is an ideal $a_0$-frame in $\gamma$. Let $J$ be a fat $A$-$B$ connector with feet $v_1, v_2$ and extreme hands $u_1, u_2$, such that $(J - v_1, A[u_1, u_2], v_2)$ is planar, $a_1, u_1, u_2, a_2$
occur on \( A \) in order, \( b_1, v_1, v_2, b_2 \) occur on \( B \) in order, and \( G \) has an \( A-B \) path from \( A(u_1, u_2) \) to \( B[b_1, v_1] \). Then there exists a separation \( (H, L) \) in \( J \) of order 4 (we allow \( H = J \) and \( L \) consists of \( u_1, u_2, v_2 \) and no edges), such that

(i) \( V(H \cap L) = \{v_1, x_1, x_2, y_2\} \), \( u_1, x_1, x_2, u_2 \) occur on \( A \) in order, \( v_1, y_2, v_2 \) occur on \( B \) in order, \( A[x_1, x_2] \cup B[v_1, y_2] \subseteq H \), and \( \{u_1, u_2, v_2\} \subseteq V(L) \);

(ii) \( (L - A, B[y_2, v_2], v_1) \) is planar, and each edge of \( L \) with exactly one end in \( A \) has its other end in \( V(B[y_2, v_2]) \cup \{v_1\} \);

(iii) \( (H - v_1, A[x_1, x_2], y_2) \) is planar, \( H - v_1 - A[x_1, x_2] \) is connected, \( x_1y_2, x_2y_2 \notin E(H) \), \( H - A(x_1, x_2) - \{v_1x_1, v_1x_2\} \) contains disjoint paths from \( v_1, y_2 \) to \( x_1, x_2 \), respectively, and disjoint paths from \( v_1, y_2 \) to \( x_2, x_1 \), respectively, and \( V(X_1 \cap X_2) = \{y_2\} \) and \( N_G(v_1) \cap V(H - A) \nsubseteq V(X_i) \) for \( i \in [2] \), where \( X_i \) is the path from \( x_i \) to \( y_2 \) on the outer walk of \( H - v_1 \) without going through \( x_{3-i} \).

Proof. Note that by Lemma 2.2.4, if we take \( H = J \) and let \( L \) consist of \( u_1, u_2, v_2 \) and no edges, then \( (H, L) \) satisfies (i) and (ii) (with \( x_i = u_i \) for \( i \in [2] \) and \( y_2 = v_2 \)). Hence, we choose \( (H, L) \) satisfying (i) and (ii) and, subject to this, \( H \) is minimal. We show that (iii) holds.

Since \( (J - v_1, A[u_1, u_2], v_2) \) is planar, \( (H - v_1, A[x_1, x_2], y_2) \) is planar. Note that \( H - v_1 - A[x_1, x_2] \) is connected; for otherwise, let \( C \) be a component of \( H - v_1 - A[x_1, x_2] \) not containing \( y_2 \), which is also a component of \( J - v_1 - A[u_1, u_2] \). Then either it contradicts the definition of frame that \( G - A \) is connected, or it contradicts Lemma 2.2.3 that all \( A-B \) bridges are fat. By the minimality of \( H \), we see that \( x_1y_2, x_2y_2 \notin E(H) \).

For \( i = 1, 2 \), let \( X_i \) denote the path in the outer walk of \( H - v_1 \) from \( y_2 \) to \( x_i \) not containing \( x_{3-i} \). Then \( V(X_1 \cap X_2) = \{y_2\} \). For, otherwise, \( H \) has a separation \( (H_1, H_2) \) such that \( |V(H_1 \cap H_2)| = 1 \), \( y_2 \in V(H_1 - H_2) \), and \( A[x_1, x_2] \subseteq H_2 \). Since \( G^* \) is 6-connected, \( V(H_1 - H_2) = \{y_2\} \). Let \( y'_2 \in V(H_1 - y_2) \). Now it is easy to check that the separation \( (H - y_2, G[L + y'_2]) \) contradicts the choice of \( (H, L) \) (that \( H \) is minimal).
Next we show that $N_G(v_1) \cap V(H - A) \not\subseteq V(X_i)$ for $i = 1, 2$. For, suppose this is false and, by symmetry, that $N_G(v_1) \cap V(H - A) \subseteq V(X_2)$. Let $y'_2 \in N_G(v_1) \cap V(X_2)$ with $X_2[y'_2, y_2]$ minimal. Let $B'$ denote the path in the outer walk of $H - A$ from $y'_2$ to $y_2$ not containing $X_2[y'_2, y_2]$. We could choose $B$ so that $B' \subseteq B$. However, this shows that $J$ is not fat, a contradiction.

It remains to show that for $j \in [2]$, $H - A(x_1, x_2) - \{v_1x_1, v_1x_2\}$ contains disjoint paths from $v_1, y_2$ to $x_{3-j}, x_j$, respectively. For, otherwise, we may assume by symmetry that $H - A(x_1, x_2) - \{v_1x_1, v_1x_2\}$ does not have disjoint paths from $v_1, y_2$ to $x_1, x_2$, respectively. Hence, $H - A(x_1, x_2) - X_2 - \{v_1x_1, v_1x_2\}$ has no path from $v_1$ to $x_1$. Since $(H - v_1, A[x_1, x_2], X_2, X_1)$ is planar, there exist $x'_1 \in V(A(x_1, x_2))$, $y'_2 \in V(X_2)$, and a 2-separation $(H_1, H_2)$ in $H - v_1$ such that $V(H_1 \cap H_2) = \{x'_1, y'_2\}$, $x_1, y_2 \in V(H_1)$, $A[x'_1, x_2] \subseteq H_2$, and $N_G(v_1) \cap V(H) \subseteq V(H_2 \cup A[x_1, x_2] \cup X_2)$. Then we see that the separation $(H_2, G[H_1 \cup L])$ of $J$ contradicts the choice of $(H, L)$. ∎

With the notation in Lemma 2.3.1, we say that $H$ is an $A$-$B$ core or a core of the fat connector $J$. Moreover, we say that $x_1, x_2$ are the extreme hands of $H$, $v_1, y_2$ are the feet of $H$, and $y_2$ is the main foot of $H$. For convenience, we write $y_1 := v_1$. By symmetry, we may always assume that $a_1, x_1, x_2, a_2$ occur on $A$ in order, and that $b_1, y_1, y_2, b_2$ occur on $B$ in order. Note that $y_1 \in V(A_0(B))$ and $G$ has a path from $a_0$ to $y_1$ internally disjoint from $B$. For $i \in [2]$, let $x'_i \in V(A(x_1, x_2))$ such that $x'_i, x_i$ are incident with some finite face of $H - y_1$, and $H - y_1$ has a path from $x'_i$ to $y_2$ and internally disjoint from $A$. And for $i \in [2]$, let $X'_i$ be the path from $y_2$ to $x'_i$ on the outer walk of $H - \{y_1, x_i\}$ without going through $x_{3-i}$.

**Lemma 2.3.2** Suppose $A, B$ is an ideal $a_0$-frame, and $H$ is an $A$-$B$ core with extreme hands $x_1, x_2$ and feet $y_1, y_2$, where $y_2$ is the main foot. Then the degree of $y_2$ in $H - y_1$ is at least 2 and, for $i \in [2]$, $|V(X_i(x_i, y_2))| \geq 1$ and $V(X_i \cap X'_{3-i}) = \{y_2\}$. Moreover, if, for some $i \in [2]$, $H$ does not contain disjoint paths from $y_1, y_2$ to $x_i, x'_i$, respectively, and internally disjoint from $A$, then the following are true:
(i) No finite face of \( H - y_1 \) is incident with both \( y_2 \) and a vertex of \( A(x_1, x_2) \).

(ii) For any \( v \in N_G(y_1) \cap V(H) \) with \( v \notin X'_{3-i} \cup A(x_1, x_{3-i}) \), there exist \( c_1 \in A(x_i, x'_{3-i}) \)
and \( c_2 \in X'_{3-i}(x'_{3-i}, y_2) \), such that \( \{c_1, c_2\} \) is a cut in \( H - \{y_1, x_{3-i}\} \) separating \( v \)
from \( x_i \), and there exist internally disjoint paths from \( v \) to \( c_1, c_2 \) in \( H - \{y_1, x_{3-i}\} \),
respectively, which are internally disjoint from \( X'_{3-i} \cup A[x_i, x'_{3-i}] \).

(iii) \( H \) has disjoint paths from \( y_1, y_2 \) to \( x_{3-i}, x'_i \), respectively, and internally disjoint from \( A \).

**Proof.** By Lemma 2.3.1, \( V(X_1 \cap X_2) = \{y_2\} \) and \( x_1y_2, x_2y_2 \notin E(H) \); so the degree of \( y_2 \) in \( H - y_1 \) is at least 2 and \( |V(X_i(x_i, y_2))| \geq 1 \). Moreover, \( V(X_i \cap X'_{3-i}) = \{y_2\} \) for \( i \in [2] \); for, suppose there exists \( c \in V(X_i \cap X'_{3-i}) - \{y_2\} \), then \( \{c, y_1, y_2, x_{3-i}\} \) is a cut in \( G \) separating \( V(X_{3-i}) \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction.

By symmetry, we may assume that \( H \) does not contain disjoint paths from \( y_1, y_2 \) to \( x_1, x'_2 \), respectively, that are internally disjoint from \( A \).

To prove (i), suppose there exists \( v_0 \in V(A(x_1, x_2)) \) such that \( v_0, y_2 \) are incident with some finite face in \( H - y_1 \). Since \( (H - y_1, A[x_1, x_2], y_2) \) is planar, \( H - y_1 \) has a separation \((H_1, H_2)\) such that \( V(H_1 \cap H_2) = \{y_2, v_0\} \), \( X_1 \subseteq H_1 \), and \( X_2 \subseteq H_2 \). Now, we further choose \( v_0 \) so that \( H_1 \) is minimal.

Now, we see that \( H_2 \) contains a path \( P_2 \) from \( y_2 \) to \( x'_2 \) and internally disjoint from \( A \); for otherwise, \( V(H_2 \cap A) = \{x_2\} \) and, hence, \( \{y_1, y_2, x_2\} \) is a cut in \( G^* \) separating \( V(X_2) \)
from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction.

Now, let \( P_1 \) be the path from \( y_1 \) to \( x_1 \) in \( H - V(A(x_1, x_2)) \cup \{y_2\} \) (by (iii) of Lemma 2.3.1).
Since \( v_0 \neq x_1, V(P_1 \cap H_2) = \emptyset \), and so \( V(P_1 \cap P_2) = \emptyset \). However, the existence of \( P_1, P_2 \)
contradicts that \( H \) does not contain disjoint paths from \( y_1, y_2 \) to \( x_1, x'_2 \), respectively, and internally disjoint from \( A \). This completes the proof of (i).

To prove (ii), let \( v \in N_G(y_1) \cap V(H) \) such that \( v \notin X'_2 \cup A(x_1, x_2) \). Since \( (H - \{y_1, x_2\}, A[x_1, x'_2] \cup X'_2[x'_2, y_2]) \) is planar and \( H - y_1 - A(x_1, x_2) \cup X'_2 \) does not have a
path from \( v \) to \( x_1 \), there exist \( c_1, c_2 \in V(A(x_1, x'_2) \cup X'_2) \) such that \( \{c_1, c_2\} \) is a cut in \( H - \{y_1, x_2\} \) separating \( v \) from \( x_1 \). We may assume \( c_1, c_2 \) occur on \( A(x_1, x'_2) \cup X'_2[x'_2, y_2] \) in order.

Note that \( c_1 \notin V(X'_2) \), to avoid the cut \( \{c_1, c_2, y_1, x_2\} \) in \( G^* \). Moreover, \( c_2 \notin A(x'_2, y_2) \); or else, \( H - V(A) \cup \{y_1\} \) is not connected, contradicting (iii) of Lemma 2.3.1.

We choose \( c_1, c_2 \) such that \( A[c_1, x'_2] \) and \( X'_2[x'_2, c_2] \) are minimal. Then \( H - \{y_1, x_2\} \) contains internally disjoint paths from \( v \) to \( c_1, c_2 \), respectively, and internally disjoint from \( A \cup X'_2 \). Moreover, by (i), \( c_2 \neq y_2 \). This completes the proof of (ii).

To prove (iii), observe that \( V(X'_1 \cap X'_2) = \{y_2\} \). For otherwise, let \( c \in V(X'_1 \cap X'_2) \) with \( c \neq y_2 \). Since \( y_2 \) has degree at least 2 in \( H - y_1 \) and \( x_1y_2, x_2y_2 \notin E(H) \), \( \{x_1, x_2, y_1, y_2, c\} \) is a cut in \( G^* \) separating \( V(X_1 \cup X_2) \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction.

Now, let \( u_2 \in V(X_2 \cap X'_2) \) such that \( X'_2[x_2, u_2] \) is minimal. Moreover, let \( v \in N_G(y_1) \cap V(H - A) \). If \( v \in V(X'_2) \) then let \( P_2 = v = c_2 \); and if \( v \notin V(X'_2) \) then by (ii), there exist \( c_1 \in V(A(x_1, x'_2)) \) and \( c_2 \in V(X'_2(x'_2, y_2)) \), such that \( \{c_1, c_2\} \) is a cut in \( H - \{y_1, x_2\} \) separating \( v \) from \( x_1 \), and there exists a path \( P_2 \) from \( v \) to \( c_2 \) in \( H - \{y_1, x_2\} \), which is internally disjoint from \( X'_2 \cup A[x_1, x'_2] \). Since \( V(X'_1 \cap X'_2) = \emptyset \) and \( (H - y_1, A[x_1, x_2] \cup X_2) \) is planar, \( P_2 \) is disjoint from \( X'_1 \). Now, \( X'_1 \) and \( y_1v \cup P_2 \cup X'_2[c_2, u_2] \cup X_2[u_2, x_2] \) are disjoint paths from \( y_2, y_1 \) to \( x'_1, x_2 \), respectively, in \( H \), which are internally disjoint from \( A \).

The next lemma describes interactions between cores from different connectors and finds a path \( B' \) so that \( A, B' \) is a good frame in \( \gamma \) which will eventually be used to form a special frame \( A', B' \) in \( \gamma \).

**Lemma 2.3.3** Let \( A, B \) be an ideal \( a_0 \)-frame in \( \gamma \), and let \( H^j, j \in [m] \), be the \( A-B \) cores in \( \gamma \) such that \( H^j \) has extreme hands \( x'_1, x'_2 \) and feet \( y'_1, y'_2 \). Then

(i) for any distinct \( i, j \in [m] \), \( A[x'_1, x'_2] \subseteq A[x'_1, x'_2] \) or \( A[x'_1, x'_2] \subseteq A[x'_1, x'_2] \),

(ii) for any \( j \in [m] \), \( H^j - A[x_1, x_2] \) has a path \( P_j \) from \( y'_1 \) to \( y'_2 \) such that \( |V(P_j)| \geq 3 \), \( H^j - P_j \) is connected, and \( P_j \) is induced in \( G - y'_1y'_2 \),

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(iii) $A, B'$ is a good $a_0$-frame and $A_0(B') = A_0(B)$, where $B'$ is obtained from $B$ by replacing $B[y_1^j, y_2^j]$ with the path $P_j$ in (ii) for $j \in [m]$, and

(iv) with $G'_0$ as the graph obtained from $G$ by deleting the component of $G - B'$ containing $A$, $(G'_0, a_0, b_1, B', b_2)$ is planar and, for any $v \in B'(y_1^j, y_2^j)$, the degree of $v$ in $G'_0$ is 2.

Proof. To prove (i), assume for some distinct $i, j \in [m]$ with $i \neq j$, we have $A[x_1^i, x_2^i] \not\subseteq A[x_1^j, x_2^j]$, and $A[x_1^i, x_2^i] \not\subseteq A[x_1^j, x_2^j]$. Without loss of generality, let $b_1, y_1^i, y_2^i, y_1^j, y_2^j, b_2$ occur on $B$ in this order, and $a_1, x_1^i, x_1^j, a_2$ occur on $A$ in this order with $x_2^i, x_2^j \in A(x_1^i, x_1^j)$. By Lemma 2.3.1, $H^i - A(x_1^i, x_2^i)$ has two disjoint $A$-$B$ paths $P_1, P_2$ from $y_1^i, y_2^i$ to $x_2^i, x_1^i$, respectively, and $H^j - A(x_1^j, x_2^j)$ has two disjoint $A$-$B$ paths $P_3, P_4$ from $y_1^j, y_2^j$ to $x_2^j, x_1^j$, respectively. Therefore, $P_1, P_2, P_3, P_4$ form a double cross in $A, B$, a contradiction.

For (ii), let $j \in [m]$. Since $H^j$ is a core, $H^j - y_1^j y_2^j - A$ has a path $T_j$ from $y_1^j$ to $y_2^j$. So by Lemma 2.1.2, $H^j - y_1^j y_2^j$ has an induced path $P_j$ from $y_1^j$ to $y_2^j$ such that $H^j - y_1^j y_2^j - P_j$ is connected and $A[x_1^j, x_2^j] \subseteq H^j - y_1^j y_2^j - P_j$.

To see (iii), we observe that $A_0(B')$, the $B'$-bridge of $G$ containing $a_0$, is the same as, $A_0(B)$, the $B$-bridge of $G$ containing $a_0$. So $A, B'$ is also a good $a_0$-frame.

To prove (iv), let $C$ denote the component of $G - B'$ containing $A$; so $G'_0 = G - C$. By Lemma 2.1.6, $(A_0(B'), a_0, b_1, B', b_2)$ is planar. Thus, to show that $(G'_0, a_0, b_1, B', b_2)$ is planar, it suffices to show that for any $A$-$B$ connector $J$ with feet $v_1, v_2$, $(J - C, B'[v_1, v_2])$ is planar. This is clear when $J$ is a slim connector. So assume $J$ is a fat connector. Then $J$ has a separation $(H, L)$ satisfying (i), (ii), and (iii) of Lemma 2.3.1. By (ii) of Lemma 2.3.1, $(L - A, B' \cap L)$ is planar. Since $H - B' \subseteq C$, we see that $(J - C, B'[v_1, v_2])$ is planar.

Moreover, for any $v \in B'(y_1^j, y_2^j)$, since $B'[y_1^j, y_2^j]$ is a path in the core $H^j$, then, combined with (ii) that $P_j$ is induced in $G - y_1 y_2$, the degree of $v$ in $G'_0$ is exactly 2.  

In the remaining parts of this section, suppose $A, B$ is an ideal frame in $\gamma$. By (i) of Lemma 2.3.3, there exists an $A$-$B$ core $H$ with extreme hands $x_1, x_2$ and feet $y_1, y_2$ (as
Lemma 2.3.5 Let \( e_3 = a_3b_3, e_4 = a_4b_4 \in E(G) \) with \( a_3, a_4 \in V(A) \) and \( b_3, b_4 \in V(B') \).

(i) If for some \( i \in [2] \), \( a_3 \in V(A[a_i, x_i]) \), \( b_3 \in V(B'[t_2, b_2]) \), \( a_4 \in V(A(a_3, x_i)) \), and \( b_4 \in V(B'[b_1, t_1]) \), then \( G'_0 \) has a 3-cut \( \{a'_0, b'_1, b'_2\} \) with \( b'_1 \in B'[b_1, b_4] \) and \( b'_2 \in B'[t_2, b_3] \), which separates \( B'[b'_1, b'_2] \) from \( \{a_0, b_1, b_2\} \) in \( G'_0 \).

(ii) If for some \( i \in [2] \), \( a_3 \in V(A[a_i, x_i]) \), \( b_3 \in V(B'(b_1, t_1)) \), \( a_4 \in V(A(a_3, x_i)) \), and \( b_4 \in V(B'(t_2, b_2)) \), then one of the following holds:
(a) $G'_0$ has a 3-cut $\{a'_0, b'_1, b'_2\}$ with $b'_1 \in B'[b_3, t_1]$ and $b'_2 \in B'[b_4, b_2]$, which separates $B'[b'_1, b'_2]$ from $\{a_0, b_1, b_2\}$ in $G'_0$.

(b) $G'_0$ has a 2-cut $\{y_1, b'_2\}$ with $b'_2 \in B'[b_4, b_2]$, which separates $B'[y_1, b'_2]$ from $\{a_0, b_1, b_2\}$ in $G'_0$.

(iii) If $a_3 \in V(A[a_1, x_1])$, $a_4 \in V(A[x_2, a_2])$, and $b_3, b_4 \in V(B'(b_1, t_1))$, then $G'_0$ has a 3-cut $\{a'_0, b'_1, b'_2\}$ with $b'_1 \in B'[b_3, b_1]$ and $b'_2 \in B'[t_2, b_2]$, which separates $B'[b'_1, b'_2]$ from $\{a_0, b_1, b_2\}$ in $G'_0$.

(iv) If $a_3 \in V(A[a_1, x_1])$, $a_4 \in V(A[x_2, a_2])$, and $b_3, b_4 \in V(B'(t_2, b_2))$, then one of the following holds:

(a) $G'_0$ has a 3-cut $\{a'_0, b'_1, b'_2\}$ with $b'_1 \in B'[b_1, t_1]$ and $b'_2 \in B'[b_3, b_4]$, which separates $B'[b'_1, b'_2]$ from $\{a_0, b_1, b_2\}$ in $G'_0$.

(b) $G'_0$ has a 2-cut $\{y_1, b'_2\}$ with $b'_2 \in B'[b_3, b_4]$, which separates $B'[y_1, b'_2]$ from $\{a_0, b_1, b_2\}$ in $G'_0$.

Proof. Suppose (i) fails. Then, since $(G'_0, a_0, b_1, B', b_2)$ is planar and $y_2$ is the main foot of $H$, there exist disjoint paths $B'_2, A'_0$ in $G'_0 - (B'[b_1, b_4] \cup B'[y_2, b_3])$ from $b_2, a_0$ to $y_1, r_1$, respectively. Now, $A[a_i, a_3] \cup e_3 \cup B'[y_2, b_3] \cup P_{3-i, 2} \cup A(x_i, a_{3-i}) \cup R_i \cup A'_0$ and $B'[b_1, b_4] \cup e_4 \cup A[a_4, x_1] \cup R_{1, 1} \cup B'_2$ show that $\gamma$ is feasible, a contradiction.

Now suppose (ii) fails. Then, since $(G'_0, a_0, b_1, B', b_2)$ is planar and $y_2$ is the main foot of $H$, $G'_0 - (B'[b_3, r_1] \cup B'[b_4, b_2])$ contains two disjoint paths $B'_1, A'_0$ from $b_1, a_0$ to $y_1, y_2$, respectively. Now $A[a_i, a_3] \cup e_3 \cup B'[b_3, r_1] \cup R_1 \cup A(x_i, a_{3-i}) \cup P_{3-i, 2} \cup A'_0$ and $B'_1 \cup R_{1, 1} \cup A[a_4, x_1] \cup e_4 \cup B'[b_4, b_2]$ show that $\gamma$ is feasible, a contradiction.

If (iii) fails then, since $(G'_0, a_0, b_1, B', b_2)$ is planar and $y_2$ is the main foot of $H$, $G'_0 - (B'[b_3, b_4] \cup B'[t_2, b_2])$ has disjoint paths $B'_1, A'_0$ from $b_1, a_0$ to $r_1, y_1$, respectively. Moreover, by Lemma 2.3.2, for some $p \in [2]$, $H$ contains disjoint paths $Y_1, Y_2$ from $x_p, x_{3-p}$ to $y_1, y_2$, respectively. Thus, $A[a_1, x_1] \cup e_3 \cup B'[b_3, b_4] \cup e_4 \cup A[x_2, a_2] \cup Y_1 \cup A'_0$ and $B'_1 \cup R_1 \cup A(x_1, x_2) \cup Y_2 \cup B'[t_2, b_2]$ show that $\gamma$ is feasible, a contradiction.
Finally, suppose \((i\, iv)\) fails. Then, since \((G'_0, a_0, b_1, B', b_2)\) is planar and \(y_2\) is the main foot of \(H, G'_0 = (B'[b_1, t_1] \cup B'[b_3, b_4])\) has disjoint paths \(B'_2, A'_0\) from \(b_2, a_0\) to \(y_2, y_1\), respectively. Thus, \(A[a_1, x_1] \cup e_3 \cup B'[b_3, b_4] \cup e_4 \cup A[x_2, a_2] \cup Y_1 \cup A'_0\) and \(B'[b_1, r_1] \cup R_1 \cup A(x_1, x_2) \cup Y_2 \cup B'_2\) show that \(\gamma\) is feasible, a contradiction. \(\square\)

**Lemma 2.3.6** \(G'_0\) does not have 3-cuts \(\{a'_0, b'_1, b_2\}\) and \(\{a''_0, b_1, b''_2\}\) with \(b'_1 \in V(B'(b_1, t_1))\) and \(b''_2 \in V(B'(t_2, b_2))\) such that \(\{a'_0, b'_1, b_2\}\) separates \(B'[b'_1, b_2]\) from \(\{a_0, b_1, b_2\}\) and \(\{a''_0, b_1, b''_2\}\) separates \(B'[b'_1, b''_2]\) from \(\{a_0, b_1, b_2\}\).

**Proof.** For, suppose both 3-cuts exist. We choose \(\{a'_0, b'_1, b_2\}\) with \(B'[b_1, b'_1]\) minimal, and choose \(\{a''_0, b_1, b''_2\}\) with \(B'[b''_2, b_2]\) minimal. Then, since \(G'_0\) has a path from \(a_0\) to \(y_1\) and internally disjoint from \(B'\), it follows from Lemma 2.1.8 that

1. (ii) or (iii) or (iv) of Lemma 2.1.8 holds (and so \(c(A, B') \geq 1\)).

By the minimality of \(B[b_1, b'_1]\) and \(B[b''_2, b_2]\), it follows from (1) and planarity of \((G'_0, a_0, b_1, B', b_2)\) that

2. \(G'_0 - B'(b_1, b'_1) - B'(b''_2, b_2)\) has disjoint paths \(B'_1, B'_2, A'_0\) from \(b_1, b_2, a_0\) to \(b'_1, b''_2, y_1\), respectively, which are internally disjoint from \(B'\).

Also by the minimality of \(B[b_1, b'_1]\) and \(B[b''_2, b_2]\), it follows from (iii) and (iv) of Lemma 2.3.5 and Lemmas 2.1.8 and 2.1.9 that

3. \(G\) has no edge from \(B'(b_1, b'_1)\) to \(A[a_1, x_1]\) or no edge from \(B'(b_1, b'_1)\) to \(A[x_2, a_2]\); and \(G\) has no edge from \(B'(b''_2, b_2)\) to \(A[a_1, x_1]\) or no edge from \(B'(b''_2, b_2)\) to \(A[x_2, a_2]\).

Next, we claim that

4. \(\alpha(A, B') \leq 1\). For, suppose \(\alpha(A, B') = 2\). Then, by (1), \(a_0 = a'_0 = a''_0\); so \(c(A, B') \geq 2\). For convenience, let \(s_1 := b'_1\) and \(s_2 := b''_2\). Now, since \(\alpha(A, B') = 2\), \(G'_0\) has a path \(A^*_i\) (for each \(i \in [2]\))
from $a_0$ to $b_i$ and internally disjoint from $B'$. Hence, since $G^*$ is 6-connected, $B'(b_i, s_i) \neq \emptyset$ for $i \in [2]$.

We claim that there do not exist $e = ab, e' = a'b' \in E(G)$, such that for some $i \in [2]$, $a, a' \in A(a_i, x_i)$, $b, b' \in B'[b_1, s_1]$, and $e \in B'(b_1, b_2)$. For, otherwise, $\alpha(A, B') = 2$ and $c(A, B') = 0$ by Lemma 2.2.1, because of the path $B'[b_1, b] \cup e \cup A[a, a'] \cup e' \cup B'[b', b_2]$ from $b_1$ to $b_2$, the path $B'_1 \cup B'[b'_1, r_1] \cup R_1 \cup A[x_i, x_{3-i}] \cup P_{1,2} \cup B'[y_2, b'_2] \cup B'_2$ from $b_1$ to $b_2$, and the path $A_i \cup P_{3-i,1} \cup A[x_{3-i}, x_{3-i}]$ from $a_0$ to $a_{3-i}$. This is a contradiction.

Since $G^*$ is 6-connected, $G$ has at least three pairwise disjoint edges from $B'(b_i, s_i)$ (for each $i \in [2]$) to $A[a_i, x_i] \cup A[x_2, a_2]$. By (3), for each $i \in [2]$, we may assume for some $j \in [2]$, $G$ has no edge from $B'(b_i, s_i)$ to $A[a_j, x_j]$. Now, by symmetry, we assume $G$ has no edge from $B'(b_1, s_1)$ to $A[x_2, a_2]$.

By Lemma 2.1.7, $G$ has no cross from $A[a_1, x_1]$ to $B'(b_1, s_1)$. So, let $f_1 = u_iv_i$ for $i \in [3]$ be pairwise disjoint edges of $G$ with $u_i \in A[a_1, x_1]$ and $v_i \in B'(b_1, s_1)$, such that $a_1, u_1, u_2, a_2$ occur on $A$ in order, and $b_1, v_1, v_2, b_2$ occur on $B'$ in order. We choose $f_1, f_2$ so that $A[u_1, u_2] \cup B'[v_1, v_2]$ is maximal.

Then $G$ has no edge from $B'(s_2, b_2)$ to $A[a_1, x_1]$. For otherwise, $G$ has no edge from $B'(s_2, b_2)$ to $A[x_2, a_2]$ and, hence, has at least three pairwise disjoint edges from $B'(s_2, b_2)$ to $A[a_1, x_1]$. Therefore, $G$ has an edge from $A[a_1, x_1]$ to $B'(s_2, b_2)$, which together with $f_3$ contradicts our claim above.

Thus, $G$ has three pairwise disjoint edges from $B'(s_2, b_2)$ to $A[x_2, a_2]$. Since $G$ has no cross from $A[x_2, a_2]$ to $B'(s_2, b_2)$ (by Lemma 2.1.7), we let $f_j = u_jv_j$ for $j \in \{4, 5, 6\}$ be pairwise disjoint edges of $G$ with $u_j \in A[x_2, a_2]$ and $v_j \in B'(s_2, b_2)$, such that $a_1, u_4, v_5, u_5, a_2$ occur on $A$ in order, and $b_1, v_4, v_5, b_2$ occur on $B'$ in order. Choose $f_4, f_5$ so that $A[u_4, u_5] \cup B'[v_4, v_5]$ is maximal.

Now by the maximality of $A[u_1, u_2]$, $G$ has an edge $f_7 = u_7v_7$ with $u_7 \in A(u_1, u_2)$ and $v_7 \in B'[f_2, b_2]$, to avoid the cut $\{u_1, u_2, b_1, s_1, a_0\}$ in $G^*$. Similarly, by the maximality of $A[u_4, u_5]$, $G$ has an edge $f_8 = u_8v_8$ with $u_8 \in A(u_4, u_5)$ and $v_8 \in B'[b_1, t_1]$. Now, by the
claim above, \( v_7 \in B'[t_2, s_2] \) and \( v_8 \in B'[s_1, t_1] \). Hence, \( f_2, f_4, f_7, f_8 \) form a double cross, contradicting Lemma 2.1.7. □

For \( i \in [2] \), let \( a'_i \in V(A[a_i, x_i]) \) with \( A[a_i, a'_i] \) minimal such that \( a'_i = x_i \) or \( G \) has an edge from \( a'_i \) to \( B'(b'_1, b_2) \). Then \( G \) has an edge \( e_4 = a_4b_4 \) with \( a_4 \in A(a'_1, x_1) \cup A[x_2, a_2] \) and \( b_4 \in B[b_1, b'_1] \); for, otherwise, \( \{a_0, a'_1, a'_2, b'_1, b_2\} \) would be a 5-cut in \( G^* \) separating \( H \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction. By symmetry, we may assume

(5) \( a_4 \in A(a'_1, x_1) \).

Let \( e_3 = a_3b_3 \in E(G) \) with \( a_3 = a'_1 \) and \( b_3 \in B'(b'_1, t_1) \cup B'[t_2, b_2] \). Since \( e_3 \), \( e_4 \) and the paths in \( H \) do not form a double cross (by Lemma 2.1.7), we have

(6) \( b_3 \in B'[t_2, b_2] \).

Let \( e = ab \in E(G) \) with \( a \in A[a_1, a_3] \) and \( b \in B'[b_3, b_2] \), such that \( B'[b, b_2] \) is minimal, and subject to this, \( A[a_1, a] \) is minimal. Further, let \( e' = a'b' \in E(G) \) with \( a' \in A[a_1, a_4] \) and \( b' \in B'[b_1, b_4] \), such that \( B'[b_1, b'] \) is minimal, and subject to this, \( A[a_1, a'] \) is minimal.

Similarly, for each \( i \in [2] \), let \( a''_i \in V(A[a_i, x_i]) \) with \( A[a_i, a''_i] \) minimal such that \( a''_i = x_i \) or \( G \) has an edge from \( a''_i \) to \( B'(b'_1, b'_2) \). Since \( G^* \) is 6-connected, there exist \( j \in [2] \) and \( e_6 = a_6b_6 \in E(G) \) such that \( a_6 \in A(a''_j, x_j) \) and \( b_6 \in B'(b'_2, b_2) \). Since \( a''_j \neq x_j \), it follows from Lemma 2.1.7 that there exists \( e_5 = a_5b_5 \in E(G) \) such that \( a_5 = a''_j \) and \( b_5 \in B'(b_1, t_1) \).

(7) \( b \in B'(b'_2, b_2) \).

For, otherwise, \( b \notin B'(b'_2, b_2) \). Then, \( j = 2 \) and \( a_6 \in A[x_2, a''_2] \) by the choice of \( e \). Hence, \( b_5 \in B'[b_1, b_4] \) to avoid the double cross \( e_3, e_4, e_5, e_6 \). So \( b_5 = b_1 \) by (3), a contradiction to \( b_5 \in B'(b_1, t_1) \). □

If \( a' \neq x_1 \) then \( \alpha(A, B') = 2 \) by Lemma 2.2.1 and the following paths: the path \( A[a_1, a'] \cup e' \cup B'[b_1, b'] \) from \( a_1 \) to \( b_1 \), the path \( A[a_1, a] \cup e \cup B'[b, b_2] \) from \( a_1 \) to \( b_2 \), the
path $B_1^* \cup B'[b_1', r_1] \cup R_1 \cup A[x_1, x_2] \cup P_{1,2} \cup B'[y_2, b_2'] \cup B_2^*$ from $b_1$ to $b_2$, and the path $A_0^* \cup P_{2,1} \cup A[x_2, a_2]$ from $a_0$ to $a_2$. This contradicts (4).

So $a' = x_1$. Hence, by the choice of $e'$ and Lemma 2.1.7, $G$ has no edge from $A[a_1, x_1]$ to $B'[b_1, t_1]$. Thus, $G$ has an edge from $a_1$ to $B'[t_2, b_2]$. So by the choice of $e$ and by Lemma 2.1.7, $a = a_1$ and, hence, $b \neq b_2$.

We claim $a_6 \in A[x_2, a_2']$. For, otherwise, $a_6 \in A(a_2', x_1)$. Then $a_5 \in A(a_1, x_1)$. Now, $e_5$ contradicts the choice of $e'$, or $e_5, e', P_{1,2}, P_{2,1}$ form a double cross, contradicting Lemma 2.1.7.

Thus, by (3), $b_6 = b_2$. Moreover, $b_5 \in B'[b_1, b']$ to avoid the double cross $e, e', e_5, e_6$. Now, by (3), we may further assume $b_5 = b_1$, a contradiction to $b_5 \in B'(b_1, t_1)$. □

**Lemma 2.3.7** Let $\{a_0', b_1', b_2'\}$ be a cut in $G_0'$ separating $B'[b_1', b_2']$ from $\{a_0, b_1, b_2\}$, with $b_1' \in B'[b_1, t_1]$ and $b_2' \in B[t_2, b_2]$. Then $b_1' = b_1$, $b_2' \neq b_2$, $a_0' = a_0$, $y_1$ is a cut vertex in $G_0'$ separating $b_2$ from $\{a_0, b_1\}$, $b_2$ has degree 1 in $G_0'$, and for some $p \in [2]$, $G$ has an edge from $b_2$ to $x_p$ and no edge from $b_2$ to $A - x_p$.

**Proof.** For $i \in [2]$, let $a_i' \in V(A[a_i, x_i])$ with $A[a_i, a_i']$ minimal such that $a_i' = x_i$ or $G$ has an edge from $a_i'$ to $B'(b_1', b_2')$. Since $G^*$ is 6-connected, there exist $i, j \in [2]$ such that $G$ has an edge $e_4 = a_4b_4$ with $a_4 \in A(a_i', x_i)$ and $b_4 \in B'[b_j', b_j']$. By symmetry, assume $i = 1$.

Then $a_1' \neq x_1$ and let $e_3 = a_3b_3 \in E(G)$ such that $a_3 = a_1'$ and $b_3 \in B'(b_1', t_1) \cup B'[t_2, b_2']$. Now $b_3 \in B'[t_3-j, b_3-j]$, to avoid the double cross formed by $e_3, e_4$ and two paths in $H$ (by Lemma 2.1.7).

First, we show that

(1) $b_1' = b_1$.

For, suppose $b_1' \neq b_1$. Choose the 3-cut $\{a_0', b_1', b_2'\}$ with $b_1' \neq b_1$, such that $B[b_2, b_2']$ is minimal and, subject to this, $B[b_1, b_1']$ is minimal.

Observe that $b_4 \in B[b_1, b_4']$. For, otherwise, $b_4 \in B(b_2', b_2')$. Then $b_3 \in B(b_1', t_1)$. Now, by Lemma 2.1.9 and (ii) of Lemma 2.3.5, $G_0'$ has a 3-cut contradicting the choice of
Then $b_3 \in B'(t_2, b_2)$. Hence, because of $e_3, e_4$, it follows from (i) of Lemma 2.3.5 that $G'_0$ has a 3-cut $\{a''_0, b'_1, b'_2\}$ with $b''_1 \in B'[b_1, b_4]$ and $b''_2 \in B'[t_2, b_3]$, separating $B'[b''_1, b''_2]$ from $\{a_0, b_1, b_2\}$. By Lemma 2.1.8 and the choice of $\{a'_0, b'_1, b'_2\}$, we have $b'_1 = b_1$.

By Lemma 2.3.6, $b'_2 \neq b_2$. Hence, by Lemma 2.1.8, there exists $a'_0 \in V(G'_0)$, such that $\{b''_0, b'_2, a'_0\}$ is a 3-cut in $G'_0$ separating $\{a_0, b_1, b_2\}$ from $B'[b''_1, b''_2]$. For $i \in [2]$, let $a''_i \in A[a_i, x_i]$ with $A[a_i, a''_i]$ minimal such that $a''_i = x_i$ or $G$ has an edge from $a''_i$ to $B'(b''_1, b''_2)$.

Since $G^*$ is 6-connected, there exist $k \in [2]$ and $e_5 = a_5b_5 \in E(G)$ with $a_5 = A(a''_k, x_k)$ and $b_5 \in B'(b'_2, b_2)$. Let $e_6 = a_6b_6 \in E(G)$ with $a_6 = a''_k$ and $b_6 \in B'(b''_1, t_1) \cup B'[t_2, b'']$. Then $b_6 \in B'(b''_1, t_1)$, to avoid the double cross formed by $e_5, e_6$ and two paths in $H$.

Because of $e_5$ and $e_6$, it follows from (ii) of Lemma 2.3.5 and the choice of $\{a'_0, b'_1, b'_2\}$ that $G'_0$ has a 2-cut $\{y_1, b''_2\}$ with $b''_2 \in B'[b_5, b_2]$, separating $B'[y_1, b''_2]$ from $\{a_0, b_1, b_2\}$. But then, by Lemma 2.1.9, $\{y_1, b''_2\}$ and $\{a'_0, b'_1, b'_2\}$ force a 3-cut in $G'_0$, which contradicts the choice of $\{a'_0, b'_1, b'_2\}$. ∎

Since $G^*$ is 6-connected, it follows from (1) that $b_2 \neq b'_2$. We choose $\{a'_0, b'_1, b'_2\}$ so that $B[b_2, b'_2]$ is minimal. Then, by (1) and (ii) of Lemma 2.3.5, $G'_0$ has a 2-cut $\{y_1, b''_2\}$ with $b''_2 \in B'[b_4, b_2]$, separating $B'[y_1, b''_2]$ from $\{a_0, b_1, b_2\}$.

Moreover, $b''_2 = b_2$; for, otherwise, by Lemma 2.1.9, $\{y_1, b''_2\}$ and $\{a'_0, b'_1, b'_2\}$ force a 3-cut in $G'_0$, which contradicts the choice of $\{a'_0, b'_1, b'_2\}$. Hence, $y_1$ is a cut vertex in $G'_0$ separating $b_2$ from $\{a_0, b_1\}$ and $\alpha(A, B') \leq 1$. And (for any choice of $\{a'_0, b'_1, b'_2\}$) $a'_0 = a_0$; or else, since $y_1$ is a cut vertex in $G'_0$ separating $b_2$ from $\{a_0, b_1\}$, $\{b_1, a'_0, b'_2, b_2\}$ is a cut in $G$ separating $a_0$ from $\{a_1, a_2\}$, a contradiction.

So by (1), $G'_0 - B'(b_1, t_1) \cup B'[y_1, b_2]$ has disjoint paths $B^*_{t_1}, A^*_0$ from $b_1, a_0$ to $t_1, y_1$, respectively, such that $A^*_0$ is internally disjoint from $B'$. By the choice of $\{a'_0, b'_1, b'_2\}$, $G'_0 - B'(b'_2, b_2)$ has a path $B^*_2$ from $b_2$ to $b'_2$.

(2) For $i \in [2]$, if $G$ has an edge from $B'(b'_2, b_2)$ to $A[a_i, x_i]$, then $G$ has no edge from
A[i, x_i] to B'[b_1, t_1].

For, suppose for some i ∈ [2], G has an edge e from b ∈ B'(b'_2, b_2) to a ∈ A[i, x_i] and an edge e' from a' ∈ A[i, x_i] to b' ∈ B'[b_1, t_1].

Then, α(A, B') = 2, by Lemma 2.2.1 and the following paths: A[i, a'] ∪ e' ∪ B'[b_1, b'] from a_i to b_1, the path A[i, a] ∪ e ∪ B'[b, b_2] from a_i to b_2, the path B_1^* ∪ R_i ∪ A[x_i, x_{3−i}] ∪ P_{i,2} ∪ B_2^* from b_1 to b_2, and the path A_0^* ∪ P_{3−i,1} ∪ A[x_{3−i}, a_3−i] from a_0 to a_3−i. This is a contradiction. □

(3) B'(b'_2, b_2) = ∅, and so b_2 has degree 1 in G_0'.

For, suppose B'(b'_2, b_2) ≠ ∅. Then, as G^* is 6-connected, G has edges from B'(b'_2, b_2) to A[a_1, x_1] ∪ A[x_2, a_2].

Indeed, G has an edge e_3 from B'(b'_2, b_2) to A[a_1, x_1], and an edge e_4 from B'(b'_2, b_2) to A[x_2, a_2]. For otherwise, there exists i ∈ [2], such that all edges of G from B'(b'_2, b_2) to A end in A[i, x_i]. Let u_1, u_2 ∈ V(A[i, x_i]), such that G has edges from B'(b'_2, b_2) to u_1, u_2, respectively, and, subject to this, A[u_1, u_2] is maximal. Now, by Lemma 2.1.7, G has no edge from A(u_1, u_2) to B'[t_2, b'_2). Moreover, by (2), G has no edge from A(u_1, u_2) to B'[b_1, t_1). But then, {t_1, u_1, u_2, b'_2, b_2} is a cut in G separating V(A[u_1, u_2] ∪ B'[b'_2, b_2]) from {a_0, a_1, a_2, b_1, b_2}, a contradiction.

Now A[a_1, x_1] ∪ e_3 ∪ B'(b'_2, b_2) ∪ e_4 ∪ A[x_2, a_2] ∪ Y_1 ∪ A_0^* and B'[b_1, r_1] ∪ R_i ∪ A(x_1, x_2) ∪ Y_2 ∪ B'[y_2, b'_2] ∪ B_2^* show that γ is feasible, a contradiction. □

(4) G has no edge from b_2 to A[a_1, x_1] ∪ A(x_2, a_2).

Suppose for some i ∈ [2], G has an edge e from b_2 to a ∈ A[i, x_i]. Let e' = a_1b' ∈ E(G) with b' ≠ t_1. Obviously, b' ∉ B'[t_2, b_2); otherwise, e, e' and two disjoint paths in H force a double cross, contradicting Lemma 2.1.7.

So b' ∈ B[b_1, t_3). Now α(A, B') = 2 by Lemma 2.2.1 and the following paths: the path e' ∪ B'[b_1, b'] from a_i to b_1, the path A[i, a] ∪ e from a_i to b_2, the path B_1^* ∪ R_i ∪
A[x_i, x_{3-i}] \cup P_{i,2} \cup B'[y_2, b_2] \text{ from } b_1 \text{ to } b_2, \text{ and the path } A_0^x \cup P_{3-i,1} \cup A[x_{3-i}, a_{3-i}] \text{ from } a_0 \text{ to } a_{3-i}. \text{ However, this is a contradiction. } \square

Now, since the degree of } b_2 \text{ in } G \text{ is at least 2, it follows from (4) that } G \text{ has an edge from } b_2 \text{ to } x_p \text{ for some } p \in [2]. \text{ If } G \text{ has no edge from } b_2 \text{ to } x_{3-p}, \text{ then we are done. So assume } b_2x_1, b_2x_2 \in E(G). \text{ Then } a_1 \neq x_1 \text{ and } a_2 \neq x_2. \text{ Now, by Lemma 2.1.7, } G \text{ has no edge from } \{a_1, a_2\} \text{ to } B'[t_2, b_2]. \text{ Since } G* \text{ is 6-connected, } G \text{ has edges } e_1, e_2 \text{ from } B'(b_1, t_1) \text{ to } a_1, a_2, \text{ respectively. But then, it follows from (iii) of Lemma 2.3.5 that } G_0' \text{ contains a 3-cut, which contradicts (1). } \square

**Lemma 2.3.8** 

*H* is the unique main A-B' core in γ.

**Proof.** Suppose for a contradiction that *H''* is a main A-B' core with *H''* ≠ *H*, and let

w_1, w_2 \text{ be the feet of } H'' \text{ (with } w_2 \text{ as the main foot). Then, by Lemma 2.1.7, } w_2 = r_1 \text{ and } b_1, w_2, w_1, y_1, y_2, b_2 \text{ occur on } B' \text{ in order.}

Recall the definition of } x'_i, X'_i \text{ before Lemma 2.3.2. For } i \in [2], \text{ let } x''_i \in V(A(x_1, x_2)) \text{ such that } x''_i, x_i \text{ are incident with some finite face of } H'' - w_1, \text{ and } H'' - w_1 \text{ has a path from } x''_i \text{ to } w_2 \text{ and internally disjoint from } A. \text{ So for } i \in [2], \text{ let } X''_i \text{ be the path from } w_2 \text{ to } x''_i \text{ on the outer walk of } H'' - \{w_1, x_i\} \text{ without going through } x_{3-i}, \text{ and, moreover, let } X''_i \text{ be the path from } x_i \text{ to } w_2 \text{ on the outer walk of } H'' - w_1 \text{ without going through } x_{3-i}. \text{ And let } A_0 \text{ be a path in } G \text{ from } a_0 \text{ to } y_1 \text{ and internally disjoint from } B'.

Suppose *H* contains disjoint paths from y_1, y_2 to x_2, x'_1, respectively, and internally disjoint from A, as well as disjoint paths from y_1, y_2 to x_1, x'_2, respectively, and internally disjoint from A. Then, by Lemma 2.1.7, for any } i \in [2], \text{ } H'' \text{ does not contain disjoint paths from } w_1, w_2 \text{ to } x_i, x''_{3-i}, \text{ respectively, and internally disjoint from } A. \text{ This contradicts (iii) of Lemma 2.3.2.}

Hence, by symmetry, we may assume that *H* contains no disjoint paths from y_1, y_2 to x_1, x'_2, respectively, and internally disjoint from A. Then by Lemma 2.3.2, *H* contains disjoint paths \(Y'_1, Y'_2\) from y_1, y_2 to x_2, x'_1, respectively, and internally disjoint from A.
Then by Lemma 2.1.7 and 2.3.2, we may further assume $H''$ contains disjoint paths $Y''_1, Y''_2$ from $w_1, w_2$ to $x_2, x''_1$, respectively, and internally disjoint from $A$, but no disjoint paths from $w_1, w_2$ to $x_1, x''_2$, respectively, and internally disjoint from $A$. Moreover, by (i) of Lemma 2.3.2, $H - \{y_1, y_2\} \cup V(A(x_1, x_2))$ contains a path $D'$ from $x_1$ to $x_2$, and $H'' - \{w_1, w_2\} \cup V(A(x_1, x_2))$ contains a path $D''$ from $x_1$ to $x_2$.

(1) There is no $A-B'$ path in $G$ from $A(x_1, x_2)$ to $B'(w_1, y_1)$.

For, suppose that $P$ is an $A-B'$ path from $p \in V(A(x_1, x_2))$ to $p' \in V(B'(w_1, y_1))$. Then $G' = B'(w_1, w_2) - B'[y_2, b_2]$ does not contain disjoint paths $B'_1, A'_0$ from $b_1, a_0$ to $p', y_1$, respectively; otherwise, $A[a_1, x_1] \cup D'' \cup A[x_2, a_2] \cup Y'_1 \cup A'_0$ and $B'_1 \cup P \cup A(x_1, x_2) \cup Y'_2 \cup B'[y_2, b_2]$ show that $\gamma$ is feasible, a contradiction. Hence, there exists $w' \in V(B'(w_2, w_1))$, $a'_0 \in V(G'_0)$, and $b'_2 \in V(B'[y_2, b_2])$, such that $\{w', a'_0, b'_2\}$ is a 3-cut in $G'_0$ separating $B'[w', b'_2]$ from $\{a_0, b_1, b_2\}$.

Now $b_1 = w_2$. For, suppose not. Since $w_1, w_2$ are feet of $H''$, $w_1, w_2$ are incident with some finite face of $G'_0$. Therefore, $\{w_2, a'_0, b'_2\}$ is a 3-cut in $G'_0$ separating $B'[w_2, b'_2]$ from $\{a_0, b_1, b_2\}$, a contradiction to Lemma 2.3.7. Similarly, by the symmetry between $H$ and $H''$, we can also prove $b_2 = y_2$.

Now, since $b'_2 \in V(B'[y_2, b_2])$, $b'_2 = b_2$. So $a'_0 = a_0$; or else, $\{b_1, a'_0, b'_2\}$ is a 3-cut in $G'_0$ separating $a_0$ from $B'(b_1, b_2)$, a contradiction. Then $a_0, b_1, w', w_1$ are incident with some finite face of $G'_0$. Similarly, by the symmetry between $H$ and $H''$, $a_0, b_2, y_1$ are incident with some finite face of $G'_0$, which implies $\alpha(A, B') = 0$.

By Lemma 2.3.2, $V(X''_2 \cap X'_1) - \{w_2\} = \emptyset$. Now $\alpha(A, B') \geq 1$ by Lemma 2.2.1 and the following paths: the path $A_0 \cup Y'_1 \cup A[x_2, a_2]$ from $a_0$ to $a_2$, the path $X''_2 \cup A(x_1, x_2) \cup Y'_2$ from $b_1$ to $b_2$, and the path $A[a_1, x_1] \cup X'_1$ from $a_1$ to $b_1$. This is a contradiction. □

(2) $a_1 = x_1$ and $a_2 = x_2$.

Recall that for $i \in [2]$, $P_{1,i}$ and $P_{2,3-i}$ are disjoint paths from $x_1, x_2$ to $y_i, y_{3-i}$, respectively,
in $H - A(x_1, x_2)$. For $i \in [2]$, let $Q_{1,i}, Q_{2,3-i}$ be disjoint paths from $x_1, x_2$ to $w_i, w_{3-i}$, respectively, in $H'' - A(x_1, x_2)$.

We claim that for $i \in [2]$, $G$ has no edge from $A[a_i, x_i]$ to $B'(b_1, w_2)$. For, suppose there exists $e' = a'b' \in E(G)$ with $a' \in A[a_i, x_i]$ and $b' \in B(b_1, w_2)$. Then $b_1 \neq w_2$. By Lemma 2.3.7, $G_0' - B'[b', w_2] - B'[y_2, b_2]$ contains disjoint paths $B_1^*, A_0^*$ from $b_1, a_0$ to $w_1, y_1$, respectively. Now $A[a_i, a'] \cup e' \cup B'[b', w_2] \cup Q_{3-i,2} \cup A[x_{3-i}, a_{3-i}] \cup P_{3-i,1} \cup A_0^*$ and $B_1^* \cup Q_{i,1} \cup P_{i,2} \cup B'[y_2, b_2]$ show that $\gamma$ is feasible, a contradiction.

Due to the symmetry between $H$ and $H''$, with the same argument above, we can show that for $i \in [2]$, $G$ has no edge from $A[a_i, x_i]$ to $B'[y_2, b_2)$. Thus, (2) follows from Lemma 2.3.4 and the assumption that $G^*$ is 6-connected. $\square$

(3) $H'' - X_1^* \cup X_2^*$ contains a path $Q''$ from $w_1$ to $A(x_1, x_2)$; and $H - X_1 \cup X_2$ contains a path $Q$ from $y_1$ to $A(x_1, x_2)$.

By the symmetry between $H$ and $H''$, we only prove the existence of $Q''$. Suppose for a contradiction that $Q''$ does not exist.

We see that $(N_G(w_1) \cap V(H'')) \subseteq V(X_2'' \cup A(x_1, x_2))$. For, otherwise, by (ii) of Lemma 2.3.2, there exists $v'' \in N_G(w_1) \cap V(H'')$, $c_1'' \in A(x_1, x_2)$, and $c_2'' \in X_2''(x_2', w_2)$, such that $v'' \notin X_2'' \cup A(x_1, x_2)$, $\{c_1'', c_2''\}$ is a cut in $H'' - \{w_1, x_2\}$ separating $v''$ from $x_1$, and there exists a path $P_1''$ from $v''$ to $c_1''$ in $H'' - w_1 - x_2$, which is internally disjoint from $X_2'' \cup A[x_1, x_2']$. But then, $w_1 v'' \cup P_1''$ is a path from $w_1$ to $A(x_1, x_2)$ in $H'' - X_1^* \cup X_2^*$, a contradiction.

Now, since $Q''$ does not exist, combined with $(N_G(w_1) \cap V(H'')) \subseteq V(X_2'' \cup A(x_1, x_2))$, we may further assume $(N_G(w_1) \cap V(H'')) \subseteq V(X_2^*)$, contradicting (iii) of Lemma 2.3.1. $\square$

(4) $b_1 = w_2$ and $b_2 = y_2$.

By the symmetry between $H$ and $H''$, we only show $b_1 = w_2$. Suppose for a contradiction that $b_1 \neq w_2$. 39
Since \( w_1, w_2 \) are incident with some finite face of \( G'_0 \), it follows from Lemma 2.3.7 that \( G'_0 = B'[w_2, w_1] - B'[y_2, b_2] \) contains disjoint paths \( B^*_1, A^*_0 \) from \( b_1, a_0 \) to \( w_1, y_1 \), respectively.

Now, \( A[a_1, x_1] \cup X^*_1 \cup X^*_2 \cup A[x_2, a_2] \cup Y^*_1 \cup A^*_0 \) and \( B^*_1 \cup Q^* \cup A(x_1, x_2) \cup Y^*_2 \cup B'[y_2, b_2] \) show that \( \gamma \) is feasible, a contradiction. \( \square \)

Note that \( G \) has no \( A-B' \) path from \( a_1 \) to \( B'(w_1, y_1) \), as such a path together with \( Y''_2, Y'_1, Y'_2 \) forms a double cross, contradicting Lemma 2.1.7. So by (1) and (4), \( \{b_1, b_2, w_1, y_1, a_2\} \) is a cut in \( G \) separating \( a_0 \) from \( a_1 \), a contradiction. \( \square \)

We now use \( A, B' \) to form a new frame \( A', B' \), called core frame.

**Lemma 2.3.9** Let \( M_0 \) denote the union of all the \( A-B' \) bridges that are disjoint from \( H - A - y_1 \). Then there exists an induced path \( A' \subseteq (A \cup M_0) - B' \) from \( a_1 \) to \( a_2 \) in \( G \), such that \( A'[a_i, x_i] = A[a_i, x_i] \) for \( i \in [2] \) and the following hold:

(i) \( A', B' \) is a good frame in \( \gamma \).

(ii) Each \( A'-B' \) bridge lying on \( B'[r_1, y_1] \) is contained in some \( A-B' \) bridge.

(iii) There exists an induced subgraph \( H^* \) in \( G \), such that \( A'[x_1, x_2] \cup H \subseteq H^* \), all \( A'-B' \) bridges not lying on \( B'[r_1, y_1] \) are contained in \( H^* \), and \( H^* \) is separated from \( \{a_0, a_1, a_2, b_1, b_2\} \) by \( V(A'[x_1, x_2]) \cup \{y_1, y_2\} \) in \( G \).

(iv) For any \( v \in (V(H^*) - V(A') \cup \{y_1\}) \), \( H^* - y_1 \) contains a path from \( v \) to \( y_2 \) and internally disjoint from \( A' \).

(v) If \( l, r \) are the extreme hands of an \( A'-B' \) bridge lying on \( B'[r_1, y_1] \) then \( \{l, r\} \neq \{x_1, x_2\} \), and \( H^* - y_1 \) does not contain a path from \( y_2 \) to \( A'(l, r) \) and internally disjoint from \( A' \).

**Proof.** We choose the induced path \( A' \) so that \( A' \subseteq A \cup M_0 - B' \) is from \( a_1 \) to \( a_2 \), such that \( A'[a_i, x_i] = A[a_i, x_i] \) for \( i \in [2] \), (i)-(iv) are satisfied, and, subject to this, \( H \) is maximal. Note that such \( A' \) exists, as \( A \) satisfies (i)-(iv).
To prove (v), let $M$ be an $A'-B'$ bridge $M$ lying on $B'[r_1, y_1]$ with extreme hands $l, r$ and feet $l', r'$. If $\{l, r\} = \{x_1, x_2\}$ then, since $M$ is contained in an $A-B'$ bridge (by (ii)), $M$ is contained in a main $A-B'$ core, a contradiction to Lemma 2.3.8. Hence, $H - y_1$ contains a path $Y_2$ from $y_2$ to $y_2' \in A'(l, r)$ and internally disjoint from $A'$.

Let $T$ be an induced path in $M - A'(l, r) \cup B'[l', r']$ from $l$ to $r$, and let $C_1, C_2, \ldots, C_n$ be the components of $M \cup A'[l, r] \cup B'[l', r'] - T$ not containing $A'(l, r)$ and not containing $B'[l', r']$. We choose $T$, such that $|T| = (\sum_{i \in [n]} |C_i|, |V(C_1)|, |V(C_2)|, \ldots, |V(C_n)|)$ is maximal with respect to the lexicographical ordering.

We claim $n = 0$. For, suppose $n > 0$. Let $l_n, r_n \in N_G(C_n) \cap V(T)$ such that $T[l_n, r_n]$ is maximal. Since $G^*$ is 6-connected, there exists another component $C$ of $M \cup A'[l, r] \cup B'[l', r'] - T$, such that $N_G(C) \cap T(l_n, r_n) \neq \emptyset$. Now, let $T'$ be an induced path in $G[T \cup C_n]$ between $l_n$ and $r_n$, such that $T' \cap T(l_n, r_n) = \emptyset$. Clearly, $|T'| > |T|$, a contradiction.

Now, let $A''$ be obtained from $A'$ by replacing $A'[l, r]$ with $T$. Clearly, $A''[a_i, x_i] = A[a_i, x_i]$ for $i \in [2]$. Since $T$ is induced, $A''$ is induced. Moreover, since $n = 0$, then any component of $G[V(M \cup A'[l, r] \cup B'[l', r']) - T$ contains $A'(l, r)$ or $B'[l', r']$, and so $G - A''$ is connected. Hence, $A'', B'$ is a frame. Since $A'_0(B') = A'_0(B') = A_0(B')$, we see that $A'', B'$ is a good frame in $\gamma$.

Next, we show that $G$ has no $A'-B'$ path from $A'(l, r)$ to $B'[b_1, y_1]$ and disjoint from $T$. For otherwise, let $S$ be an $A'-B'$ path from $s \in A'(l, r)$ to $s' \in B'[b_1, y_1]$ and disjoint from $T$. Then $A''$ and $B'[b_1, s'] \cup S \cup A'[s, y_2'] \cup Y_2 \cup B'[y_2, b_2']$ are disjoint paths from $a_1, b_1$ to $a_2, b_2$, respectively, in $G - (A_0(B') - Y_1)$, a contradiction to (i) of Lemma 2.2.2.

Hence, there does not exist an $A'-B'$ bridge $N$ lying on $B'[r_1, y_1]$, such that $N \neq M$, $N \cap A'(l, r) \neq \emptyset$, and $N \cap B'[b_1, y_1] \neq \emptyset$. So each $A''-B'$ bridge lying on $B'[r_1, y_1]$ must be contained in some $A'-B'$ bridge and, hence, contained in some $A-B'$ bridge. So $A'', B'$ satisfies (ii).

And $V(A''[x_1, x_2]) \cup \{y_1, y_2\}$ is a cut in $G$ separating $V(H)$ from $\{a_0, a_1, a_2, b_1, b_2\}$. Now, we let $V''$ be the set of vertices of $A'' \cup B'[b_1, y_1] \cup B'[y_2, b_2]$-bridge of $G$ containing
A'(l, r), and let \( H'' := G[V'' \cup V(A''[x_1, x_2])] \). Then clearly (iii) and (iv) holds for \( A'', B' \).

However, \( H'' \) properly contains \( H \), a contradiction. \( \square \)

### 2.4 Inside the main \( A'-B' \) core

We use the notation of the previous section, in particular, Lemma 2.3.3 and 2.3.9: \( \gamma \) is infeasible, \( A', B' \) is a core frame, and let \( H' := H^* - \{x_1y_2, x_2y_2\} \), where \( B', t_1, t_2, R_1, r_1 \) are defined as in or after Lemma 2.3.3, \( A', H^*, x_1, x_2, y_1, y_2 \) are defined as in Lemma 2.3.9.

We also say that \( H' \) is the main \( A'-B' \) core in \( \gamma \) with extreme hands \( x_1, x_2 \) and feet \( y_1, y_2 \) (such that \( y_2 \) is the main foot).

We now study the structure of \( G \) inside \( H' \).

**Lemma 2.4.1** \( (H' - y_1, A'[x_1, x_2], y_2) \) is planar, the degree of \( y_2 \) in \( H' - y_1 \) is at least 2, and \( H' - y_1 - A'(x_1, x_2) \) contains disjoint paths from \( y_1, y_2 \) to \( x_i, x_{3-i} \), respectively, for \( i \in [2] \). Moreover, for \( i \in [2] \), let \( X_i \) be the path from \( x_i \) to \( y_2 \) on the outer walk of \( H' - y_1 \) without going through \( x_{3-i} \), then \( N_G(y_1) \cap V(H' - y_1 - A') \not\subseteq V(X_i) \) for \( i \in [2] \).

**Proof.** We can apply the same proof in Lemma 2.2.4, and show that \( (H' - y_1, A'[x_1, x_2], y_2) \) is planar, and \( N_G(y_1) \cap V(H' - y_1 - A') \not\subseteq V(X_i) \) for \( i = 1, 2 \).

Moreover, since \( V(H - y_1) \subseteq V(H' - y_1) \), then, by (iii) of Lemma 2.3.1, the degree of \( y_2 \) in \( H' - y_1 \) is at least 2, and \( H' - A'(x_1, x_2) - \{y_1x_1, y_1x_2\} \) contains disjoint paths from \( y_1, y_2 \) to \( x_1, x_2 \), respectively, as well as disjoint paths from \( y_1, y_2 \) to \( x_2, x_1 \), respectively. \( \square \)

**Lemma 2.4.2** Let \( R \) be an \( A'-B' \) path from \( r \in V(A'(x_1, x_2)) \) to \( r' \in V(B'[r_1, y_1]) \) such that \( B'[r_1, r'] \) is minimal. If \( r' \neq r_1 \) then the following conclusions hold:

(i) There exists an \( A-B \) core \( H_1 \) with \( r_1 \) as a foot.

(ii) Let \( r_2 \) be the other foot of \( H_1 \), then there exists an \( A'-B' \) bridge with \( r_1 \) as a foot, intersecting \( A' \) only at \( x_j \) for some \( j \in [2] \), and lying on \( B'[r_1, r_2] \).
Lemma 2.3.9). Hence, since \( r_j \), which must have \( r \) 

Therefore, since \( r \) 

Lemma 2.3.1, \( G \) respectively, in \( r \) since \( G \) containing an \( G \) 

Proof. To prove (i), assume that \( r_1 \) is not a foot of any \( A-B \) core. Then by the definition of \( r_1 \), \( G \) has an edge from \( r_1 \) to \( a' \in V(A(x_1, x_2)) \). Since \( r' \neq r_1 \), \( a' \notin A'(x_1, x_2) \). Moreover, \( a' \) is not contained in any \( A'-B' \) bridge lying on \( B'[r_1, y_1] \), as any such \( A'-B' \) bridge is contained in an \( A-B' \) bridge (by (ii) of Lemma 2.3.9). So \( a' \in V(H' - y_1) \setminus V(A') \). Hence, by (iv) of Lemma 2.3.9, \( H' - y_1 \) has a path \( Y_2 \) from \( a' \) to \( y_2 \) and internally disjoint from \( A' \). Therefore, \( A' \) and \( B'[b_1, r_1] \cup r_1 a' \cup Y_2 \cup B'[y_2, b_2] \) are disjoint paths from \( a_1, b_1 \) to \( a_2, b_2 \), respectively, in \( G - V(A'_0(B') - B') \cup \{ y_1 \} \), contradicting (i) of Lemma 2.2.2.

Now, we prove (ii). By Lemma 2.3.4, \( r_2 \) is the main foot of \( H_1 \). Hence, by (iii) of Lemma 2.3.1, \( r_1 \) has two neighbors \( u_1, u_2 \) in \( H_1 - r_2 - A \). Since \( B'[r_1, r_2] \) is induced in \( G - \{ r_1 r_2 \} \) (by Lemma 2.3.3), \( u_p \notin B' \) for some \( p \in [2] \). Moreover, \( u_p \notin A'(x_1, x_2) \) 

since \( r' \neq r_1 \). Thus, \( u_p \) must be contained in some \( A'-B' \) bridge \( M_0 \) lying on \( B'[r_1, r_2] \), which must have \( r_1 \) as a foot and cannot have both \( x_1 \) and \( x_2 \) as extreme hands (by (v) of Lemma 2.3.9). Hence, since \( r' \neq r_1 \), this \( A'-B' \) bridge intersect \( A' \) only at \( x_j \) for some \( j \in [2] \).

Obviously, since \( G^* \) is 6-connected, \( r' \in B'(r_1, r_2) \) to avoid the cut \( \{ r_1, r_2, x_1, x_2 \} \) in \( G^* \) separating \( V(H_1) \) from \( \{ a_0, a_1, a_2, b_1, b_2 \} \). Let \( l'_0, r'_0 \) be the feet of \( M_0 \) with \( l'_0 = r_1 \) and \( r'_0 \in B'[r_1, r_2] \). For, suppose (iii) fails. Then \( r' \in B'[r'_0, r_2] \). Since \( x_{3-j} \notin V(H_1 \cap A') \) (by Lemma 2.3.8), then by the definition of \( r' \), \( \{ x_j, r_1, r' \} \) is a cut in \( G \) separating \( M_0 \) from \( \{ a_0, a_1, a_2, b_1, b_2 \} \), a contradiction.

To prove (iv), we observe that \( B'[r_1, r_2] \) is on the boundary of a finite face of \( G'_0 \). Therefore, since \( r' \in B'(r_1, r_2) \), \( a'_0 \) and \( r_1 \) are also incident with that finite face. Suppose \( r_1 \neq b_1 \) or \( a'_0 \neq a_0 \). Then \( \{ a'_0, r_1, b_2 \} \) is a 3-cut in \( G'_0 \) separating \( B'[r_1, b_2] \) from \( \{ a_0, b_1, b_2 \} \).
By Lemma 2.3.7, \( r_1 = b_1 \). So \( a'_0 \neq a_0 \). Then, by Lemma 2.3.7, \( \{a'_0, b_1, b'_2, b_2\} \) is a cut in \( G \) separating \( a_0 \) from \( \{a_1, a_2\} \), a contradiction. So, \( r_1 = b_1 \) and \( a'_0 = a_0 \). Hence, \( G'_0 \) has no path that is from \( a_0 \) to \( b_1 \) and internally disjoint from \( B' \). In particular, \( \alpha(A', B') \leq 1 \). \( \square \)

Since \( G^* \) is 6-connected, \( G \) has two disjoint \( A' - B' \) paths \( P, Q \) from \( p, q \in V(A'(x_1, x_2)) \) to \( p', q' \in V(B'[r_1, y_1]) \), respectively. We choose \( P, Q \) so that

(i) \( A'[p, q] \) is maximal,

(ii) subject to (i), \( B'[b_1, p'] \cap B'[b_1, q'] \) is minimal, and

(iii) subject to (ii), \( B'[p', q'] \) is maximal.

By the symmetry between \( a_1 \) and \( a_2 \), we may relabel \( a_1, x_1, x_2, a_2 \) so that

- \( a_1, x_1, p, q, x_2, a_2 \) occur on \( A' \) in order, and \( b_1, r_1, p', q', y_1, b_2 \) occur on \( B' \) in order.

**Lemma 2.4.3** Any \( A' - B' \) path from \( B'[r_1, p'] \) to \( A'(x_1, x_2) \) must be disjoint from \( P, Q \), and end in \( A'(p, q) \). Moreover, if \( H' - y_1 \) contains a path from \( u \in A'[q, x_2] \) to \( y_2 \) and internally disjoint from \( A' \), then all \( A' - B' \) paths from \( A'(u, x_2) \) to \( B'[r_1, y_1] \) and internally disjoint from \( H' - y_1 \) are edges ending in \( \{r', y_1\} \).

**Proof.** First, assume \( S \) is an \( A' - B' \) path from \( s' \in V(B'[r_1, p']) \) to \( s \in V(A'(x_1, x_2)) \). Then \( V(S \cap (P \cup Q)) = \emptyset \); for otherwise, let \( v \in V(S \cap (P \cup Q)) \) with \( S[s', v] \) minimal then \( P' := S[s', v] \cup P[v, p] \) and \( Q \) (when \( v \in V(P) \)) or \( P \) and \( Q' := S[s', v] \cup Q[v, q] \) (when \( v \in V(Q) \)) contradict the choice of \( P, Q \). Hence, \( s \in A'(p, q) \) as otherwise \( S, P \) or \( S, Q \) contradict the choice of \( P, Q \).

Now let \( Y_2 \) be a path in \( H' - y_1 \) from \( u \in V(A'[q, x_2]) \) to \( y_2 \) and internally disjoint from \( A' \). We first see that \( G \) has no path from \( A'(u, x_2) \) to \( B'[r_1, y_1] - p' \). For, suppose not. Let \( S \) be an \( A' - B' \) path from \( s \in V(A'(u, x_2)) \) to \( s' \in V(B'[r_1, y_1] - p') \). Then \( V(S \cap P) \neq \emptyset \), or else, \( P, S \) contradict the choice of \( P, Q \). Since \( s' \neq p' \), \( S, P \) are contained in an \( A' - B' \) bridge. However, by \( u \in A'(p, s) \), the existence of \( Y_2 \) contradicts (v) of Lemma 2.3.9.
Now let $S$ be an arbitrary $A'-B'$ path from $s \in A'(u, x_2)$ to $s' \in B'[r_1, y_1]$. Suppose $S$ has length at least 2. Then $S$ is contained in some $A'-B'$ bridge $N$ with feet $n'_1, n'_2$ and extreme hands $n_1, n_2$. Then $n'_1, n'_2 \in \{p', y_1\}$. By (v) of Lemma 2.3.9 and the existence of $S$ and $Y_2$, $A'[n_1, n_2] \subseteq A[u, x_2]$. Let $h_1, h_2 \in A'[x_1, x_2]$, such that $A'[n_1, n_2] \subseteq A'[h_1, h_2]$, $H' - y_1$ does not contain a path from $A'(h_1, h_2)$ to $y_2$ and internally disjoint from $A'$, and subject to this, $A'[h_1, h_2]$ is maximal. Clearly, $A'(h_1, h_2) \subseteq A'(u, x_2)$, and for $i \in [2]$, $H' - y_1$ contains a path from $h_i$ to $y_2$ and internally disjoint from $A'$. By (v) of Lemma 2.3.9, $\{h_1, h_2, p', y_1\}$ is a cut in $G^*$ separating $V(N)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Thus, $S$ must be an edge. To complete the proof, we need to show $r' = p'$. For, suppose $r' \neq p'$. By (i), $R$ is disjoint from $P, Q$ with $r \in A'(p, q)$, and so $R, P, S, Y_2$ force a double cross in $A, B$, contradicting Lemma 2.1.7.

Let $R = P$ if $r' = p'$, and if $r' \neq p'$ then by Lemma 2.4.3, $R$ is disjoint from $P, Q$ with $r \in A'(p, q)$ (seen at Figure 2.8). By Lemma 2.4.1, for $i \in [2]$, we let $P_{1,i}, P_{2,3-i}$ be disjoint paths from $x_1, x_2$ to $y_i, y_{3-i}$, respectively, in $H' - y_1 - A'(x_1, x_2)$.

![Figure 2.8: A core frame](image)

We now use the structure inside $H'$ to derive further structure outside $H'$.
Lemma 2.4.4  
(i) $G$ has no edge from $B'(b_1, r_1)$ to $A'(x_2, a_2)$ and no edge from $B'[y_2, b_2]$ to $A'[a_1, x_1]$.

(ii) $G$ has no edge from $b_1$ to $A'[a_1, x_1]$ and no edge from $b_2$ to $A'[x_2, a_2]$.

(iii) $r_1 = b_1$ implies $x_1 = a_1$, and $y_2 = b_2$ implies $x_2 = a_2$.

(iv) If $y_2 \neq b_2$ and $y_2$ is a cut vertex of $G'_0$ separating $b_2$ from $\{a_0, b_1\}$, then $N_G(b_2) = \{y_2, x_1\}$, $a_1 \neq x_1$, and $a_2 = x_2$.

Proof. By Lemma 2.3.7 and (iv) of Lemma 2.4.2, we may assume that

(1) when $b_1 \neq r_1$, $G'_0 - B'(b_1, r') - B'[y_2, b_2]$ contains disjoint paths $B_1^*, A_0^*$ from $b_1, a_0$ to $q', y_1$, respectively.

(2) $G$ has no edge from $A'(x_2, a_2)$ to $B'(b_1, r_1)$.

For, let $e = ab \in E(G)$ with $a \in A'(x_2, a_2)$ and $b \in B'(b_1, r_1)$. Then $b_1 \neq r_1$; so $B_1^*, A_0^*$ exist by (1). Now $A'[a_1, r] \cup R \cup B'[b_1, r'] \cup e \cup A'[a_2] \cup P_{1,1} \cup A_0^* \cup B_1^* \cup Q \cup A'[q, x_2] \cup P_{2,2} \cup B'[y_2, b_2]$ show that $\gamma$ is feasible, a contradiction. □

(3) $G$ has no edge from $b_2$ to $A'[x_2, a_2]$.

For, let $e = ab_2 \in E(G)$ with $a \in A'[x_2, a_2]$. Then $a \neq a_2$ and let $e' = a_2b' \in E(G)$ with $b' \in B'(b_1, b_2)$. Now $b' \notin B'[y_2, b_2]$ to avoid the double cross $e, e', P_{1,2}, P_{2,1}$. Hence, $b' \in B'(b_1, r_1)$, contradicting (2). □

(4) $G$ has no edge from $A'[a_1, x_1]$ to $B'[y_2, b_2]$.

Otherwise, let $e = ab \in E(G)$ with $a \in A'[a_1, x_1)$ and $b \in B'[y_2, b_2)$. Then $G$ has no edge from $b_2$ to $\{x_1, x_2\}$; as such an edge must be $b_2x_1$ by (3), which forms a double cross with $e, P_{1,1}$ and $P_{2,2}$, contradicting Lemma 2.1.7.

Hence, by Lemma 2.3.7 and (iv) of Lemma 2.4.2, $G'_0 - B'[b_1, r'] - B'[y_2, b]$ has disjoint paths $B_2, A_0$ from $b_2, a_0$ to $y_1, q'$, respectively. But then, $A'[a_1, a] \cup e \cup B'[y_2, b] \cup P_{2,2} \cup$
$A'[q, a_2] \cup Q \cup A_0$ and $B'[b_1, r'] \cup R \cup A'[x_1, r] \cup P_{1,1} \cup B_2$ show that $\gamma$ is feasible, a contradiction. \hfill \Box

(5) (i) and (ii) hold.

For, suppose not. Then $G$ has an edge $e = b_1a$ with $a \in A'[a_1, x_1] \cup A'[x_2, a_2]$. Suppose $a \in A'[a_1, x_1]$. Then $a \neq a_1$, and let $e' = a_1b' \in E(G)$ with $b' \in B'(b_1, b_2)$. Now $b' \notin B'(b_1, r_1]$ to avoid the double cross $e, e', P_{1,2,1}$. So $b' \in B'(y_2, b_2)$, contradicting (4).

Hence, $a \in A'[x_2, a_2]$. Then $a \neq a_2$, and let $e' = a_2b' \in E(G)$ with $b' \in B'(b_1, b_2)$. Now $b' \notin B'(b_1, r_1]$ to avoid the double cross $e, e', P_{1,1,2}$. Hence, $b' \in B'(y_2, b_2)$.

If $G$ has an edge $e_3$ from $b_2$ to $\{x_1, x_2\}$ then, by (3), it ends with $x_1$. So $a_1 \neq x_1$, and $G$ has an edge $e_4$ from $a_1$ to $B'(b_1, b_2)$. But now, $e, e', e_3, e_4$ force a double cross, a contradiction.

So $G$ has no edge from $b_2$ to $\{x_1, x_2\}$. Hence, by Lemma 2.3.7, $G'_0 - B'[b_1, r_1] - B'[y_2, b'_2]$ has disjoint paths $B_2, A_0$ from $b_2, a_0$ to $y_1$, $q'$, respectively. But then, $A'[a_1, q] \cup P_{1,2} \cup B'[y_2, b'] \cup e' \cup Q \cup A_0$ and $e \cup A'[x_2, a] \cup P_{2,1} \cup B_2$ show that $\gamma$ is feasible, a contradiction. \hfill \Box

Since $G^*$ is 6-connected, it follows from (2) and (4) that (iii) holds. It remains to prove (iv). So assume $y_2 \neq b_2$ and $y_2$ is a cut vertex of $G'_0$ separating $b_2$ form $\{a_0, b_1\}$. Then $\alpha(A', B') \leq 1$.

Suppose $B'(y_2, b'_2) \neq \emptyset$. Then, since $G^*$ is 6-connected, it follows from (4) that $G$ has edges from $B'(y_2, b'_2)$ to distinct $u_1, u_2 \in V(A'[x_2, a_2])$, and we choose $u_1, u_2$ so that $A'[u_1, u_2]$ is maximal. Now, by (2) and (3), $\{u_1, u_2, y_2, b_2, x_1\}$ is a cut in $G^*$ separating $B'(y_2, b'_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

So $B'(y_2, b'_2) = \emptyset$. Then $a_2 = x_2$; for otherwise, since $G^*$ is 6-connected, $G$ has an edge from $a_2$ to $B'(b_1, r_1)$, contradicting (2). We may assume that there exists $e = b_2a \in E(G)$ with $a \in A'(a_1, x_1)$; as otherwise, (iv) holds. Let $e' = a_1b' \in E(G)$ with $b' \in B'(b_1, b_2)$.
Then $b' \in B'(b_1, r_1)$ by (4); so $b_1 \neq r_1$, and $B_1^b, A_0^b$ exist by (1). Now, by Lemma 2.2.1, we derive $\alpha(A', B') = 2$ with the following paths: the path $e' \cup B'[b_1, b']$ from $a_1$ to $b_1$, the path $A'[a_1, a] \cup e$ from $a_1$ to $b_2$, the path $B_1^a \cup Q \cup A'[x_1, q] \cup P_{1,2} \cup B'[y_2, b_2]$ from $b_1$ to $b_2$, and the path $A_0^a \cup P_{2,1}$ from $a_0$ to $a_2$. This contradicts $\alpha(A', B') \leq 1$ as $A', B'$ is a good frame. 

Let $H_0$ denote the minimal union of blocks of $H' - y_1 - A'(q, x_2)$ containing $X_1$, let $W$ denote the path between $x_1$ and $y_2$, such that $W$ is contained in the outer walk of $H_0$, and for any vertex $v \in V(W - A')$, there exists a vertex $u \in V(A'[q, x_2])$, such that $u, v$ are incident with a finite face of $H' - y_1$, and let $w_1 \in V(A' \cap W)$ with $A'[x_1, w_1]$ maximal. We further study the structure inside $H'$.

**Lemma 2.4.5**  
(i) $H_0 = H' - y_1 - A(w_1, x_2)$, and each vertex in $W(w_1, y_2)$ has at most two neighbors on $A'[q, x_2]$, inducing a subpath of $A'$ with at most two vertices.

(ii) $H' - \{y_1, y_2\} - A'(x_1, x_2)$ contains a path from $x_1$ to $x_2$.

**Proof.** Suppose (i) is not true. Then $H' - y_1$ has a $(H_0 \cup A'[q, x_2])$-bridge $J$ which has exactly one vertex in $W(w_1, y_2)$ (by definition of $H_0$ and since $G - A'$ is connected) or some vertex $w \in V(W(w_1, y_2))$ has two neighbors on $A'[q, x_2]$ such that the subpath of $A'$ between them has at least three vertices. In the first case, let $w \in V(J \cap H_0)$ and $u, v \in V(J \cap A')$ such that $J \cap A' \subseteq A'[u, v]$; and in the second case, let $u, v$ be the neighbors of $w$ on $A'[q, x_2]$ such that $A'[u, v]$ is maximal. Then by Lemma 2.4.3, $\{u, v, w, y_1, r'\}$ is a cut in $G^*$, a contradiction.

Now suppose (ii) is not true. Then there exists $v_0 \in V(A'(x_1, x_2))$ such that $y_2, v_0$ are incident with a finite face of $H' - y_1$. We further choose $v_0$ so that $A'[v_0, x_2]$ is minimal, and let $(L_1, L_2)$ be a separation in $H' - y_1$ such that $V(L_1 \cap L_2) = \{y_2, v_0\}$, $x_1 \in V(L_1)$, and $x_2 \in V(L_2)$.

By Lemma 2.4.1, for each $j \in [2]$, $H' - A'(x_1, x_2)$ contains disjoint paths from $y_1, y_2$ to $x_j, x_{3-j}$, respectively. So for $j \in [2]$, $G[V(L_j) \cup \{y_1\}] - y_2$ contains a path $T_j$ from $y_1$
to $x_j$ and internally disjoint from $A'$.

We see that $y_2,v_0$ are not incident with some finite face of $H_0$. For otherwise, $v_0 \in A'(x_1,w_1)$, $x_1 \neq w_1$, and $W[w_1,y_2] \subseteq L_2$. Hence, $T_1, W[w_1,y_2]$, $P$ and $Q$ are disjoint, which form a double cross, a contradiction to Lemma 2.1.7.

Now, by the minimality of $A'[v_0,x_2]$ and planarity of $H'-y_1, v_0 \in A'[q,x_2]$. Therefore, by Lemma 2.4.3, $\{v_0,x_2,r',y_1,y_2\}$ is a cut in $G^*$ separating $V(L_2)$ from $\{a_0,a_1,a_2,b_1,b_2\}$, a contradiction. \hfill $\square$

**Lemma 2.4.6** $w_1 \neq x_1$, and $H_0$ is 2-connected.

**Proof.** Suppose this is false. Let $z \in V(H_0)$ such that $z = x_1$ (when $x_1 = w_1$) or $z$ is a cut vertex of $H_0$ and, subject to this, $W[x_1,z]$ is maximal. Then $V(W[z,y_2] \cap X_1) = \{z,y_2\}$. Note that $z \in X_1[x_1,y_2]$.

Let $w \in W(z,y_2)$ and $u \in N_G(w) \cap V(A'[q,x_2])$ such that $A'[u,x_2] \cup W[w,y_2]$ is maximal. Moreover, let $K$ denote the $\{z,u\}$-bridge of $H'-y_1$ containing $A'[u,x_2] \cup X_2$, and let $K^* := G[V(K) \cup \{y_1\}]$.

By (v) of Lemma 2.3.9 and by the existence of $W[y_2,w] \cup wu$,

(1) no $A'-B'$ bridge outside $H'$ has one extreme hand in $A'[x_1,u]$ and the other in $A'[u,x_2]$.

Thus, since $\{y_1,y_2,u,x_2\}$ is not a cut in $G^*$ separating $K$ from $\{a_0,a_1,a_2,b_1,b_2\}$, $G$ has an $A'-B'$ path from $A'(u,x_2)$ to $B'[r_1,y_1]$ and internally disjoint from $H'$. By Lemma 2.4.3,

(2) all $A'-B'$ paths from $A'(u,x_2)$ to $B'[r_1,y_1]$ and internally disjoint from $H'$ are edges from $A'(u,x_2)$ to $\{r',y_1\}$.

So let $e = ar' \in E(G)$ with $a \in A'[u,x_2]$, and choose $a$ such that $A'[u,a]$ is minimal. Let $L$ denote the path on the outer walk of $K$ between $y_2$ and $u$ not going through $x_2$, and let $L_0 := L \cup A'[u,a]$. Then
First, suppose there exists \( v \in V(L_0 \cap X_2) \), such that \( v \neq y_2 \). Then \( \{v, y_1, u, x_2, r'\} \) is a cut in \( G^* \) separating \( V(A'(u, x_2)) \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction.

Now suppose there exist \( v \in N_G(y_1) \cap V(K) \) such that \( v \notin V(L_0) \). We claim that \( K^* - L_0 \) has a path \( Y_1 \) from \( y_1 \) to \( x_2 \). For otherwise, by the planar structure of \( K \), there exist \( c_1, c_2 \in V(L_0) \), such that \( c_1, c_2 \) are incident with a finite face of \( K \), and \( \{c_1, c_2\} \) is a 2-cut in \( K \) separating \( v \) from \( x_2 \). Thus, by (2) and the choice of \( a \), \( \{c_1, c_2, y_1, u, z\} \) is a cut in \( G^* \) separating \( v \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction.

If \( G \) has an \( A'-B' \) path \( T \) from \( A'(x_1, u) \) to \( B'(r', y_1) \) and internally disjoint from \( H' \), then \( T, e, L, Y_1 \) force a double cross, a contradiction. So \( T \) does not exist. Then \( u = q \) and, by (1), \( \{x_1, u, z, r'\} \) is a cut in \( G^* \) separating \( r \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction. \( \square \)

We will need the following claim.

(4) \( G_0' \) contains a path \( A_0^* \) from \( B'(r', y_1) \) to \( a_0 \) and internally disjoint from \( B' \).

For otherwise, there exists \( b_1' \in V(B'[b_1, r']) \), such that \( \{b_1', y_1\} \) is a 2-cut in \( G_0' \) separating \( B'[b_1', y_1] \) from \( \{a_0, b_1, b_2\} \). Furthermore, \( \{b_1', y_1, y_2\} \) is a 3-cut in \( G_0' \) separating \( B'[b_1', y_2] \) from \( \{a_0, b_1, b_2\} \). We choose \( b_1' \) so that \( B'[b_1, b_1'] \) is minimal. By Lemma 2.3.7 and (iv) of Lemma 2.4.2, \( b_1' = b_1 \), and \( \{b_1, y_1, y_2, b_2\} \) is a cut in \( G^* \) separating \( a_0 \) from \( \{a_1, a_2\} \), a contradiction. \( \square \)

Let \( y_1', y_1'' \in V(L_0) \cap N_G(y_1) \) such that \( a, y_1', y_1'', y_2 \) occur on \( L_0 \) in order and, subject to this, \( L_0[y_1', y_1''] \) is maximal.

(5) \( y_1'' \in L_0[z, u] \).

For, otherwise, \( y_1'' \notin L_0(z, y_2) \). Then \( y_1' \notin L_0[z, y_2] \); otherwise, \( G \) has a separation \( (G_1, G_2) \), such that \( V(G_1 \cap G_2) = \{r', u, z, y_1, y_2, x_2\} \), \( \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1) \), \( G_2 = K^* \), and \( (G_2, r', u, z, y_1, y_2, x_2) \) is planar, which contradicts Lemma 2.1.3.
We claim that $K - L_0[y_1', a] \cup L_0[y_2, y_1'']$ contains a path $X'$ from $x_2$ to $z$. For otherwise, by (3) and the planar structure of $K$, there exist $c_1 \in V(L_0[y_1', a])$ and $c_2 \in V(L_0[y_2, y_1''])$, such that $c_1, c_2$ are incident with a finite face of $K$, and $\{c_1, c_2\}$ is a 2-cut in $K$ separating $x_2$ from $z$. If $c_1 \in A'[u, a]$ then $\{c_1, c_2, y_2, x_2, r'\}$ is a cut in $G^*$ separating $V(X_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. So $c_1 \notin A'[u, a]$. Then $G$ has a separation $(G_1, G_2)$, such that $V(G_1 \cap G_2) = \{r', u, c_1, c_2, y_2, x_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(A'[u, x_2] \cup X_2) \subseteq V(G_2)$, and $(G_2, r', u, c_1, c_2, y_2, x_2)$ is planar. This contradicts Lemma 2.1.3.

Now, the following paths give a contradiction to (i) of Lemma 2.2.2: the path $A'[a_1, x_1] \cup X_1[x_1, z] \cup X' \cup A'[x_2, a_2]$ from $a_1$ to $a_2$, the path $B'[b_1, r'] \cup e \cup L_0[a, y_1'] \cup y_1'y_1 \cup y_1'y_1' \cup L_0[y_1', y_2] \cup B'[y_2, b_2]$ from $b_1$ to $b_2$, and the path $A_0'$ from $B'(r', y_1)$ to $a_0$. □

Now $y_1' \in A'(u, a)$. For, otherwise, $y_1', y_2'' \notin L_0[z, u]$. Now, $G$ has a separation $(G_1, G_2)$, such that $V(G_1 \cap G_2) = \{r', u, y_1, z, y_2, x_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $G_2 = K^*$, and $(G_2, r', u, y_1, z, y_2, x_2)$ is planar. This contradicts Lemma 2.1.3.

Moreover, $K - L_0[y_1', a] \cup L_0[y_2, y_1'']$ contains a path $X'$ from $x_2$ to $u$. For otherwise, by (3) and the planar structure of $K$, there exist $c_1 \in V(L_0[y_1', a])$ and $c_2 \in V(L_0[y_2, y_1''])$, such that $c_1, c_2$ are incident with a finite face of $K$, and $\{c_1, c_2\}$ is a 2-cut in $K$ separating $x_2$ from $u$. If $c_2 \in L_0[y_2, z]$ then $\{c_1, c_2, y_2, x_2, r'\}$ is a cut in $G^*$ separating $V(X_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. So $c_2 \notin L_0[y_2, z]$. Then $G$ has a separation $(G_1, G_2)$, such that $V(G_1 \cap G_2) = \{r', c_1, c_2, z, y_2, x_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(A'[c_1, x_2] \cup X_2) \subseteq V(G_2)$, and $(G_2, r', c_1, c_2, z, y_2, x_2)$ is planar. This contradicts Lemma 2.1.3.

Hence, the following paths contradict (i) of Lemma 2.2.2: the path $A'[a_1, u] \cup X' \cup A'[x_2, a_2]$ from $a_1$ to $a_2$, the path $B'[b_1, r'] \cup e \cup L_0[a, y_1'] \cup y_1'y_1 \cup y_1'y_1'' \cup L_0[y_1'', y_2] \cup B'[y_2, b_2]$ from $b_1$ to $b_2$, and the path $A_0'$ from $B'(r', y_1)$ to $a_0$. □

**Lemma 2.4.7** Let $z_1, z_2 \in V(W)$ with $W[z_1, z_2]$ is maximal, such that $x_1, z_1, z_2, y_2$ occur on $W$ in order, and for each $i \in [2]$, $G[H_0 + y_1]$ has a path $Z_i$ from $y_1$ to $z_i$ and internally disjoint from $W$. Then, $N_G(y_1) \cap V(X_1[x_1, y_2]) = \emptyset$ and $Z_1 \cap (X_1 \cup X_2) = \emptyset$.
Lemma 2.4.8  \( a_2 = x_2 \), and if \( y_2 \neq b_2 \) then \( y_1, y_2 \) are cut vertices in \( G'_0 \) separating \( b_2 \) from \( \{a_0, b_1\} \). \( N_G(b_2) = \{y_2, x_1\} \), and \( a_1 \neq x_1 \). Moreover, one of the following holds:

(i) there exists a 2-cut \( \{z'_1, z'_2\} \) in \( H_0 \) with \( x_1, z'_1, z_1, z_2, z'_2, y_2 \) on \( W \) in order such that \( W(z'_1, z'_2) \neq \emptyset \) and \( z'_1, z'_2 \) are incident with a finite face of \( H_0 \), or

(ii) \( N_G(y_1) \cap V(H_0) \subseteq V(W[w_1, y_2]) \) and, for any \( i \in [m] \), \( w_i \notin W(z_1, z_2) \).

Proof. By Lemma 2.4.6, \( w_1 \neq x_1 \) and \( H_0 \) is 2-connected. If \( y_2 = b_2 \), then by (iii) of Lemma 2.4.4, we have \( a_2 = x_2 \).

Now assume \( y_2 \neq b_2 \). We claim that \( G'_0 \) has a 3-cut \( \{a'_0, b'_1, y_2\} \) with \( b'_1 \in B'[b_1, r_1] \), which separates \( B'[b'_1, y_2] \) from \( \{a_0, b_1, b_2\} \). For otherwise, by (iv) of Lemma 2.4.2, \( G'_0 - B'[b_1, r'] - y_2 \) contains disjoint paths \( A_0, B_2 \) from \( a_0, b_2 \) to \( q', y_1 \), respectively. Let \( Y_1 \) be a path in \( Z_1 \cup W[z_1, w_1] \cup A'[w_1, r'] \) from \( y_1 \) to \( r \). Note that \( r \notin A'[q, x_2] \) and, by Lemma 2.4.7, \( Y_1 \cap (A'[q, x_2] \cup X_1 \cup X_2) = \emptyset \). Now, \( A'[a_1, x_1] \cup X_1 \cup X_2 \cup A'[q, a_2] \cup Q \cup A_0 \) and \( B'[b_1, r'] \cup R \cup Y_1 \cup B_2 \) show that \( \gamma \) is feasible, a contradiction.

Thus, when \( y_2 \neq b_2 \), we may apply Lemma 2.3.7 (with \( b'_2 = y_2 \)), and conclude that \( b'_1 = b_1, a'_0 = a_0, \) and \( y_1, y_2 \) are cut vertices in \( G'_0 \) separating \( b_2 \) from \( \{a_0, b_1\} \). By (iv) of Lemma 2.4.4, we have \( N_G(b_2) = \{y_2, x_1\}, a_1 \neq x_1, \) and \( a_2 = x_2 \).
We now show (i) or (ii) holds. First, suppose $z_1 = z_2$. Then $N_G(y_1) \cap V(H_0) = \{ z_1 \}$; or else, there exists $v \in N_G(y_1) \cap V(H_0)$ with $v \neq z_1$, and $\{ z_1, y_1 \}$ is a cut in $G$ separating $v$ from $\{ a_0, a_1, a_2, b_1, b_2 \}$, a contradiction. Clearly, $z_1 \in V(W(w_1, y_2))$, and so (ii) holds.

So we may assume $z_1 \neq z_2$. Now suppose $W(z_1, z_2) \cap \{ w_1, \ldots, w_m \} = \emptyset$. Then (ii) holds or there exists $v \in N_G(y_1) \cap V(H_0)$ such that $v \notin V(W)$. In the latter case, there exist $c_1, c_2 \in V(W(x_1, y_2))$, such that $\{ c_1, c_2 \}$ is a 2-cut in $H_0$ separating $v$ from $x_1$; since, otherwise, $H_0 - W(x_1, y_2)$ contains a path $T$ from $v$ to $x_1$, and $y_1 \cup T, W[w_1, y_2], R, Q$ force a double cross, contradicting Lemma 2.1.7. Now, $\{ y_1, c_1, c_2 \}$ is a cut in $G^*$, a contradiction.

Hence, we may assume $W(z_1, z_2) \cap \{ w_1, \ldots, w_m \} \neq \emptyset$. Now suppose (i) fails. Then by the planar structure of $H_0$, $H_0 - W(x_1, z_1) - W(z_2, y_2)$ contains a path $X'$ from $x_1$ to $W(z_1, z_2)$ and internally disjoint from $W$.

We claim that $X'$ must be disjoint from $Z_1, Z_2$. For otherwise, let $x^* \in V(X' \cap Z_j)$ for some $j \in [2]$. As $X', Z_1, Z_2$ are all internally disjoint from $W$, $Z_j[s_j, x^*] \cup X'[x^*, x_1]$ implies that $z_1 = x_1$, contradicting Lemma 2.4.7 that $V(Z_1 \cap (X_1 \cup X_2)) = \emptyset$.

We claim $w_1 \in W(z_1, z_2)$. For otherwise, $w_i \in W(z_1, z_2)$ for some $i \geq 2$. Let $v_i \in N_G(w_i) \cap V(A'[q_2, x_2])$ with $A'[v_i, x_2]$ minimal. By Lemma 2.3.7 and (iv) of Lemma 2.4.2, there exists a path $A'_0$ in $G'$ from $a_0$ to $B'(r', y_1)$, which is internally disjoint from $B'$. Now $A'[a_1, x_1] \cup X' \cup W(z_1, z_2) \cup w_i v_i \cup A'[q, a_2] \cup Q \cup B'(r', y_1) \cup A'_0$ and $B'[b_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, z_1] \cup Z_1 \cup Z_2 \cup W[z_2, y_2] \cup B'[y_2, b_2]$ show that $\gamma$ is feasible, a contradiction.

So $z_1 \in A'(x_1, w_1)$. Moreover, $r \notin A'(x_1, z_1)$; otherwise, $A'[a_1, x_1] \cup X' \cup W(z_1, z_2) \cup A'[w_1, a_2] \cup Q \cup B'(r', y_1) \cup A'_0$ and $B'[b_1, r'] \cup R \cup A'[r, z_1] \cup Z_1 \cup Z_2 \cup W[z_2, y_2] \cup B'[y_2, b_2]$ show that $\gamma$ is feasible, a contradiction. But now, $A'[a_1, z_1] \cup Z_1 \cup B'(r', y_1) \cup A'_0 \cup Q \cup A'[q, a_2]$ and $B'[b_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, y_2] \cup B'[y_2, b_2]$ show that $\gamma$ is feasible, a contradiction.

\[\square\]

**Lemma 2.4.9** Suppose (i) of Lemma 2.4.8 holds, and the 2-cut $\{ z'_1, z'_2 \}$ in $G_0$ is chosen with $W[z'_1, z'_2]$ maximal. Then $z'_1 \in A'[x_1, w_1]$ (seen at Figure 2.9).
Proof. For, suppose $z'_1 \notin A'[x_1, w_1]$. By Lemma 2.4.5, let $u', u'' \in V(A'[q, x_2])$ and $v', v'' \in V(W(z'_1, z'_2))$ such that $x_1, u', u'', x_2$ occur on $A'$ in order, $u'v', u''v'' \in E(G)$, and, subject to this, $A'[u', u'']$ is maximal and then $W[v', v'']$ is maximal. Then $H' - y_1$ has a separation $(K, K')$ such that $V(K \cap K') = \{u', u'', z'_1, z'_2\}$, $W[z'_1, z'_2] \cup A'[u', u''] \subseteq K$, and $W[x_1, z'_1] \cup X_1 \subseteq K'$.

By (v) of Lemma 2.3.9 and by the existence of paths from $y_2$ to $u', u''$, respectively, in $H' - y_1$ that are internally disjoint from $A'$,

(1) no $A'-B'$ bridge outside $H'$ has $u'$ or $u''$ as internal vertex of the subpath of $A'$ between its extreme hands.

Therefore, since $\{y_1, z'_1, z'_2, u', u''\}$ does not separate $K$ from $\{a_0, a_1, a_2, b_1, b_2\}$ in $G^*$.

(2) $A'(u', u'') \neq \emptyset$, and $G$ has an $A'-B'$ path from $A'(u', u'')$ to $B'[r_1, y_1]$ and internally disjoint from $H' - y_1$.

Recall from Lemma 2.4.3 that

(3) all $A'-B'$ paths from $A'(u', u'')$ to $B'[r_1, y_1]$ and internally disjoint from $H' - y_1$ are edges from $A'(u', u'')$ to $\{r', y_1\}$.
By (2) and (3), let \( e = ar' \in E(G) \) with \( a \in V(A'[u', u'']) \) and \( A'[u', a] \) minimal. Note that

(4) \( G'_0 \) contains a path \( A'_0 \) from \( B'(r', y_1) \) to \( a_0 \) and internally disjoint from \( B' \).

For otherwise, there exists \( b'_1 \in B'[b_1, r'] \), such that \( \{b'_1, y_1\} \) is a 2-cut in \( G'_0 \) separating \( B'[b'_1, y_1] \) from \( \{a_0, b_1, b_2\} \). Furthermore, \( \{b'_1, y_1, y_2\} \) is a 3-cut in \( G'_0 \) separating \( B'[b'_1, y_2] \) from \( \{a_0, b_1, b_2\} \). By Lemma 2.3.7 and (iv) of Lemma 2.4.2, \( b'_1 = b_1 \), and \( \{b_1, y_1, y_2, b_2\} \) is a cut in \( G \) separating \( a_0 \) from \( \{a_1, a_2\} \), a contradiction. \( \square \)

Since \( z'_1 \notin A'[x_1, w_1] \), no finite face of \( K' \) incident with \( z'_2 \) is incident with a vertex of \( A'[x_1, w_1] \). Thus,

(5) \( K' - A'[x_1, u'] \) contains a path \( Y \) from \( y_2 \) to \( z'_1 \) and internally disjoint from \( A' \).

Let \( L \) denote the path on the outer walk of \( K \) from \( z'_1 \) to \( u' \) without going through \( u'' \), and let \( L_0 := L \cup A'[u', a] \). Note that \( z'_2 \notin V(L_0) \).

(6) \( N_G(y_1) \cap V(K) \not\subseteq V(L_0) \cup \{z'_2\} \).

For, suppose \( N_G(y_1) \cap V(K) \subseteq V(L_0) \cup \{z'_2\} \). Then \( V(L_0) \cap N_G(y_1) \neq \emptyset \); otherwise, \( \{u', u'', z'_1, z'_2, r'\} \) is a cut in \( G^* \) separating \( K \) from \( \{a_0, a_1, b_1, b_2\} \), a contradiction.

Let \( y'_1, y''_1 \in V(L_0) \cap N_G(y_1) \), such that \( a, y'_1, y''_1, z'_1 \) occur on \( L_0 \) in order and \( L_0[y'_1, y''_1] \) is maximal.

We first claim \( y'_1 \in L_0(u', a) \). For otherwise, \( y'_1, y''_1 \in V(L_0[z'_1, u']) \). Now, \( G \) has a separation \( (G_1, G_2) \), such that \( V(G_1 \cap G_2) = \{r', u', y_1, z'_1, z'_2, u''\} \), \( \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1), V(K) \subseteq V(G_2) \), and \( (G_2, r', u, y'_1, z'_1, z'_2, u'') \) is planar, contradicting Lemma 2.1.3.

Next, \( y''_1 \in L_0[z'_1, u'] \). For, suppose \( y''_1 \notin L_0[z'_1, u'] \). Then \( y''_1 \in L_0[u', a] \). Now, \( G \) has a separation \( (G_1, G_2) \), such that \( V(G_1 \cap G_2) = \{r', y_1, u', z'_1, z'_2, u''\} \), \( \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1), V(K) \subseteq V(G_2) \), and \( (G_2, r', u, y'_1, z'_1, z'_2, u'') \) is planar, contradicting Lemma 2.1.3.

We further claim \( K - z'_2 - L_0[z'_1, y''_1] - L_0[y'_1, a] \) contains a path \( X' \) from \( u'' \) to \( u' \). For otherwise, by the planar structure of \( K \), there exist \( c_1 \in V(L_0[y'_1, a]), c_2 \in V(L_0[z'_1, y''_1]) \cup \)
\{z'_2\}$, such that $c_1, c_2$ are incident with some finite face of $K$, and $\{c_1, c_2\}$ is a 2-cut in $K$ separating $u'$ from $u''$. By the existence of the path $u'' \cup W[v'', v'] \cup v'u'$ from $u''$ to $u'$, we may assume $c_2 = v'$. Moreover, $v' \neq v''$; otherwise, $\{v', u', u'', r', y_1\}$ is a cut in $G^*$ separating $A'(u', u'')$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. Now $G$ has a separation $(G_1, G_2)$, such that $V(G_1 \cap G_2) = \{r', c_1, v', z'_1, z'_2, u''\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(A'[c_1, u'']) \cup \{v''\} \subseteq V(G_2)$, and $(G_2, r', c_1, v', z'_1, z'_2, u'')$ is planar, which contradicts Lemma 2.1.3.

Now, the path $A'[a_1, u'] \cup X' \cup A'[u'', a_2]$ from $a_1$ to $a_2$, the path $B'[b_1, r'] \cup e \cup L_0[a, y'_1] \cup y'_1 y_1 \cup y_1 y'' \cup L_0[y''_1, z'_1] \cup Y \cup B'[y_2, b_2]$ from $b_1$ to $b_2$, and the path $A'_0$ from $B'(r', y_1)$ to $a_0$ contradict (i) of Lemma 2.2.2. □

(7) $G[K + y_1] - V(L_0) \cup \{z'_2\}$ contains a path $Y_1$ from $y_1$ to $u''$.

Note that, by (6), there exists $v \in N_G(y_1) \cap V(K)$ such that $v \notin V(L_0) \cup \{z'_2\}$. So if (7) fails then, $K - z'_2 - L_0$ has no path from $v$ to $u''$; so there exist $c_1, c_2 \in V(L_0) \cup \{z''_2\}$, such that $c_1, c_2$ are incident with some finite face of $K$, and $\{c_1, c_2\}$ is a 2-cut in $K$ separating $v$ from $u''$. Thus, by (3) and the choice of $a$, $\{c_1, c_2, y_1, u', z'_1\}$ is a cut in $G^*$ separating $v$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

(8) $b_1 = r_1 = r'$, and $G$ has no $A' - B'$ path from $A'[a_1, u']$ to $B'(r', y_1)$ and internally disjoint from $H'$.

First, $G$ has no $A' - B'$ path from $A'[a_1, u']$ to $B'(r', y_1)$ and internally disjoint from $H'$, to avoid forming a double cross with $e, Y \cup L, Y_1$.

Next we show $b_1 = r_1$ (and so $a_1 = x_1$ by (iii) of Lemma 2.4.4). For, suppose $b_1 \neq r_1$.

By Lemma 2.3.7 and (iv) of Lemma 2.4.2, $G'_0 - r' - B'[y_2, b_2]$ contains disjoint paths $B_1, A_0$ from $b_1, a_0$ to $q', y_1$, respectively. Now, $A'[a_1, r] \cup R \cup e \cup A'[a_2] \cup Y_1 \cup A_0$ and $B_1 \cup Q \cup A'[q, u'] \cup L \cup Y \cup B'[y_2, b_2]$ show that $\gamma$ is feasible, a contradiction.

Moreover, $r_1 = r'$. For, suppose $r_1 \neq r'$. By (iii) of Lemma 2.4.2, there exists an $A' - B'$ bridge $M$ with feet $l^*, r^*$, such that $M$ is internally disjoint from $R$, and $r' \in B'(l^*, r^*)$. Let
Let $P^*$ be the path from $l^*$ to $r^*$ in $M$ and internally disjoint from $A', B'$, and let $A'_0$ be the path from $a_0$ to $y_1$ in $G'_0$ and internally disjoint from $B'$. Then $A'[a_1, r] \cup R \cup e \cup A'[a_2] \cup Y \cup A_0'$ and $B'[b_1, l'] \cup P^* \cup B'[r', q'] \cup Q \cup A'[q, u'] \cup L \cup Y \cup B'[y_2, b_2]$ show that $\gamma$ is feasible, a contradiction. \hfill $\square$

Now, by (1), (3), (8), Lemma 2.4.3, and Lemma 2.4.8, \{b_1, u', a_2, y_1, b_2\} is a cut in $G^*$ separating $a_0$ from $a_1$, a contradiction. \hfill $\square$

**Lemma 2.4.10** \ y_1 is a cut vertex in $G'_0$ separating $b_2$ from $\{a_0, b_1\}$, $\alpha(A', B') = 1$, and $G'_0 - B'(b_1, r') - A'_0$ has a path $B'_1$ from $b_1$ to $q'$, where $A'_0$ is the path from $a_0$ to $y_1$, which is in the outer walk of $G'_0$ and disjoint from $B' - y_1$.

**Proof.** Recall the path $Z_1$ from Lemma 2.4.7. We claim that $H' - \{y_1, y_2\}$ contains a path $X_0$ from $x_1$ to $x_2$ and disjoint from $Z_1 \cup W[z_1, w_1] \cup A'(x_1, x_2)$. For otherwise, by the planar structure of $H' - y_1$, there exists a vertex $v \in V(Z_1 \cup W[z_1, w_1] \cup A'(x_1, x_2))$, such that $y_2, v$ are incident with some finite face of $H_0$. By Lemma 2.4.5, $v \notin A'(x_1, x_2)$, and so $v \in V(Z_1 \cup W[z_1, w_1])$. If $v \in W[z_1, w_1]$ then (i) of Lemma 2.4.8 holds and the 2-cut $\{z'_1, z'_2\}$ can be chosen with $z'_2 = y_2$; so $z'_1 \in A'[x_1, w_1]$ by Lemma 2.4.9, contradicting Lemma 2.4.5. So $v \in Z_1 - z_1$, which implies that $y_1$ has a neighbor in $H_0 - W$; so (i) of Lemma 2.4.8 holds and the 2-cut $\{z'_1, z'_2\}$ still can be chosen with $z'_2 = y_2$. Again, $z'_1 \in A'[x_1, w_1]$ by Lemma 2.4.9, contradicting Lemma 2.4.5.

Now suppose $y_1$ is not a cut vertex in $G'_0$ separating $b_2$ from $\{a_0, b_1\}$. Then $y_2 = b_2$ by Lemma 2.4.8. If $G'_0 - B'[b_1, r'] - B'(y_1, b_2)$ contains disjoint paths $A_0, B_2$ from $a_0, b_2$ to $q', y_1$, respectively, then $A'[a_1, x_1] \cup X_0 \cup A'[a_2] \cup Q \cup A_0$ and $B'[b_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, z_1] \cup Z_1 \cup B_2$ show that $\gamma$ is feasible, a contradiction. Thus, such paths do not exist. Then by planarity, $G'_0$ has a 3-cut $\{a'_0, b'_1, b'_2\}$ with $b'_1 \in B'[b_1, r']$ and $b'_2 \in B'(y_1, b_2)$, which separates $B'(b'_1, b'_2)$ from $\{a_0, b_1, b_2\}$. Since $y_1, b_2, b'_2$ are incident with some finite face of $G'_0$, then $a'_0, b_2$ are incident with some finite face of $G'_0$, and so $\{b'_1, a'_0, b_2\}$ is a 3-cut in $G'_0$. Moreover, since $y_1$ is not a cut vertex in $G'_0$, then $a'_0 \neq a_0$. But now, by (iv)
of Lemma 2.4.2, \( b'_1 \notin B'(r_1, r') \), and therefore, \( b'_1 \in B'[b_1, r_1] \). Now, by Lemma 2.3.7, \( b'_1 = b_1 \). Then \( \{b_1, b_2, a'_0\} \) is a cut in \( G^* \) separating \( a_0 \) from \( \{a_1, a_2\} \), a contradiction.

Thus, \( y_1 \) is a cut vertex in \( G'_0 \) and, hence, \( \alpha(A', B') \leq 1 \). Indeed, \( \alpha(A', B') = 1 \). To see this, let \( A'_0 \) be the path from \( a_0 \) to \( y_1 \), which is in the outer walk of \( G'_0 \) and disjoint from \( B' \). When \( y_2 = b_2 \), let \( B^* := A'[a_1, x_1] \cup X_1 \); when \( y_2 \neq b_2 \), by Lemma 2.4.8, \( x_1b_2 \in E(G) \), and we let \( B^* := A'[a_1, x_1] \cup x_1b_2 \). Then by Lemma 2.2.1, the following paths show \( \alpha(A', B') = 1 \): the path \( A'_0 \cup B'[q', y_1] \cup Q \cup A'[q, a_2] \) from \( a_0 \) to \( a_2 \), the path \( B'[b_1, r'] \cup R \cup A'[r, w] \cup W[w, y_2] \cup B'[y_2, b_2] \) from \( b_1 \) to \( b_2 \), and the path \( B^* \) from \( a_1 \) to \( b_2 \).

Finally, suppose \( G'_0 - B'(b_1, r') - A'_0 \) has no path \( B'_1 \) from \( b_1 \) to \( q' \). Then by planarity, \( G'_0 \) has a 2-cut \( \{a'_0, b'_1\} \) with \( a'_0 \in V(A'_0) \), \( b'_1 \in V(B'(b_1, r')) \), and \( a'_0, b'_1 \) cofacial, which separates \( b_1 \) from \( q' \). Hence, \( \{a'_0, b'_1, b_2\} \) is a 3-cut in \( G'_0 \) separating \( B'[b'_1, b_2] \) from \( \{a_0, b_1, b_2\} \).

By Lemma 2.3.7, \( b'_1 \notin B'(b_1, r_1) \), and so \( b'_1 \in (r_1, r') \). But, by (iv) of Lemma 2.4.2, \( r_1 = b_1 \), \( a'_0 = a_0 \), and \( G'_0 \) has no path from \( a_0 \) to \( b_1 \) and internally disjoint from \( B' \). Therefore, \( \alpha(A', B') = 0 \), a contradiction. \( \Box \)

**Lemma 2.4.11** Suppose (i) of Lemma 2.4.8 does not hold and (ii) of Lemma 2.4.8 holds. Then \( N_G(y_1) \cap V(H_0) \subseteq V(W[w_1, w_2]) \) (seen at Figure 2.10).

**Proof.** Note that in this case, \( y_1z_1, y_2z_2 \in E(G) \). Since \( z_1 \notin V(X_2) \) (by Lemma 2.4.7), \( z_1 \notin W[w_m, y_2] \); so (ii) of Lemma 2.4.8 implies the existence of \( j \in [m - 1] \) with \( z_1, z_2 \in W[w_j, w_{j+1}] \) and \( z_2 \neq w_j \). We may assume \( j \geq 2 \) as otherwise the assertion holds. Thus, since (i) of Lemma 2.4.8 does not hold, \( H_0 - W[x_1, w_1] - W[z_2, w_m] \) contains a path \( Y_2 \) from \( y_2 \) to \( w_2 \). Recall from Lemma 2.4.8 that \( a_2 = x_2 \), and recall paths \( B'_1, A'_0 \) from Lemma 2.4.10.

(1) \( b_2 = y_2 \).

For, suppose \( b_2 \neq y_2 \). Then by Lemma 2.4.8, \( G \) has an edge from \( b_2 \) to \( x_1 \), and \( a_1 \neq x_1 \).

Let \( a_1b \in E(G) \) with \( b \in V(B'(b_1, r_1)) \). Now \( \alpha(A', B') = 2 \) by applying Lemma 2.2.1

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with the following paths: the path $A'_0 \cup y_1 z_2 \cup W[z_2, w_m] \cup w_m a_2$ from $a_0$ to $a_2$, the path $B'_1 \cup Q \cup A'[w_1, q] \cup W[w_1, w_2] \cup Y_2 \cup B'[y_2, b_2]$ from $b_1$ to $b_2$, the path $a_1 b \cup B'[b_1, b]$ from $a_1$ to $b_1$, and the path $A'[a_1, x_1] \cup x_1 b_2$ from $a_1$ to $b_2$ show that $\alpha(A', B') = 2$, contradicting Lemma 2.4.10. □

Let $u_2 \in N_G(w_2) \cap V(A')$ with $A'[u_2, a_2]$ is maximal. Then

(2) $u_2 \neq x_2$.

For, suppose $u_2 = x_2$. Then $G$ has an $A'-B'$ path $T$ from $t \in V(A'[a_1, w_1])$ to $t' \in V(B'[b_1, y_1])$ and internally disjoint from $H'$; as otherwise, $\{a_1, w_1, x_2, y_1, y_2\}$ is a cut in $G^*$ separating $H_0$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. We choose $T$ so that $B'[b_1, t']$ is minimal and, subject to this, $A'[a_1, t]$ is minimal.

Then $t' \in B'[b_1, r']$ and $G$ has no $A'-B'$ path from $A'[a_1, t]$ to $B'[b_1, y_1]$ and internally disjoint from $H'$. For, if $t' \in B'(r', y_1)$ then, by the choice of $T$, we have $T \cap R = \emptyset$ and $r \in A'[w_1, q]$; now $T, R, y_1 z_2 \cup W[z_2, w_m] \cup w_m x_2$, and $Y_2 \cup W[w_2, w_1]$ form a double cross, a contradiction. Now if $G$ has an $A'-B'$ path $S$ from $s \in A'[a_1, t]$ to $s' \in B'[b_1, y_1]$ and internally disjoint from $H'$, then by the choice of $T$, $T \cap S = \emptyset$ and $s \in B'(t', y_1)$; so $T, S, y_1 z_2 \cup W[z_2, w_m] \cup w_m x_2$, and $Y_2 \cup W[w_2, w_1]$ form a double cross, a contradiction.
Now \( V(T \cap Q) = \emptyset \). Otherwise, \( T, Q \) are contained in a same \( A'-B' \) bridge. Since \( w_1 \in A'(t, q) \), the path from \( w_1 \) to \( y_2 \) in \( H' - y_1 \) contradicts (v) of Lemma 2.3.9.

Next, we show that \( H_0 - (A'[x_1, t] \cup X_1(x_1, y_2) \cup W[z_1, w_j]) \) contains a path \( Y_2' \) from \( y_2 \) to \( w_1 \). For otherwise, by the planar structure of \( H_0 \), there exist \( c_1 \in V(W[z_1, w_j]) \) and \( c_2 \in V(A'[x_1, t]) \cup V(X_1(x_1, y_2)) \), such that \( \{c_1, c_2\} \) is a cut in \( H_0 \) separating \( y_2 \) from \( w_1 \). Recall that \( j < m \) and \( z_1 \notin V(X_2) \), and so \( z_1 \in W[w_j, w_m] \). In fact, \( c_2 \in A'(x_1, t) \); as otherwise \( \{c_1, c_2, y_1, y_2, x_2\} \) is a cut in \( G^* \) separating \( w_m \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction. Hence, \( t \in A'(x_1, w_1) \). Since \( G \) has no \( A'-B' \) path from \( A'[a_1, t] \) to \( B'[b_1, y_1] \) and internally disjoint from \( H' \), \( G \) has a separation \((G_1, G_2)\) such that \( V(G_1 \cap G_2) = \{x_1, y_2, x_2, y_1, c_1, c_2\}, \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1), V(X_1) \cup V(X_2) \subseteq V(G_2) \), and \((G_2, x_1, y_2, x_2, y_1, c_1, c_2)\) is planar, which contradicts Lemma 2.1.3.

Hence, by Lemma 2.2.1, the path \( A_0' \cup z_1y_1 \cup W[z_1, w_j] \cup w_ja_2 \) from \( a_0 \) to \( a_2 \), the path \( B_1' \cup Q \cup A'[w_1, q] \cup Y_2' \) from \( b_1 \) to \( b_2 \), the path \( A'[a_1, t] \cup T \cup B'[b_1, t'] \) from \( a_1 \) to \( b_1 \), and the path \( A'[a_1, x_1] \cup X_1 \) from \( a_1 \) to \( b_2 \) show that \( \alpha(A', B') = 2 \), contradicting Lemma 2.4.10. \( \square \)

(3) \( G \) has no \( A'-B' \) path from \( A'(u_2, a_2) \) to \( B'(b_1, r') \).

For, suppose \( G \) has an \( A'-B' \) path \( S \) from \( s \in A'(u_2, a_2) \) to \( s' \in B'(b_1, r') \). Then, \( A'[a_1, r] \cup R \cup B'[s', r'] \cup S \cup A'[s, a_2] \cup x_2w_m \cup W[w_m, z_2] \cup z_2y_1 \cup A_0' \) and \( B_1' \cup Q \cup A'[q, u_2] \cup u_2w_2 \cup Y_2 \) show that \( \gamma \) is feasible, a contradiction. \( \square \)

(4) \( G \) has no disjoint \( A'-B' \) paths \( C, D \) from \( c, d \in V(A'[x_1, x_2]) \) to \( c', d' \in V(B'[b_1, y_1]) \) and internally disjoint from \( H' \), such that \( a_1, c, d, a_2 \) occur on \( A' \) in order, and \( b_1, d', c', y_1 \) occur on \( B' \) in order.

For, suppose such \( C, D \) exist. Then \( c \notin A'[a_1, u_2] \); otherwise, \( C, D, y_1z_2 \cup W[z_2, w_m] \cup w_mx_2 \), and \( Y_2 \cup w_2u_2 \) form a double cross, a contradiction. So \( d \in A'(u_2, x_2) \).

Then, by Lemma 2.4.3, \( D = dd' \) and \( d' = r' \). Moreover, by (3), \( b_1 = r' \).
Now, $G$ has no $A'-B'$ path from $A'[a_1, u_2]$ to $B'(b_1, y_1]$ and internally disjoint from $H'$; otherwise, replace $C$ by this path we have a contradiction to our claim that $c \notin A'[a_1, u_2]$. But then, by Lemma 2.4.3, $\{b_1, b_2, y_1, u_2, a_2\}$ is a cut in $G^*$ separating $a_1$ from $a_0$, a contradiction. □

(5) $H_0 - A'(x_1, w_1) - W[z_2, y_2]$ has a path $X'$ from $x_1$ to $w_j$.

For otherwise, by planarity of $H_0$, there exist $c_1 \in V(A'(x_1, w_1))$ and $c_2 \in V(W[z_2, y_2])$, such that $\{c_1, c_2\}$ is a cut in $H_0$ separating $x_1$ from $w_j$. But then, (i) of Lemma 2.4.8 holds, a contradiction. □

(6) $H_0 - (A'[x_1, w_1] \cup X_1[x_1, y_2] \cup W[z_2, w_m])$ contains a path $Y_2^*$ from $y_2$ to $w_2$.

For otherwise, by planarity of $H_0$, there exist $c_1 \in V(W[z_2, w_m])$ and $c_2 \in V(A'[x_1, w_1]) \cup V(X_1[x_1, y_2])$, such that $\{c_1, c_2\}$ is a 2-cut in $H_0$ separating $y_2$ from $w_2$. Now $c_2 \in X_1[x_1, y_2]$; as otherwise $c_2 \notin A'[x_1, w_1]$ and (i) of Lemma 2.4.8 holds, a contradiction.

Let $w_i \in W(c_1, y_2)$ such that $i$ is minimum, and let $u_i \in N_G(w_i) \cap V(A')$ with $A'[u_2, u_i]$ minimum. Then $G$ has an $A'-B'$ path $S$ from $s \in V(A'(u_i, x_2))$ to $s' \in V(B'[b_1, y_1])$ and internally disjoint from $H'$; otherwise, $\{u_i, c_1, c_2, y_2, x_2\}$ is a cut in $G^*$ separating $w_m$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

By Lemma 2.4.3, $S$ is an edge with $s' \in \{r', y_1\}$. If $s' = r'$ then $S, Q$ contradict (4). So $s' = y_1$. Then $A'[a_1, w_1] \cup W[w_1, z_1] \cup z_1y_1 \cup A_0' \cup s's \cup A'[s, a_2]$ and $B'[b_1, q'] \cup Q \cup A'[q, u_i] \cup u_iw_i \cup W[w_i, y_2]$ show that $\gamma$ is feasible, a contradiction. □

(7) $z_1, x_2$ are incident with some finite face of $H' - y_1$.

For otherwise, there exist $k \in \{j + 1, \cdots, m\}$ and a vertex $u_k \in V(A'[u_2, x_2])$, such that $w_ku_k \in E(G)$. We choose $k$ with $k$ minimum and choose $u_k$ so that $A'[u_k, a_2]$ is maximal. Clearly, $k = j + 1$ or $k = j + 2$.

Suppose $G$ has an $A'-B'$ path $S$ from $a_2$ to $s' \in V(B'[b_1, y_1])$. By (3), $s' \notin B'(b_1, r')$. Moreover, $s' \notin B'(r', y_1)$; otherwise, $S, R, u_kw_k \cup W[w_k, y_2]$, and $X' \cup W[w_j, z_1] \cup z_1y_1$
force a double cross. So \( s' = b_1 \). Note that \(|V(S)| \geq 3\) as \( a_2b_1 \notin E(G) \); so \( S \) is contained in an \( A'-B' \) bridge \( N \) and let \( n_1, n_2 \) be the extreme hands of \( N \). Since we forced \( s' = b_1 \), we see that \( b_1 \) is the only foot of \( N \). By Lemma 2.4.3, \( V(N \cap A'(u_2, x_2)) = \emptyset \). By (v) of Lemma 2.3.9, \( n_1 \notin A'[a_1, u_2] \), and so \( n_1 = u_2 \). But then, \( \{n_1, n_2, b_1\} \) is a cut in \( G \) separating \( V(N) \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction.

Then \( G \) has no \( A'-B' \) path from \( a_2 \) to \( B'[b_1, y_1] \) and internally disjoint from \( H' \). Since the degree of \( a_2 \) in \( G \) is at least 4, \( G \) has an edge from \( a_2 \) to some \( w \in V(W[w_k, w_m]) \).

We derive \( \alpha(A', B') = 2 \) by Lemma 2.2.1 and the following paths: the path \( A'[a_1, r] \cup R \cup B'[b_1, r'] \) from \( a_1 \) to \( b_1 \), the path \( A'[a_1, x_1] \cup X_1 \) from \( a_1 \) to \( b_2 \), the path \( B'[b_1 \cup Q \cup A'[q, u_2] \cup u_2w_2 \cup Y_2^s \) from \( b_1 \) to \( b_2 \), and the path \( a_2w \cup W[w, z_2] \cup z_2y_1 \cup A_0 \) from \( a_2 \) to \( a_0 \). This contradicts Lemma 2.4.10. □

(8) Let \( v_j \in N_G(w_j) \cap V(A') \) with \( A'[v_j, a_2] \) is minimal. Then \( G \) has two disjoint \( A'-B' \) paths from \( A'(x_1, v_j) \) to \( B'[b_1, y_1] \) and internally disjoint from \( H' \).

For otherwise, there exists \( v \in V(G) \) such that \( G - v \) does not contain any \( A'-B' \) path from \( A'(x_1, v_j) \) to \( B'[b_1, y_1] \) and internally disjoint from \( H' \). But then, combined with (6), \( G \) has a separation \( (G_1, G_2) \), such that \( V(G_1 \cap G_2) = \{v, x_1, y_2, x_2, u, v_j\} \) with \( u = y_1 \) (when \( z_1 \neq z_2 \)) or \( u = z_1 \) (when \( z_1 = z_2 \)), \( \{a_0, a_1, a_2, b_1, b_2\} \cup V(A'[v_j, x_2]) \subseteq V(G_1) \), and \( A'[x_1, v_j] \cup X_1 \subseteq G_2 \).

By Lemma 2.1.3, \( (G_2, v, x_1, y_2, x_2, u, v_j) \) is not planar. So, clearly, \( v \notin A' \), and there exists an \( A'-B' \) bridge \( N \) with feet \( n'_1, n'_2 \) and extreme hands \( n_1, n_2 \), such that \( v \in N \). By (v) of Lemma 2.3.9, \( H' - y_1 \) does not contain a path from \( A'(n_1, n_2) \) to \( y_2 \) and internally disjoint from \( A' \). Suppose \( v \notin B' \). Then \( N \) has a separation \( (N', N'') \) of order 1, such that \( V(N' \cap N'') = \{v\}, n_1, n_2 \in V(N' - N''), \) and \( n'_1, n'_2 \in V(N'' - N') \). Now \( V(N') = \{n_1, n_2, v\} \); or else, \( \{n_1, n_2, v\} \) is a cut in \( G \) separating \( V(N') - \{n_1, n_2, v\} \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction. This implies that \( (G_2, v, x_1, y_2, x_2, u, v_j) \) is planar, a contradiction. So \( v \in B' \). But then, by (v) of Lemma 2.3.9 and the definition of \( v, n'_1 = n'_2 = v \) and there
exist \( n_1^* \in A'[a_1, n_1] \) and \( n_2^* \in A'[n_2, a_2] \), such that \( \{n_1^*, n_2^*, v\} \) is a cut in \( G^* \) separating \( V(N) \) from \( \{a_0, a_1, a_2, b_1, b_2\} \), a contradiction. \( \Box \)

By (8), let \( T_1, T_2 \) be disjoint \( A'-B' \) paths from \( t_1, t_2 \in A'(x_1, v_j) \) to \( t_1', t_2' \in B'[b_1, y_1] \), respectively, which are internally disjoint from \( H' \), such that \( B'[t_1', t_2'] \) is maximal and, subject to this, \( A'[t_1, t_2] \) is maximal. We may choose notation so that \( a_1, t_1, t_2, a_2 \) occur on \( A' \) in order. Then by (4), \( b_1, t_1', t_2', b_2 \) occur on \( B' \) in order.

\( 9 \) \( t_1' \in B'[b_1, r'] \), and there exist \( c_1 \in V(B'[b_1, t_1']) \) and \( c_2 \in V(B'[t_2', y_1]) \) such that \( c_1, c_2 \) are incident with some finite face of \( G'_0 \).

First, suppose such \( \{c_1, c_2\} \) does not exist. Then \( G'_0 \) contains a path from \( a_0 \) to \( B'(t_1', t_2') \) and internally disjoint from \( B' \). This contradicts Lemma 2.2.2 along with the path \( A'[a_1, x_1] \cup X' \cup w_j v_j \cup A'[v_j, a_2] \) from \( a_1 \) to \( a_2 \) and the path \( B'[b_1, t_1'] \cup T_1 \cup A'[t_1, t_2] \cup T_2 \cup B'[t_2', y_1] \cup y_1 z_2 \cup W[z_2, y_2] \) from \( b_1 \) to \( b_2 \).

Now suppose \( t_1' \notin B'[b_1, r'] \). Then \( t_1' \in B'(r', y_1) \). First, assume \( R \) is internally disjoint from \( T_1, T_2 \). If \( r \in A'(t_1, x_2) \) then \( R, T_1 \) contradict (4). So \( r \in A'[a_1, t_1] \) and, then, \( R, T_2 \) contradict the choice of \( T_1, T_2 \). So there exists \( v \in V(R \cap (T_1 \cup T_2)) \), and we choose \( v \) so that \( R[r', v] \) is minimal. If \( v \in V(T_1) \), then \( R[r', v] \cup T_1[v, t_1], T_2 \) contradict the choice of \( T_1, T_2 \); if \( v \in V(T_2) \), then \( T_1, R[r', v] \cup T_2[v, t_2] \) form a cross, contradicting (4). \( \Box \)

Now, we further choose \( c_1, c_2 \) in (9) so that \( B'[c_1, c_2] \) is maximal.

\( 10 \) \( G'_0 - A'_0 - B'(b_1, c_1) \cup B'[c_2, y_1] \) contains a path \( B'_0 \) from \( b_1 \) to \( c_1 \), and \( G'_0 - A'_0 - B'(b_1, c_2) \cup B'([c_2, y_1]) \) contains a path \( B''_0 \) from \( b_1 \) to \( c_2 \).

Suppose \( B'_0 \) does not exist. Then \( B'(b_1, c_1) \neq \emptyset \) and, by planarity of \( G'_0 \), there exist \( b_1' \in V(B'(b_1, c_1)) \) and \( a_0' \in V(B'[c_2, y_1]) \cup V(A'_0) \) such that \( b_1', a_0' \) are incident with some finite face of \( G'_0 \). If \( a_0' \in B'[c_2, y_1] \) then \( b_1', a_0' \) contradict the choice of \( c_1, c_2 \); if \( a_0' \in A'_0 \) then \( \{b_1', a_0', b_2\} \) is a 3-cut in \( G'_0 \), contradicting Lemma 2.3.7.
Now suppose $B'_0$ does not exist. Then by planarity of $G'_0$, there exist $b'_1 \in V(B'(b_1,c_2))$ and $a'_0 \in V(B'(c_2,y_1)) \cup V(A'_0)$, such that $b'_1, a'_0$ are incident with some finite face of $G'_0$. Now, if $a'_0 \in V(B'(c_2,y_1))$ then $b'_1, a'_0$ or $c_1, a'_0$ contradict the choice of $c_1, c_2$. So $a'_0 \in V(A'_0)$. Then $b'_1 \in B'(c_1, c_2)$ and $b_1 = c_1$; otherwise, $\{b'_1, a'_0, b_2\}$ or $\{c_1, a'_0, b_2\}$ is a 3-cut in $G'_0$, contradicting Lemma 2.3.7. But now, $a_0, b_1, b'_1, c_2$ are incident with some finite face of $G'_0$, so $\alpha(A', B') = 0$, a contradiction to Lemma 2.4.10. □

(11) $G$ has no $A'-B'$ path from $B'(b_1, c_1)$ to $A'$, but $G$ has an $A'-B'$ path $T$ from $t' \in B'(c_2, y_1)$ to $t \in A'[x_1, x_2]$.

Note that $c_1 \in B'[b_1, r_1]$, since $c_1 \in B'[b_1, t'_1]$ and $t'_1 \in B'[b_1, r_1]$. Thus, if $G$ has an $A'-B'$ path from $B'(b_1, c_1)$ to $A'$, it should be an edge $ab$ with $b \in V(B'(b_1, c_1))$ and $a \in V(A'[a_1, x_1]) \cup \{a_2\}$. By (3), $a \in A'[a_1, x_1]$. Now by Lemma 2.2.1, the following paths show $\alpha(A', B') = 2$: the path $A'[a_1, a] \cup ab \cup B'[b_1, b]$ from $a_1$ to $b_1$, the path $A'[a_1, x_1] \cup X_1$ from $a_1$ to $b_2$, the path $A'_0 \cup B'[q', y_1] \cup Q \cup A'[a_2]$ from $a_0$ to $a_2$, and the path $B'_0 \cup B'[c_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, y_2]$ from $b_1$ to $b_2$. This contradicts Lemma 2.4.10.

Now the path $T$ must exist; otherwise $\{b_1, c_1, c_2, y_1, b_2\}$ is a cut in $G^*$ separating $a_0$ from $\{a_1, a_2\}$, a contradiction. □

We choose $T$ in (11) so that $A'[t, a_2]$ is minimal. Then

(12) $t \neq a_2$, $T$ is internally disjoint from $T_1, T_2$, and $t = u_2 = v_j$.

First, suppose there exists $v \in V(T \cap (T_1 \cup T_2))$, and choose $v$ with $T[v, t']$ minimal. If $v \in T_1$ then $T_1[v, t'] \cup T_1 \cup T[v, t']$, $T_2$ contradict (4); if $v \in T_2$ then $T_1 \cup T_2[v, t] \cup T[v, t']$ contradict the choice of $T_1, T_2$. So $T$ is internally disjoint from $T_1, T_2$.

Now suppose $t = a_2$. By Lemma 2.2.1, the following paths show $\alpha(A', B') = 2$: the path $A'[a_1, t_i] \cup T_1 \cup B'[b_1, t'_1]$ from $a_1$ to $b_1$, the path $A'[a_1, x_1] \cup X_1$ from $a_1$ to $b_2$, the path $T \cup B'[t', y_1] \cup A'_0$ from $a_2$ to $a_0$, and the path $B'_0 \cup B'[t_2, c_2] \cup T_2 \cup A'[t_2, u_2] \cup u_2 w_2 \cup W[w_2, y_2]$ from $b_1$ to $b_2$. This contradicts Lemma 2.4.10.

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By (4), \( t \in A'[t_2, a_2] \). By the choice of \( T_1, T_2, t \notin A'[t_2, v_j] \). By Lemma 2.4.3, we have \( t \notin A'(u_2, a_2) \), and so \( t = u_2 = v_j \). □

(13) \( t_1 \in A'(a_1, w_1) \).

For otherwise, \( t_1 \in A'[w_1, v_j] \). Suppose that \( G \) has no \( A'-B' \) path from \( A'(x_1, w_1) \) to \( B'[b_1, y_1] \) and internally disjoint from \( H' \). By (7) and \( u_2 = v_j \) in (12), \( G \) has a separation \((G_1, G_2)\), such that \( V(G_1 \cap G_2) = \{x_1, w_1, u_2, u, x_2, y_2\} \) with \( u = y_1 \) (when \( z_1 \neq z_2 \)) or \( u = z_1 \) (when \( z_1 = z_2 \)), \( \{a_0, a_1, a_2, b_1, b_2\} \cup V(A'[u_2, x_2]) \subseteq V(G_1) \), \( X_1 \cup X_2 \subseteq G_2 \), and \( (G_2, x_1, w_1, u_2, u, x_2, y_2) \) is planar. This contradicts Lemma 2.1.3.

So \( G \) has an \( A'-B' \) path \( T_0 \) from \( t_0 \in A'(x_1, w_1) \) to \( t'_0 \in B'[b_1, y_1] \) and internally disjoint from \( H' \). If \( T_0 \) is disjoint from \( T_1, T_2 \) then either \( T_0, T_2 \) contradict the choice of \( T_1, T_2 \), or \( T_0, T_1 \) contradict (4). So there exists \( v \in V(T_0 \cap (T_1 \cup T_2)) \), and we choose \( v \) with \( T_0[v, t'_0] \) minimal. If \( v \in T_1 \) then \( T_1[t_1, v] \cup T_0[v, t'_0], T_2 \) contradict the choice of \( T_1, T_2 \); if \( v \in T_2 \) then \( T_1, T_2[t_2, v] \cup T_0[v, t'_0] \) contradict (4). □

Now, by (13) and Lemma 2.2.1, the following paths show \( \alpha(A', B') = 2 \): the path from \( A'[a_1, t_1] \cup T_1 \cup B'[b_1, t'_1] \) from \( a_1 \) to \( b_1 \), the path \( A'[a_1, x_1] \cup X_1 \) from \( a_1 \) to \( b_2 \), the path \( A'[t, a_2] \cup T \cup B'[t', y_1] \cup A'_0 \) from \( a_2 \) to \( a_0 \), and the path \( B''_0 \cup B'[t_2, c_2] \cup T_2 \cup A'[w_1, t_2] \cup W[w_1, y_2] \) from \( b_1 \) to \( b_2 \). This contradicts Lemma 2.4.10. □

**Lemma 2.4.12** There is no fat \( A'-B' \) connector in \( \gamma \).

**Proof.** For, otherwise, (i) or (ii) of Lemma 2.4.8 holds. Then

(a) if (i) of Lemma 2.4.8 holds then, by Lemma 2.4.9, we may choose the 2-cut \( \{z'_1, z'_2\} \)
so that \( z'_1 \in A'[x_1, w_1] \).

(b) if (i) of Lemma 2.4.8 does not hold but (ii) of Lemma 2.4.8 holds then, by Lemma 2.4.11,
\[ N_G(y_1) \cap V(H_0) \subseteq V(W[w_1, w_2]) \]
and let \( z'_1 := w_1 \) and \( z'_2 := z_1 \).

(1) \( z'_2 \notin V(X_2) \).
For, suppose $z'_2 \in V(X_2)$. Since $z_1 \notin V(X_2)$ by Lemma 2.4.7, (a) holds. Then $z'_1 = x_1$; 

or else, it contradicts Lemma 2.4.1 that $H' - A'(x_1, x_2)$ contains disjoint paths from $y_1, y_2$ to $x_1, x_2$, respectively. But now, $\{x_1, y_2, z'_2\}$ is a cut in $G^*$ separating $X_1(x_1, y_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

By (1), $w_m \in W(z'_2, y_2)$. Let $h \in \{2, \cdots, m\}$ be minimum with $w_h \in W(z'_2, y_2)$, and let $u_h \in N_G(w_h) \cap V(A'[q, x_2])$ with $A'[q, u_h]$ minimal. Let $Y_2 := W[y_2, w_h] \cup w_h u_h$, which is a path from $y_2$ to $u_h$.

(2) $G[H_0 + y_1] - A'(x_1, w_1)$ contains a path $Y_1$ from $y_1$ to $x_1$ and disjoint from $Y_2$.

Let $v \in N_G(y_1) \cap V(H_0)$ such that $v \notin A'$. Then $v \notin W[w_h, y_2]$. If $H_0 - W[w_h, y_2] - A'(x_1, w_1)$ contains a path $Y$ from $v$ to $x_1$ then $Y \cup v y_1$ gives the desired $Y_1$. So assume such $Y$ does not exist. Then, by the planar structure of $H_0$, there exist $z''_1 \in V(A'[x_1, z'_1]), z''_2 \in V(W[w_h, y_2])$ such that $z''_1, z''_2$ are incident with some finite face of $H_0$, and $\{z''_1, z''_2\}$ is a 2-cut in $H_0$. But then, $\{z''_1, z''_2\}$ contradicts the choice of $\{z'_1, z'_2\}$. □

Let $Y'_1 := Z_2 \cup W[z_2, w_m] \cup w_m v_m$, which is a path from $y_1$ to $x_2$. Then

(3) $H_0 - Y'_1$ has a path $Y'_2$ from $y_2$ to $z'_1$ and internally disjoint from $A'$.

For otherwise, by the planar structure of $H_0$, we may assume there exist $z''_1 \in V(A'[x_1, z'_1]), z''_2 \in V(W[z_2, w_m])$ such that $z''_1, z''_2$ are incident with some finite face of $H_0$, and $\{z''_1, z''_2\}$ is a 2-cut in $H_0$. But $\{z''_1, z''_2\}$ contradicts the choice of $\{z'_1, z'_2\}$. □

Now, the following statement holds to avoid forming a double cross with $Y'_1, Y'_2$:

(4) $G$ has no disjoint $A'-B'$ paths from $c, d \in V(A')$ to $c', d' \in V(B'[b_1, y_1])$, respectively, and internally disjoint from $A' \cup B' \cup H'$, such that $c \in V(A'[a_1, z'_1]), d \in V(A'(c, x_2))$, and $b_1, d', c', y_1$ occur on $B'$ in order.

(5) If $u_h \neq x_2$ and $G$ has an $A'-B'$ path $S$ from $s \in A'(u_h, x_2)$ to $s' \in B'[b_1, y_1]$ and internally disjoint from $H'$, then $b_1 = r_1 = r' = s'$ and $S$ is an edge from $s$ to $s'$. 

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Firs, $S \cap R = \emptyset$; otherwise, $S, R$ are contained in some $A'-B'$ bridge, which contradicts (v) of Lemma 2.3.9 due to the path $u_h w_h \cup W[w_h, y_2]$ from $u_h$ to $y_2$. Now, $s' \in B'[b_1, r']$; otherwise, $S, R, Y_1, Y_2$ form a doublecross as we assume $u_h \neq x_2$. Thus, $G$ has no $A'-B'$ path from $A'(u_h, x_2)$ to $B'(r', y_1)$, which further implies that $S \cap Q = \emptyset$.

We claim $b_1 = r_1$ and so $a_1 = x_1$ by (iii) of Lemma 2.4.4. For, suppose $b_1 \neq r_1$. Then $s' \neq b_1$; otherwise, $s = x_2 = a_2$, and $S = a_2 b_1$, a contradiction. But then, $A'[a_1, r] \cup R \cup B'[s', r'] \cup S \cup A'[s, a_2] \cup Y_1 \cup A'_0$ and $B'_1 \cup Q \cup A'[q, u_h] \cup Y_2 \cup B'[y_2, b_2]$ show that $\gamma$ is feasible, a contradiction. (Recall $B'_1, A'_0$ from Lemma 2.4.10.)

Now suppose $r_1 \neq r'$. By Lemma 2.4.2, there exist an $A-B$ core $H''$ with feet $r_1, r_2$ and $r' \in B'(r_1, r_2)$, and an $A'-B'$ bridge $M$ with extreme hands $l_0, r_0$ and feet $l'_0, r'_0$, such that $R$ is internally disjoint from $M$, $l_0 = r_0 = x_i$ for some $i \in [2]$, and $r' \in B'(l'_0, r'_0)$.

Since $G$ has no $A'-B'$ path from $A'(u_h, x_2)$ to $B'(r', y_1)$, then $i = 1$, $x_1$ is an extreme hand of $H''$, and $S$ is internally disjoint from $M$. If $s' = r'$ then let $P^*$ be the path from $l'_0$ to $r'_0$ in $M$ and internally disjoint from $A', B'$; now $A'[a_1, r] \cup R \cup S \cup A'(u_h, a_2) \cup Y_1 \cup A'_0$ and $B'[b_1, l'_0] \cup P^* \cup B'[r'_0, q'] \cup Q \cup A'[q, u_h] \cup Y_2$ show that $\gamma$ is feasible, a contradiction. Thus, $s' \in B'(r_1, r')$ and $s = x_2$ (by the definition of $r'$). Now, we see that $S$ is not contained in an $A'-B'$ bridge. For otherwise, by (ii) of Lemma 2.3.9, $S$ is contained in $H''$, which further implies $x_2$ is an extreme hand of $H''$. So $H''$ is a main core of $A, B$, a contradiction to Lemma 2.3.8. So $S = x_2 s'$. If $s' \in B'(r_1, r')$ then $S \in E(H'')$, which implies that $x_2$ is an extreme hand of $H''$, still a contradiction to Lemma 2.3.8. So $s' = r_1$ and $S = x_2 b_1$, which implies $a_2 \neq x_2$, a contradiction to Lemma 2.4.8.

Therefore, $b_1 = r_1 = r' = s'$. To complete the proof of (5), we need to prove that $S = ss'$. For, suppose $S \neq ss'$. Then $S$ is contained in some $A'-B'$ bridge $N$, and let $n_1, n_2$ be the extreme hands of $N$. Note that $V(N \cap B') \subseteq \{b_1\}$, as $b_1 = r_1 = s' = r'$ for any choice of $S$. Moreover, by Lemma 2.4.3, $V(N \cap A'(u_h, x_2)) = \emptyset$. Hence, $n_1 \in A'[x_1, u_h]$ and $n_2 = x_2$. By (v) of Lemma 2.3.9, $H'-y_1$ does not have a path from $A'(n_1, n_2)$ to $y_2$ and internally disjoint from $A'$. So, by the existence of path $Y_2$, $n_1 \notin A'[x_1, u_2]$. So $n_1 = u_h$. 67
But then, \(\{n_1, n_2, b_1\}\) is a cut in \(G^*\) separating \(N\) from \(\{a_0, a_1, a_2, b_1, b_2\}\), a contradiction. \(\square\)

(6) \(x_1 \neq z'_1\), and \(b_2 = y_2\).

First, suppose \(x_1 = z'_1\). Since \(w_1 \neq x_1\) then (a) holds. Now \(G\) has an \(A'-B'\) path from \(A'(u_h, x_2)\) to \(B'[b_1, y_1]\) internally disjoint from \(H' - y_1\); otherwise, \(\{x_1, z'_2, u_h, x_2, y_2\}\) is a cut in \(G^*\) separating \(\{a_0, a_1, a_2, b_1, b_2\}\) from \(V(X_1 \cup X_2)\), a contradiction. Hence, \(A'(u_h, x_2) \neq \emptyset\) and, by (5), \(b_1 = r_1 = r'\) and \(a_1 = x_1\) (by (iii) of Lemma 2.4.4). But then, \(G\) has a separation \((G_1, G_2)\) of order 6, such that \(V(G_1 \cap G_2) = \{x_1, z'_2, u_h, x_2, y_2, b_1\}\), \(\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)\), \(V(X_1 \cup X_2) \subseteq V(G_2)\), and \((G_2, x_1, y_2, x_2, b_1, u_h, z'_2)\) is planar, a contradiction to Lemma 2.1.3.

Now suppose \(b_2 \neq y_2\). By Lemma 2.4.8, \(N_G(b_2) = \{y_2, x_1\}\) and \(a_1 \neq x_1\). Let \(a_1b' \in E(G)\) with \(b' \in V(B'(b_1, r_1)) \cup V(B'[y_2, b_2])\). By (i) of Lemma 2.4.4, \(b' \in B'(b_1, r_1)\). Since \(x_1 \neq z'_1\), we have \(\alpha(A', B') = 2\) by Lemma 2.2.1 and the following paths: the path \(A'_0 \cup Y'_1 \cup A'[x_2, a_2]\) from \(a_0\) to \(a_2\), the path \(B'_1 \cup Q \cup A'[z'_1, q] \cup Y'_2 \cup B'[y_2, b_2]\) from \(b_1\) to \(b_2\), the path \(a_1b' \cup B'[b_1, b']\) from \(a_1\) to \(b_1\), and the path \(A'[a_1, x_1] \cup e\) from \(a_1\) to \(b_2\). This contradicts Lemma 2.4.10. \(\square\)

(7) \(G\) has an \(A'-B'\) path from \(A'[a_1, z'_1]\) to \(B'(b_1, y_1)\) and internally disjoint from \(H'\).

For, suppose (7) fails. Then by Lemma 2.4.10 and by (5) and (6) \((b_2 = y_2)\), if (a) holds then \(\{b_1, b_2, z'_1, z'_2, u_h\}\) is a cut in \(G^*\) separating \(a_1, a_2\) from \(a_0\), a contradiction; if (b) holds then \(\{b_1, b_2, z'_1, y_1, u_h\}\) (when \(z_1 \neq w_2\)) or \(\{b_1, b_2, z'_1, z_1, u_h\}\) (when \(z_1 = w_2\)) is a cut in \(G\) separating \(a_1, a_2\) from \(a_0\), a contradiction. \(\square\)

(8) If \(u_h \neq x_2\), then \(G\) has no \(A'-B'\) path from \(A'(u_h, x_2)\) to \(B'[b_1, y_1]\) and internally disjoint from \(H'\).

For, otherwise, it follows from (5) that \(b_1 = r_1 = r' = s'\) and \(G\) has an edge \(sb_1\) with \(s \in V(A(u_h, x_2))\). So \(s \neq a_2\). Now \(sb_1\) and a path from (7) contradict (4). \(\square\)
(9) $G$ has disjoint $A'-B'$ paths from $A'[a_1, z'_1]$ to $B'[b_1, y_1]$ and internally disjoint from $H'$.

For otherwise, there exists a vertex $v \in V(G)$ such that $G - v$ has no $A'-B'$ path from $A'[a_1, z'_1]$ to $B'[b_1, y_1]$ and internally disjoint from $H'$. Then by (8), there exists a separation $(G_1, G_2)$ in $G$ such that $V(G_1 \cap G_2) = \{v, z'_1, u, u_h\}$ (with $u = z'_2$ if (a) holds and $u = y_1$ if (b) holds), $b_1, a_0 \in V(G_1)$, and $a_1, a_2, b_2 \in V(G_2)$.

Suppose $(G_2, v, z'_1, u, u_h, a_1, b_2, a_1)$ is planar. If $v = a_1, u_h = a_2$ then $\{v, z'_1, u, u_h, b_2\}$ is a cut in $G^*$ separating $V(X_1 \cup X_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction; if $v \neq a_1, u_h = a_2$ or $v = a_1, u_h \neq a_2$, then Lemma 2.1.3 applies; if $v \neq a_1, u_h \neq a_2$, then Lemma 2.1.4 applies.

So $(G_2, v, z'_1, u, u_h, a_2, b_2, a_1)$ is not planar. Clearly, $v \notin A'$, and there exists an $A'-B'$ bridge $N$ with feet $n'_1, n'_2$ and extreme hands $n_1, n_2$, such that $v \in N$. By (v) of Lemma 2.3.9, $H' - y_1$ does not contain a path from $A'(n_1, n_2)$ to $y_2$ and internally disjoint from $A'$. Suppose $v \notin B'$. Then $N$ has a separation $(N', N'')$ of order 1, such that $V(N' \cap N'') = \{v\}$, $n_1, n_2 \in V(N' - N'')$, and $n'_1, n'_2 \in V(N'' - N')$. Now $V(N') = \{n_1, n_2, v\}$; or else, $\{n_1, n_2, v\}$ is a cut in $G$ separating $V(N') - \{n_1, n_2, v\}$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. This implies that $(G_2, v, z'_1, u, u_h, a_2, b_2, a_1)$ is planar, a contradiction. So $v \in B'$. But then, by (v) of Lemma 2.3.9 and the definition of $v, n'_1 = n'_2 = v$ and there exist $n^*_1 \in A'[a_1, n_1]$ and $n^*_2 \in A'[n_2, a_2]$, such that $\{n^*_1, n^*_2, v\}$ is a cut in $G^*$ separating $V(N)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

By (9), let $T_1, T_2$ be disjoint $A'-B'$ paths from $t_1, t_2 \in A'[a_1, z'_1]$ to $t'_1, t'_2 \in B'[b_1, y_1]$, such that $a_1, t_1, t_2, a_2$ occur on $A'$ in order, $T_1, T_2$ are internally disjoint from $H'$ and, subject to this, $A'[t_1, t_2] \cup B'[t'_1, t'_2]$ are maximal. Then by (4), $b_1, t'_1, t'_2, y_1$ occur on $B'$ in order.

(10) $t'_1 \in B'[b_1, r'], t'_2 \notin B'(q', y_1)$, and $Q$ is internally disjoint from $T_1, T_2$.

Suppose $Q$ is not internally disjoint from $T_j$ for some $j \in [2]$, then $Q, T_j$ are contained in
some $A'-B'$ bridge. But then, the existence of the path from $z'_1$ to $y_2$ in $H' - y_1$ contradicts (v) of Lemma 2.3.9.

So $Q$ is internally disjoint from $T_1, T_2$. Hence, by (4), $t'_2 \notin B'(q', y_1)$. Now suppose $t'_1 \in B'(r', t'_2)$. If $R \cap (T_1 \cup T_2) = \emptyset$, then $R, T_2$ contradict the choice of $T_1, T_2$ (when $r \in A'[a_1, t_1]$) or $T_1, R$ contradict (4) (when $r \in A'(t_1, q)$). So there exists $u \in V(R \cap (T_1 \cup T_2))$, and we choose $u$ so that $R[r', u]$ is minimal. If $u \in T_1$, then $R[r', u] \cup T_1[u, t_1], T_2$ contradict the choice of $T_1, T_2$. If $u \in T_2$, then $T_1, R[r', u] \cup T_2[u, t_2]$ contradict (4). □

We let $Q_0$ be an $A'-B'$ path from $q_0 \in A'[z'_1, a_2]$ to $q'_0 \in B'[b_1, y_1]$ and internally disjoint from $H'$, such that $B'[q'_0, y_1]$ is minimal. By the existence of $Q, q'_0 \in B'[q', y_1]$.

(11) No finite face of $G'_0$ is incident with both a vertex of $B'[b_1, t'_1]$ and a vertex of $B'[q'_0, y_1]$.

For, suppose $c_1 \in V(B'[b_1, t'_1])$ and $c_2 \in V(B'[q'_0, y_1])$ such that $c_1, c_2$ are incident with a finite face of $G'_0$. We choose $c_1, c_2$ so that $B'[c_1, c_2]$ is maximal. Since $t'_1 \in B'[b_1, r']$, $c_1 \in B'[b_1, r']$. We may further assume $c_1 \in B'[b_1, r_1]$; otherwise, $r' \neq r_1, c_1 \in B'(r_1, r')$, and by (iii) of Lemma 2.4.2, $r' \in B'(r_1, r_2)$ for some $r_2 \in V(B'(r', y_1))$ and $r', r_1, r_2$ are incident with some finite face of $G'_0$, implying $c_1 \in B'[b_1, r_1]$ by the choice of $c_1, c_2$, a contradiction.

Note that $G$ has an $A'-B'$ path $T_3$ from $t'_3 \in B'(b_1, c_1) \cup B'(c_2, y_1)$ to $t_3 \in A'$, to avoid the cut $\{b_1, b_2, c_1, c_2, y_1\}$ in $G^*$, separating $a_0$ from $\{a_1, a_2\}$.

Note that $t'_3 \in B'(c_2, y_1)$. For, suppose $t'_3 \in B'(b_1, c_1)$. Then $t'_3 \in B'(b_1, r_1)$ and, by the choice of $T_1, T_2$ and by (4) and (8), we have $t_3 = u_h = a_2$. Thus, $A'[a_1, t_1] \cup T_1 \cup B'[t'_3, t'_1] \cup T_3 \cup Y'_1 \cup A'_0$ and $B'_1 \cup Q \cup A'[z'_1, q] \cup Y'_2$ show that $\gamma$ is feasible, a contradiction.

Moreover, $t_3 = z'_1$, as $t_3 \notin A'(z'_1, a_2)$ (by the choice of $Q_0$), and $t_3 \notin A'[a_1, z'_1]$ (so that $T_3, Q_0$ do not contradict (4)).

If $G'_0 - B'[t'_1, q'_0] - A'_0$ contains a path $B'_3$ from $b_1$ to $t'_3$, then $A'[a_1, t_1] \cup T_1 \cup B'[t'_1, q'_0] \cup Q_0 \cup A'[q_0, a_2] \cup Y'_1 \cup A'_0$ and $B'_3 \cup T_3 \cup Y'_2$ show that $\gamma$ is feasible, a contradiction.
So such $B_3^*$ does not exist. Then, by the maximality of $B'[c_1, c_2]$, there exists $c_3 \in V(A'_0)$ such that $\{c_2, c_3\}$ is a cut in $G'_0$ separating $b_1$ from $t'_1$, and there does not exist any $A'-B'$ bridge with one foot in $B'[b_1, c_2)$ and another in $B'(c_2, y_1)$. Hence, $\{z'_1, c_2, c_3, y_1\}$ is a cut in $G^*$ separating $t'_3$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

(12) $G'_0 - B'(b_1, t'_1) - (B'[q'_0, y_1] \cup A'_0)$ contains a path $B'_1$ from $b_1$ to $B'(t'_1, q'_0)$.

For otherwise, $b_1 \neq t'_1$, and there exist $c_1 \in V(B'(b_1, t'_1))$ and $c_2 \in V(B'[q'_0, y_1]) \cup V(A'_0)$ such that $c_1, c_2$ are incident with a finite face of $G'_0$. By (11), $c_2 \in A'_0$. By Lemma 2.3.7, $c_1 \notin B'(b_1, r_1)$. So $c_1 \in B'(r_1, r')$ as $t'_1 \in B'[b_1, r']$. Hence, by (iv) of Lemma 2.4.2, $c_2 = a_0$, $b_1 = r_1$, and $\alpha(A', B') = 0$, contradicting Lemma 2.4.10. □

(13) If (a) holds then $H' - y_1 - z'_2 - X_1[x_1, y_2]$ has a path $Y_2^*$ from $z'_1$ to $y_2$ and internally disjoint from $A'$.

For otherwise, there exists $u \in V(A'[x_1, z'_1] \cup X_1[x_1, y_2])$, such that $u, z'_2$ are incident with a finite face of $H' - y_1$. By the choice of $\{z'_1, z'_2\}$, $u \in V(X_1[x_1, y_2])$. Now $\{u, z'_2, u_h, x_2, y_2\}$ is a cut in $G^*$ separating $X_2$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

(14) If (a) holds then $H' - y_1 - A'(x_1, z'_1) - W[z'_2, y_2]$ has a path $X^*$ from $x_1$ to $z'_1$; if (b) holds then $H' - y_1 - A'(x_1, z'_1) - W[z'_2, y_2]$ has a path $X^*$ from $x_1$ to $z'_1$.

For otherwise, let $v = z'_2$ when (a) holds; and let $v = z_2$ when (b) holds. Then there exist $z''_1 \in V(A'(x_1, z'_1))$ and $z''_2 \in V(W[v, y_2])$ such that $z''_1, z'_2$ are incident with a finite face of $H_0$. Hence, (a) holds, and $\{z''_1, z''_2\}$ contradicts the choice of $\{z'_1, z'_2\}$. □

(15) $G - T_1 - Q_0$ has no $A'-B'$ path from $A'(t_1, z'_1)$ to $B'(t'_1, q'_0)$.

For, suppose $G - T_1 - Q_0$ has an $A'-B'$ path $T$ from $t \in V(A'(t_1, z'_1))$ to $t' \in V(B'(t'_1, q'_0))$. When (a) holds, we let $B^*$ be the path from $b_1$ to $b_2$ in $B_1 \cup B'(t'_1, q'_0) \cup T \cup A'[t, z'_1] \cup Y_2^*$; when (b) holds, we let $B^*$ be the path from $b_1$ to $b_2$ in $B_1^* \cup B'(t'_1, q'_0) \cup T \cup W[t, y_2]$. By
Lemma 2.2.1, the following paths show that $\alpha(A', B') = 2$: the path $B^*$ from $b_1$ to $b_2$, the path $A'[q_0, a_2] \cup Q_0 \cup B'[q_1, y_1] \cup A'_0$ from $a_2$ to $a_0$, the path $A'[a_1, t_1] \cup T_1 \cup B'[b_1, t'_1]$ from $a_1$ to $b_1$, and the path $A'[a_1, x_1] \cup X_1$ from $a_1$ to $b_2$. This contradicts Lemma 2.4.10. □

(16) $t'_2 = q'_0$, $t'_1 = r'$, and $G$ has an $A'-B'$ path $R^*$ from $r'$ to $A'(x_1, z'_1)$.

For, suppose $t'_2 \neq q'_0$. By (15), $T_2, Q_0$ are contained in an $A'-B'$ bridge. But the existence of the path from $z'_1$ to $y_2$ in $H' - y_1$ contradicts (v) of Lemma 2.3.9.

Note that $G$ has an $A'-B'$ path from $r'$ to $A'(x_1, z'_1)$; for otherwise, $R \cap T_2 = \emptyset$, and $R, T_2$ contradicts (4).

Next $t'_1 = r'$. For otherwise, $r' \in B'(t'_1, q'_0)$. Now, by (15), $R^* \cap (T_1 \cup Q_0) \neq \emptyset$. By the definition of $r'$, $R^* \cap T_1 = \emptyset$. Thus, $R^*, Q_0$ are contained in some $A'-B'$ bridge. But then, the path from $z'_1$ to $y_2$ in $H' - y_1$ contradicts (v) of Lemma 2.3.9. □

Now, the path $A'[a_1, x_1] \cup X^* \cup A'[z'_1, a_2]$ from $a_1$ to $a_2$ and the path $B'[b_1, r'] \cup R \cup A'[r, t_2] \cup T_2 \cup B'[t'_2, y_1] \cup Z_2 \cup W[z_2, y_2]$ from $b_1$ to $b_2$ show that $G'_0$ does not contain a path from $B'(t'_1, t'_2)$ to $a_0$ and internally disjoint from $B'$; or else, it contradicts (i) of Lemma 2.2.2. So, there exist $c_1 \in B'[b_1, t'_1]$ and $c_2 \in B'[t'_2, y_2]$, such that $c_1, c_2$ are incident with some finite face of $G'_0$, a contradiction to (11). □

2.5 Slim connectors

In this section, we let $\gamma := (G, a_0, a_1, a_2, b_1, b_2)$, and assume that $\gamma$ is infeasible and no ideal frame in $\gamma$ admits a fat connector (see at Figure 2.11).

Recall that $b_1 b_2 \notin E(G)$, $a_i b_j \notin E(G)$ for $i = 0, 1, 2$ and $j = 1, 2$, and $G^* := G + b_1 b_2 + \{a_i b_j : i = 0, 1, 2$ and $j = 1, 2\}$ is 6-connected. Let $A, B$ be an ideal $a_0$-frame in $\gamma$. Let $G_0 := G - A$. By Lemma 2.1.6 and the structure of slim connectors, $G_0$ has a disk representation with $B$ and $a_0$ occurring on the boundary of the disk, and any $A-B$ path in $G$ is induced by a single edge.

**Lemma 2.5.1** Let $a_{-1} := a_2$ and $a_3 := a_0$. Then the following statements hold:
(i) \( G \) cannot be obtained from a planar graph \( H \) by identifying two vertices of \( H \), such that \( b_1, b_2 \) and two of \( \{a_0, a_1, a_2\} \) are incident with a face of \( H \).

(ii) For any \( i \in \{0, 1, 2\} \), \((G - a_{i-1}, a_i, b_1, a_{i+1}, b_2)\) or \((G - a_{i+1}, a_i, b_1, a_{i-1}, b_2)\) is not planar.

(iii) There do not exist a permutation \( \pi \) of \( \{0, 1, 2\} \), a graph \( H \) and distinct vertices \( s, t, s', t' \in V(H) \), such that \((H, a_{\pi(0)}, b_1, a_{\pi(1)}, s, t, s', t', a_{\pi(2)}, b_2)\) is planar, and \( G \) is obtained from \( H \) by identifying \( s \) with \( s' \) and \( t \) with \( t' \), respectively.

Proof. Let \( n = |V(G)| \). Since \( G^* \) is 6-connected, \( |E(G)| \geq 3n - 7 \). First, we see that (i) holds. For, otherwise, there exist \( i \in \{0, 1, 2\} \), graph \( H \) with \((H, a_{i-1}, b_1, a_{i+1}, b_2)\) planar, and distinct \( s, s' \in V(H) \), such that \( G \) is isomorphic to the graph obtained from \( H \) by identifying \( s \) with \( s' \). Then \( |E(H)| \geq |E(G)| \geq 3n - 7 \), and \( H' := H + \{a_{i-1}b_1, a_{i-1}b_2, a_{i+1}b_1, a_{i+1}b_2, b_1b_2\} \) is planar. However, \( |E(H')| \geq 3n - 2 = 3|V(H')| - 5 \), a contradiction.

Now suppose (ii) fails. Then for some \( i \in \{0, 1, 2\} \), both \((G - a_{i-1}, a_i, b_1, a_{i+1}, b_2)\) and \((G - a_{i+1}, a_i, b_1, a_{i-1}, b_2)\) are planar. Without loss of generality, we assume \( i = 0 \) and that
Since \( G \) is 6-connected, \( d_{G'}(a_2) \geq d_G(a_1) + 2 \), \( d_{G'}(a_0) \geq 6 \), \( d_{G'}(b_j) \geq 5 \) for \( j \in [2] \), and 
\[
|E(G' - a_1)| = (6(n - 5) + 6 + 5 + 5 + 2)/2 = 3n - 6 = 3|V(G' - a_1)| - 3,
\]
contradicting the planarity of \( G' - a_1 \).

Finally, suppose (iii) fails. So there exists a permutation \( \pi \) of \( \{0,1,2\} \), a graph \( H \) and distinct vertices \( s,t,s',t' \in V(H) \), such that \( (H, a_{\pi(0)}, b_1, a_{\pi(1)}, s, t, s', t', a_{\pi(2)}, b_2) \) is planar, and \( G \) is obtained from \( H \) by identifying \( s \) with \( s' \) and \( t \) with \( t' \), respectively. Now \[
|E(H)| \geq |E(G)| \geq 3n - 7, a_{\pi(0)}a_{\pi(1)}, a_{\pi(0)}a_{\pi(2)}, a_{\pi(0)}t, a_{\pi(0)}s' \notin E(H), \quad \text{and} \quad H' := H + \{b_1a_{\pi(0)}, b_1a_{\pi(1)}, b_2a_{\pi(0)}, b_2a_{\pi(2)}, a_{\pi(0)}a_{\pi(1)}, a_{\pi(0)}a_{\pi(2)}, a_{\pi(0)}t, a_{\pi(0)}s' \} \text{ is planar. Thus,} 
\]
\[
|V(H')| = n + 2 \text{ and } |E(H')| \geq 3n + 1 = 3(n + 2) - 5, \text{ contradicting planarity of } H'. \quad \square
\]

We now investigate the edges between \( A \) and \( B \). Let \( a'b', a''b'' \in E(G) \) with \( a', a'' \in V(A) \) and \( b', b'' \in V(B) \) all distinct. We say that \( a'b', a''b'' \) form a cross (w.r.t. \( A, B \)) if \( a_1, a', a'', a_2 \) occur on \( A \) in order, and \( b_1, b'', b', b_2 \) occur on \( B \) in order. We say that \( a'b', a''b'' \) are parallel if \( a_1, a', a'', a_2 \) occur on \( A \) in order, and \( b_1, b', b'', b_2 \) occur on \( B \) in order.

Two sets of edges of \( G \) between \( A \) and \( B \) play critical roles in the remainder of this section. For \( i = 5, 6, 7 \), let \( e_i = a_ib_i \in E(G) \) with \( a_i \in V(A) \) to \( b_i \in V(B) \); we say that \( (e_5, e_6, e_7) \) is a 3-edge configuration if \( b_6 \in B(b_5, b_7) \) and \( a_1, a_2, a_6 \notin A[a_5, a_7] \). For \( i = 3, 4, 5, 6, 7 \), let \( e_i = a_ib_i \in E(G) \) with \( a_i \in V(A) \) and \( b_i \in V(B) \); we say that \( (e_3, e_4, e_5, e_6, e_7) \) is a 5-edge configuration (seen at Figure 2.12) if

- \((e_5, e_6, e_7)\) is a 3-edge configuration,

- \(A[a_5, a_7] \subseteq A(a_3, a_4)\), and

- \(b_3, b_4 \in B(b_j, b_5) \cap B(b_j, b_7)\) for some \( j \in [2]\).
Lemma 2.5.2 There exists a 5-edge configuration.

Proof. (1) For \( i \in [2] \), \( G \) has a cross from \( A - a_i \) to \( B \).

For, suppose \( G \) has no cross from \( A - a_i \) to \( B \) and, without loss of generality, let \( i = 2 \). Let \( a'b' \in E(G) \) with \( a' \in V(A[a_1, a_2]) \) and \( b' \in V(B[b_1, b_2]) \), such that \( B[b', b_2] \) is minimal. Then \( G \) has an edge from \( a_2 \) to \( B[b_1, b'] \), as otherwise, \( (G, a_1, a_2, b_2, a_0, b_1) \) is planar, contradicting (i) of Lemma 2.5.1. Let \( a_2 u_i \in E(G) \), with \( u_i \in V(B[u_1, u_2]) \) for \( i \in [2] \), such that \( B[u_1, u_2] \) is maximal and \( b_1, u_1, u_2, b_2 \) occur on \( B \) in order.

Then there exists \( ab \in E(G) \) with \( b \in V(B(u_1, u_2)) \) and \( a \in V(A[a_1, a_2]) \). For, otherwise, let \( H \) be obtained from \( G \) by splitting \( a_2 \) to \( s, s' \), such that \( H \) has no edge from \( B[u_1, u_2] \) to \( s' \) and no edge from \( B[b', b_2] \) to \( s \). Now \( (H, a_1, b_2, a_0, b_1) \) is planar and \( G \) can be obtained from \( H \) by identifying \( s \) and \( s' \), contradicting (i) of Lemma 2.5.1.

We see that \( a = a_1 \). For, otherwise, let \( a_1 b^* \in E(G) \) with \( b^* \neq b \). Since \( G \) has no cross from \( A - a_2 \) to \( B \), \( b^* \in B(b_1, b) \). Now, \( (a_1 b^*, u_1 a_2, ab, u_2 a_2, a'b') \) is a 5-edge configuration.

So all edges from \( B(u_1, u_2) \) to \( A[a_1, a_2] \) end with \( a_1 \). But now, \( (G - a_1, a_2, b_2, a_0, b_1) \) and \( (G - a_2, a_1, b_2, a_0, b_1) \) are planar, contradicting (ii) of Lemma 2.5.1. \( \square \)
We let \( b'_1, b'_2 \in B[b_1, b_2] \), such that \( b_1, b'_1, b'_2, b_2 \) occur on \( B \) in order, \( G \) has an edge from \( b'_i \) to \( A \) for each \( i \in [2] \), and subject to this, \( B[b'_1, b'_2] \) is maximal. By relabelling notation, we may assume that

(2) \( G \) has no edge from \( b'_i \) to \( A(a_1, a_2) \), and has an edge \( e_3 := b'_1a_1 \).

First, suppose there exist \( b'_ia'_i \in E(G) \) with \( a'_i \in V(A(a_1, a_2)) \) for each \( i \in [2] \). Since \( d_G(a_i) \geq 4 \) for \( i \in [2] \), there exists \( a_ib''_i \in E(G) \) with \( b''_i \in V(B(b'_1, b'_2)) \). Now \( b'_ia'_i, b'_2a'_2, b''_ia_1, b''_2a_2 \) form a double cross in \( \gamma \), a contradiction.

Thus, for some \( i \in [2] \), \( G \) has no edge from \( b'_i \) to \( A(a_1, a_2) \). By symmetry, we may assume \( i = 1 \) and \( b'_1a_1 \in E(G) \). □

By (1), there exist \( e_4 = a_4b_4, e_5 = a_5b_5 \in E(G) \) with \( a_4, a_5 \in V(A(a_1, a_2)) \) and \( b_4, b_5 \in V(B[b'_1, b'_2]) \), such that \( e_4, e_5 \) form a cross, and \( b_1, b_4, b_5, b_2 \) occur on \( B \) in order. We further choose \( e_4, e_5 \) so that \( B[b'_1, b'_4] \cup A[a_1, a_5] \) is minimal and, subject to this, \( B[b_5, b_2] \cup A[a_4, a_2] \) is minimal. Then

(3) \( G \) has no edge from \( B[b_1, b_4] \) to \( A(a_5, a_2) \), no edge from \( A(a_1, a_5) \) to \( B(b_4, b_2) \), no edge from \( b_4 \) to \( A(a_4, a_2) \), and no edge from \( a_5 \) to \( B(b_5, b_2) \).

To avoid forming a double cross with \( e_4, e_5 \),

(4) \( G \) has no cross from \( B[b_1, b_4] \) to \( A[a_1, a_5] \) or from \( B[b_5, b_2] \) to \( A[a_4, a_2] \).

(5) \( G \) has no edge \( B(b_5, b_2) \) to \( A(a_1, a_4) \), or no edge from \( B(b_4, b_5) \) to \( A(a_1, a_4) - a_5 \).

For, suppose there exists \( ab, a'b' \in E(G) \) with \( b \in V(B(b_5, b_2)) \), \( a \in V(A(a_1, a_4)) \), \( b' \in V(B(b_4, b_5)) \) to \( a' \in V(A(a_1, a_4) - a_5) \). By (3), \( a, a' \in A(a_5, a_4) \). Now \( (e_3, e_4, b'a', e_5, ba) \) is a 5-edge configuration. □

Let \( e'_5 = a_5b'_5 \in E(G) \) with \( b'_5 \in V(B[b_4, b_5]) \) such that \( B[b'_5, b_2] \) is maximal. If \( G \) has an edge \( e \) from \( B(b'_5, b_3) \) to \( A - a_5 \), then \( (e_3, e_4, e'_5, e, e_5) \) is a 5-edge configuration. Hence, we may assume that
(6) \(G\) has no edge from \(B(b'_5, b_5)\) to \(A - a_5\).

We may also assume that

(7) \(G\) has no cross from \(B[b'_5, b_2]\) to \(A(a_5, a_2)\) not involving the possible edge \(a_4b'_5\).

For, suppose \(G\) has a cross \(e' = a'b', e'' = a''b''\) avoiding \(a_4b'_5\), with \(a', a'' \in V(A(a_5, a_2))\), \(b', b'' \in V(B[b'_5, b_2])\), and \(a_5, a', a'', a_2\) on \(A\) in order. Then \(a'' \in A(a_5, a_4)\), to avoid the double cross \(e_4, e'_5, e', e''\). Hence, we may assume \(b'' = b'_5\); as otherwise, \((e_3, e_4, e'_5, e'', e')\) is a 5-edge configuration. Then \(a'' \in A(a_5, a_4)\), as \(e'' \neq a_4b'_5\).

Let \(e^* = a''b^* \in E(G)\) with \(b^* \in V(B[b_1, b_2])\). Since \(G^*\) is 6-connected, we can choose \(e^*\) so that \(b^* \notin \{b', b'', b_1\}\). Now \(b^* \in B[b_4, b_2]\), to avoid the double cross \(e^*, e', e_4, e'_5\).

If \(b^* \in B(b_4, b'_5)\) then \((e_3, e_4, e^*, e'_5, e')\) is a 5-edge configuration. If \(b^* \in B(b'_5, b')\) then \((e_3, e_4, e'_5, e^*, e')\) is a 5-edge configuration. If \(b^* \in B(b', b_2)\) then \((e_3, e_4, e'', e', e^*)\) is a 5-edge configuration. \(\square\)

If \(a_4 \neq a_2\) then there exist \(e_i^* = a_i^*b_i^* \in E(G), i \in [2]\), with \(a_i^* \in A(a_4, a_2)\) and \(b_i^* \in V(B(b_4, b_2))\), and we choose them so that \(B[b_1^*, b_2^*]\) is maximal, and \(b_1, b_1^*, b_2^*, b_2\) occur on \(B\) in order.

(8) If \(a_4 \neq a_2\), then \(G\) has no edge from \(B(b_1^*, b_2^*)\) to \(a_5\).

We show that if (8) fails, then the desired 5-edge configuration exists, or splitting \(a_5\) or \(b_5\) results in a graph \(H\) such that \((H, a_1, b_2, a_0, b_1)\) is planar, contradicting (i) of Lemma 2.5.1.

So assume \(a_4 \neq a_2\) and that \(G\) has an edge \(e_5^*\) from \(b_5^* \in B(b_1^*, b_2^*)\) to \(a_5\). We see that \(b_2^* \neq b_2\). For otherwise, \(b_2^* = b_2\) and \(a_2^* \neq a_2\). By (3), \(G\) has no edge from \(a_2\) to \(B[b_1, b_4]\), and so \(G\) has an edge from \(a_2\) to \(B(b_4, b_2)\), which together with \(e_4, e_2^*, e_5^*\) forms a double cross.

We may assume that \(G\) has no edge from \(B(b_4, b_1)\) to \(A[a_1, a_2] - a_4\). For otherwise, let \(e = ab \in E(G)\) with \(b \in V(B(b_4, b_1))\) and \(a \in V(A[a_1, a_2] - a_4)\). Then by the definition
of $b_1^*, b_2^*$, we have $a \notin A(a_4, a_2)$. Moreover, $a \neq a_1$ to avoid the double cross $e_1, e_4, e_5^*, e_1^*$. But then $a \in A(a_1, a_4)$, and so $(e_3, e_4, e, e_1^*, e_5^*)$ is a 5-edge configuration.

Hence, by (3) and (4), we may assume that $G$ has no edge from $B[b_1, b_1^*]$ to $A(a_4, a_2)$ and $G$ has no cross from $B[b_1, b_1^*]$ to $A[a_1, a_2]$.

We may also assume that $G$ has no edge from $B(b_2^*, b_2)$ to $A[a_1, a_2]$. For, suppose $G$ has an edge $e$ from $b \in B(b_2^*, b_2)$ to $a \in A[a_1, a_2]$. Note $a \neq a_4$ to avoid the double cross $e_4, e_5^*, e, e_1^*$. If $a = a_1$ then $(e, e_4, e_5^*, e_1^*, e_1^*)$ is a 5-edge configuration. If $a \in A(a_1, a_4)$ then $(e_3, e_4, e_5^*, e_2)$ is a 5-edge configuration.

Moreover, we may assume that $G - \{a_5, b_2^*\} - a_4 b_1^*$ has no edge from $B[b_1, b_2^*]$ to $A[a_1, a_4]$. For, suppose there exists $e = ab \in E(G)$ with $e \neq a_4 b_1^*$, $a \in V(A[a_1, a_4] - a_5)$, and $b \in V(B[b_1, b_2^*] - b_5^*)$. First, assume $b \in B(b_2^*, b_2^*)$. Then $a \in A(a_1, a_5)$ to avoid the double cross $e_4, e_5^*, e, e_1^*$. Hence $a = a_1$ by (3), and $(e_2^*, e, e_5^*, e_1^*, e_4)$ is a 5-edge configuration. So $b \in B[b_1, b_2^*]$. Then $a \in A(a_5, a_4)$ to avoid the double cross $e_4, e_5^*, e, e_1^*$. We may assume $b = b_1^*$; or else, $b \in B(b_1^*, b_2^*)$, and $(e_2^*, e_5^*, e, e_1^*, e_4)$ is a 5-edge configuration. Since $e \neq a_4 b_1^*$, $a \in A(a_5, a_4)$. Let $e_0 = ab_0 \in E(G)$ with $b_0 \in V(B[b_1, b_2]) \{b_4, b_1^*, b_5^*\}$ (as $d_G(a) \geq 6$). By (3), $b_0 \notin B[b_1, b_4^*]$. Now $b_0 \notin B(b_1^*, b_2^*)$ as $b = b_1^*$, and $b_0 \notin B(b_2^*, b_2^*)$ as $G$ has no edge from $B(b_2^*, b_2)$ to $A[a_1, a_2]$. So $b_0 \in B(b_4^*, b_1^*)$, and $(e_3, e_4, e_0, e_1^*, e_5^*)$ is a 5-edge configuration.

We may further assume that $G$ has no cross from $A(a_4, a_2)$ to $B[b_1^*, b_2^*] \cup B(b_2^*, b_2^*)$. For, suppose $G$ has a cross $e' = a'b', e'' = a''b''$ with $a', a'' \in A(a_4, a_2)$ and $b', b'' \in B[b_1^*, b_2^*] \cup B(b_2^*, b_2^*)$, such that $a_1, a', a'', a_2$ occur on $A$ in order. Then $b' \in B[b_1^*, b_5^*]$ to avoid the double cross $e_4, e_5^*, e', e''$, and so $b'' \in B[b_1^*, b_5^*]$. Moreover, $a_2^* \in A[a'', a_2]$ to avoid the double cross $e_4, e_5^*, e', e''$. But now, $(e_2^*, e_5^*, e', e'', e_4)$ is a 5-edge configuration.

Let $e' = a'b', e'' = a''b'' \in E(G)$ with $b' \in V(B[b_1^*, b_5^*])$, $b'' \in V(B[b_2^*, b_5^*])$, and $a', a'' \in V(A(a_4, a_2))$, such that $B[b', b'']$ is minimal. Then there exists $e_0 = b_5^* a_0 \in E(G)$ with $a_0 \in V(A(a_1, a')) \cup V(A(a'', a_2)) \{a_5\}$; for otherwise, by (6) and above claims, we can split $a_5$ to obtain a graph $H$ from $G$ such that $(H, a_1, b_2, a_0, b_1)$ is planar, contradicting

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(i) of Lemma 2.5.1. In fact, \( a_0 \in A[a_1, a'] \) to avoid the double cross \( e^*, e_0, e'', e_4 \).

We may assume that \( G \) has no edge from \( a_5 \) to \( B(b_4, b_2) - b^*_5 \) (and, hence, \( b_5 = b^*_5 \)). For, suppose \( e = a_5b \in E(G) \) with \( b \in V(B(b_4, b_2) - b^*_5) \). If \( b \in B(b^*_5, b^*_2) \) by (6) and \( a_0 \in A(a_5, a') \) to avoid the double cross \( e_0, e, e', e'' \); now \( (e^*_5, e_0, e', e_4) \) is a 5-edge configuration. We may thus assume \( b \in B(b_4, b^*_5) \). Then \( a_0 \in A[a_1, a_5] \) to avoid the double cross \( e, e_0, e_4, e' \). Let \( e_6 = a_5b_6 \in E(G) \) with \( b_6 \notin \{b_4, b', b^*_5\} \). Then \( b_6 \notin B(b^*_5, b_2) \) to avoid the double cross \( e_6, e_0, e_4, e' \). Moreover, \( b_6 \notin B(b', b^*_5) \); or else, \( (e^*_5, e_0, e_6, e', e_4) \) is a 5-edge configuration. By (6), \( b_6 \notin B(b_4, b') \). So \( b_6 \in B[b_1, b_4] \). But then \( (e^*_5, e_0, e, e_4, e_6) \) is a 5-edge configuration.

Hence, by above claims, we can obtain a new graph \( H \) from \( G \) by splitting \( b^*_5 \) such that \( (H, a_1, b_2, a_0, b_1) \) is planar, which contradicts (i) of Lemma 2.5.1. \( \square \)

We let \( u_1, u_2 \in B[b_1, b_2] \), such that \( b_1, u_1, u_2, b_2 \) occur on \( B \) in order, \( G \) has an edge \( f_i \) from \( a_2 \) to \( u_i \) for \( i \in \{2\} \), and subject to this, \( B[u_1, u_2] \) is maximal. By \( d_G(a_2) \geq 4, u_1 \neq u_2 \).

(9) If \( a_4 \neq a_2 \), then \( G \) has an edge from \( a_2 \) to \( B(b_5, b_2) \).

For, suppose \( a_4 \neq a_2 \) and \( G \) has no edge from \( a_2 \) to \( B(b_5, b_2) \). By the choice of \( e_4, u_1, u_2 \in B(b_4, b_5) \).

We may assume that \( G \) has no edge from \( B(u_1, u_2) \) to \( A[a_1, a_2] \). For, suppose there exists \( ab \in E(G) \) with \( b \in V(B(u_1, u_2)) \) and \( a \in V(A[a_1, a_2]) \). Then \( a \neq a_5 \) by (8), and \( a \in A(a_5, a_2) \) to avoid the double cross \( e, e_4, e_5, f_1 \). If \( b_5 \neq b_2 \) then \( (e_5, f_2, e, f_1, e_4) \) is a 5-edge configuration. So \( b_5 = b_2 \). Then \( u_2 \neq b_5 \) and \( (e_3, f_1, e, f_2, e_5) \) is a 5-edge configuration.

We may also assume that \( G \) has no cross from \( A[a_1, a_2] \) to \( B[b_1, u_1] \). For, suppose there exist \( e' = a'b', e'' = a''b'' \in E(G) \) with \( a', a'' \in A[a_1, a_2] \) and \( b', b'' \in B[b_1, u_1] \), such that \( e', e'' \) form a cross, and \( a_1, a', a'', a_2 \) occur on \( A \) in order. If \( b'' \in B[b_1, b_4] \) then by the choice of \( e_4, e_5 \), we have \( a'' \in A[a_1, a_5] \) and \( a' = a_1 \); now \( e', e'', e_4, e_5 \) form a double cross, a contradiction. So \( b'' \in B[b_4, u_1] \). Let \( f \) denote an edge from \( a_2 \) to \( B(u_1, u_2) \). Then \( a' \neq a_1 \) to avoid the double cross \( e', f, e_4, e_5 \). Now \( (e_3, e'', e', f, e_5) \) is a 5-edge configuration.
By (i) of Lemma 2.5.1, \((G, a_1, b_2, a_0, b_1)\) is not planar. So there exist \(e' = a'b', e'' = a''b'' \in E(G)\) with \(a', a'' \in V(A[a_1, a_2])\) and \(b', b'' \in V(B[u_2, b_2])\), such that \(e', e''\) are parallel, and \(a_1, a', a'', a_2\) occur on \(A\) in order. Now \(a' \in A[a_4, a_2]\) to avoid the double cross \(e', e'', e_4, f_1\), and \(b'' \in B[u_2, b_5]\) to avoid the double cross \(e_5, e'', e_4, f_1\). We may assume \(b_5 = b_2\); otherwise, \((e_5, e'', e_4, f_1, e_4)\) is a 5-edge configuration. So \(u_2 \neq b_5\). Now, let \(e = a''b \in E(G)\) with \(b \notin \{b', b'', b_5\}\). Then \(b \notin B[b_1, u_1]\) to avoid the double cross \(e, e'', f_2, e'\). We may assume \(b \notin B[u_2, b']\); otherwise, \((e_3, f_1, e, e', e'')\) is a 5-edge configuration. Since \(G\) has no edge from \(B(u_1, u_2)\) to \(A[a_1, a_2]\), \(b \in B(b', b_5)\). But now, \((e_3, f_1, e', e, e_5)\) is a 5-edge configuration. □

(10) \(G\) has no edge from \(B(b_5, b_2)\) to \(A(a_1, a_4)\).

For, suppose there exists \(e = ab \in E(G)\) with \(b \in V(B(b_5, b_2))\) and \(a \in V(A(a_1, a_4))\). We choose \(e\) so that \(B[b, b_2]\) is minimal. By (3), \(a \in A(a_5, a_4)\). By (5), \(G\) has no edge from \(B(b_4, b_5)\) to \(A(a_1, a_4) - a_5\). Moreover, since the degree of \(a\) in \(G\) is at least 6, then we let \(e_0 = ab_0\) with \(b_0 \in B[b_1, b_2]\) and \(b_0 \notin \{b_4, b_5, b\}\). Now, by (3) and (5), and by the definition of \(b\), we have \(b_0 \in B(b_5, b)\).

\(G\) has no edge from \(A(a_4, a_2)\) to \(B[b_1, b]\). For, suppose there exists \(e' = a'b' \in E(G)\) with \(a' \in A(a_4, a_2)\) and \(b' \in B[b_1, b]\). Then by (3), \(b' \notin B[b_1, b_4]\). So \(b' \in B(b_4, b)\). But then, \(e, e', e_4, e_5\) form a double cross.

\(G\) has no edge from \(b_4\) to \(A(a_5, a_4)\) or no edge from \(a_4\) to \(B(b_4, b)\); otherwise, such two edges together with \(e_5, e\) form a double cross, a contradiction.

Now, we see that \(G\) has an edge \(e'\) from \(a_1\) to \(b' \in B(b_4, b_2)\); otherwise, since \(G\) has no edge from \(b_4\) to \(A(a_5, a_4)\) or no edge from \(a_4\) to \(B(b_4, b)\), then combined with (3), (4), (6), and (7), we can obtain a new graph \(H\) from \(G\) by splitting \(a_4\) or \(b_4\) as \(s, s'\), such that \((H, a_1, a_2, b_2, a_0, b_1)\) is planar, a contradiction to (i) of Lemma 2.5.1.

We also see that \(G\) has no edge from \(a_1\) to \(B(b_5', b)\); otherwise, such an edge together with \(e_3, e_4, e_5', e\) forms a 5-edge configuration, a contradiction.
Hence, \( b' \in B(b_4, b'_5) \cup B(b, b_2) \). We further choose \( e' \) so that \( B[b', b_2] \) is maximal. Moreover, we let \( e'' = a_1b'' \in E(G) \) with \( b'' \in B(b_4, b'_5) \cup B(b, b_2) \) so that \( B[b'', b_2] \) is minimal.

Now, assume \( b'' \in B(b_4, b'_5) \). Then by the choice of \( e'' \), \( G \) has no edge from \( a_1 \) to \( B[b, b_2] \). Moreover, \( G \) has no edge from \( B[b_1, b_4] \) to \( A(a_1, a_2) \); otherwise, by (3), such an edge must end in \( A(a_1, a_5) \), which together with \( e', e_4, e_5 \) forms a double cross. Hence, \( G \) has an edge \( e_6 \) from \( a_4 \) to \( b_4 \in B(b_4, b_5) \); or else, we can obtain a new graph \( H \) from \( G \) by splitting \( b_4 \) as \( s, s' \), such that \((H, a_1, a_2, b_2, a_0, b_1)\) is planar, a contradiction to (i) of Lemma 2.5.1. Now, \( G \) has no edge from \( b_4 \) to \( A(a_1, a_4) \); or else, such an edge together with \( e_5, e', e_6 \) forms a double cross. So we may assume \( a_2 \neq a_4 \); otherwise, 
\[(G - a_1, a_2, b_2, a_0, b_1) \) and \((G - a_2, a_1, b_2, a_0, b_1) \) are planar, a contradiction to (ii) of Lemma 2.5.1. Then \( u_2 \in B[b, b_2] \) (by (7) and (9)). Moreover, \( b_4 \notin B(b', b_5) \); otherwise, 
\[(f_2, e, e_6, e', e_4) \) is a 5-edge configuration. So \( G \) has no edge from \( a_4 \) to \( B(b', b_5) \). Therefore, 
we can obtain a new graph \( H \) from \( G \) by splitting \( a_4 \) as \( s, s' \), such that \((H, a_1, a_2, b_2, a_0, b_1)\) is planar, a contradiction to (i) of Lemma 2.5.1.

So we may assume \( b'' \in B[b, b_2] \). Now, \( a_2 = a_4 \); otherwise, \( u_2 \in B[b, b_2] \) (by (7) and (9)) and 
\[(f_2, e'', e_0, e_5, e_4) \) is a 5-edge configuration.

We also claim that \( G \) has an edge \( e_6 \) from \( a_6 \in A(a_1, a_2) \) to \( b_6 \in B[b_1, b_4] \); otherwise, 
\[(G - a_1, a_2, b_2, a_0, b_1) \) and \((G - a_2, a_1, b_2, a_0, b_1) \) are planar, a contradiction to (ii) of 
Lemma 2.5.1.

Then \( b_6 \notin B[b_1, b_4] \); otherwise, \( a_6 \in A(a_1, a_5) \), and 
\[(e, e'', e_5, e_4, e_6) \) is a 5-edge configuration. Hence, \( b_6 = b_4 \), and \( G \) has no edge from \( a_5 \) to \( B[b_1, b_4] \), which further implies 
\( b'_5 \neq b_5 \) (as the degree of \( a_5 \) in \( G \) is at least 6).

Now, we may assume \( u_2 \notin B[b, b_2] \). For, suppose not. Then \( G \) has no edge from 
\{\(a_1, a_2\)\} to \(B(b_4, b_5)\); otherwise, such an edge together with \(f_2, e'', e_5, e_6\) forms a 5-edge configuration. Moreover, \(a_6 \notin A(a_5, a_2)\); otherwise, 
\[(f_2, e'', e_0, e_5, e_6) \) is a 5-edge configuration. But now, 
\((G - a_1, a_2, b_2, a_0, b_1) \) and 
\((G - a_2, a_1, b_2, a_0, b_1) \) are planar, a contradiction
to (ii) of Lemma 2.5.1.

Since \( u_2 \notin B[b, b_2] \), then \( G \) has no edge from \( a_2 \) to \( B[b, b_2] \). By (7), \( G \) has no edge from \( a_2 \) to \( B(b'_5, b) \). By (3), \( G \) has no edge from \( a_2 \) to \( B(b_1, b_4) \). Since the degree of \( a_2 \) in \( G \) is at least 4, then \( G \) has an edge \( e'_2 \) from \( a_2 \) to \( B(b_1, b'_5) \). Now, \( a_6 \notin A(a_5, a_2) \); otherwise, \( e_6, e_5, e, e'_2 \) form a double cross. Moreover, \( b' \notin B(b_4, b) \) to avoid the double cross \( e', e'_2, e_6, e \). Hence, combined with (6), we can obtain a new graph \( H \) from \( G \) by splitting \( a_2 \) as \( s, s' \), such that \( (H, a_1, b_2, a_0, b_1) \) is planar, a contradiction to (i) of Lemma 2.5.1. \( \square \)

Now, by (3), (8), (9), and (10), we have

(11) \( G \) has no edge from \( A(a_1, a_5) \cup A(a_4, a_2) \) to \( B(b_4, b_5) \) and no edge from \( B(b_1, b_4) \cup B(b_5, b_2) \) to \( A(a_5, a_4) \).

We may assume that

(12) \( G - \{a_5b_4, a_4b_5\} \) has no parallel edges from \( A[a_5, a_4] \) to \( B[b_4, b_5] \).

For, otherwise, let \( e' = a'b', e'' = a''b'' \in E(G) \) be parallel with \( a', a'' \in V(A[a_5, a_4]) \) and \( b', b'' \in V(B(b_4, b_5)) \), such that \( a_1, a', a'', a_2 \) occur on \( A \) in order, \( e' \neq a_5b_4 \), and \( e'' \neq a_4b_5 \).

We may further assume \( b' = b_4 \) for any choice of \( e', e'' \). For, suppose \( b' \neq b_4 \). If \( b'' \neq b_5 \) then \( (e_3, e_4, e', e'', e_5) \) is a 5-edge configuration. So assume \( b'' = b_5 \). Then \( a'' \neq a_4 \). Since \( d_G(a'') \geq 6 \), there exists \( e = a''b \in E(G) \) with \( b \in V(B[b_1, b_2]) \setminus \{b_4, b', b_5\} \). By (11), \( b' \in B(b_4, b_5) \) to avoid the double cross \( e', e'' \). If \( b \in B(b_4, b'_5) \) then \( (e_3, e_4, e, e', e'') \) is a 5-edge configuration. If \( b \in B(b', b_5) \) then \( (e_3, e_4, e', e, e_5) \) is a 5-edge configuration.

Thus, \( G - a_4b_5 \) has no parallel edges from \( B(b_4, b_5) \) to \( A[a_5, a_4] \). Now, since \( e'' \neq a_4b_5 \) and \( d_G(a'') \geq 6 \), then by (11), we may choose \( e'' \) so that \( b'' \in B(b_4, b_5) \). Since \( e' \neq a_5b_4 \), \( a' \in A(a_5, a_4) \). Moreover, since \( d_G(a') \geq 6 \), there exists \( e = a'b \in E(G) \) with \( b \in V(B[b_1, b_2]) \setminus \{b_4, b'_4, b_5\} \). By (11), \( b \in B(b_4, b_5) \). If \( b \in B(b_4, b''_5) \) then \( (e_3, e_4, e, e'', e_5) \) is a 5-edge configuration. So assume \( b \in B(b'', b_5) \).

We may assume that \( G \) has no edge from \( a_2 \) to \( B[b_5, b_2] \); otherwise, \( (f_2, e_5, e, e'', e') \) is a 5-edge configuration. Hence, \( a_4 = a_2 \) (by (9)). Moreover, \( G \) has no edge from \( a_1 \) to
For, suppose $B(b_4, b_5)$, to avoid forming a double cross with $e', e_5, e''$. Therefore, since $G - a_4 b_5$ has no parallel edges from $B(b_4, b_5)$ to $A[a_5, a_4]$, it follows from (3), (4), and (11) that there is no cross from $B[b_1, b_3]$ to $A$ and no parallel edges from $B(b_4, b_2)$ to $A$. Now $(G, a_1, b_2, a_0, b_1)$ is planar, contradicting (i) of Lemma 2.5.1. □

If $G$ has no edge from $a_1$ to $B(b_4, b_2)$ then by (3), (4), (11), and (12), we can split $a_5, a_4$ to $s, t, t'$, respectively, in $G$ to obtain a graph $H$ such that $(H, a_0, b_1, a_1, s, t, s', t', a_2, b_2)$ is planar, contradicting (iii) of Lemma 2.5.1. So let $e_0 = a_1 b_0$ with $b_0 \in V(B(b_4, b_2))$. Choose $e_0$ with $B[b_0, b_2]$ maximal, and let $e'_0 = a_1 b'_0 \in E(G)$ with $b'_0 \in B(b_4, b_2)$ so that $B[b'_0, b_2]$ is minimal.

(13) $a_4 = a_2$ implies $A(a_5, a_2) \neq \emptyset$.

For, suppose $a_4 = a_2$ and $A(a_5, a_2) = \emptyset$. Then there exists $e = ab \in E(G)$ with $b \in V(B[b_1, b_4])$ and $a \in V(A(a_1, a_5))$; or else, by (3), (4) and (6), $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, contradicting (ii) of Lemma 2.5.1.

Suppose there exists $e' = a_2 b' \in E(G)$ with $b' \in V(B(b_4, b_5))$. Then $G$ has no edge from $a_1$ to $B(b_4, b_5)$, as such an edge would form a double cross with $e, e', e_5$. So $b_0 \in B[b_5, b_2]$. Now $G$ has an edge $e^*$ from $a_2$ to $B(b'_5, b_2)$; otherwise, by (3), (4) and (6), $(G, a_1, b_2, a_0, b_1)$ is planar, contradicting (i) of Lemma 2.5.1. Hence, $(e^*, e_0, e'_5, e', e)$ is a 5-edge configuration.

So assume that $G$ has no edge from $a_2$ to $B(b_4, b_5)$. Then, since $d_G(a_2) \geq 4$, $u_2 \in B(b_5, b_2)$.

Assume $b_0 \in B(b_4, b_5)$. Then $b \notin B[b_1, b_4]$ to avoid the double cross $e_0, e, e_4, e_5$. Since $d_G(a_5) \geq 6$, then $b'_5 \neq b_5$, and there exists $e''_5 = a_5 b''_5 \in E(G)$ with $b''_5 \in V(B(b'_5, b_5))$. By (6), $b_0 \in B(b_4, b'_5)$. We may assume that $G$ has no edge from $a_1$ to $B[b_5, b_2]$; otherwise, such an edge together with $f_2, e''_5, e_0, e$ forms a 5-edge configuration. Hence, by (3), (4) and (6), we can obtain a new graph $H$ from $G$ by splitting $b_4$ such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, contradicting (i) of Lemma 2.5.1.
Therefore, $b_0 \notin B(b_4, b_5)$ for any choice of $b_0$. Then $G$ has an edge from $B[b_1, b_4]$ to $A(a_1, a_5)$; otherwise, $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, contradicting (ii) of Lemma 2.5.1. Hence, we may choose $e$ so that $b \in B[b_1, b_4]$. If $b_0' \in B(b_5, b_2)$ or $b_5' \neq b_5$ then $(f_2, e_0', e_5', e_4, e)$ is a 5-edge configuration. So assume $b_0 = b_0' = b_5$. Then we can obtain a new graph $H$ from $G$ by splitting $b_5$ such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, contradicting (i) of Lemma 2.5.1. □

(14) We may assume $a_4 \neq a_2$.

For, suppose $a_4 = a_2$. By (13), let $a_6 \in V(A(a_5, a_2))$. Since $d_G(a_6) \geq 6$, there exist distinct $e_6' = a_6b_6', e_6'' = a_6b_6'' \in E(G)$ with $b_6', b_6'' \in V(B) \setminus \{b_4, b_5\}$ such that $B[b_6', b_6'']$ is maximal. Without loss of generality, assume $b_1, b_6', b_6'', b_2$ occur on $B$ in order. By (11), $b_6', b_6'' \in B(b_4, b_5)$. Suppose there exists $e'' = b''a'' \in E(G)$ with $b'' \in V(B[b_1, b_4])$ and $a'' \in V(A(a_1, a_5))$.

Then $b_0 \notin B(b_4, b_0')$ to avoid the double cross $e_0, e_6'', e_5, e_6''$. We may assume $b_0 \notin B(b_6', b_5)$; otherwise, $(e_3, e_4, e_6', e_0, e_5)$ is a 5-edge configuration. Hence, $b_0 \in B[b_5, b_2]$ and $G$ has no edge from $a_1$ to $B(b_4, b_5)$. We also see that $G$ has no edge from $a_1$ to $B[b_5, b_2]$ or no edge from $a_2$ to $B(b_5, b_2)$; otherwise, such two edges form a 5-edge configuration with $e_5, e_6', e''$. By (3), (4), (11), and (12), we can obtain a graph $H$ from $G$ by splitting $a_2$ such that $(H, a_1, b_2, a_0, b_1)$ is planar, contradicting (i) of Lemma 2.5.1.

Thus, we may assume that $G$ has no edge from $B[b_1, b_4]$ to $A(a_1, a_5)$. Hence, by (11) and (12), $(G - a_1, a_2, b_2, a_0, b_1)$ is planar. Now, by (ii) of Lemma 2.5.1, $(G - a_2, a_1, b_2, a_0, b_1)$ is not planar; hence, there exist $e = a_1b, e' = a'b' \in E(G)$ with $b \in V(B(b_4, b_5)), b' \in V(B[b_1, b])$, and $a' \in V(A(a_1, a_2))$. We may assume $b \notin B(b_6', b_5)$, as otherwise $(e_3, e_4, e_6', e, e_5)$ is a 5-edge configuration. Moreover, $G$ has no edge from $a_2$ to $B(b_4, b_5)$, as such an edge would form a double cross with $e, e', e_5$. Since $d_G(a_2) \geq 4$, $u_2 \in B[b_5, b_2]$. But now, $(f_2, e_5, e_6'', e, e')$ is a 5-edge configuration. □

Now, by (9) and (14), $u_2 \in B(b_5, b_2)$. By (3), (11) and (14), $G$ has no edge from $a_2$ to $B[b_1, b_5]$, and so $u_1 \in B[b_5, b_2]$. 84
(15) \( b_0 \in B(b_4, b_5) \).

For, otherwise, \( b_0 \in B(b_5, b_2) \). Note that \( b'_0 \neq b_5 \); otherwise, \( b_0 = b'_0 = b_5 \), and by (3), (4), (11), (12), and (14), we can obtain a new graph \( H \) from \( G \) by splitting \( a_4 \) such that \((H, a_1, a_2, b_2, a_0, b_1)\) is planar, contradicting (i) of Lemma 2.5.1.

We may assume that \( G \) has no edge from \( B[b_1, b_4] \) to \( A(a_1, a_5) \), as such an edge forms a 5-edge configuration with \( f_2, e'_0, e_5, e_4 \). Hence, \( A(a_1, a_5) = \emptyset \) and, since \( d_G(a_5) \geq 6 \), \( b'_5 \neq b_5 \). We may thus assume that \( G \) has no edge from \( B[b_5, b'_0] \) to \( A[a_1, a_2] \), as such an edge forms a 5-edge configuration with \( f_2, e'_0, e'_5, e_4 \). We may also assume that if \( b_4 a_5 \in E(G) \) then \( G \) has no edge from \( B(b_4, b_5) \) to \( A(a_5, a_2) \), as such an edge forms a 5-edge configuration with \( f_2, e'_0, e_5, b_4 a_5 \).

Suppose \( u_1 \notin B[b_5, b'_0] \). Then by definition, \( G \) has no edge from \( B[b_5, b'_0] \) to \( A[a_1, a_2] \). Now, by (3), (4), (11), (12), and our previous statements, we can obtain a new graph \( H \) from \( G \) by splitting \( a_1, a_4 \) as \( s, s' \) and \( t, t' \), respectively, such that \((H, a_0, b_1, a_1 = s, t, s', t', a_2, b_2)\) is planar, contradicting (ii) of Lemma 2.5.1.

So \( u_1 \in B[b_5, b'_0] \) and, hence, \( G \) has no edge from \( B[b_5, b_2] \) to \( A[a_4, a_2] \). By (3), (4), (11), (12), and our previous statements, \((G - a_1, a_2, b_2, a_0, b_1)\) and \((G - a_2, a_1, b_2, a_0, b_1)\) are planar, contradicting (ii) of Lemma 2.5.1. \( \square \)

Suppose there exists \( a \in V(A(a_5, a_4)) \). Since \( d_G(a) \geq 6 \) and because of (11), there exists \( e = ab \in E(G) \) with \( b \in V(B[b_4, b_5]) \setminus \{b_4, b_5, b_0\} \). If \( b \in B(b_4, b_0) \) then \((e_3, e_4, e, e_0, e_5)\) is a 5-edge configuration; if \( b \in B(b_0, b_5) \) then \((f_2, e_5, e, e_0, e_4)\) is a 5-edge configuration.

So we may assume \( A(a_5, a_4) = \emptyset \). Then \( G \) has no edge from \( A(a_1, a_5) \) to \( B[b_1, b_4] \), as such an edge would form a double cross with \( e_0, e_4, e_5 \).

Then we may assume that \( G \) has no edge from \( B(b_0, b'_0) \) to \( A(a_1, a_2) \). For, suppose there exists \( e = ab \in E(G) \) with \( b \in V(B(b_0, b'_0)) \) and \( a \in V(A(a_1, a_2)) \). If \( b'_0 \in B(b_5, b_2) \), then \((f_2, e'_0, e_5, e_0, e_4)\) is a 5-edge configuration. So assume \( b'_0 \notin B(b_4, b_5) \). Then \( b \in B(b_4, b_5) \) and, by (11), \( a \in A[a_5, a_4] \). But then, \((f_2, e'_0, e, e_0, e_4)\) is a 5-edge configuration.
If $G$ has no edge from $a_4$ to $B(b_4, b_5)$ then, by (3), (4), (6), (11), and our previous statements, we can obtain a new graph $H$ from $G$ by splitting $b_4$ such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, contradicting (i) of Lemma 2.5.1. So let $e = a_4b \in E(G)$ with $b \in V(B(b_4, b_5))$. We may assume $b \notin B(b_0, b_5)$; otherwise $(f_2, e_5, e, e_0, e_4)$ is a 5-edge configuration. Moreover, $G$ has no edge from $b_4$ to $a_5$, to avoid forming a double cross with $e_5, e_0, e$. Now by (3), (4), (6), (11), and our previous statements, we can obtain a new graph $H$ from $G$ by splitting $a_4$ such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, contradicting (i) of Lemma 2.5.1. □

**Lemma 2.5.3** Suppose $(e_3, e_4, e_5, e_6, e_7)$ is a 5-edge configuration in an ideal $a_0$-frame $A, B$ in $\gamma$ with $b_1, b_3, b_4, b_5, b_6, b_7, b_2$ on $B$ in order. Let $G_0 := G - A$, where $(G_0, a_0, b_1, B, b_2)$ is planar. Then $G_0$ has a separation $(G_1, G_2)$ with $|V(G_1) \cap V(G_2)| \leq 3$, $\{a_0, b_1, b_2\} \subseteq V(G_1)$, and $B[b_1', b_2'] \subseteq G_2$, $|V(G_1 - G_2)| \geq 1$, such that one of the following holds for $b_1', b_2' \in V(G_1) \cap V(G_2)$:

(i) $|V(G_1) \cap V(G_2)| = 3$, $b_1' \in B[b_3, b_4]$, $b_2' \in B[b_7, b_2]$, and $G_0$ has a path from $a_0$ to $B(b_1', b_2')$ and internally disjoint from $B$.

(ii) $|V(G_1) \cap V(G_2)| = 2$, $b_1' \in B[b_3, b_4]$, and $b_2' \in B[b_7, b_2]$.

(iii) $|V(G_1) \cap V(G_2)| = 2$, $b_1' \in B[b_3, b_4]$, and $b_2' \in B[b_6, b_7]$.

(iv) $|V(G_1) \cap V(G_2)| = 2$, $b_1' \in B[b_4, b_5]$, and $b_2' \in B[b_7, b_2]$.

**Proof.** By planarity of $G_0$, it is easy to see that if the assertion fails then $G_0 - (B[b_3, b_4] \cup B[b_7, b_2])$ contains disjoint paths $B_1, A_0$ from $b_1, a_0$ to $b_5, b_6$, respectively. Now $(A - A[a_5, a_7]) \cup e_3 \cup B[b_3, b_4] \cup e_4 \cup e_6 \cup A_0$ and $B_1 \cup e_5 \cup A[a_5, a_7] \cup e_7 \cup B[b_7, b_2]$ show that $\gamma$ is feasible, a contradiction. (See Figure 2.13.) □

In the remainder of this section, we will assume the following: $\mathcal{P} := (e_3, e_4, e_5, e_6, e_7)$ is a 5-edge configuration in $A, B$, where $e_i = a_ib_i \in E(G)$ with $a_i \in V(A)$ and $b_i \in V(B)$ for $i = 3, 4, 5, 6, 7$, such that $a_1, a_3, a_4, a_2$ occur on $A$ in order, $b_1, b_3, b_4, b_5, b_6, b_7, b_2$ occur on $B$ in order, and the following are satisfied in order listed:
• $B[b_4, b_7]$ is maximal,

• $B[b_6, b_7]$ is minimal,

• $B[b_4, b_5]$ is minimal,

• $A[a_5, a_7]$ is minimal,

• $A[a_3, a_4]$ is maximal,

• $B[b_1, b_3]$ is minimal, and

• $A[a_6, a_5] \cap A[a_6, a_7]$ is maximal.

**Lemma 2.5.4** Suppose $a_7 \in A[a_1, a_5]$, $a_6 \in A(a_5, a_2)$, and $G$ has no edge from $B(b_4, b_5)$ to $A[a_1, a_5]$ or from $B[b_7, b_2]$ to $A(a_5, a_2)$. Then $G_0$ admits no separation $(G_1, G_2)$ such that
\[V(G_1 \cap G_2) = \{b_1^*, b_2^*\} \text{ with } b_1^* \in V(B[b_1, b_4]) \text{ and } b_2^* \in V(B[b_6, b_2]), \{a_0, b_1, b_2\} \subseteq V(G_1), B[b_1^*, b_2^*] \subseteq G_2, \text{ and } |V(G_1 - G_2)| \geq 1.\]

**Proof.** For, suppose such a separation does exist. Then we choose such \((G_1, G_2)\) so that \(B[b_1^*, b_2^*]\) is maximal. Note that \(G\) has no parallel edges from \(B[b_6, b_2]\) to \(A[a_1, a_5]\), as such edges and \(e_5, e_6\) would form a double cross.

Next, we show that all edges from \(A(a_5, a_2)\) to \(B\) must end in \(B[b_4, b_6]\). For, suppose there exists \(e = ab \in E(G)\) with \(a \in V(A(a_5, a_2))\) and \(b \in V(B) \setminus V(B[b_4, b_6])\). Then \(b \in B[b_1, b_4]\); for, otherwise, \(b \in B(b_6, b_7)\) (as \(G\) has no edge from \(B[b_7, b_2]\) to \(A(a_5, a_2)\)) and, hence, \((e_3, e_4, e_5, e, e_7)\) contradicts the choice of \(\mathcal{P}\). If \(a \in A(a_5, a_4)\) then \(b \in B[b_3, b_4]\) to avoid the double cross \(e, e_3, e_4, e_5\); thus \(b_3 \neq b_4\) and \((e_3, e, e_5, e_6, e_7)\) contradicts the choice of \(\mathcal{P}\). Hence, \(a \in A(a_4, a_2)\). Then \(b = b_1\) as, otherwise, \((e_3, e, e_5, e_6, e_7)\) contradicts the choice of \(\mathcal{P}\). Now \(a \neq a_2\) and there exists \(e' = a_2b' \in E(G)\) with \(b' \in V(B) - \{b_1, b_2\}\). Note that \(b' \notin B[b_7, b_2]\) as \(G\) has no edge from \(B[b_7, b_2]\) to \(A(a_5, a_2)\). But then \(e, e', e_3, e_7\) form a double cross, a contradiction.

Let \(e_8 = a_8b_8 \in E(G)\) with \(a_8 \in V(A(a_1, a_5))\) and \(b_8 \in V(B(b_1^*, b_2^*))\), so that \(A[a_1, a_8]\) is minimal. Since \(G^*\) is 6-connected, there exists \(e^* = a^*b^* \in E(G)\) with \(a^* \in A(a_8, a_2)\) and \(b^* \in B - B[b_1^*, b_2^*]\). Since all edges from \(A(a_5, a_2)\) to \(B\) end in \(B[b_4, b_6]\), \(a^* \in A(a_8, a_5)\) and, hence, \(a_8 \in A(a_1, a_5)\).

Moreover, \(b_8 \in B(b_1^*, b_4) \cup B[b_6, b_2^*]\). For otherwise, \(b_8 \in B(b_4, b_6)\). Since \(a_8 \in A(a_1, a_5)\) and \(G\) has no edge from \(B(b_4, b_5)\) to \(A(a_1, a_5)\) (by assumption), \(b_8 \in B(b_5, b_6)\). Then \(a_8 \in A(a_7, a_5)\) to avoid the double cross \(e_5, e_6, e_7, e_8\). Since \(a^* \in A(a_8, a_5)\), we have \(b^* \in B[b_1, b_4^*]\) to avoid the double cross \(e_8, e^*, e_5, e_6\), and \(b^* \notin B[b_1, b_3]\) to avoid the double cross \(e_3, e^*, e_6, e_7\). Hence, \(b_3, b^* \in B(b_1, b_4)\), and \((e_3, e^*, e_8, e_6, e_7)\) contradicts the choice of \(\mathcal{P}\).

**Case 1.** \(b_8 \in B[b_6, b_2^*]\). So \(b^* \in B[b_1, b_1^*]\) to avoid the double cross \(e_8, e^*, e_5, e_6\).

We claim that \(G\) has no edge from \(B[b_1^*, b_4]\) to \(A[a_1, a_5]\). For suppose \(e = ab \in E(G)\) with \(a \in A[a_1, a_3]\) and \(b \in B[b_1^*, b_4]\). Note that \(b_1^*\) and \(b_2^*\) are feet of some connector \(J\), and
Let \( u_1, u_2 \) denote the extreme hands for \( J \). Note that \( e^* \) is from \( A(x_1, x_2) \) to \( B[b_1^*, b_2^*] \); so we know \( (J - b_1^*, u_1, A(u_1, u_2), u_2, b_2^*) \) is planar by Lemma 2.2.4. But this cannot be the case because of \( e, e_4, e_5 \).

Let \((G'_1, G'_2)\) be a separation in \( G_0 \) such that \( V(G'_1 \cap G'_2) = \{b'_1, b'_2\} \) with \( b'_1, b'_1, b_4, b_6, b_2, b_2^* \) on \( B \) in order, \( B[b'_1, b'_2] \subseteq G'_1 \), and \( \{a_0, b'_1, b'_2\} \subseteq V(G'_2) \). (Possibly \( G'_i = G_i \) for \( i = 1, 2 \).) We choose \((G'_1, G'_2)\) such that \( B[b_0, b_2^*] \) is minimal and, subject to this, \( B[b_1^*, b'_1] \) is minimal.

Let \( e'_8 = a'_8 b'_8 \in E(G) \) with \( a'_8 \in A[a_1, a_5] \) and \( b'_8 \in B(b'_1, b'_2) \), and choose \( e'_8 \) so that \( A[a_1, a'_8] \) is minimal. Since \( G^* \) is 6-connected, there exists \( e' = a'b' \in E(G) \) with \( a' \in A(a'_8, a_2) \) and \( b' \in B - B[b'_1, b'_2] \). Then \( b'_8 \in B[b_0, b_2^*] \) (by the claim above) and \( b' \in B[b_1, b_1^*] - b'_1 \) to avoid the double cross \( e_5, e_6, e'_8, e' \). So \((e'_8, e_6, e_5, e_4, e')\) is a 5-edge configuration. By Lemmas 2.1.9 and 2.5.3, \( G_0 \) has a cut that contradicts the choice of \((G_1, G_2)\) or \((G'_1, G'_2)\).

Case 2. \( b_8 \in B(b_2^*, b_4) \). Then \( b^* \in B(b_2^*, b_2) \) to avoid the double cross \( e_8, e^*, e_4, e_5 \).

We claim that \( G \) has no edge from \( B[b_0, b_2^*] \) to \( A[a_1, a_5] \). For suppose \( e = ab \in E(G) \) with \( a \in V(A[a_1, a_5]) \) and \( b \in V(B[b_0, b_2^*]) \). Note that \( b_1^* \) and \( b_2^* \) are feet of some connector \( J \), and \( B[b_1^*, b_2^*] \subseteq J \). Let \( u_1, u_2 \) denote the extreme hands for \( J \). Note that \( e^* \) is from \( A(u_1, u_2) \) to \( B(b_2^*, b_2) \); so we know \((J - b_2^*, u_1, A(u_1, u_2), u_2, b_1^*)\) is planar by Lemma 2.2.4. But this cannot be the case because of \( e, e_5, e_6 \).

Let \((G'_1, G'_2)\) be a separation in \( G_0 \) such that \( V(G'_1 \cap G'_2) = \{b'_1, b'_2\} \) with \( b'_1, b'_1, b_4, b_6, b_2, b_2^* \) on \( B \) in order, \( B[b'_1, b'_2] \subseteq G'_1 \), and \( \{a_0, b'_1, b'_2\} \subseteq V(G'_2) \). We choose \((G'_1, G'_2)\) such that \( B[b'_1, b_1] \) is minimal and, subject to this, \( B[b'_2, b_2^*] \) is minimal.

Let \( e'_8 = a'_8 b'_8 \in E(G) \) with \( a'_8 \in A[a_1, a_5] \) and \( b'_8 \in B(b'_1, b'_2) \), and choose \( e'_8 \) so that \( A[a_1, a'_8] \) is minimal. Since \( G^* \) is 6-connected, there exists \( e' = a'b' \in E(G) \) with \( a' \in A(a'_8, a_2) \) and \( b' \in B - B[b'_1, b'_2] \). Then \( b'_8 \in B(b'_1, b_1) \) (by the above claim) and \( b' \in B[b'_2, b_2] - b'_2 \) to avoid the double cross \( e', e'_8, e_5 \). So \((e'_8, e_4, e_5, e_6, e')\) is a 5-edge configuration. By Lemmas 2.1.9 and 2.5.3, \( G_0 \) has a separation that contradicts choice of
Lemma 2.5.5 Suppose $G_0$ has a 2-cut $\{b'_1, b'_2\}$ with $b'_1 \in B[b_1, b_4]$ and $b'_2 \in B[b_6, b_7]$ separating $B[b'_1, b'_2]$ from $\{a_0, b_1, b_2\}$. Then $G_0$ has a separation $(G_1, G_2)$ with $|V(G_1 \cap G_2)| \leq 3$ and $b'_1, b'_2 \in V(G_1 \cap G_2) \cap V(B)$ such that $b'_1 \in B[b_1, b_4]$, $b'_2 \in B[b_6, b_2]$, \{a_0, a_1, a_2\} $\subseteq V(G_1)$, $B[b'_1, b'_2] \subseteq G_2$, and if $b'_2 \in B[b_6, b_7]$ then $|V(G_1 \cap G_2)| = 2$ and $G$ has no edge from $B(b'_2, b_7)$ to $A - a_7$.

Proof. We choose $\{b'_1, b'_2\}$ such that $\{b'_1, b'_2\}$ is a 2-cut (with $b'_1 \in B[b_1, b_4]$ and $b'_2 \in B[b_6, b_2]$) separating $B[b'_1, b'_2]$ from $\{a_0, b_1, b_2\}$, subject to this, $B[b'_1, b_4]$ is minimal and, subject to this, $B[b'_2, b_2]$ is minimal.

Clearly, we may assume $b'_2 \in B[b_6, b_7]$, and there exists $e_8 = a_8b_8 \in E(G)$ with $a_8 \in V(A - a_7)$ and $b_8 \in V(B(b'_2, b_7))$. We choose $e_8$ so that $A[a_8, a_5]$ is minimal. Note that $a_8 \in A[a_5, a_7]$, for otherwise, $(e_3, e_4, e_5, e_8, e_7)$ contradicts $P$.

Case 1. $a_5 \in A(a_7, a_2)$.

Then $G$ has no edge from $A(a_8, a_5)$ to $B[b_1, b_3]$ to avoid forming a double cross with $e_3, e_8, e_4$. Also $G$ as no edge from $A(a_5, a_2)$ to $B(b_1, b'_1)$; for suppose $e$ is such an edge then $(e_3, e, e_5, e_6, e_7)$ contradicts the choice of $P$.

1) $G$ has no edge from $A(a_8, a_2)$ to $B(b'_2, b_2) + b_1$.

For, suppose there exists $e = ab \in E(G)$ with $a \in A(a_8, a_2)$ and $b \in B(b'_2, b_2) + b_1$. If $b = b_1$ then $a \neq a_2$ and there exists $e_2 = a_2b' \in E(G)$ with $b' \in B(b_1, b_2)$; now $b' \in B[b_7, b_2]$ to avoid the double cross $e, e_3, e_7, e_2$ and, hence, $(e_2, e_7, e_5, e_3, e)$ contradicts the choice of $P$.

Thus, $b \in B(b'_2, b_2)$. In fact $b \in B[b_7, b_2]$, otherwise, $a \in A(a_5, a_2)$ (by the minimality of $A[a_8, a_5]$) and $(e_3, e_4, e_5, e, e_7)$ contradicts the choice of $P$. Now $a \in A(a_5, a_2)$, as otherwise $(e_3, e_4, e_5, e_6, e)$ contradicts the choice of $P$. Hence, $(e, e_7, e_8, e_6, e_5)$ is a 5-edge configuration. By Lemmas 2.1.9 and 2.5.3 and by the choice of $\{b'_1, b'_2\}$, $G_0$ has the desired separation. □
(2) \( G \) has no edge from \( A(a_7, a_2) \) to \( b_2 \).

For, let \( e = ab_2 \in E(G) \) with \( a \in A(a_7, a_2) \). Then \( a \neq a_2 \). Moreover, \( a \in A(a_5, a_2) \); as otherwise, \( (e_3, e_4, e_5, e_6, e) \) contradicts the choice of \( \mathcal{P} \).

Suppose \( a \in A(a_4, a_2) \). Then let \( e_2 = a_2b_2' \in E(G) \) with \( b_2' \in V(B) - \{b_1, b_2\} \). Now \( b_2' \in B(b_1, b_4) \) to avoid the double cross \( e_2, e, e_4, e_8 \). So \( (e_3, e_2, e_5, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \).

Thus \( a \in A(a_5, a_4) \). Now \( b_7 = b_2 \), or else \( (e_3, e_4, e_5, e_7, e) \) contradicts the choice of \( \mathcal{P} \). Moreover, \( a_8 = a_5 \), or else \( (e_3, e_4, e_5, e_8, e) \) contradicts the choice of \( \mathcal{P} \).

Suppose \( a_6 \in A(a_1, a_7) \). Let \( e_7' = a_7b_7' \in E(G) \) with \( b_7' \in V(B - b_7) \). Then \( b_7' \notin B(b_1, b_6) \) to avoid the double cross \( e_6, e_7', e_7, e_8 \). If \( b_7' = b_6 \) then \( (e_3, e_4, e_7', e_8) \) contradicts the choice of \( \mathcal{P} \). If \( b_7' \in B(b_6, b_2) \) then \( (e_3, e_4, e_5, e_7', e_7) \) contradicts the choice of \( \mathcal{P} \).

So \( a_6 \in A(a_5, a_2) \) for all choices of \( e_6 \). Then \( a_6 \in A(a_4, a_2) \), or else \( (e_3, e_4, e_6, e_8, e) \) contradicts the choice of \( \mathcal{P} \). Let \( e' = ab_6' \in E(G) \) with \( b_6' \in V(B - b_2) \). Then \( b_6' \neq b_6 \) as \( a_6 \in A(a_4, a_2) \) for all choices of \( e_6 \). So \( b_6' \in B(b_6, b_2) \) to avoid the double cross \( e_8, e_6, e, e' \). But then \( (e_3, e_4, e_5, e', e_7) \) contradicts the choice of \( \mathcal{P} \). □

(3) There exists \( e_9 = a_9b_9 \in E(G) \) with \( a_9 \in A(a_1, a_8) \) and \( b_9 \in B(b_1', b_2') \).

For, suppose such an edge does not exist. Then \( a_6 \in A(a_5, a_2) \) and \( G \) has no edge from \( B(b_4, b_5) \) to \( A(a_1, a_5) \) by the choice of \( \mathcal{P} \). Note that we have \( a_5 \neq a_7 \) and \( a_7 \in A(a_1, a_5) \) and that, by (1) and (2), \( G \) has no edge from \( B(b_7, b_2) \) to \( A(a_5, a_2) \). This contradicts Lemma 2.5.4. □

(4) \( b_9 \in B(b_4, b_2) \) and \( a_9 = a_3 \); so all edges from \( B(b_1', b_2') \) to \( A(a_1, a_8) \) must be from \( B(b_4, b_2') \) to \( a_3 \).

First, suppose \( b_9 \in B(b_1', b_4) \). Then \( (e_9, e_4, e_5, e_6, e_8) \) is 5-edge configuration. Thus, by Lemma 2.1.9 and 2.5.3 and by the choice of \( \{b_1', b_2'\} \), \( G_0 \) has the desired separation.
So we may assume \( b_0 \in B(b_4, b_2') \). Suppose \( a_9 \neq a_3 \). Then \( a_9 \in A(a_3, a_4) \), to avoid the double cross \( e_3, e_9, e_5, e_7 \). But now \( (e_3, e_4, e_9, e_8, e_7) \) is a 5-edge configuration contradicting the choice of \( \mathcal{P} \). \( \square \)

Suppose \( a_4 \neq a_2 \). Let \( e_2^* = a_2b_2^* \in E(G) \) with \( b_2^* \in V(B) \). Then \( b_2^* \in B(b_1, b_4) \) to avoid the double cross \( e_2^*, e_4, e_9, e_8 \). Now \( (e_3, e, e_5, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \).

Thus, \( G \) has no edge \( e \) from \( B[b_1, b_1'] \) to \( v \in V(A(a_8, a_2)) \); for, if \( v \neq a_2 \) then \( e, e_9, e_8, e_4 \) would form a double cross, and if \( v = a_2 \) then \( (e_3, e, e_5, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \).

Hence, by (1) and (4), \( G \) has a 5-separation \( (H_1, H_2) \) such that \( V(H_1 \cap H_2) = \{b_1', b_2', a_8, a_3, a_2\}, V(A[a_8, a_2]) \cup V(B[b_1', b_2']) \cup \{a_3\} \subseteq V(H_1) \), and \( V(A[a_3, a_8]) \cup \{a_0, a_1, a_2, b_1, b_2\} \)
\( \subseteq V(H_2) \), a contradiction as \( G^* \) is 6-connected.

**Case 2.** \( a_5 \in A[a_1, a_7] \).

Then \( a_6 \notin A(a_4, a_2) \) to avoid the double cross \( e_4, e_6, e_5, e_7 \), and \( a_6 \notin A(a_7, a_4) \) as, otherwise, \( (e_3, e_4, e_6, e_8, e_7) \) contradicts the choice of \( \mathcal{P} \). Hence, \( a_6 \in A[a_1, a_5] \) or \( a_6 = a_4 \).

(1) For some \( v \in \{a_4, b_1\} \), all edges from \( A(a_8, a_2) \) to \( B(b_1', b_2') \) are incident with \( v \).

To prove (1), we first claim that \( G \) has no edge from \( A(a_8, a_2) - a_4 \) to \( B(b_1', b_2') - b_4 \). For otherwise, suppose there exists \( e_9 = a_9b_9 \in E(G) \) with \( a_9 \in A(a_8, a_2) - a_4 \) to \( b_9 \in B(b_1', b_2') - b_4 \). If \( b_9 \in B(b_1', b_4) \) then \( a_9 \in A(a_4, a_2) \) to avoid the double crossing \( e_9, e_4, e_7, e_8 \); so \( (e_3, e_9, e_5, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \). Hence, \( b_9 \in B(b_4, b_2') \). Then \( a_9 \in A(a_8, a_4) \) to avoid the double cross \( e_4, e_9, e_8, e_7 \). Now \( (e_3, e_4, e_9, e_8, e_7) \) contradicts the choice of \( \mathcal{P} \).

Next, observe that, by the choice of \( \mathcal{P} \), any edge from \( b_1 \) to \( A(a_8, a_2) - a_4 \) must end in \( A(a_8, a_4) \), and any edge from \( a_4 \) to \( B(b_1', b_2') - b_4 \) must end in \( B(b_4, b_2') \). Thus, \( G \) has no edge from \( b_4 \) to \( A(a_8, a_2) - a_4 \) or no edge from \( a_4 \) to \( B(b_1', b_2') - b_4 \); as such two edges and \( e_7, e_8 \) would form a double cross, a contradiction. \( \square \)

Define \( a_1' \in V(A[a_1, a_8]) \) such that \( G \) has no edge from \( A[a_1, a_1'] \) to \( B(b_1', b_2') \) and, subject to this, \( A[a_1, a_1'] \) is maximal. By the definition of \( a_1' \), there exits \( e_1 = a_1'b \in E(G) \) with \( b \in B(b_1', b_2') \).

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We claim that $a'_1 \in A[a_3, a_8]$. For, suppose $a'_1 \in A[a_1, a_3]$. Then $b \in B(b_1, b_3)$ to avoid the double cross $e_1, e_3, e_4, e_8$. Now $(e_1, e_4, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

(2) $G$ has no edge from $A(a'_1, a_8)$ to $B - B[b'_1, b'_2]$.

For, otherwise, $a'_1 \neq a_8$, and there exists $e_9 = a_9b_9 \in E(G)$ with $a_9 \in V(A(a'_1, a_8))$ to $b_9 \in V(B) \setminus V(B[b'_1, b'_2])$. Then $b_9 \notin B[b_1, b'_1]$ to avoid the double cross $e_1, e_9, e_4, e_7$.

We claim $b_9 = b_2$ and $a_9 \notin A[a_5, a_8]$. For, if $b_9 \in B(b'_2, b_7)$ then $a_9 \in A(a'_1, a_5)$ by the choice of $e_8$ (that $A[a_5, a_8]$ is minimal); now $(e_3, e_4, e_5, e_9, e_7)$ contradicts the choice of $\mathcal{P}$. Hence, $b_9 \in B[b_7, b_2]$. Thus, $a_9 \notin A[a_5, a_8]$; as otherwise $(e_3, e_4, e_5, e_9, e_9)$ contradicts the choice of $\mathcal{P}$. Now suppose $b_9 \neq b_2$. Then $(e_7, e_9, e_8, e_6, e_5)$ is a 5-edge configuration. Thus, by Lemmas 2.1.9 and 2.5.3 and by the choice of $\{b'_1, b'_2\}$, $G_0$ has the desired separation.

Now $a_8 = a_5$; otherwise, $(e_3, e_4, e_5, e_8, e_9)$ contradicts the choice of $\mathcal{P}$. Moreover, $a_4 = a_2$; for otherwise, $G$ has an edge $e'$ from $a_2$ to $B$, then either $(e_3, e', e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$ or $e', e_4, e_5, e_7$ form a double cross.

Next, we claim that all edges from $A(a_8, a_2)$ to $B$ must end in $\{b_1, b_2\}$. Note that $G$ has no edge from $A(a_8, a_2)$ to $b_1$, to avoid forming a double cross with $e_7, e_3, e_4$. $G$ has no edge from $A(a_8, a_2)$ to $B(b_1, b_4)$; otherwise, such an edge together with $e_3, e_5, e_1, e_9$ forms a 5-edge configuration contradicting the choice of $\mathcal{P}$. $G$ has no edge from $A(a_8, a_2)$ to $B(b_4, b_8)$; otherwise, such an edge together with $e_3, e_4, e_8, e_7$ forms a 5-edge configuration contradicting the choice of $\mathcal{P}$. $G$ has no edge from $A(a_8, a_2)$ to $B(b_8, b_2)$; otherwise, such an edge together with $e_3, e_4, e_5, e_9$ forms a 5-edge configuration contradicting the choice of $\mathcal{P}$.

Therefore, since $a_7 \in A(a_8, a_2)$, $\{a_2, a_8, b_2, b_4\}$ is a 4-cut in $G$ separating $a_7$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction as $G^*$ is 6-connected. $\square$

By (1) and (2), $G$ has a separation $(H_1, H_2)$ such that $V(H_1 \cap H_2) = \{b'_1, b'_2, a_8, a'_1, v\}$, $b_5 \in V(H_2 - H_1)$, and $\{a_0, a_1, a_2, b_1, b_2\} \subseteq H_1$, a contradiction as $G^*$ is 6-connected. $\square$
Lemma 2.5.6 Suppose $G_0$ has a 2-cut $\{b'_1, b'_2\}$ separating $B[b'_1, b'_2]$ from $\{a_0, a_1, a_2\}$ with $b'_1 \in B(b_4, b_5)$ and $b'_2 \in B(b_7, b_2)$. Then $G_0$ has a separation $(G_1, G_2)$ with $|V(G_1 \cap G_2)| \leq 3$ and $b'_1, b'_2 \in V(G_1 \cap G_2) \cap V(B)$ such that $b'_1 \in B[b_1, b_6]$, $b'_2 \in B[b_7, b_2]$, $\{a_0, a_1, a_2\} \subseteq V(G_1)$, $B[b'_1, b'_2] \subseteq G_2$, and if $b'_1 \in B(b_4, b_5)$ then $|V(G_1 \cap G_2)| = 2$ and $G$ has no edge from $B(b_4, b'_1)$ to $A - a_4$.

Proof. We choose $\{b'_1, b'_2\}$ such that $\{b'_1, b'_2\}$ is a 2-cut (with $b'_1 \in B[b_1, b_6]$ and $b'_2 \in B[b_7, b_2]$) separating $B[b'_1, b'_2]$ from $\{a_0, b_1, b_2\}$, and, subject to this, $B[b'_1, b'_2]$ is maximal. Clearly, we may assume $b'_1 \in B(b_4, b_5)$, and there exists $e_8 = a_8 b_8 \in E(G)$ with $a_8 \in V(A - a_4)$ and $b_8 \in V(B(b_4, b'_1))$.

We claim that $a_8 \in A[a_1, a_3] \cup A(a_4, a_2)$. For, suppose $a_8 \in A(a_3, a_4)$. Then $a_6 \in A[a_7, a_8]$ and $a_8 \notin A[a_7, a_5]$; for otherwise $(e_3, e_4, e_8, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. Therefore, $a_5 \notin A[a_6, a_8]$ (since $a_6 \notin A[a_5, a_7]$). So $(e_3, e_4, e_8, e_5, e_6)$ is a 5-edge configuration. Thus, by Lemmas 2.1.9 and 2.5.3 and by the choice of $\{b'_1, b'_2\}$, $G_0$ has the desired separation.

Case 1. $a_8 \in A(a_4, a_2)$.

Choose $e_8$ so that $A[a_8, a_2]$ is minimal. Note that $a_6 \in A[a_8, a_2]$ and $a_7 \in A[a_3, a_5]$, since, otherwise, $e_4, e_8$ and two of $\{e_5, e_6, e_7\}$ force a double cross.

(1) $G$ has no edge from $A(a_5, a_2)$ to $B[b_1, b_4] \cup B(b_6, b_2]$.

For, let $e = ab \in E(G)$ with $a \in A(a_5, a_2)$ and $b \in B(b_1, b_4) \cup B(b_6, b_2)$.

Suppose $b \in B(b_6, b_2]$. Then $a \in A[a_8, a_2]$ to avoid the double cross $e, e_4, e_5, e_8$. So $b \in B[b_7, b_2]$, or else $(e_3, e_4, e_5, e, e_7)$ contradicts the choice of $\mathcal{P}$. If $b = b_2$ then $a \neq a_2$ and there exists $e' = a_2 b' \in E(G)$ with $b' \in V(B(b_1, b_2))$; $e_4, e_5, e, e'$ form a double cross (when $b' \in B(b_4, b_2)$) or $(e_3, e', e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$ (when $b' \in B(b_1, b_4)$).

Thus, $b \neq b_2$. Now $(e, e_7, e_5, e_8, e_4)$ is a 5-edge configuration. Hence, by Lemmas 2.1.9 and 2.5.3 and by the choice of $\{b'_1, b'_2\}$, $G_0$ has the desired separation.
Thus, \( b \in B[b_1, b_4] \) for every choice of \( e = ab \). If \( a \in A(a_5, a_4) \) then either \( e_3, e_4, e_5, e \) form a double cross, or \( (e_3, e_5, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \). So \( a \in A[a_4, a_2] \). Then \( b = b_1 \), or else, \( (e, e_3, e_5, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \). Now, since \( G \) has no edge from \( B(b_6, b_2) \) to \( A(a_5, a_2) \), \( G \) has an edge from \( a_2 \) to \( B(b_1, b_7) \), which forms a double cross with \( e, e_3, e_7 \). \( \square \)

(2) \( G \) has no edge from \( B(b_1, b_3) \) to \( A \).

For otherwise, let \( e = ab \in E(G) \) with \( a \in A \) and \( b \in B(b_1, b_3) \). If \( a \in A[a_1, a_3] \), then \( (e, e_4, e_5, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \); if \( a \in A(a_3, a_4) \), then \( e, e_3, e_4, e_7 \) form a double cross; if \( a \in A[a_4, a_2] \), then \( (e_3, e, e_5, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \). \( \square \)

(3) \( b'_2 = b_2 \) and \( G_0 \) has a separation \((G'_1, G'_2)\) that \( V(G'_1 \cap G'_2) = \{b_1, b'_2, a_0\}, b''_2 \in B(b'_1, b'_2), B[b_1, b''_2] \subseteq G'_1, \) and \( \{a_0, b_1, b_2\} \subseteq V(G'_2) \).

First, suppose \( b_5 \in B(b'_1, b'_2) \) and there exist \( e'_5 = a_5 b'_5, e''_5 = a'_5 b_5 \in E(G) \) with \( a'_5 \in A[a_1, a_8] \) and \( b'_5 \in B(b'_1, b'_2) \) such that \( a'_5 \neq a_5 \) and \( b'_5 \neq b_5 \). Then \( e'_5, e''_5 \) form a cross to avoid the double cross \( e'_5, e''_5, e_4, e_8 \). Hence, \( b'_5 \in B(b_5, b'_2) \) by the choice of \( \mathcal{P} \), and so \( (e_6, e'_5, e''_5, e_8, e_4) \) is a 5-edge configuration. By Lemmas 2.1.9 and 2.5.3 and by the choice of \( \{b'_1, b'_2\} \), we see that (3) or the assertion of the lemma holds.

So the above case will not happen. Then we claim that there exists \( v \in \{a_5, b_5\} \) such that all edges from \( B(b'_1, b'_2) \) to \( A[a_1, a_8] \) in \( G \) contain \( v \). For, otherwise, there exists \( e = ab \in E(G) \) such that \( a \in V(A[a_1, a_8] - a_5) \) and \( b \in V(B(b'_1, b'_2) - b_5) \). Suppose \( b \in B(b'_1, b'_5) \). Then \( a \in A(a_5, a_8) \) to avoid the double cross \( e, e_5, e_4, e_8 \), and, hence, \( (e_6, e_5, e, e_8, e_4) \) is a 5-edge configuration. Now by Lemmas 2.1.9 and 2.5.3 and by the choice of \( \{b'_1, b'_2\} \), (3) or the assertion of the lemma holds. So assume \( b \in B(b_5, b'_2) \). Then \( a \notin A(a_5, a_8) \) to avoid the double cross \( e_4, e_5, e_8, e \). Hence, \( (e, e_6, e_5, e_8, e_4) \) is a 5-edge configuration. Again by Lemmas 2.1.9 and 2.5.3 and by the choice of \( \{b'_1, b'_2\} \), (3) or the assertion of the lemma holds.
Now, since \( \{v, a_8, a_2, b'_1, b'_2\} \) is not a cut in \( G^* \), there exists \( e = ab \in E(G) \) with \( a \in V(\alpha(a_8, a_2)) \) and \( b \in V(B - B[b'_1, b'_2]) \). By (1), \( b \in B[b_4, b'_1] \). Now \( b = b_4 \) by the choice of \( e_8 \). Hence, \( (e_3, e, e_5, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \). \( \square \)

By (3), \( \alpha(A, B) \leq 1 \). Choose \( b''_2 \) so that \( B[b''_2, b_7] \) is minimal. We may assume

\[
(4) \quad b''_2 \notin B[b_7, b_2], \text{ and either } b_7 = b_2 \text{ (in which case let } B_0 = B[b''_2, b_2]) \text{ or } b_7 \neq b_2 \text{ and } G_0 - (B[b_1, b''_2] \cup B[b_7, b_2]) \text{ has a path } B_0 \text{ from } b''_2 \text{ to } b_2.
\]

Clearly, \( b''_2 \notin B[b_7, b_2] \) as otherwise the conclusion of the lemma holds. Now suppose \( b_7 \neq b_2 \) and the desired path \( B_0 \) in \( G_0 = (B[b_1, b''_2] \cup B[b_7, b_2]) \) does not exist. Then there exist \( b''_2 \in V(B[b_7, b_2]) \) and a separation \((H_1, H_2)\) in \( G_0 \) such that \( V(H_1 \cap H_2) = \{b_1, b''_2, a_0\} \); so the conclusion of this lemma holds. \( \square \)

\[(5) \quad G \text{ has two nonadjacent edges from } B(b'_1, b_2) \text{ to } A[a_1, a_5].\]

For otherwise, \( b'_1 = b_5 \), and there exists \( v \in \{a_7, b_7\} \) such that all edges in \( G \) from \( B(b'_1, b_2) \) to \( A[a_1, a_5] \) are incident with \( v \). Then \( G \) has no edge from \( B(b'_1, b_0) \) to \( A(a_5, a_8) \), to avoid forming a double cross with \( e_4, e_5, e_8 \). Since \( \{v, b'_1, b_2, a_8, a_2\} \) is not a cut in \( G^* \), it follows form (1) that there exists \( e = ab \in E(G) \) with \( b \in V(B[b_4, b'_1]) \) and \( a \in V(A(a_8, a_2)) \). By the choice of \( e_8 \), \( b = b_4 \). But then, \( (e_3, e, e_5, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \). \( \square \)

Note that no two edges of \( G \) from \( B(b'_1, b_2) \) to \( A[a_1, a_4] \) can be parallel, as such edges would form a double cross with \( e_4, e_8 \). Therefore, by (5), \( G \) has two nonadjacent edges \( e'_9 = a'_9b'_9, e''_9 = a''_9b''_9 \) with \( a'_9, a''_9 \in A[a_1, a_5] \) and \( b'_9, b''_9 \in B(b'_1, b_2) \) such that \( b_1, b'_9, b''_9 \) occur on \( B \) in order, and \( a_1, a'_9, a''_9 \) occur on \( A \) in order. We further choose \( e'_9, e''_9 \) so that \( A[a'_9, a_2] \cup B[b''_9, b_2] \) is minimal. Because of \( e_7 \), we have \( a'_9 \in A[a_7, a_2] \) and \( b''_9 \in B[b_7, b_2] \).

\[(6) \quad G \text{ has two parallel edges } e' = a'b', e'' = a''b'' \text{ with } b', b'' \in V(B(b_3, b'_1)), a', a'' \in V(A[a_4, a_2]), \text{ and } b_1, b', b'', b_2 \text{ on } B \text{ in order.}\]

We may assume \( b_3 = b_4 \); as otherwise \( e_4, e_8 \) give the desired edges for (6). Let \( e = a_1b \in E(G) \) with \( b \notin \{b_1, b_2, b_3, b_7\} \). Then \( b \notin B(b_1, b_3) \); otherwise, \( (e, e_4, e_5, e_6, e_7) \)
contradicts the choice of $\mathcal{P}$. Moreover, $b \notin B(b_3, b_7)$ to avoid the double cross $e, e_4, e_7, e_8$. So $b \in B(b_7, b_2)$.

Now, since $(e, e_6, e_5, e_8, e_4)$ is a 5-edge configuration, $b''_2 \in B[b_6, b_7]$; or else, the desired separation of $G_0$ follows from Lemmas 2.1.9 and 2.5.3, the choice of $\{b'_1, b'_2\}$, and the choice of $b''_2$.

Now, let $a^* \in A[a_1, a_2]$, such that $G$ has an edge $e^*$ from $b^* \in B(b''_2, b_7) \cup B(b_7, b_2)$ to $a^*$, subject to this, $A[a^*, a_2]$ is minimal, and subject to this, $B[b''_2, b^*]$ is minimal.

We claim that $a^* \notin A(a_5, a_2)$. For otherwise, suppose $a^* \in A(a_5, a_2)$. Now, if $b^* \in B(b''_2, b_7)$, then $(e_3, e_4, e_5, e^*, e_7)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. So $b^* \in B(b_7, b_2)$. If $a^* \in A(a_5, a_8)$, then $e_4, e_5, e_8, e^*$ form a double cross; if $a^* \in A[a_8, a_2]$, then $(e, e^*, e_5, e_8, e_4)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$.

We further claim that $G$ has no edge from $A(a_1, a^*)$ to $B(b_1, b_3) \cup B(b_3, b''_2)$. (Recall that $b_3 = b_4$.) For otherwise, let $e' = a'b' \in E(G)$ with $a' \in A(a_1, a^*)$ and $b' \in B[b_1, b_3] \cup B(b_3, b''_2)$. Then $b' \notin B(b_3, b''_2)$ to avoid the double cross $e_4, e_8, e', e^*$. So $b' \in B[b_1, b_3]$. But then $a' \notin A(a_5, a^*)$ to avoid the double cross $e_3, e_4, e', e_7$. So $a' \in A[a_1, a_3]$, and $(e', e_4, e_5, e_6, e_7)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$.

We may assume $G$ has an edge $e'_7$ from $b_7$ to $a'_7 \in A(a^*, a_2)$ and an edge $e'_3$ from $b_3$ to $a'_3 \in A(a_1, a^*)$. For otherwise, $G$ has a separation $(H_1, H_2)$ of order 5, such that $V(H_1 \cap H_2) = \{a_1, a^*, v, b''_2, b_2\}, v \in \{b_3, b_7\}, \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(H_1)$, and $V(A[a_1, a^*] \cup B[b''_2, b_2]) \subseteq V(H_2)$, a contradiction.

Then $G$ has a separation $(H_1, H_2)$ of order 6, such that $V(H_1 \cap H_2) = \{a_1, a^*, b_3, b'_2, b_7, b_2\}, \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(H_1)$, and $V(A[a_1, a^*] \cup B[b''_2, b_2]) \subseteq V(H_2)$. Any two edges from $A[a_1, a^*]$ to $B[b''_2, b_2]$ are not parallel; or else, such two edges together with $e_4, e_8$ form a double cross. Moreover, by the choice of $\mathcal{P}$, we can further assume $a'_7 \in A(a_5, a_2)$.

Now, assume $b^* \notin B(b''_2, b_7)$. Then since any two edges from $A[a_1, a^*]$ to $B[b''_2, b_2]$ are not parallel, then, combined with the choice of $e^*$, we have $(H_2, a_1, b_3, a^*, b_7, b''_2, b_2)$ is planar, a contradiction to Lemma 2.1.3.

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So \( b^* \in B(b''_2, b_7) \). But then \((e'_7, e, e^*, e_6, e'_3)\) is a 5-edge configuration. Now, by Lemmas 2.1.9 and 2.5.3 and by the choice of \( b''_2, G_0 \) has the desired separation. \( \Box \)

We choose \( e', e'' \) in (6) such that \( B[b_3, b'] \) is minimal and, subject to this, \( B[b''_2, b'_1] \) is minimal.

Suppose \( G_0 - B(b_1, b_3) - B(b'_1, b_2) \) has disjoint paths \( P_1, P_2 \) from \( b_1, a_0 \) to \( b', b'' \), respectively. Let \( A' := P_2 \cup e'' \cup A[a'', a_2] \) and \( B' := P_1 \cup e' \cup A[a'_9, a'_7] \cup e'_9 \cup B[b'_9, b''_2] \cup B_0 \). Now, since \( A, B \) is a good frame, the existence of \( A', B', A[a_1, a'_9] \cup e'_9 \cup B[b'_9, b''_2] \), and \( A[a_1, a_3] \cup e_3 \cup B[b_1, b_3] \) shows \( \alpha(A, B) = 2 \), a contradiction.

Thus, such \( P_1, P_2 \) do not exist. Then \( G_0 \) has a separation \( (H_1, H_2) \) with \( V(H_1 \cap H_2) = \{b^*_1, b^*_2\} \) such that \( b^*_1 \in B(b_1, b_3), B[b^*_1, b'] \subseteq H_1 \), and \( \{a_0, b_1, b_2\} \subseteq H_2 \). We may assume \( b^*_2 \in B[b'', b'_1] \) as otherwise \( G_0 \) has the desired separation.

Since \( G^* \) is 6-connected, \( \{b_1, b^*_1, b_2, b'_1, a_0\} \) is not a cut in \( G \); so there exists \( e_0 = a_0 b_0 \in E(G) \) with \( b_0 \in V(B(b^*_2, b'_1)) \) and \( a_0 \in V(A) \). By the choice of \( e', e'', a_0 \in A[a_4, a''_4] \). So \((e_3, e'', e_0, e_6, e_7)\) is a 5-edge configuration. Now, by Lemma 2.1.9 and 2.5.3, and by the choice of \( \{b'_1, b_2\} \) and the existence of \( \{b^*_1, b^*_2\} \), \( G_0 \) has the desired separation.

**Case 2.** \( a_8 \in A[a_1, a_3] \).

Note that if \( b_3 = b_4 \) we have symmetry between \( e_3 \) and \( e_4 \); so by Case 1, we may assume that if \( b_3 = b_4 \) then there exists \( e_9 = a_3 b_9 \in E(G) \) with \( b_9 \in B(b_4, b'_1) \). Next, \( G \) has no edge from \( B(b_3, b_7) \) to \( A[a_1, a_3] \), to avoid the double cross \( e_3, e_9, e', e_7 \) (when \( b_3 = b_4 \) or \( e_3, e_4, e', e_7 \) (when \( b_3 \neq b_4 \)). So \( a_8 = a_3 \), and all edges from \( B(b_4, b'_1) \) to \( A \) must end in \( \{a_3, a_4\} \). Moreover, \( G \) has no edge from \( B(b_4, b_7) \) to \( A[a_4, a_2] \) to avoid forming a double cross with \( e_4, e_7, e_8 \). So \( a_6 \notin A(a_4, a_2) \).

(1) For some \( v \in \{a_4, b_4\} \), all edges from \( B[b_1, b'_1] \) to \( A[a_3, a_2] \) are incident to \( v \).

Now, we claim that \( G \) has no edge from \( B[b_1, b_4] \) to \( A[a_3, a_2] \). For, let \( e = ab \in E(G) \) with \( b \in B[b_1, b_4] \) and \( a \in A[a_3, a_2] \). Then \( a \in A[a_4, a_2] \), to avoid the double cross \( e, e_4, e_5, e_8 \). So \( b = b_1 \) by the choice of \( \mathcal{P} \). Then \( a \neq a_2 \); so \( G \) has an edge \( e_2 = a_2 b' \) with
\( b' \in B(b_1, b_2) \). Then \( b' \in B[b_7, b_2] \) to avoid the double cross \( e_2, e_7, e_8, e' \). If \( b_3 \neq b_4 \) then \((e_2, e_7, e_4, e_3, e)\) contradicts the choice of \( \mathcal{P} \). So \( b_3 = b_4 \). Then \( e_9 \) is defined by (2.2.1).

Hence, \((e_2, e_7, e_9, e_3, e)\) contradicts the choice of \( \mathcal{P} \).

Thus, suppose (1) fails, since all edges from \( B(b_4, b'_1) \) to \( A \) must end in \( \{a_3, a_4\} \), then there exist \( e' = a_4b' \), \( e'' = a''b \) with \( a'' \in A(a_3, a_2) - a_4 \) and \( b' \in B(b_4, b'_1) \). By the choice of \( \mathcal{P} \), \( a'' \in A(a_3, a_4) \). So \( e_8, e', e'', e_7 \) form a double cross, a contradiction. \( \square \)

(2) \( a_1 = a_3 \).

For, suppose \( a_1 \neq a_3 \). Then there exists \( e_1 = a_1b \in E(G) \) with \( b \in V(B(b_1, b_2)) \). Note that \( b \notin B(b_3, b_7) \) by observation above (1), and \( b \notin B(b_1, b_4) \) as otherwise \((e_1, e_4, e_5, e_6, e_7)\) contradicts the choice of \( \mathcal{P} \). So \( b \in B[b_7, b_2] \). Moreover, \( b_3 = b_4 \), for, otherwise, \((e_7, e_1, e_8, e_4, e_3)\) contradicts the choice of \( \mathcal{P} \). Thus the edge \( e_9 \) is defined, and hence \( v = a_4 \).

Now \( G \) has no edge from \( B(b'_1, b_7) \) to \( A(a_1, a_7) \). For such an edge and \( e_1, e_7, e_9, e_3 \) form a 5-edge configuration. Hence, by Lemmas 2.1.9 and 2.5.3 and by the choice of \( \{b'_1, b'_2\} \), \( G_0 \) has the desired separation.

Thus, \( a_6 \in A(a_7, a_4) \) by (1). So \((e_6, e_1, e_5, e_9, e_3)\) is a 5-edge configuration. If \( b'_2 \neq b_2 \) then by Lemmas 2.1.9 and 2.5.3 and by the choice of \( \{b'_1, b'_2\} \), \( G_0 \) has the desired separation.

So \( b'_2 = b_2 \). Since \( G^* \) is 6-connected, \( \{b_1, b_2, b'_1, a_3, a_4\} \) is not a cut in \( G \). Hence, there exists \( e^* = a^*b^* \in E(G) \) with \( a^* \in V(A[a_1, a_2]) - \{a_3, a_4\} \) and \( b^* \in V(B(b_1, b'_1)) \). By (1) and by the existence of \( e_9 \), \( a^* \in A[a_1, a_3] \). Then \( b^* \notin B(b_1, b_3) \); otherwise, \((e^*, e_4, e_5, e_6, e_7)\) contradicts the choice of \( \mathcal{P} \). But then, \( b^* \in B(b_3, b'_1) \), and \( e^*, e_3, e_6, e_7 \) form a double cross. \( \square \)

Let \( e_2 = a_2b' \in E(G) \) with \( a_2' \in V(A) \) and \( b' \in V(B(b'_1, b'_2)) \), such that \( A[a_2, a_2'] \) is minimal. Since \( G^* \) is 6-connected, \( \{b'_1, b'_2, a_1, a_2'\} \) is not a cut in \( G \); so there exists \( e_0 = a_0b_0 \in E(G) \) with \( a_0 \in V(A(a_1, a_2')) \) and \( b_0 \in V(B - B[b'_1, b'_2]) \).

We claim that \( b_0 \in B[b_1, b'_1] \) for every choice of \( e_0 \). For, otherwise, \( b_0 \in B(b'_2, b_2) \).
Then \( a_0 \in A(a_1, a_4) \) to avoid the double cross \( e_4, e_8, e_2, e_0 \). Also, \( a_6 \in A[a_5, a_0] \); otherwise \( (e_3, e_4, e_5, e_6, e_0) \) contradicts the choice of \( \mathcal{P} \). Moreover, \( a_7 \in A[a_6, a_0] \); or else \( (e_3, e_4, e_6, e_7, e_0) \) contradicts the choice of \( \mathcal{P} \). But this shows that \( a_6 \in A[a_5, a_7] \), a contradiction.

Therefore, by (1), \( \{a_1, a_2', b_1', b_2, v\} \) is a cut in \( G^* \) separating \( \{a_0, a_1, a_2, b_1, b_2\} \) from \( A[a_1, a_2'] \cup B[b_1', b_2'] \), a contradiction. \[ \square \]

Thus by Lemmas 2.5.3, 2.5.5, and 2.5.6, \( G_0 \) has a separation \((G_1, G_2)\) with \( |V(G_1) \cap V(G_2)| \leq 3, |V(G_1 - G_2)| \geq 1, \{a_0, a_1, a_2\} \subseteq V(G_1), \) and \( B[b_1', b_2'] \subseteq G_2 \) where \( b_1', b_2' \in V(G_1) \cap V(G_2), \) such that one of the following holds:

(a) \( |V(G_1) \cap V(G_2)| = 3, b_1' \in B[b_1, b_4], b_2' \in B[b_7, b_2], \) and \( G_0 \) has a path from \( a_0 \) to \( B(b_1', b_2') \) and internally disjoint from \( B \). In this case, let \( t_1 := b_1', t_2 := b_2', \) and \( a'_0 = t_0 \in V(G_1 \cap G_2) \setminus \{b_1', b_2'\} \).

(b) \( |V(G_1) \cap V(G_2)| = 2, b_1' \in B[b_1, b_4], \) and \( b_2' \in B[b_7, b_2] \). In this case, let \( t_0 = t_1 := b_1', \) and \( t_2 := b_2' \).

(c) \( |V(G_1) \cap V(G_2)| = 2, b_1' \in B[b_1, b_4], b_2' \in B[b_6, b_7], \) and \( G \) has no edge from \( B(b_2', b_7) \) to \( A - a_7 \). In this case, let \( t_1 := b_1' \) and \( t_0 = b_2' \). Moreover, if \( G \) has no edge from \( B(b_2', b_7) \) to \( a_7 \) then let \( t_2 := b_7, \) and if \( G \) has an edge \( f_7 \) from \( b_7^* \in B(b_2', b_7) \) to \( a_7 \) then let \( t_2 := a_7, B(t_1, t_2) := B(b_1', b_2') \) and \( B(t_2, b_2) := B(b_7, b_2) \).

(d) \( |V(G_1) \cap V(G_2)| = 2, b_1' \in B[b_4, b_5], b_2' \in B[b_7, b_2], \) and \( G \) has no edge from \( B(b_4, b_1') \) to \( A - a_4 \). In this case, let \( t_0 := b_1' \) and \( t_2 := b_2' \). Moreover, if \( G \) has no edge from \( B(b_4, b_1') \) to \( a_4 \) then let \( t_1 := b_4, \) and if \( G \) has an edge \( f_4 \) from \( b_4^* \in B(b_4, b_1') \) to \( a_4 \) then let \( t_1 := a_4, B(t_1, t_2) := B[b_1', b_2') \) and \( B(b_1, t_1) := B[b_1, b_4] \).

We choose \( b_1', b_2' \) so that \( b_1', b_2' \) satisfy (a) or (b) whenever possible, subject to this, \( B[b_1, b_1'] \) is minimal, and subject to this, \( B[b_7, b_2'] \) is minimal.
Let $f_i = a_i^*b_i^* \in E(G)$, $i \in [2]$, with $a_i^* \in V(A)$ and $b_i^* \in V(B(t_1, t_2))$ such that $A[a_1^*, a_2^*]$ is maximal. Then $A[a_5, a_6] \subseteq A[a_1^*, a_2^*]$. Without loss of generality, we may assume that $a_1, a_1^*, a_2^*, a_2$ occur on $A$ in order.

**Lemma 2.5.7** $G - e_4$ has an edge from $B[b_1, t_1]$ to $A(a_1^*, a_2^*)$.

**Proof.** For, suppose $G-e_4$ has no edge from $B[b_1, t_1]$ to $A(a_1^*, a_2^*)$. Then, since $\{t_0, t_1, t_2, a_1^*, a_2^*\}$ is not a cut in $G^*$ separating $A(a_1^*, a_2^*) \cup B(t_1, t_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, there exists $e_8 = a_8b_8 \in E(G)$ with $b_8 \in V(B(t_2, b_2))$ and $a_8 \in V(A(a_1^*, a_2^*) - t_2)$. Obviously, $b_8 \in B(b_1', b_2) \cap B(b_7, b_2)$.

We claim that $a_8 \in A(a_3, a_4)$. For, otherwise, $a_8 \in A(a_1, a_3) \cup A(a_4, a_2)$. If $a_8 \in A(a_1, a_3)$, then $a_1^* \in A(a_1, a_3)$, and so $e_3, f_1, e_5, e_8$ force a double cross, or $(f_1, e_4, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. Therefore, $a_8 \in A(a_4, a_2)$. Then $b_2^* \in B(b_1', b_4)$; otherwise $e_4, e_5, f_2, e_8$ force a double cross. But now, $(e_3, f_2, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

If $b_8 \in B(b_7, b_2)$ then $(e_3, e_4, e_5, e_6, e_8)$ (when $a_6 \notin A[a_5, a_8]$) or $(e_3, e_4, e_6, e_7, e_8)$ (when $a_6 \in A[a_5, a_8]$) contradicts the choice of $\mathcal{P}$.

Hence $b_8 = b_7$ and, thus, $t_2 = a_7 \neq a_8$ and $G$ has an edge $f_7 = a_7b_7^*$ with $b_7^* \in V(B(b_2', b_7))$. Let $e = a_8b \in E(G)$ with $b \in V(B(b_1, b_2)) \setminus \{b_4, b_7\}$, which exists as $G^*$ is 6-connected.

We claim that $b \in B[b_1, b_4]$. Note that $b \notin B(b_2', b_7)$ (as $t_2 = a_7$) and $b \notin B(b_7, b_2)$ (as $b_8 = b_7$). So if the claim fails then $b \in B(b_4, b_2)$; now $(e_3, e_4, e, f_7, e_8)$ contradicts the choice of $\mathcal{P}$.

Thus, $a_8 \in A(a_3, a_7)$ to avoid the double cross $e, e_4, f_7, e_8$. Then $a_7 \in A[a_1, a_5]$; otherwise, $(e_3, e_4, e_5, f_7, e_8)$ contradicts the choice of $\mathcal{P}$. Now $a_6 \in A(a_3, a_2)$, for, if $a_6 \in A[a_1, a_8]$ then $e_4, e_6, e_8, e$ form a double cross, and if $a_6 \in A[a_8, a_7]$ then $(e_3, e_4, e_6, f_7, e_8)$ contradicts the choice of $\mathcal{P}$.

Suppose there exists $e_9 = a_9b_9 \in E(G)$ with $a_9 \in V(A[a_1, a_5])$ and $b_9 \in V(B(b_4, b_5))$. Then $a_9 \notin A[a_1, a_8]$ to avoid the double cross $e, e_4, e_8, e_9$. Moreover, $a_9 \notin A[a_8, a_7]$, or else
(e_3, e_4, e_9, f_7, e_8) contradicts the choice of \( \mathcal{P} \). So \( a_9 \in A[a_7, a_5] \). Now \( (e_3, e_4, e_9, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \).

Hence, \( G \) has no edge from \( B(b_4, b_5) \) to \( A(a_1, a_5) \). By Lemma 2.5.4 and by the existence of \( \{b'_1, b'_2\} \), there exists \( e_9 = a_9b_9 \in E(G) \) with \( a_9 \in V(A(a_5, a_2)) \) and \( b_9 \in V(B[b_7, b_2]) \). Then \( (e_9, e_8, f_7, e_6, e_5) \) is a 5-edge configuration. Then \( G_0 \) has a cut \( \{b''_1, b''_2\} \) or \( \{b''_1, b''_2, a'_9\} \) satisfying the conclusion of Lemma 2.5.3 (with respect to \( (e_9, e_8, f_7, e_6, e_5) \)), such that \( b_1, b''_1, b''_2, b_2 \) occur on \( B \) in order. But then, by Lemma 2.1.9, \( G_0 \) has a cut that would contradict the choice of \( \{b'_1, b'_2\} \).

Thus, by Lemma 2.5.7, there exists \( e_8 = a_8b_8 \in E(G - e_4) \) with \( b_8 \in V(B[b_1, t_1]) \) and \( a_8 \in V(A(a_1^*, a_2^*)) \). Note that \( b_8 \in B[b_1, b_4] \cap B[b_1, b'_1] \).

**Lemma 2.5.8** \( a_8 \in A(a_1^*, a_5) \).

**Proof.** For otherwise, \( a_8 \in A(a_5, a_2^*) \), and we choose \( e_8 \) so that \( A[a_8, a_2] \) is maximal. Then

1. \( b_8 \not\in B(b_1, b_4) \) for all choices of \( b_8 \).

First, suppose \( b_8 \in B(b_1, b_4) \). Then \( a_8 \not\in A(a_5, a_7) \) to avoid the double cross \( e_8, e_4, e_5, e_7 \). Now, \( b_3 = b_4 \) and \( a_8 \in A[a_1, a_4] \); otherwise, \( (e_3, e_8, e_5, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \). But then, \( e_3, e_4, e_7, e_8 \) form a double cross.

Now assume \( b_8 = b_4 \). Then \( t_1 = a_4 \) and there exists \( f_4 = a_4b_4^* \in E(G) \) with \( b_4^* \in V(B(b_4, b'_1)) \). Note that \( a_8 \in A(a_5, a_4) \); otherwise, by \( e_8 \neq e_4 \), we have \( a_8 \in A(a_4, a_2) \) and \( (e_3, e_8, e_5, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \).

\( G \) has no edge from \( A(a_5, a_4) \) to \( B(b_5, b_2) \), to avoid forming a double cross with \( e_5, e_8, f_4 \). Hence, \( a_7 \in A(a_3, a_5) \) and \( a_6 \not\in A(a_5, a_4) \). Moreover, \( a_6 \not\in A[a_1, a_7] \) to avoid the double cross \( e_6, e_7, e_8, f_4 \). So \( a_6 \in A[a_4, a_2] \).

Since \( d_G(a_8) \geq 6 \), there exists \( e'_8 = a_8b'_8 \in E(G) \) with \( b'_8 \in V(B[b_1, b_2]) - \{b_1, b_1, b_5\} \). Since \( b_8 \not\in B(b_1, b_4) \) and \( G \) has no edge from \( A(a_5, a_4) \) to \( B(b_5, b_2) \), then \( b'_8 \in B(b_4, b_5) \). But then, \( (e_3, e_4, e'_8, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \). \( \square \)
Hence, \( b_8 = b_1 \) and \( b'_1 \neq b_1 \). Now, \( a_8 \in A(a_4, a_5^*) \) to avoid the double cross \( e_8, e_4, e_3, e_7 \). And \( b_2' \in B[b_7, b_2'] \) to avoid the double cross \( e_8, f_2, e_3, e_7 \). Then \( b_3 = b_4 \); otherwise, \((f_2, e_7, e_4, e_3, e_8)\) contradicts the choice of \( \mathcal{P} \).

Note that \( a_5 \in A(a_1, a_7) \), or else \((f_2, e_7, e_5, e_3, e_8)\) contradicts the choice of \( \mathcal{P} \). Moreover, \( a_6 \in A(a_1, a_5) \), as, otherwise, \( e_8, e_6, e_3, e_7 \) (when \( a_6 \in A(a_8, a_2) \)) would form a double cross, or \((f_2, e_7, e_6, e_3, e_8)\) (when \( a_6 \in A[a_5, a_8] \)) contradicts the choice of \( \mathcal{P} \).

(2) \( G \) has no cross from \( B[b_6, b_2] \) to \( A[a_5, a_2] \) and \( G \) has no edge from \( B(b_6, b_2) \) to \( A[a_1, a_5] \).

Note that \( G \) has no cross from \( B[b_6, b_2] \) to \( A[a_5, a_2] \), to avoid forming a double cross with \( e_5, e_6 \). Now suppose there exists \( e = ab \in E(G) \) with \( b \in V(B(b_6, b_2)) \) and \( a \in V(A[a_1, a_5]) \). Then \( b = b_2 \); or else, \((e_3, e_4, e_5, e, e_7)\) (when \( b \notin B(b_6, b_7) \)) or \((f_2, e, e_5, e_3, e_8)\) (when \( b \in B(b_7, b_2) \)) contradicts the choice of \( \mathcal{P} \). But then \( a \neq a_1 \), and \( e, e_8 \) and two edges from \( a_1, a_2 \) to \( B(b_1, b_2) \) would form a double cross. \( \square \)

(3) \( G \) has no edge from \( B(b_1, b_3) \) to \( A \).

For, otherwise, let \( e = ab \in E(G) \) with \( a \in A \) and \( b \in B(b_1, b_3) \). Then \( a \in A[a_4, a_8] \); or else, \((f_2, e_7, e_4, e, e_8)\) contradicts the choice of \( \mathcal{P} \). But now, \((e, e_3, e_5, e_6, e_7)\) contradicts the choice of \( \mathcal{P} \). \( \square \)

(4) \( G \) has no edge from \( A(a_4, a_2) \) to \( B(b_1, b_7) \).

For, otherwise, let \( e = ab \in E(G) \) with \( a \in A(a_4, a_2) \) and \( b \in B(b_1, b_7) \). Then \( b \notin B(b_4, b_7) \) to avoid the double cross \( e_4, e_6, e_7, e \). But then \( b \in B(b_1, b_4) \), and \((e, e_3, e_5, e_6, e_7)\) contradicts the choice of \( \mathcal{P} \). \( \square \)

Let \( e^* = a_2b^* \in E(G) \), such that \( b^* \in B(b_1, b_2) \), and \( B[b^*, b_2] \) is minimal. Then by (2) and (4), \( b^* \in B[b_7, b_2] \) and \( G \) has no edge from \( B(b^*, b_2) \) to \( A \).

Let \( e' = a'b' \in E(G) \) with \( a' \in A(a_8, a_2) \) and \( b' \in B(b_6, b_2) \), such that \( B[b', b_2] \) is maximal. Note that \( e' \) exists because of \( e^* \). And \( b' \in B[b_7, b^*] \) by (2).
Now, by (2), (4), and the choice of $e^*, e'$, we have

(5) $G$ has no edge from $B(b^*, b_2]$ to $A$ and no edge from $B(b_1, b')$ to $A(a_8, a_2]$.

(6) $G$ has no edge from $b_1$ to $A[a_1, a_8]$.

For, suppose there exists $e = ab_1 \in E(G)$ with $a \in V(A[a_1, a_8])$. Then, by the choice of $e_8$, $a \notin A(a_5, a_8)$. Hence, $a \in A[a_1, a_5]$. Since $a \neq a_1$, there exists $e_0 = a_1b_0 \in E(G)$ with $b_0 \in V(B(b_1, b_2))$. Then $b_0 \in B[b_7, b_2]$ to avoid the double cross $e_0, e_4, e_7, e$. So $(e_0, e^*, e_5, e_4, e)$ contradicts the choice of $\mathcal{P}$. □

(7) If there exists $f_b' = a_8'b_8' \in E(G)$ with $a_8' \in V(A[a_5, a_2])$ and $b_8' \in V(B(b_5, b_2))$, then $G$ has no edge from $B(b_4, b_8')$ to $A(a_8', a_2]$.

For, suppose such $f_b'$ exists, and let $f_9' = a_9'b_9' \in E(G)$ with $a_9' \in A(a_8', a_2]$ and $b_9' \in B(b_4, b_5)$. Then $b_9' \notin B(b_5, b_8')$ to avoid the double cross $e_5, e_6, f_b', f_9'$. So $b_9' \in B(b_4, b_5)$. Moreover, $b_9' \notin B(b_4, b_5)$; otherwise, $(e_3, e_4, f_b', e_6, e_7)$ contradicts the choice of $\mathcal{P}$. So $b_9' = b_5$. Now, we see that $a_7 \in A[a_5, a_9']$; or else, $(e_3, e_4, f_b', e_6, e_7)$ contradicts the choice of $\mathcal{P}$. But then $(e^*, e_7, f_b', e_3, e_8)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. □

(8) There do not exist $b'' \in V(B[b_6, b'])$ and a cut $S$ of $G_0$ such that $|S| \leq 3$, $\{b_3, b''\} \subseteq S$, and $S$ separates $B[b_3, b'']$ from $\{a_0, b_1, b_2\}$.

For, suppose such $b''$ and $S$ do exist. Let $f_9' = a_9'b_9' \in E(G)$, such that $a_9' \in V(A[a_1, a_2])$, $b_9' \in V(B(b_3, b''))$, and subject to this, $A[a_9', a_2]$ is minimal. Then $a_9' \in A[a_5, a_2]$, by the existence of $e_5$.

We claim that $a_9' \notin A(a_8, a_2]$, and so by (6), $G$ has no edge from $b_1$ to $A[a_1, a_9']$. For otherwise, $b_9' \notin B(b_3, b_7)$ to avoid the double cross $e_6, e_7, e_8, f_9'$. But then $b_9' \in B[b_7, b']$, and $f_9'$ contradicts the choice of $e'$.

By (2) and (7), $G$ has no edge from $B(b', b_2]$ to $A[a_1, a_9']$. Thus, $S \cup \{a_1, a_9'\}$ is a cut in $G^*$ separating $A[a_1, a_9'] \cup B[b_3, b'']$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □
Since \((e', e_6, e_5, e_3, e_8)\) is a 5-edge configuration, \(G_0\) has a cut \(S' := \{b'_1, b'_2\}\) or \(S' := \{b''_1, b''_2, a'_0\}\) satisfying the conclusion of Lemma 2.5.3 (with respect to \((e', e_6, e_5, e_3, e_8)\)), such that \(b_1, b'_1, b'_2, b_2\) occur on \(B\) in order.

**Case 1.** Conclusions (i), or (ii), or (iii) of Lemma 2.5.3 holds for \(S'\) and \((e', e_6, e_5, e_3, e_8)\).

Since \((e_3, e_4, e_5, e_6, e_7)\) is a 5-edge configuration, \(G_0\) has a cut \(S^\# := \{b^\#_1, b^\#_2\}\) or \(S^\# := \{b^\#_1, b^\#_2, a^\#_0\}\) satisfying the conclusion of Lemma 2.5.3 (with respect to \((e_3, e_4, e_5, e_6, e_7)\)), such that \(b_1, b^\#_1, b^\#_2, b_2\) occur on \(B\) in order.

We may assume conclusion (iv) of Lemma 2.5.3 holds for \(S^\#\) and \((e_3, e_4, e_5, e_6, e_7)\), and so \(b^\#_1 \in B(b_4, b_5)\) and \(b^\#_2 \in B(b_7, b_2)\). For otherwise, assume conclusions (i), or (ii), or (iii) of Lemma 2.5.3 holds for \(S^\#\) and \((e_3, e_4, e_5, e_6, e_7)\). Then by the choice of \(\{b^\#_1, b^\#_2\}\) and \(b'_1 \neq b_1\), and by Lemma 2.1.8 and 2.1.9, we could find a cut \(\{b_3, b''\}\) or \(\{b_3, b''_2, a''\}\) with \(b'' \in B(b_3, b')\) in \(G_0\), which separates \(B(b_3, b'')\) from \(\{a_0, b_1, b_2\}\), a contradiction to (8).

Suppose conclusion (i) of Lemma 2.5.3 holds for \(\{b''_1, b''_2, a''_0\}\) and \((e', e_6, e_5, e_3, e_8)\). Then \(b''_2 \in B(b_0, b_7)\) by \(b'_1 \neq b_1\) and the choice of \(\{b'_1, b'_2\}\). Moreover, by Lemma 2.1.9, \(b''_2 = b_2\), and \(b''_1, b''_2, b_2, a_0\) are incident with a finite face of \(G_0\). Let \(f'_8 = a'_8b'_8 \in E(G)\) with \(a'_8 \in V(A[a_1, a_2])\) and \(b'_8 \in V(B(b''_2, b_2))\), such that \(A[a'_8, a_2]\) is maximal. Now, by (2), (3), and (7), \(G\) has a separation \((H_1, H_2)\), such that \(V(H_1 \cap H_2) = \{b_1, b_2, b_4, b''_2, a'_8\}\), \(\{a_0, a_1, b_1, b_2\} \subseteq V(H_1)\), and \(V(A[a'_8, a_2] \cup B[b''_2, b_2]) \subseteq V(H_2)\), a contradiction.

Now suppose conclusion (ii) of Lemma 2.5.3 holds for \(\{b''_1, b''_2\}\) and \((e', e_6, e_5, e_3, e_8)\). So \(b''_1 = b_1\) and \(b''_2 \in B(b_0, b')\). Then by Lemma 2.1.9, \(\{b''_1, b''_2\}\) is a cut in \(G_0\) separating \(B[b_1, b'_2]\) from \(\{b_1, b_2, a_0\}\), which contradicts the choice of \(\{b'_1, b'_2\}\) (as \(b'_1 \neq b_1\)).

So conclusion (iii) of Lemma 2.5.3 holds for \(\{b''_1, b''_2\}\) and \((e', e_6, e_5, e_3, e_8)\). Now \(b''_1 \in B(b_1, b_3)\) and \(b''_2 \in B(b_0, b')\). Then by Lemma 2.1.9, \(\{b''_1, b''_2\}\) is a cut in \(G_0\) separating \(B[b''_1, b''_2]\) from \(\{b_1, b_2, a_0\}\). Let \(f''_9 = a''_9b''_9 \in E(G)\), with \(a''_9 \in V(A[a_4, a_2])\) and \(b''_9 \in V(B[b_4, b''_2))\), such that \(A[a''_9, a_2]\) is minimal. If \(G\) has no edge from \(B[b''_2, b_2]\) to \(A[a_1, a'_9]\) then \(\{b_1, b''_1, b''_2, a''_9\}\) is a cut in \(G^*\) separating \(\{a_0, a_2, b_1, b_2\}\) from \(A[a_1, a'_9] \cup B[b''_1, b''_2]\),
a contradiction. So there exists \( f'_8 = a'_8b'_8 \in E(G) \) with \( a'_8 \in V(A[a_1, a'_9]) \) and \( b'_8 \in V(B(b^#, b_2)) \). Then \( a'_8 \notin A[a_5, a_4] \); or else, \((e_3, e_4, e_5, e_6, e'_8)\) contradicts the choice of \( \mathcal{P} \).

So \( a'_8 \in A[a_1, a_2] \) by (2), and \( b'_9 = b_4 \) by (7). But now, \((e_3, f'_9, e_5, e_6, e_7)\) contradicts the choice of \( \mathcal{P} \).

**Case 2.** Conclusion (iv) of Lemma 2.5.3 holds for \( S' \) and \((e', e_6, e_5, e_3, e_8)\).

Then \( b''_2 \in B[b_5, b_6], b'_1 = b_1, \) and \( \{b_1, b''_2\} \) is a cut in \( G_0 \) separating \( B[b_1, b''_2] \) from \( \{a_0, b_1, b_2\} \). By Lemma 2.1.9, the choice of \( \{b_1, b''_2\} \), and \( b' \neq b_1 \), we have \( b''_2 = b_2, b'_1 \in B(b_1, b_3) \), and \( b'_1, b''_2 \) are cut vertices of \( G_0 \) separating \( b_1 \) from \( \{a_0, b_2\} \). So \( \alpha(A, B) \leq 1 \).

Recall \( e^* = a_2b^* \) with \( B[b^*, b_2] \) minimal. If \( b^* = b_7 \), then, by (4) and (5), \( \{b_1, b_7, a_4\} \) is a cut in \( G^* \) separating \( \{a_0, a_1, b_1, b_2\} \) from \( A(a_4, a_2) \), a contradiction. So \( b^* \neq b_7 \). Then \( b^* \in B(b_7, b_2) \). Note that no finite face of \( G_0 \) is incident with both \( b''_2 \) and some vertex \( u \in B[b^*, b_2] \); or else, \( \{b_1', b''_2, u\} \) is a 3-cut in \( G_0 \) separating \( B[b_1', u] \) from \( \{a_0, b_1, b_2\} \), contradicting the choice of \( \{b_1', b''_2\} \).

We claim that \( G_0 - B[b_1, b''_2] - B[b^*, b_2] \) has disjoint paths \( B_2, A_0 \) from \( b_2, a_0 \) to \( b_7, b_0 \), respectively. For otherwise, since we may assume that Case 1 does not hold, it follows from the planar structure of \( G_0 \) and the choice of \( \{b_1', b''_2\} \) that there exist \( u_0 \in V(G_0), u_2 \in B[b^*, b_2] \), such that \( \{b''_2, u_0, u_2\} \) is a cut in \( G_0 \) separating \( B[b'_1, b''_2] \cup B[b''_2, u_2] \) from \( \{a_0, b_2\} \).

By (5), \( \{b''_2, u_0, u_2\} \) is a cut in \( G^* \) separating \( \{a_0, a_2\} \) from \( \{a_1, a_2, b_1\} \), a contradiction.

Now, let \( A' := A[a_1, a_0] \cup e_6 \cup A_0 \) and \( B' := B[b_1, b_5] \cup e_5 \cup A[a_5, a_7] \cup e_7 \cup B_2 \). Then the existence of \( A', B', e_8 \cup A[a_8, a_2] \), and \( e^* \cup B[b', b_2] \) implies \( \alpha(A, B) = 2 \) (by Lemma 2.2.1), a contradiction. \( \square \)

Thus by Lemma 2.5.8, \( a_8 \in A(a'_1, a_5) \) for all choices of \( e_8 \). Choose \( e_8 \) so that \( A[a_8, a_5] \) is minimal and, subject to this, \( B[b_8, b'_1] \) is minimal. Then \( G \) has no edge from \( B[b_1, b_4] \cap B[b_1, b'_1] \) to \( A(a_8, a_2^*) \).

(1) \( G \) has no cross from \( B[b_1, b_4] \) to \( A[a_1, a_5] \); so \( b_8 \in B[b_3, b_4] \).

For, such a cross would form a double cross with \( e_4, e_5 \). \( \square \)
(2) $G$ has no edge from $B(b_8, b_7)$ to $A[a_1, a_8] \cap A[a_1, a_7]$; so $b_1^* \in B[b_7, b_2] \cap A[a_1, a_7]$. 

For, such an edge would form a double cross with $e_4, e_7, e_8$ (when $b_8 \neq b_4$) or $f_4, e_7, e_8$ (when $t_1 = a_4$ and $b_8 = b_4$). □

(3) $a_7 \in A[a_1, a_5]$. 

For, suppose $a_7 \in A(a_5, a_2)$. Then $b_1^* \in B[b_7, b_2]$ by (2). So $b_7 \neq b_2$ (as $b_1^* \neq b_2$). Now, we may assume $t_1 = a_4$ and $b_8 = b_4$; otherwise, $b_8 \in B[1, b_4]$ and $(f_1, e_7, e_5, e_4, e_8)$ contradicts the choice of $P$. But then $(f_1, e_7, e_5, f_4, e_8)$ is a 5-edge configuration. So by Lemmas 2.1.9 and 2.5.3, $G_0$ has a cut contradicting the choice of $\{b_1', b_2'\}$. □

(4) $G$ has no edge from $B(b_5, b_7)$ to $A[a_1, a_7]$, and so $a_6 \in A(a_5, a_2)$. 

For, otherwise, let $e_9 = a_9b_9 \in E(G)$ with $a_9 \in V(A[a_1, a_7])$ and $b_9 \in V(B(b_5, b_7))$. Then $a_8 \in A[a_1, a_9]$ and $b_1^* \in B[b_7, b_2]$ by (2). So $b_7 \neq b_2$ (as $b_1^* \neq b_2$) and $(f_1, e_7, e_9, f_4, e_8)$ is a 5-edge configuration. Now, $t_1 = a_4$ and $b_8 = b_4$; otherwise, $(f_1, e_7, e_9, e_4, e_8)$ contradicts the choice of $P$. So by Lemmas 2.1.9 and 2.5.3, $G_0$ has a cut contradicting the choice of $\{b_1', b_2'\}$. □

(5) $G$ has no edge from $B(b_4, b_5)$ to $A[a_1, a_5]$. 

For, otherwise, let $e_9 = a_9b_9 \in E(G)$ with $a_9 \in V(A[a_1, a_5])$ and $b_9 \in V(B(b_4, b_5))$. Then $a_9 \notin A[a_7, a_5]$; otherwise, $(e_3, e_4, f_9, e_6, e_7)$ contradicts the choice of $P$. Moreover, $a_8 \in A[a_1, a_9]$ and $b_1^* \in B[b_7, b_2]$ by (2). So $b_7 \neq b_2$ (as $b_1^* \neq b_2$) and $(f_1, e_7, e_9, f_4, e_8)$ is a 5-edge configuration. Now, $t_1 = a_4$ and $b_8 = b_4$; otherwise, $(f_1, e_7, e_9, e_4, e_8)$ contradicts the choice of $P$. But then, $b_9 = b_5$ and by Lemmas 2.1.9 and 2.5.3, $G_0$ has a cut contradicting the choice of $\{b_1', b_2'\}$. □

(6) $G$ has no edge from $B(b_6, b_2)$ to $A(a_5, a_2)$.
For, otherwise, let $e_9 = a_9b_9 \in E(G)$ with $a_9 \in V(A(a_5, a_2))$ and $b_9 \in V(B(b_6, b_2))$. Then $b_9 \in B[b_7, b_2]$; or else, $(e_3, e_4, e_5, e_9, e_7)$ contradicts the choice of $\mathcal{P}$.

Suppose $b_9 = b_2$. Then $a_9 \neq a_2$ and let $e = a_2b \in E(G)$ with $b \in V(B(b_1, b_2))$ and $b \neq b_4$. If $b \in B(b_1, b_4)$ then $(e_3, e_5, f_1, e_9)$ contradicts the choice of $\mathcal{P}$; if $b \in B(b_4, b_2)$ then $e_8, e_9, f_1, e$ form a double cross, a contradiction.

So $b_9 \in B[b_7, b_2]$ and $b_7 \neq b_2$. So $(e_9, e_7, e_5, e_4, e_8)$ (when $a_7 \in A[a_1, a_8]$) or $(e_9, f_1, e_5, e_4, e_8)$ (when $a_8 \in A[a_1, a_7]$ and by (2)) is a 5-edge configuration. Hence, by the choice of $\mathcal{P}$, $t_1 = a_4$ and $b_8 = b_4$. Now by Lemma 2.1.9 and 2.5.3, $G_0$ has a cut contradicting the choice of $\{b'_1, b'_2\}$. \(\square\)

Now, by (3)–(6) and by Lemma 2.5.4,

(7) $G_0$ does not contain a cut $\{b''_1, b''_2\}$ separating $B[b''_1, b''_2]$ from $\{a_0, b_1, b_2\}$ with $b''_1 \in B[b_1, b_4]$ and $b''_2 \in B[b_6, b_2]$.

By (7), we have

(8) (b) and (c) do not hold.

(9) $G$ has no edge from $B[b_1, b_4]$ to $A(a_5, a_2)$.

For, suppose there exists $e = ab \in E(G)$ with $b \in V(B[b_1, b_4])$ and $a \in V(A(a_5, a_2))$. If $b \in B(b_1, b_4)$ then $a \in A(a_3, a_4)$ and $b_3 \in B(b, b_4)$; or else, $(e_3, e, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. But then $e_3, e_4, e_5$ form a double cross.

So $b = b_1$ and, hence, $a \neq a_2$. Let $e_0 = a_2b_0 \in E(G)$ with $b_0 \in V(B(b_1, b_2))$. By (6), $b_0 \in B(b_1, b_7)$. But then $e_0, e, e_3, e_7$ form a double cross, a contradiction. \(\square\)

(10) $G$ has no parallel edges from $A[a_1, a_8]$ to $B[b_4, b_2]$ and no parallel edges from $A[a_1, a_5]$ to $B[b_6, b_2]$.

For, such parallel edges would form a double cross with $e_4, e_8$ or $e_5, e_6$. \(\square\)

Let $e'_7 = a'_7b'_7 \in E(G)$ with $a'_7 \in A[a_1, a_7]$ and $b'_7 \in B[b_7, b_2]$, such that $A[a_1, a'_7] \cup B[b'_7, b_2]$ is minimal. Then
For, suppose (d) holds and the cut \( \{a_7^\prime \in A[a_1, a_8], \text{ and } G \text{ has no edge from } B(b_7^\prime, b_2] \) to \( A \).

For, if \( a_7^\prime \notin A[a_1, a_8] \) then, since \( a_7^\prime \in A[a_1, a_8], b_7^\prime \in B(b_8, b_7^\prime) \) by the choice of \( e_7^\prime; \) so \( e_8, e_4, f_1, e_7^\prime \) form a double cross, a contradiction. Thus, by (6) and (10) and by the choice of \( e_7^\prime, G \) has no edge from \( B(b_7^\prime, b_2] \) to \( A \). \( \blacksquare \)

Let \( e^\prime = ab^\prime \in E(G) \) with \( a^\prime \in A[a_1, a_5] \) and \( b^\prime \in B[b_1, t_1] \), such that \( A[a_1, a^\prime] \cup B[b_1, b^\prime] \) is minimal. By (1) and (9) and by the choice of \( e^\prime \), we have

\[
(12) \quad e^\prime, e_8 \text{ do not form a cross, and } G \text{ has no edge from } B[b_1, b^\prime] \text{ to } A, \text{ and no edge from } B(b^\prime, b_8) \text{ to } A[a_1, a^\prime] \cup A(a_8, a_2].
\]

\[
(13) \quad \text{If (d) holds then there does not exist a } 3\text{-cut } \{b_1^\prime, b_2^\prime, a_0^\prime\} \text{ in } G_0 \text{ with } b_1^\prime \in B[b_1, b_4] \text{ and } b_2^\prime \in B[b_5, b_2], \text{ which separates } B[b_1^\prime, b_2^\prime] \text{ from } \{a_0, b_1, b_2\}.
\]

For, suppose (d) holds and the cut \( \{b_1^\prime, b_2^\prime, a_0^\prime\} \) in (13) exists. Then \( b_1^\prime \in B(b_4, b_5), b_2^\prime \in B[b_7, b_2] \), and \( G \) has no edge from \( B[b_4, b_1] \) to \( A - a_4 \). Now, by the choice of \( \{b_1^\prime, b_2^\prime\} \) and by Lemma 2.1.9, \( b_1^\prime = b_1, b_2^\prime \in B(b_5, b_7), a_0^\prime = a_0, b_2^\prime = b_2, \) and \( \alpha(A, B) \leq 1 \).

By the choice of \( \{b_1^\prime, b_2^\prime\} \) and by the planar structure of \( G_0, G_0 - a_0 - B[b_7^\prime, b_2] \) contains a path \( B_2 \) from \( b_2 \) to \( b_2^\prime \). Let \( e_4^\prime = a_4 b_4^\prime \in E(G) \) with \( b_4^\prime \in B[b_4, b_1] \) such that \( B[b_4^\prime, b_1] \) is minimal. Since \( b_8 \in B[b_1, t_1] \), then \( b_8 \neq b_4^\prime \).

We claim that if \( b_4^\prime \neq b_4 \) then \( G \) has no edge from \( B[b_1, b_4^\prime] \) to \( A(a_5, a_2) - a_4 \). For, suppose \( b_4^\prime \in B(b_4, b_1^\prime) \) and there exists \( e = ab \in E(G) \) from \( b \in V(B[b_1, b_4^\prime]) \) to \( a \in V(A(a_5, a_2) - a_4) \). Now \( b = b_4 \) by (9) and (d). So \( a \in A(a_5, a_4) \) by the choice of \( \mathcal{P} \). Let \( e_0 = ab_0 \in E(G) \) with \( b_0 \in V(B[b_1, b_2]) \setminus \{b_4, b_5\} \). Then \( b_0 \notin B[b_1, b_4] \) by (9). Moreover, \( b_0 \notin B(b_5, b_2) \) to avoid the double cross \( e, e_0, e_4^\prime, e_5 \). So \( b_0 \in B(b_4, b_5) \). If \( a_6 \in A(a_5, a_4) \) then \( e, e_6, e_4^\prime, e_5 \) form a double cross; if \( a_6 \in A[a_4, a_2] \) then \( (e_3, e_4, e_0, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \).

Hence, by the choice of \( e_8, (1), (9), \) and (d), if \( b_4^\prime = b_4 \), then \( G \) has no edge from \( B(b_8, b_4^\prime) \) to \( A \); if \( b_4^\prime \neq b_4 \), then \( G \) has no edge from \( B(b_8, b_4^\prime) \) to \( A - a_4 \).
Now $e'$ is not adjacent with $e_8$. For, suppose $v$ is a vertex incident with both $e'$ and $e_8$. Then, by (12), (d), and our previous analysis, $\{b_1, v, b_4, b'_1, b_2\}$ (when $b'_4 = b_4$) or $\{b_1, v, a_4, b'_1, b_2\}$ (when $b'_4 \neq b_4$) is a cut in $G^*$ separating $a_0$ from $A$, a contradiction.

$G_0 - B(b_1, b') - B(b'_1, b_2)$ contains disjoint paths $B_1, A_0$ from $b_1, a_0$ to $b_8, b'_4$, respectively. For, suppose there exists a cut vertex $v$ in $G_0 - B(b_1, b') - B(b'_1, b_2)$ separating $\{b_1, a_0\}$ from $\{b_8, b'_4\}$. Then $v \notin B[b', b_8]$; otherwise, $v$ and $b'_1$ are incident with some finite face of $G_0$, and so $\{v, b'_1, b_2\}$ is a 3-cut in $G_0$ separating $B[v, b'_2]$ from $\{a_0, b_1, b_2\}$, contradicting the choice of $\{b'_1, b_2\}$. Moreover, $v \notin B[b'_4, b'_1]$; for otherwise, there exists $v_1 \in V(B(b_1, b')]$ such that $v_1, v$ are incident with some finite face of $G_0$ and, by (12), (d), and the choice of $e'_4$, $\{v_1, v, b'_4\}$ is a cut in $G$ separating $\{a_0, b_1\}$ from $\{a_1, a_2, b_2\}$, a contradiction. Hence, $v \notin V(B)$ and there exists $v_1 \in V(B(b_1, b')]$ such that $v_1, v$ are incident with some finite face of $G_0$, and $v, b'_1$ are incident with some finite face of $G_0$. But then, by (12), $\{v_1, v, b'_1\}$ is still a cut in $G$ separating $\{a_0, b_1\}$ from $\{a_1, a_2, b_2\}$, a contradiction.

Now, by Lemma 2.2.1, we have $\alpha(A, B) = 2$ by the following paths: the path $B_1 \cup e_8 \cup A[a_8, a_5] \cup e_5 \cup B[b_5, b'_2] \cup B_2$ from $b_1$ to $b_2$, the path $A[a_4, a_2] \cup e'_4 \cup A_0$ from $a_2$ to $a_0$, the path $A[a_1, a'_1] \cup e'_1 \cup B[b_1, b']$ from $a_1$ to $b_1$, and the path $A[a_1, a'_2] \cup e'_7 \cup B[b'_7, b_2]$ from $a_1$ to $b_2$. This is a contradiction. □

(14) (a) holds, $b_8 \neq b_1$, and $G$ has no edge from $B[b_1, b'_1]$ to $A(a_8, a_2)$.

First, (a) holds. For, otherwise, (d) holds by (8). So $b'_1 \in B(b_4, b_5)$ and $b_2 \in B[b'_7, b_2]$. By (1) and (5), $b_1' \in B(b_5, b_2)$.

Hence, $(f_1, e_6, e_5, e_4, e_8)$ (when $t_1 = b_4$) or $(f_1, e_6, e_5, f_4, e_8)$ (when $t_1 = a_4$) is a 5-edge configuration. However, by Lemma 2.1.9 and 2.5.3, $G_0$ has a cut contradicting (13) or the choice of $\{b'_1, b'_2\}$.

Thus, $b'_1 \in B[b_1, b_4]$. Since $b_8 \in B[b_1, b'_1], b_8 \neq b_4$. By (9), $G$ has no edge from $B[b_1, b'_1]$ to $A(a_5, a_2)$. Now, by the choice of $e_8, G$ has no edge from $B[b_1, b'_1]$ to $A(a_8, a_2)$.

□

(15) $G$ has no edge from $B(b_8, b_0)$ to $A(a_1, a_8)$, and so $(f_1, e_6, e_5, e_4, e_8)$ is a 5-edge con-

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configuration with \( b_1^* \subseteq B[b_6, b_2] \).

First, suppose there exists \( e = ab \in E(G) \) with \( b \in V(B(b_8, b_6)) \) and \( a \in V(A[a_1, a_8]) \). Then \( a_7 \in A(a_1, a) \) to avoid the double cross \( e_4, e_7, e_8, e \). But now, since \( a_3 \in A[a_1, a_7] \), then \( b_3 \in B(b_1, b_8) \) by (1), and so \( (e_3, e_8, e, e_6, e_7) \) contradicts the choice of \( \mathcal{P} \).

Thus, \( b_1^* \subseteq B[b_6, b_2] \) and, hence, \((f_1, e_6, e_5, e_4, e_8)\) is a 5-edge configuration. □

We choose \( f_1 \) so that \( B[b_6, b_1^*] \) is minimal. Moreover, we let \( e_5' = a_5 b_5' \in E(G) \) with \( a_5' \in A(a_1, a_8) \) and \( b_5' \in B[b_5, b_6) \) so that \( B[b_5', b_6] \) is minimal. Now, since \((f_1, e_6, e_5', e_4, e_8)\) is a 5-edge configuration (by (15)), \( G_0 \) has a cut \( S^* := \{b_1^*, b_2^*\} \) or \( S^* := \{b_1^*, b_2^*, a_0^*\} \) satisfying the conclusion of Lemma 2.5.3 (with respect to \((f_1, e_6, e_5', e_4, e_8)\)), such that \( b_1, b_1^*, b_2^*, b_2 \) occur on \( B \) in order.

By (7), we have

(16) Conclusions \((ii)\) and \((iii)\) of Lemma 2.5.3 do not hold for \( S^* \) and \((f_1, e_6, e_5', e_4, e_8)\).

Case 1. \((i)\) of Lemma 2.5.3 does not hold for \( S^* \) and \((f_1, e_6, e_5', e_4, e_8)\).

Then \( b_1^* \subseteq B[b_1, b_8] \) and \( b_2^* \subseteq B[b_5, b_6) \). By Lemma 2.1.9 and by the choice of \( \{b_1^*, b_2^*\} \), we have \( b_1^* = b_1, b_2^* = b_2, a_0 = a_0' \), and \( \alpha(A, B) \leq 1 \). We further choose \( \{b_1^*, b_2^*\} \) so that \( B[b_2^*, b_2] \) is minimal.

By the choice of \( \{b_1^*, b_2^*\} \) and the planar structure of \( G_0 \), \( G_0 - a_0 - B(b_1, b_1^*) \) contains a path \( B_1 \) from \( b_1 \) to \( b_1^* \). Let \( e_6' = a_6' b_6' \in E(G) \) with \( a_6' \in A(a_5, a_2) \) and \( b_6' \in B(b_2^*, b_6) \), such that \( A[b_6', b_2] \) is maximal.

Now \( G \) has no edge from \( B(b_5', b_6') \) to \( A \). For, suppose \( G \) has an edge from \( B(b_5', b_6') \) to some \( a \in V(A) \). Then \( a \in A[a_1, a_5'] \) by the choice of \( e_6' \), and \( a \notin A(a_1, a_6) \) by the choice of \( e_5' \). So \( a \in A[a_1, a_5'] \), contradicting (15).

Let \( A_0 \) be the path from \( a_0 \) to \( b_6' \) on the boundary of \( G_0 - B[b_1, b_2^*] \) without going through \( b_2 \). Since we are in Case 1, \( A_0 \cap B(b_6, b_2] = \emptyset \) by the choice of \( \{b_1^*, b_2^*\} \).

Note that there exists \( e = ab \in E(G) \) with \( a \in V(A[a_1, a_8]) \) and \( b \in V(B[b_1^*, b_2]) \) \{b_6\}, such that \( e \) and \( e_5' \) are nonadjacent. For, otherwise, by (1) and (10), there exist \( u \in
\[\{a_7', b_7'\}\] and a separation \((G_1, G_2)\) in \(G\), such that \(V(G_1 \cap G_2) = \{b_1, b_1', a_8, b_6, u, a_1\}\), 
\(A[a_1, a_8] \cup B[b_1, b_1'] \subseteq G_1\), \(\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_2)\), and \((G_1, b_1, b_1', a_8, b_6, u, a_1)\) is planar. This contradicts Lemma 2.1.3.

Then there exists \(e_7'' = a_7''b_7'' \in E(G)\) with \(a_7'' \in V(A(a_7', a_8))\) and \(b_7'' \in V(B(b_6, b_7'))\).
In fact, \(b \notin B(b_6, b_6)\) by (15) and, hence, \(b \in B(b_6, b_2)\). Thus, by (10) and the choice of \(e_7'\), 
\(a \in A(a_7', a_8)\) and \(b \in B(b_6, b_7')\). So \(e\) gives the desired \(e_7''\).

We further choose \(e_7'' = a_7'' \in A(a_7', a_8)\) and \(b_7'' \in B(b_6, b_7')\) so that \(A[a_1, a_7'']\) is maximal. Then \(a_7'' \in A(a', a_8)\). For otherwise, \(a_7'' \in A[a_1, a']\). By (10), (15), and the choice of 
\(e_7'\), \(\{b_1, b_1', a', a_8, b_6\}\) is a cut in \(G^*\) separating \(A[a', a_8] \cup B[b_1, b_1']\) from \(\{a_0, a_1, a_2, b_1, b_2\}\), a contradiction.

Note that \(G_0 - A_0 - B[b_7', b_2]\) contains a path \(B_2\) from \(b_2\) to \(b_7''\). For otherwise, \(b_7' \neq b_2\), and there exist \(v_1 \in V(A_0)\) and \(v_2 \in V(B[b_7', b_2])\), such that \(v_1, v_2\) are incident with some finite face in \(G_0\). If \(v_1 = a_0\) then \(\{v_1, v_2, b_2\}\) is a cut in \(G^*\) separating \(N_G(b_2)\) from \(\{a_0, a_1, a_2, b_1, b_2\}\), a contradiction; if \(v_1 \neq a\) then by (11), \(\{b_1, b_2^#, v_1, v_2, b_2\}\) is a cut in \(G^*\) separating \(a_0\) from \(\{a_1, a_2\}\), a contradiction.

Hence, \(\alpha(A, B) = 2\) by Lemma 2.2.1 and the following paths: the path \(B_1 \cup B[b_1', b_5] \cup e_5 \cup A[a_7', a_5] \cup e_7'' \cup B_2\) from \(b_1\) to \(b_2\), the path \(A[a_6', a_2] \cup e_6' \cup A_0\) from \(a_2\) to \(a_0\), the path 
\(A[a_1, a'] \cup e' \cup B[b_1, b']\) from \(a_1\) to \(b_1\), and the path \(A[a_1, a_7'] \cup e_7' \cup B[b_7', b_2]\) from \(a_1\) to \(b_2\).
This is a contradiction. \(\square\)

**Case 2.** (i) of Lemma 2.5.3 holds for \(S^\#: = \{b_1^#, b_2^#, a_0^\#\}\) and \((f_1, e_6, e_8^', e_4, e_8)\).

Then \(b_1^# \in B[b_1, b_8]\) and \(b_2^# \in B[b_6, b_1']\). Moreover, we choose \(\{b_1^#, b_2^#\}\) so that 
\(B[b_1^#, b_2^#]\) is maximal. By (7), \(G_0\) contains a path from \(a_0\) to \(B(b_4, b_6)\) and internally disjoint from \(B\). Then by Lemma 2.1.8 and the choice of \(\{b_1', b_2'\}\), we have \(b_1^# = b_1, b_2^# = b_2\), and one of the following holds:

(N1) \(a_0 = a_0' = a_0^\#\), and so \(c(A, B) \geq 2\).

(N2) \(a_0^\# = a_0, b_2^#\) is a cut vertex of \(G_0\) separating \(b_2\) from \(\{a_0, b_1\}\), \(a_0', a_0^\#, b_2^#, b_2'\) are
incident with some finite face of $G_0$; so $\alpha(A, B) \leq 1$.

(N3) $a'_0 = a_0, b'_1$ is a cut vertex of $G_0$ separating $b_1$ from $\{a_0, b_2\}$, $a'_0, a_0^*, b_1^*, b'_1$ are incident with some finite face of $G_0$; so $\alpha(A, B) \leq 1$.

In particular, there exists a vertex $a'_0 \in \{a_0', a_0^*\}$, such that $\{b'_1, b_2^*, a_0^*\}$ is a 3-cut in $G_0$ separating $B[b'_1, b_2^*]$ from $\{a_0, b_1, b_2\}$. Let $e_9 = a_9b_9 \in E(G)$ with $b_9 \in B(b'_1, b_2^*)$ and $a_9 \in A[a_1, a_2]$, such that $A[a_1, a_9]$ is minimal. There also exists $e'_9 = a'_9b'_9 \in E(G)$ with $a'_9 \in V(A(a_9, a_2))$ and $b'_9 \in V(B(b'_1, b_1')) \cup V(B(b_2^*, b_2))$; for otherwise, $\{a_0^*, b'_1, b_2^*, a_9, a_2\}$ is a cut in $G$ separating $A[a_9, a_2] \cup B[b'_1, b_2^*]$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Note that $a_9 \notin A[a_1, a_8]$; for otherwise, $b_9 \notin B(b_8, b_6)$ by (15) and, hence, $b_9 \in B[b_6, b_2^*]$, contradicting the choice of $f_1$. Next, $b'_9 \notin B(b_2^*, b_2]$; as otherwise, $a'_9 \notin A(a_5, a_2)$ by (9) and, hence, $a'_9 \in A(a_9, a_5)$, contradicting the choice of $e_8$. By (6), $a'_9 \notin A(a_5, a_2)$; so $a'_9 \in A(a_9, a_5)$. Furthermore, $b'_9 \in B(b_2^*, b_7]$; or else, $(e_3, e_4, e_5, e_6, e'_9)$ contradicts the choice of $P$.

Now, since $a'_9 \in A(a_9, a_5)$, $a_9 \neq a_5$; so $a_9 \in A(a_8, a_5)$. Moreover, $b_9 \notin B(b_5, b_2^*)$ to avoid the double cross $e'_9, e_5, e_6, e_9$. By (5), $b_9 \notin B(b_4, b_3)$. So $b_9 \in B(b'_1, b_4]$.

We choose $e'_9$ so that $B[b_2^*, b'_9]$ is minimal. Since $a'_9 \in A(a_9, a_5)$, $a_5 \neq a_9$. Then we will derive a contradiction by showing that $\alpha(A, B) = 2$.

Subcase 2.1. (N1) holds.

By the choice of $\{b'_1, b_2\}$ and the planar structure of $G_0$, $G_0 - B(b_1, b'_1) - a_0$ contains a path $B_1$ from $b_1$ to $b'_1$. Moreover, by the choice of $\{b_1^*, b_2^*\}$ and by planar structure of $G_0$, $G_0 - B(b_2^*, b_2) - a_0$ contains a path $B_2$ from $b_2^*$ to $b_2$.

Note that there exist $f_8 = a_8b_8^*, f_9 = a_9b_9^* \in E(G)$ with $a_8^*, a_9^* \in V(A(a_1, a_8))$ and $b_8^*, b_9^* \in V(B(b'_1, b_2))$ such that $a_8^* \neq a_9^*$ and $b_8^* \neq b_9^*$. For otherwise, there exist $v \in V(G)$ and a separation $(G_1, G_2)$ in $G$, such that $V(G_1 \cap G_2) = \{b'_1, a_0, b_1, a_1, v, a_8\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $A(a_1, a_8) \cup B(b_1, b'_1) \subseteq G_2$, and $(G_2, b'_1, a_0, b_1, a_1, v, a_8)$ is planar. This contradicts Lemma 2.1.3.
Now \( b_8^*, b_9^* \in B[b_6, b_2] \) by (15), and \( f_8, f_9 \) form a cross by (10). So \( a_1, a_8^*, a_9^*, a_2 \) occur on \( A \) in order, and \( b_1, b_5^*, b_8^*; b_2 \) occur on \( B \) in order. We further choose \( f_8, f_9 \) with \( A[a_8^*, a_9^*] \) maximal. By the existence of \( c_9^* \) and by (10), \( b_8^* \in B(b_2^*, b_2) \).

There exists \( f_5 = a_5^*b_5^* \) with \( b_5^* \in V(B[b_1, b_1']) \) and \( a_5^* \in V(A(a_1, a_6)) \). For otherwise, all edges from \( B[b_1, b_1'] \) will end in \( \{ a_1 \} \cup V(A[a_5^*, a_8]) \). By the choice of \( f_8, f_9, G \) has no edge from \( A[a_5^*, a_8] \) to \( B(b_8, b_2) \). Hence, \( G \) has a separation \((G_1, G_2)\), such that \( V(G_1 \cap G_2) = \{ b_1', a_0, b_1, a_1, a_5^*, a_8 \}, \{ a_0, a_1, a_2, b_1, b_2 \} \subseteq V(G_1), A(a_5^*, a_8) \cup B(b_1, b_1') \subseteq G_2, \) and \((G_2, b_1', a_0, b_1, a_1, a_5^*, a_8)\) is planar. By Lemma 2.1.3, \( |V(G_2 - G_1)| = 1 \). So \( V(G_2 - G_1) = \{ b_5 \}, \) and \( G \) has edges from \( b_8 \) to \( b_1', a_0, b_1, a_1, a_5^*, a_8 \), respectively. But then, \( b_1 \) has degree 1 in \( G \), a contradiction.

By (7), there exists a path \( A_0 \) from \( a_0 \) to \( B(b_4, b_6) \) in \( G_0 \) and internally disjoint from \( B \). Now, \( \alpha(A, B) = 2 \) and \( c(A, B) = 0 \) by Lemma 2.2.1 and the following paths: the path \( B_1 \cup B[b_1', b_9] \cup e_9 \cup A[a_5^*, a_9] \cup f_9 \cup B[b_5^*, b_2^*] \cup B_2 \) from \( b_1 \) to \( b_2 \), the path \( B[b_1, b_5^*] \cup f_5 \cup A[a_5^*, a_9] \cup f_8 \cup B[b_8^*, b_2] \) from \( b_1 \) to \( b_2 \), and the path \( A_0 \cup B(b_4, b_6) \cup e_5 \cup A[a_5, a_2] \) from \( a_0 \) to \( a_2 \). This is a contradiction.

**Subcase 2.2.** (N2) holds.

Then there exists \( e_7'' = a_7''b_7'' \in E(G) \) with \( a_7'' \in V(A[a_1, a_8]) \) and \( b_7'' \in V(B(b_1, b_2)) \) such that \( a_7'' \neq a_7' \) and \( b_7'' \neq b_7' \). For otherwise, by (1), (10) and (15), \( G \) has a separation \((G_1, G_2)\), such that \( V(G_1 \cap G_2) = \{ v, a_8, b_1', a_6 \} \) with \( v \in \{ a_7'', b_7'' \}, a_0, a_1, b_1 \subseteq V(G_2), |V(G_2 - G_1)| \geq 4, a_2, b_2 \subseteq V(G_1), \) and \((G_2, a_0, b_1, a_1, v, a_8, b_1', a_6)\) is planar. This contradicts Lemma 2.1.3 (when \( v = a_7'' = a_1 \)) or Lemma 2.1.4 (when \( v \neq a_1 \)).

By (10) and (15), \( a_7'' \in A(a_1, a_8) \) and \( b_7'' \in B[b_6, b_7'] \). We further choose \( e_7'' \) so that \( A[a_1, a_7''] \) is maximal. Then \( a_7'' \in A(a', a_8) \). For otherwise, \( a_7'' \in A[a_1, a'] \) and, by the choice of \( e_7'' \), \( G \) has no edge from \( A(a', a_8) \) to \( B(b_7', b_2) \). Hence, \( G \) has a separation \((G_1, G_2)\) such that \( V(G_1 \cap G_2) = \{ a', a_8, b_1', a_6, a_0, b_1 \}, \{ a_0, a_1, a_2, b_1, b_2 \} \subseteq V(G_1), \) and \((G_2, a', a_8, b_1', a_6, a_0, b_1)\) is planar. This contradicts Lemma 2.1.3.

By the choice of \( \{ a_0^#, b_1^#, b_2^# \} \) and the planar structure of \( G_0, G_0 - B[b_7', b_2] \) contains a
path $B_2$ from $b_2$ to $b_2^\#$. Let $A_0$ be the path from $a_0$ to $B(b_4, b_6)$ in $G_0$, which is internally disjoint from $B$. Moreover, we further choose $A_0$ such that $A_0[a_0, a_0']$ is on the boundary of $G_0$ without going through $b_1$.

Then $G_0 - B(b_1, b') - A_0$ contains a path $B_1$ from $b_1$ to $b'_1$. For otherwise, $b'_1 \neq b_1$ and there exist $v_1 \in V(A_0[a_0, a_0'])$ and $v_2 \in V(B(b_1, b'))$, such that $v_1, v_2$ are incident with some finite face of $G_0$. Now, by (12), $\{b_1, v_1, v_2, b_2\}$ (if $v_1 \neq a_0$) is a cut in $G^*$ separating $a_0$ from $\{a_1, a_2\}$, or $\{v_1, v_2, b_1\}$ (if $v_1 = a_0$) is a cut in $G^*$ separating $N_G(b_1)$ from $\{a_0, a_1, a_2, b_1, b_2\}$. This is a contradiction.

Hence, $\alpha(A, B) = 2$ by Lemma 2.2.1 and the following paths: the path $B_1 \cup B[b_1', b_0] \cup e_9 \cup A[a_7', a_9] \cup e_9' \cup B[b_7', b_2^\#] \cup B_2$ from $b_1$ to $b_2$, the path $A_0 \cup B(b_4, b_6) \cup e_{5} \cup A[a_5, a_2]$ from $a_0$ to $a_2$, the path $A[a_1, a'] \cup e' \cup B[b_1, b']$ from $a_1$ to $b_1$, and the path $A[a_1, a_7'] \cup e_7' \cup B[b_7', b_2^\#]$. This is a contradiction.

Subcase 2.3. (N3) holds.

Then there exists $e_7'' = a_7''b_7'' \in E(G)$ with $a_7'' \in V(A(a', a_8))$ and $b_7'' \in V(B(b_1', b_2))$, such that $a_7'' \neq a_7'$ and $b_7'' \neq b_7'$. For otherwise, by (10) and (15), there exist $v \in \{a_7', b_7'\}$ and a separation $(G_1, G_2)$ in $G$, such that $V(G_1 \cap G_2) = \{v, a', a_8, b_1, b_1'\}, A[a', a_8] \cup B[b_1, b_1'] \subseteq G_1$, and $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_2)$, a contradiction.

By (10) and (15), $a_7'' \in A(a_7', a_8)$ and $b_7'' \in B(b_6, b_7')$. By the choice of $\{a_0, b_1', b_2\}$ and the planar structure of $G_0$, $G_0 - B(b_1, b')$ contains a path $B_1$ from $b_1$ to $b'_1$. Let $A_0$ be the path from $a_0$ to $B(b_4, b_6)$ in $G_0$, which is internally disjoint from $B$, and we choose $A_0$ such that $A_0[a_0, a^\#_0]$ is on the boundary of $G_0$ without going through $b_2$.

Then $G_0 - B[b_7', b_2] - A_0$ contains a path $B_2$ from $b_2$ to $b_2^\#$. For otherwise, $b_7' \neq b_2$, and there exist $v_1 \in V(A_0[a_0, a^\#_0])$ and $v_2 \in V(B(b_7', b_2))$, such that $v_1, v_2$ are incident with some finite face of $G_0$. Now, by (11), $\{b_1, v_1, v_2, b_2\}$ (if $v_1 \neq a_0$) is a cut in $G^*$ separating $a_0$ from $\{a_1, a_2\}$, or $\{v_1, v_2, b_2\}$ (if $v_1 = a_0$) is a cut in $G^*$ separating $N_G(b_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$. This is a contradiction.

Now, $\alpha(A, B) = 2$ by Lemma 2.2.1 and the following paths: the path $B_1 \cup B[b_1', b_0] \cup
e_9 \cup A[a''_7, a_9] \cup e''_7 \cup B[b''_7, b''_2] \cup B_2 \text{ from } b_1 \text{ to } b_2, \text{ the path } A_0 \cup B(b_4, b_6) \cup e_5 \cup A[a_5, a_2] \text{ from } a_0 \text{ to } a_2, \text{ the path } A[a_1, a'_1] \cup e' \cup B[b_1, b'_1] \text{ from } a_1 \text{ to } b_1, \text{ and the path } A[a_1, a''_7] \cup e''_7 \cup B[b''_7, b_2].

This is a contradiction. \qed
3.1 A characterization of two-three linked graphs

In fact, Robertson and Seymour asked for a characterization of two-three linked graphs. Here, we believe we have such a characterization, although it is quite complicated (even to state) and its proof is longer.

We say that \((G, a_0, a_1, a_2, b_1, b_2)\) is reducible, if one of the following holds:

(R1) \(G\) has an edge \(e\) with one end in \(\{a_0, a_1, a_2\}\) and one end in \(\{b_1, b_2\}\).

(R2) There exists a separation \((G_1, G_2)\) in \(G\) of order at most 1.

(R3) There exists a separation \((G_1, G_2)\) in \(G\) of order 2, satisfying one of the following properties:

(a) \(\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)\) and \(V(G_2 - G_1) \neq \emptyset\); or

(b) \(|V(G_2 - G_1) \cap \{a_0, a_1, a_2, b_1, b_2\}| = 1\) and \(|E(G_2)| \geq 3\); or

(c) for some \(i \in \{0, 1, 2\}\) and some \(j \in \{1, 2\}\), \(V(G_1 \cap G_2) = \{c_1, c_2\}\), \(a_i, b_j \in V(G_2 - G_1)\), \(\{a_0, a_1, a_2, b_1, b_2\} - \{a_i, b_j\} \subseteq V(G_1)\), and \((G_2, a_i, b_j, c_2, c_1)\) is planar; or

(d) for some \(j \in \{1, 2\}\) and some permutation \(\pi\) of \(\{0, 1, 2\}\), \(V(G_1 \cap G_2) = \{c_1, c_2\}\), \(a_{\pi(0)}, a_{\pi(1)}, b_j \in V(G_2 - G_1)\), \(a_{\pi(2)}, b_{3-j} \in V(G_1)\), and \((G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_2, c_1)\) is planar; or

(e) for some \(i \in \{0, 1, 2\}\), \(V(G_1 \cap G_2) = \{c_1, c_2\}\), \(a_i, b_1, b_2 \in V(G_2 - G_1)\), \(\{a_0, a_1, a_2, b_1, b_2\} - \{a_i, b_1, b_2\} \subseteq V(G_1)\), and \((G_2, b_1, a_i, b_2, c_2, c_1)\) is planar.
(R4) There exists a separation \((G_1, G_2)\) in \(G\) of order 3, satisfying one of the following properties:

(a) \(\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)\) and \(V(G_2 - G_1) \neq \emptyset\); or

(b) \(V(G_1 \cap G_2) = \{c_1, c_2, c_3\}\), \(\{d\} = \{a_0, a_1, a_2, b_1, b_2\} \cap V(G_2 - G_1)\), \((G_2, d, c_3, c_2, c_1)\) is planar, and \(|V(G_2 - G_1)| \geq 2\); or

(c) for some \(i \in \{0, 1, 2\}\) and some \(j \in \{1, 2\}\), \(V(G_1 \cap G_2) = \{c_1, c_2, c_3\}\), \(a_i, b_j \in V(G_2 - G_1)\), \(\{a_0, a_1, a_2, b_1, b_2\} - \{a_i, b_j\} \subseteq V(G_1)\), \((G_2, a_i, b_j; c_1, c_2, c_3)\) is planar, and \(|V(G_2 - G_1)| \geq 3\); or

(d) for some permutation \(\pi\) of \(\{0, 1, 2\}\), \(V(G_1 \cap G_2) = \{c_1, c_2, c_3\}\), \(a_{\pi(0)}, a_{\pi(1)}, b_j \in V(G_2 - G_1)\), \(a_{\pi(2)}, b_{3-j} \in V(G_1)\), and \((G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_3, c_2, c_1)\) is planar; or

(e) for some \(i \in \{0, 1, 2\}\), \(V(G_1 \cap G_2) = \{c_1, c_2, c_3\}\), \(b_1, a_i, b_2 \in V(G_2 - G_1)\), \(\{a_0, a_1, a_2\} - \{a_i\} \subseteq V(G_1)\), and \((G_2, b_1, a_i, b_2, c_3, c_2, c_1)\) is planar.

(R5) There exists a separation \((G_1, G_2)\) in \(G\) of order 4, satisfying one of the following properties:

(a) let \(W\) be a graph with \(V(W) = \{w_0, w_1, w_2, w_3, w_4\}\), \(E(W) = \{w_0w_i; i = 1, 2, 3, 4\} \cup \{w_1w_2, w_1w_3\}\), then \(a_0, a_1, a_2, b_1, b_2 \in V(G_1)\), \(V(G_2 - G_1) \neq \emptyset\), and \(G_2\) is not a subgraph of \(W\); or

(b) \(V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\}\), \(a_0, a_1, a_2, b_1, b_2 \in V(G_1)\), \(V(G_2 - G_1) = \{c\}\), \(G\) has edges from \(c\) to \(c_1, c_2, c_3, c_4\), \(G\) has edges from \(c_1\) to \(c_2, c_3\), and for some \(i \in \{0, 1, 2\}\) and some \(j \in \{1, 2\}\), \(a_i, b_j \in V(G_1 \cap G_2); or

(c) for some \(i \in \{0, 1, 2\}\) and some \(j \in \{1, 2\}\), \(V(G_1 \cap G_2) = \{c_1, c_2, a_i, b_j\}\), \(a_0, a_1, a_2, b_1, b_2 \in V(G_1)\), \(V(G_2 - G_1) = \{c\}\), \(G\) has edges from \(c\) to \(c_1, c_2, a_i, b_j\), and \(G\) has an edge from \(c_1\) to \(c_2\); or
(d) \( V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\}, a_0, a_1, a_2, b_1, b_2 \in V(G_1), V(G_2 - G_1) = \{c\}, \)

\(G\) has edges from \(c\) to \(c_1, c_2, c_3, c_4\), \(G\) has an edge from \(c_1\) to \(c_2\), and for some permutation \(\pi\) of \(\{0, 1, 2\}\), \(\{a_{\pi(0)}, a_{\pi(1)}\} \subseteq V(G_1 \cap G_2)\) and \(\{a_{\pi(0)}, a_{\pi(1)}\} \cap \{c_1, c_2\} \neq \emptyset\); or

(e) for some \(i \in \{0, 1, 2\}\), \(\{a_i\} = V(G_2 - G_1) \cap \{a_0, a_1, a_2, b_1, b_2\}, V(G_1 \cap G_2) = \{b_1, b_2, c_1, c_2\}, (G_2, a_i, b_1, c_1, c_2, b_2)\) is planar, and \(|V(G_2 - G_1)| \geq 2\); or

(f) for some permutation \(\pi\) of \(\{0, 1, 2\}\) and some \(j \in \{1, 2\}\), \(\{b_j\} = V(G_2 - G_1) \cap \{a_0, a_1, a_2, b_1, b_2\}, V(G_1 \cap G_2) = \{a_{\pi(1)}, a_{\pi(2)}, c_1, c_2\}, (G_2, a_{\pi(0)}, b_j, c_1, c_2, a_{\pi(2)})\)
is planar, and \(|V(G_2 - G_1)| \geq 2\); or

(g) for some permutation \(\pi\) of \(\{0, 1, 2\}\) and some \(j \in \{1, 2\}\), \(\{a_{\pi(0)}\} = V(G_2 - G_1) \cap \{a_0, a_1, a_2, b_1, b_2\}, V(G_1 \cap G_2) = \{b_j, a_{\pi(1)}, c_1, c_2\}, (G_2, a_{\pi(0)}, b_j, c_1, a_{\pi(1)}, c_2)\)
is planar, and \(|V(G_2 - G_1)| \geq 2\); or

(h) for some permutation \(\pi\) of \(\{0, 1, 2\}\), \(V(G_1 \cap G_2) = c_1, c_2, c_3, a_{\pi(0)}\), \(a_{\pi(1)}, b_j \in V(G_2 - G_1), a_{\pi(2)}, b_{3-j} \in V(G_1), (G_2, c_1, c_2, a_{\pi(0)}, c_3, a_{\pi(1)}, b_j)\) planar, and \(|V(G_2 - G_1)| \geq 3\); or

(i) for some permutation \(\pi\) of \(\{0, 1, 2\}\), \(V(G_1 \cap G_2) = c_1, c_2, c_3, a_{\pi(0)}\), \(a_{\pi(1)}, b_j \in V(G_2 - G_1), a_{\pi(2)}, b_{3-j} \in V(G_1), (G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_3, c_2, c_1)\) planar, and \(|V(G_2 - G_1)| \geq 3\); or

(j) for some \(i \in \{0, 1, 2\}\) and some \(j \in \{1, 2\}\), \(V(G_1 \cap G_2) = c_1, c_2, c_3, b_j\),

\(a_i, b_{3-j} \in V(G_2 - G_1), \{a_1, a_2, a_3\} - a_i \subseteq V(G_1), (G_2, b_{3-j}, a_i, b_j, c_3, c_2, c_1)\)
is planar, and \(|V(G_2 - G_1)| \geq 3\); or

(k) for some permutation \(\pi\) of \(\{0, 1, 2\}\), \(V(G_1 \cap G_2) = c_1, c_2, c_3, c_4\), \(a_{\pi(0)}, a_{\pi(1)}, b_j \in V(G_2 - G_1), a_{\pi(2)}, b_{3-j} \in V(G_1), (G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_4, c_3, c_2, c_1)\)
is planar, and \(|V(G_2 - G_1)| \geq 4\); or

(l) \(V(G_1 \cap G_2) = c_1, c_2, c_3, c_4\), \(a_i, b_1, b_2 \in V(G_2 - G_1), \{a_1, a_2, a_3\} - a_i \subseteq V(G_1), (G_2, b_1, a_i, b_2, c_4, c_3, c_2, c_1)\) planar, and \(|V(G_2 - G_1)| \geq 4\); or
(m) for some permutation \( \pi \) of \( \{0, 1, 2\} \), \( a_{\pi(0)}, a_{\pi(1)}, b_1, b_2 \in V(G_1), \{a_{\pi(0)}, a_{\pi(1)}, b_1, b_2\} \cap V(G_2) \neq \emptyset, a_{\pi(2)} \in V(G_2) - V(G_1) \), and \( G_1 \) has a disk representation in which \( a_{\pi(0)}, b_1, a_{\pi(1)}, b_2 \) occur on the boundary of the disk in the order listed and the vertices in \( V(G_1) \cap V(G_2) \) are incident with a common finite face.

(R6) There exists a separation \((G_1, G_2)\) in \( G \) of order 5, satisfying one of the following properties:

(a) \( V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4, c_5\}, \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1), E(G[\{c_1, c_2, c_3, c_4, c_5\}]) \subseteq E(G_1), (G_2, c_1, c_2, c_3, c_4, c_5) \) is planar, and \( |V(G_2 - G_1)| \geq 2; \) or

(b) \( V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4, c_5\}, \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1), \) and for some permutation \( \pi \) of \( \{0, 1, 2\} \), \( G_1 \) has a disk representation with the vertices \( a_{\pi(0)}, b_1, a_{\pi(1)}, b_2, a_{\pi(2)}, c_1, c_2, c_3, c_4, c_5 \) drawn on the boundary of the disk in the order listed; or

(c) for some permutation \( \pi \) of \( \{0, 1, 2\} \), \( V(G_1 \cap G_2) = \{c_1, c_2, b_1, b_2, a_{\pi(1)}\}, a_{\pi(2)} \in V(G_1 - G_2), a_{\pi(0)} \in V(G_2 - G_1), (G_2, b_1, c_1, a_{\pi(1)}, c_2, b_2, a_{\pi(0)} \} \) is planar, and \( |V(G_2 - G_1)| \geq 4; \) or

(d) for some \( j \in \{1, 2\} \) and some permutation \( \pi \) of \( \{0, 1, 2\} \), \( V(G_1 \cap G_2) = \{c_1, c_2, c_3, a_{\pi(1)}, b_j\}, a_{\pi(2)} \in V(G_1 - G_2), a_{\pi(0)} \in V(G_2 - G_1), (G_2, a_{\pi(1)}, c_1, c_2, c_3, b_j, a_{\pi(0)}, b_{3-j}) \) is planar, and \( |V(G_2 - G_1)| \geq 3. \)

Actually, we can prove that if \((G, a_0, a_1, a_2, b_1, b_2)\) is reducible, then we could either easily determine whether or not \((G, a_0, a_1, a_2, b_1, b_2)\) is feasible, or reduce \((G, a_0, a_1, a_2, b_1, b_2)\) to \((G', a_0', a_1', a_2', b_1', b_2')\) with \(|V(G)|, |E(G)| > (|V(G')|, |E(G')|)\) in lexicographic order, such that \((G, a_0, a_1, a_2, b_1, b_2)\) is feasible iff \((G', a_0', a_1', a_2', b_1', b_2')\) is feasible.

With all these, we can state our main result.

**Theorem 3.1.1** Let \((G, a_0, a_1, a_2, b_1, b_2)\) be a rooted graph. Then one of the following conclusions holds:

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(C1) There exists a cluster \( \{X_1, X_2\} \) in \( G \) such that \( \{a_0, a_1, a_2\} \subseteq X_1 \) and \( \{b_1, b_2\} \subseteq X_2 \).

(C2) \( (G, a_0, a_1, a_2, b_1, b_2) \) is reducible.

(C3) For some \( i \in \{0, 1, 2\} \), \( G - a_i \) has no cluster \( \{X_1, X_2\} \) such that \( \{a_0, a_1, a_2\} - \{a_i\} \subseteq X_1 \) and \( \{b_1, b_2\} \subseteq X_2 \).

(C4) There exist a permutation \( \pi \) of \( \{0, 1, 2\} \), a graph \( H \) and vertices \( s, t, s', t' \in V(H) \) such that \( G \) is obtained from \( H \) by identifying \( s \) with \( s' \) and \( t \) with \( t' \), respectively, and \( H \) has a disk representation with the vertices \( a_{\pi(0)}, b_1, a_{\pi(1)}, b_2, a_{\pi(2)}, s, t, s', t' \) drawn on the boundary of the disk in the order listed.

(C5) \( G \) has a separation \( (G_1, G_2) \) in \( G \) of order 4, such that \( V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\} \), \( a_0, a_1, a_2, b_1, b_2 \in V(G_1) \), and there exist a permutation \( \pi \) of \( \{0, 1, 2\} \), a graph \( H \) and vertices \( c'_2, c''_2 \in V(H) \), where \( G_1 \) is obtained from \( H \) by identifying \( c'_2 \) with \( c''_2 \), \( (H, a_{\pi(1)}, b_1, a_{\pi(0)}, b_2, a_{\pi(2)}, c'_2, c_4, c_3, c'_2, c_1) \) is planar, and \( c_2 \in V(G_1) \) is the vertex obtained by identifying \( c'_2 \) with \( c''_2 \).

3.2 Clarifying (C3)

Note that if (C4) or (C5) holds, then (C1) will not hold. However, if (C3) holds, \( (G, a_0, a_1, a_2, b_1, b_2) \) may be feasible or may be infeasible. Although by using 2-linkage algorithms, it is easy to judge whether \( (G, a_0, a_1, a_2, b_1, b_2) \) admits (C3), we want to give a more precise characterization of feasible rooted graphs when (C3) holds.

We will still assume \( G \) is not reducible. So by applying Seymour’s version of 2-linkage theorem in [37], when (C3) holds, there exists \( i \in \{0, 1, 2\} \), such that \( (G - a_i, a_{i+1}, b_1, a_{i-1}, b_2) \) is planar. So \( G \) actually is an apex graph.
3.3 A practical algorithm

Another possible future work is to develop a practical polynomial time algorithm for the two-three linkage problem.

Note that the existence of such an algorithm with polynomial running time is guaranteed by the work of Robertson and Seymour in [40]: Given a graph $G$ and $k \geq 1$ pairs of vertices $\{s_i, t_i\}, i = 1, \cdots, k$ of $G$ with $k$ fixed, there exists a polynomial time algorithm for deciding if there are $k$ mutually internally vertex-disjoint paths in $G$ joining $s_i$ and $t_i$, $i = 1, \cdots, k$. In fact, to resolve the two-three linkage problem, we just need to check:

(i) whether for some $i \in \{0, 1, 2\}$, $G$ contains 3 mutually internally vertex-disjoint paths joining the pairs $\{b_1, b_2\}, \{a_{i-1}, a_i\}$ and $\{a_i, a_{i+1}\}$; or

(ii) whether for some vertex $v \in V(G) - \{a_0, a_1, a_2, b_1, b_2\}$, $G$ contains 4 mutually vertex-disjoint paths to join the pairs $\{b_1, b_2\}, \{v, a_0\}, \{v, a_1\}$ and $\{v, a_2\}$.

Clearly, the answer is yes iff $(G, a_0, a_1, a_2, b_1, b_2)$ is feasible. The disjoint paths algorithm of Robertson and Seymour has running time $O(|V(G)|^3)$. So the above algorithm runs $O(|V(G)|^4)$ time.

However, the disjoint paths algorithm of Robertson and Seymour is not practical, since it involves an enormous constant. Hence, it is meaningful to come up with a practical algorithm for the two-three linkage problem. In fact, to the best of our knowledge, Tholey [41] found the $O(|E(G)| + |V(G)|\alpha(|V(G)|, |V(G)|))-time algorithm, the currently best known nearly linear time bound, of 2-linkage problem, where $\alpha$ denotes the inverse of the Ackermann function. By repeatedly using 2-linkage algorithm, we expect to obtain a $O(|V(G)|^3)$-time two-three linkage algorithm.

3.4 A related conjecture

A graph $G$ is apex if $G - v$ is planar for some vertex $v \in V(G)$. Jørgensen [34] conjectured that every 6-connected graph with no $K_6$-minor is apex.
In the two-three linkage problem, we only consider finding disjoint connected subgraphs $G_1, G_2$ such that $\{a_0, a_1, a_2\} \subseteq V(G_1)$ and $\{b_1, b_2\} \subseteq V(G_2)$. However, it is also natural to ask whether we can find such disjoint connected subgraphs $G_1, G_2$ satisfying additional properties. For example, we have the following conjecture.

**Conjecture 3.4.1** Any 6-connected non-apex graph $G$ with distinct vertices $a_0, a_1, a_2, b_1, b_2 \in V(G)$ contains disjoint connected subgraphs $G_1, G_2$ such that $\{a_0, a_1, a_2\} \subseteq V(G_1)$, $\{b_1, b_2\} \subseteq V(G_2)$, and the following properties hold:

(P1) there exists a vertex $v \in V(G_1) - \{a_0, a_1, a_2\}$ such that $G_1$ has three internally disjoint paths from $v$ to $a_0, a_1, a_2$, respectively;

(P2) for each vertex $v \in G_1$, $\{a_0, a_1, a_2\} - \{v\}$ are contained in one component of $G_1 - v$.

One observation is that if $(G - a_0, a_1, b_1, a_2, b_2)$ is planar, then there do not exist disjoint connected subgraphs $G_1, G_2$ in $G$ such that $\{a_0, a_1, a_2\} \subseteq V(G_1)$, $\{b_1, b_2\} \subseteq V(G_2)$, and $G_1$ satisfies (P1) and (P2). Note that such $G$ is apex, and $G$ can be 6-connected.

If Conjecture 3.4.1 is true, we may prove that given a 6-connected graph $G$ and triangles $a_i b_1 b_2 a_i$ for $i = 0, 1, 2$, $G - b_1 b_2 - \{a_i b_j : i = 0, 1, 2$ and $j = 1, 2\}$ contains disjoint connected subgraphs $G_1, G_2$ such that $\{a_0, a_1, a_2\} \subseteq V(G_1)$, $\{b_1, b_2\} \subseteq V(G_2)$, and $G_1$ satisfies (P1) and (P2). Such properties could be useful in resolving Jørgensen’s conjecture for 6-connected graph in which some edge is contained in three triangles.
REFERENCES


