SMALL TORSION GENERATING SETS FOR MAPPING CLASS GROUPS

A Dissertation
Presented to
The Academic Faculty

By

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In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in the
School of Mathematics

Georgia Institute of Technology

May 2020

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Nourish beginnings, let us nourish beginnings.

Not all things are blessed, but the seeds of all things are blessed.

The blessing is in the seed.

*Muriel Rukeyser*
To my parents.
ACKNOWLEDGEMENTS

Pertaining to the work presented in the first five chapters of this thesis: I would like to thank Dan Margalit for his guidance, support, feedback, and encouragement. I would like to thank Shane Scott for feedback and helpful conversations. Thanks also go to Martin Kassabov, both for his comments and for encouraging the author to sharpen the main result to three elements. I also thank Edgar Bering IV, Allen Broughton, John Dixon, John Etnyre, Benson Farb, Bill Harvey, Naoyuki Monden, Balázs Strenner, Tom Tucker, and Josephine Yu for comments on a draft of the paper containing this work. Finally, the author thanks the anonymous referee for a number of useful suggestions.

I would like to thank the National Science Foundation for supporting my graduate studies under Grant No. DGE - 1650044.

I would like to thank Dan Margalit, for serving as my advisor, for working with me as a collaborator, and for giving me encouragement and guidance throughout my time in graduate school.

I have been fortunate to learn and live as a part of many exemplary educational institutions, and I thank everyone who I learned from at the McNeese Lab School, T. S. Cooley Elementary Magnet School, St. Margaret Catholic School, St. Louis Catholic High School, St. John’s College, the PROMYS for Teachers program, The Calverton School, Saint Ann’s School, Hunter College, NYU, Princeton Learning Cooperative, and Georgia Tech.

I also would like to thank the members of several virtual communities that have been an important part of my mathematical education: the MathTwitterBlogosphere, the Virtual Reading Group, and Mod Squad.

I would like to thank the faculty and staff of the School of Mathematics at Georgia Tech, who have helped to make my years in graduate school progress smoothly and fruitfully. I especially thank Mitchell Everett, Klara Grodzinsky, and Chris Jankowski; and John Etnyre, Mohammad Ghomi, and Xingxing Yu, each of whom served as Graduate Director.
I am grateful to the many teachers, mentors, colleagues, collaborators, and friends who have been a part of my math journey. These include Santana Afton, Ara Basmajian, Max Bean, Igor Belegradek, Jim Belk, Ben Blum-Smith, Tara Brendle, Aaron Calderon, Sylvain Cappell, Lei Chen, Solly Coles, Al Cuoco, Diana Davis, Moon Duchin, John Etnyre, Benson Farb, Nir Gadish, Dave Glickenstein, Elaine Greenman, Joel Hammon, Joseph Hebert, Debra Herpin, Cvetelina Hill, Jen Hom, Sudipta Kolay, Sarah Koch, Kisun Lee, Paul Lockhart, Marissa Loving, Joe Macfarland, Ryota Matsuura, Andre Oliveira, Avery Pickford, Teresa Powers, Paul Salomon, Sylvia Schroll, Shane Scott, Elizabeth Sheridan Rossi, Joyce Steen, Glenn Stevens, Balázs Strenner, Coco Van Meerendonk, Nick Vlamis, Anna Weltman, Becca Winarski, Josephine Yu, Charles Zbley, and many others.

Finally, I would like to thank my family, especially my siblings, my grandparents, and my parents, Mary and Rickey Lanier. Your encouragement, your support, your faith, and your love have meant everything to me.
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SUMMARY

A surface of genus $g$ has many symmetries. These form the surface’s mapping class group $\text{Mod}(S_g)$, which is finitely generated. The most commonly used generating sets for $\text{Mod}(S_g)$ are comprised of infinite order elements called Dehn twists; however, a number of authors have shown that torsion generating sets are also possible. For example, Brendle and Farb showed that $\text{Mod}(S_g)$ is generated by six involutions for $g \geq 3$. We will discuss our extension of these results to elements of arbitrary order: for $k > 5$ and $g$ sufficiently large, $\text{Mod}(S_g)$ is generated by three elements of order $k$.

Generalizing this idea, in joint work with Margalit we showed that for $g \geq 3$ every non-trivial periodic element that is not a hyperelliptic involution normally generates $\text{Mod}(S_g)$. This result raises a question: does there exist an $N$, independent of $g$, so that if $f$ is a periodic normal generator of $\text{Mod}(S_g)$, then $\text{Mod}(S_g)$ is generated by $N$ conjugates of $f$? We show that in general there does not exist such an $N$, but that there do exist universal bounds when additional conditions are placed on $f$.

In Chapter 1 we give an introduction and overview of our work. In Chapter 2 we show how to construct elements of order $k$ in $\text{Mod}(S_g)$ for sufficiently large values of $g$. In Chapter 3 we show how to write a Dehn twist as a product in elements of order $k$. In Chapters 4 and 5 we construct generating sets for $\text{Mod}(S_g)$ that are comprised of elements of fixed finite order. In Chapter 6 we show that that in general that there is no universal upper bound on the number of conjugates of a periodic normal generator required to generate $\text{Mod}(S_g)$, although universal bounds do exist when additional conditions are placed on $f$. 
Let $S_g$ be a closed, connected, and orientable surface of genus $g$. The mapping class group $\text{Mod}(S_g)$ is the group of homotopy classes of orientation-preserving homeomorphisms of $S_g$. In this paper, we construct small generating sets for $\text{Mod}(S_g)$ where all of the generators have the same finite order.

**Theorem 1.1.** Let $k \geq 6$ and $g \geq (k-1)^2 + 1$. Then $\text{Mod}(S_g)$ is generated by three elements of order $k$. Also, $\text{Mod}(S_g)$ is generated by four elements of order 5 when $g \geq 8$.

Theorem 1.1 follows from a stronger but more technical result that we prove as Theorem 4.1. Our generating sets for $\text{Mod}(S_g)$ are constructed explicitly. In addition, the elements in any particular generating set are all conjugate to each other. Of course, attempting to construct generating sets consisting of elements of a fixed order $k$ only makes sense if $\text{Mod}(S_g)$ contains elements of order $k$ in the first place. A construction of Tucker [1] guarantees an element of any fixed order $k$ in $\text{Mod}(S_g)$ whenever $g$ is sufficiently large, as described in Section 2.

Later in the introduction we describe prior work by several authors on generating $\text{Mod}(S_g)$ with elements of fixed finite orders 2, 3, 4, and 6. In each case, the authors show that the number of generators required is independent of $g$. Set alongside this prior work, a new phenomenon that emerges in our results is that the sizes of our generating sets for $\text{Mod}(S_g)$ are not only independent of the genus of the surface, but they are also independent of the order of the elements.

Our generators are all finite-order elements that can be realized by rotations of $S_g$ embedded in $\mathbb{R}^3$. There are values of $g$ and $k$ where there exist elements of order $k$ in $\text{Mod}(S_g)$, but where these cannot be realized as rotations of $S_g$ embedded in $\mathbb{R}^3$. For instance, there are elements of order 7 in $\text{Mod}(S_3)$ that cannot be realized in this way.
**Problem 1.2.** Extend Theorem 4.1 to cases where elements of order \( k \) exist in \( \text{Mod}(S_g) \) but cannot be realized as rotations of \( S_g \) embedded in \( \mathbb{R}^3 \).

Our results in Chapter 6 provide an answer to this problem.

We can also seek smaller generating sets for \( \text{Mod}(S_g) \) consisting of elements of order \( k \). We note that any such sharpening of Theorem 4.1 would seem to demand a new approach. Our proofs hinge on applications of the lantern relation, and a lantern has only a limited number of symmetries.

**Problem 1.3.** For fixed \( k \geq 3 \) and any \( g \geq 3 \) where elements of order \( k \) exist, can \( \text{Mod}(S_g) \) be generated by two elements of order \( k \)? What about three elements for orders 4 and 5?

**Background and prior results**

The most commonly-used generating sets for \( \text{Mod}(S_g) \) consist of Dehn twists, which have infinite order. Dehn [2] showed that \( 2g(g-1) \) Dehn twists generate \( \text{Mod}(S_g) \), and Lickorish [3] showed that \( 3g + 1 \) Dehn twists suffice. Humphries [4] showed that only \( 2g + 1 \) Dehn twists are needed, and he also showed that no smaller set of Dehn twists can generate \( \text{Mod}(S_g) \). The curves for these Dehn twists are depicted in Figure 1.1.

There have also been many investigations into constructing generating sets for \( \text{Mod}(S_g) \) that include or even consist entirely of periodic elements. For instance, Maclachlan [5] showed that \( \text{Mod}(S_g) \) is normally generated by a set of two periodic elements that have orders \( 2g + 2 \) and \( 4g + 2 \), and McCarthy and Papadopoulos [6] showed that \( \text{Mod}(S_g) \) is normally generated by a single involution (element of order 2) for \( g \geq 3 \). Korkmaz [7] showed that \( \text{Mod}(S_g) \) is generated by two elements of order \( 4g + 2 \) for \( g \geq 3 \).

Luo [8] explicitly constructed a finite generating set for \( \text{Mod}(S_g) \) consisting of \( 6(2g+1) \) involutions, given that \( g \geq 3 \). Luo asked whether there exists a universal upper bound (that is, independent of \( g \)) for the number of involutions required to generate \( \text{Mod}(S_g) \). Brendle and Farb [9] showed that six involutions suffice to generate \( \text{Mod}(S_g) \), again for \( g \geq 3 \). Kassabov [10] sharpened this result by showing that only five involutions are needed for
$g \geq 5$ and only four are needed for $g \geq 7$. Monden [11] showed that $\text{Mod}(S_g)$ can be generated by three elements of order 3 and by four elements of order 4, each for $g \geq 3$. Recently Yoshihara [12] has shown that $\text{Mod}(S_g)$ can be generated by three elements of order 6 when $g \geq 10$ and by four elements of order 6 when $g \geq 5$.

![Figure 1.1: The $2g + 1$ Humphries curves in $S_g$.](image)

Much work has been done to establish when elements of a particular finite order exist in $\text{Mod}(S_g)$. In his paper, Monden noted that for all $g \geq 1$, $\text{Mod}(S_g)$ contains elements of orders 2, 3, and 4. Aside from order 6, elements of larger orders do not always exist in $\text{Mod}(S_g)$. For example, $\text{Mod}(S_3)$ contains no element of order 5 and $\text{Mod}(S_4)$ contains no element of order 7.

Determining the orders of the periodic elements in $\text{Mod}(S_g)$ for any particular $g$ is a solved problem, at least implicitly. In fact, this is even true for determining the conjugacy classes of periodic elements in $\text{Mod}(S_g)$. Ashikaga and Ishizaka [13] listed necessary and sufficient criteria for determining the conjugacy classes in $\text{Mod}(S_g)$ for any particular $g$. The criteria are number theoretic and consist of the Riemann–Hurwitz formula, an upper bound of $4g + 2$ on the order of elements due to Wiman, an integer-sum condition on the valencies of the ramification points due to Nielsen, and several conditions on the least common multiple of the ramification indices that are due to Harvey.

Ashikaga and Ishizaka also gave lists of the conjugacy classes of periodic elements in $\text{Mod}(S_1)$, $\text{Mod}(S_2)$, and $\text{Mod}(S_3)$. Hirose [14] gave a list of the conjugacy classes of periodic elements in $\text{Mod}(S_4)$. Broughton [15] listed criteria for determining actions of finite groups on $S_g$, and hence for determining conjugacy classes of finite subgroups of...
Mod($S_g$). Broughton also gave a complete classification of actions of finite groups on $S_2$ and $S_3$. Kirmura [16] gave a complete classification for $S_4$.

Several results have been proved about guaranteeing the existence of elements of order $k$ in Mod($S_g$) for sufficiently large $g$. Harvey [17] showed that Mod($S_g$) contains an element of order $k$ whenever $g \geq (k^2 - 1)/2$. Glover and Mislin [18] showed that Mod($S_g$) contains an element of order $k$ whenever $g > (2k)^2$. A fundamental result in this direction was shown by Kulkarni [19]: for any finite group $G$, the $g$ for which $G$ acts faithfully on $S_g$ all fall in some infinite arithmetic progression; and further, all but finitely many values in the arithmetic progression are admissible $g$.

Tucker [1] gave necessary and sufficient conditions for the existence of an element of order $k$ in Mod($S_g$) that can be realized as a rotation of $S_g$ embedded in $\mathbb{R}^3$. Using this characterization, Tucker showed that for any $k$ and for sufficiently large $g$, Mod($S_g$) contains an element of order $k$ that is realizable by a rotation of $S_g$ embedded in $\mathbb{R}^3$. We give a proof of this fact in Lemma 2.1.
In this section we construct elements of order \(k\) in \(\text{Mod}(S_g)\) whenever \(g\) is sufficiently large. We will use elements that are conjugate to these elements when we build our generating sets.

The following result gives sufficient conditions for the existence of elements of order \(k\) in \(\text{Mod}(S_g)\) that can be realized by a rotation of \(S_g\) embedded in \(\mathbb{R}^3\). The result was proved by Tucker [1], who additionally showed that these sufficient conditions are in fact necessary. We include a proof of the result in order to establish conventions about the geometric realizations of the elements it guarantees, as these geometric realizations will be important in proving our main result.

**Lemma 2.1.** Let \(k \geq 2\). Then \(\text{Mod}(S_g)\) contains an element of order \(k\) that can be realized as a rotation of \(S_g\) embedded in \(\mathbb{R}^3\) whenever \(g > 0\) can be written as \(ak + b(k-1)\) with \(a, b \in \mathbb{Z}_{\geq 0}\) or as \(ak + 1\) with \(a \in \mathbb{Z}_{>0}\).

**Proof.** In Figure 2.1 we depict two ways of embedding a surface in \(\mathbb{R}^3\) so that it has \(k\)-fold rotational symmetry. First, we can embed a surface of genus \(k\) in \(\mathbb{R}^3\) so that it has a rotational symmetry of order \(k\) by evenly spacing \(k\) handles about a central sphere. We can also embed a surface of genus \(k - 1\) in \(\mathbb{R}^3\) so that it has a rotational symmetry of order \(k\), as follows. Arrange two spheres along an axis of rotation and remove \(k\) disks from each sphere, evenly spaced along the equator of each. Then connect pairs of boundary components, one from each sphere, with a cylinder. This can be done symmetrically so that a rotation by \(2\pi/k\) permutes the cylinders cyclically.

We can use these two types of embeddings to construct embeddings of surfaces of higher genus that also have rotational symmetry of order \(k\). Whenever \(g = ak + b(k-1)\),
we can construct an embedding of $S_g$ in $\mathbb{R}^3$ by taking a connected sum of surfaces of genus $k$ and $k - 1$ along their axis of rotational symmetry. See the left of Figure 2.2 for an example. Rotating a surface embedded in this way by $2\pi/k$ produces an element of $\text{Mod}(S_g)$ of order $k$ for any genus $g = ak + b(k - 1)$. That an element so formed does not have order less than $k$ can be seen by the element’s action on homology.

In order to produce elements of order $k$ in the case where $g = ak + 1$, we first construct a surface of genus $ak$ with $k$-fold rotational symmetry by the above construction. We can modify this surface to increase its genus by 1 while preserving its symmetry as follows. See the right of Figure 2.2. The axis of a genus $ak$ surface intersects the surface
at two points—at the top and the bottom. Removing an invariant disk around each of these points creates two boundary components. Connecting the two boundary components with a cylinder yields an embedding of a surface of genus $ak + 1$ with $k$-fold symmetry.

By way of some elementary number theory, we show that all sufficiently large integers have either the form $ak + b(k - 1)$ or the form $ak + 1$.

**Lemma 2.2.** If $k \geq 5$ and $g \geq (k - 1)(k - 3)$, then $g$ can either be written in the form $ak + b(k - 1)$ with $a, b \in \mathbb{Z}_{\geq 0}$ or in the form $ak + 1$ with $a \in \mathbb{Z}_{>0}$.

**Proof.** All integers at least $(k - 1)(k - 2)$ can be written in the form $ak + b(k - 1)$ with $a, b \in \mathbb{Z}_{\geq 0}$ by the solution to the Frobenius coin problem. Further, every number from $(k - 1)(k - 3)$ to $k(k - 3)$ can also be written as a sum of $k$’s and $k - 1$’s. Start with $k - 3$ copies of $k - 1$ and replace the $k - 1$’s one at a time by $k$’s. Finally, $k(k - 3) + 1 = (k - 1)(k - 2) - 1$ is of the form $ak + 1$.

In addition to producing elements of order $k$ in the stable range $g \geq (k - 1)(k - 3)$, we note that the construction given in Lemma 2.1 is also valid for approximately half of the values of $g$ less than $(k - 3)(k - 1)$. Specifically, $(k^2 - 3k - 4)/2$ of these $k^2 - 4k + 2$ smaller values of $g$ are either of the form $ak + b(k - 1)$ or $ak + 1$. This amount is simply $\sum_{i=3}^{k-2} i$, since $\{k - 1, k, k + 1\}$ is the first run of numbers of the given forms and $\{(k - 4)(k - 1), \ldots, (k - 4)k + 1\}$ is the last run less than $(k - 1)(k - 3)$.

Also, note that Lemma 2.1 includes the cases where $k$ is 2, 3, or 4. However, the construction we use to create the generating sets of Theorem 4.1 does not work for these small orders. However, these values of $k$ are those already treated by Luo, Brendle and Farb, Kassabov, and Monden in their work on generating sets for $\text{Mod}(S_g)$ consisting of elements of fixed finite order.
CHAPTER 3
BUILDING A DEHN TWIST

In this section, we show that a Dehn twist in $\text{Mod}(S_g)$ about a nonseparating curve may be written as a product in four elements whenever these elements act on a small collection of curves in a specified way. In fact, even fewer than four elements will suffice as long as products in these elements act on the collection of curves as specified. In our proof, we follow the argument that Luo [8] gave for writing a Dehn twist as a product of involutions, as well as the pair swap argument made by Brendle and Farb [9].

We write $T_c$ for the (left) Dehn twist about the curve $c$. Recall the lantern relation that holds among Dehn twists about seven curves arranged in a sphere with four boundary components, called a lantern. In the left of Figure 3.1 we depict a lantern $L$ that is a subsurface of $S_g$. Singling out this particular lantern is convenient for our proof of Theorem 4.1. Note that $S_g \setminus L$ is connected. Seven curves lie in $L$ in a lantern arrangement, and several of these are Humphries curves. We will call these seven curves lantern curves. We have the following lantern relation:

$$T_{\alpha_1}T_{\alpha_2}T_{x_1}T_{\gamma_2} = T_{\gamma_1}T_{x_3}T_{x_2}.$$ 

Recall also that for a Dehn twist $T_c$ and a mapping class $f$, we have $fT_c f^{-1} = T_{f(c)}$.

**Lemma 3.1.** Suppose we are given the subsurface $L$ in $S_g$ and elements $f$, $g$, and $h$ in $\text{Mod}(S_g)$ such that

$$f(\gamma_1) = \gamma_2$$
$$g(x_3, x_1) = (\gamma_1, \gamma_2)$$
$$h(x_2, \alpha_2) = (\gamma_1, \gamma_2).$$
Then the Dehn twist $T_{\alpha_1}$ may be written as a product in $f, g, h$, an element conjugate to $f$, and their inverses.

While this lemma is stated for a specific Dehn twist by way of a specific lantern, the result holds for other Dehn twists by the change of coordinates principle: if two collections of curves on a surface $S_g$ are given by the same topological data, then there exists a homeomorphism of $S_g$ to itself that maps the first collection of curves to the second. Details are given by Farb and Margalit [20].

**Proof.** Since Dehn twists about nonintersecting curves commute, one form of the lantern relation for $L$ is

$$T_{\alpha_1} = (T_{\gamma_1} T_{\gamma_2}^{-1})(T_{x_3} T_{x_1}^{-1})(T_{x_2} T_{\alpha_2}^{-1}).$$

Applying the assumptions on the elements $g$ and $h$ yields

$$T_{\alpha_1} = (T_{\gamma_1} T_{\gamma_2}^{-1})(g^{-1}(T_{\gamma_1} T_{\gamma_2}^{-1})g)(h^{-1}(T_{\gamma_1} T_{\gamma_2}^{-1})h).$$

Applying the assumption on the element $f$ and regrouping yields

$$T_{\alpha_1} = ((f^{-1}T_{\gamma_2} f)T_{\gamma_2}^{-1})(g^{-1}(f^{-1}T_{\gamma_2} f)T_{\gamma_2}^{-1} g)(h^{-1}(f^{-1}T_{\gamma_2} f)T_{\gamma_2}^{-1} h)$$

$$= (f^{-1}(T_{\gamma_2} f T_{\gamma_2}^{-1}))(g^{-1} f^{-1}(T_{\gamma_2} f T_{\gamma_2}^{-1}) g)(h^{-1} f^{-1}(T_{\gamma_2} f T_{\gamma_2}^{-1}) h).$$

We have written $T_{\alpha_1}$ as a product in $f, g, h, T_{\gamma_2} f T_{\gamma_2}^{-1}$, and their inverses.

Note that if $f$ has order $k$, then so does $T_{\gamma_2} f T_{\gamma_2}^{-1}$ since it is a conjugate of $f$. Finally, notice that we required very little of $f, g, h$ in this argument—only that they map one specific curve or one specific pair of curves to another. We will take advantage of this flexibility in the proof of Theorem 4.1.
Figure 3.1: On the left, the subsurface $L$ and five of the lantern curves. Two lantern curves are omitted for clarity. On the right, the subsurface $L$ and all seven lantern curves.
CHAPTER 4
GENERATING Mod(Sg) WITH FOUR ELEMENTS OF ORDER k

In this section and in the following section we prove the two parts of the following theorem, which is our main technical result.

Theorem 4.1. (1) Let \( k \geq 5 \) and let \( g > 0 \) be of the form \( ak + b(k - 1) \) with \( a, b \in \mathbb{Z}_{\geq 0} \) or of the form \( ak + 1 \) with \( a \in \mathbb{Z}_{>0} \). Then \( \text{Mod}(S_g) \) is generated by four elements of order \( k \).

(2) Let \( k \geq 8 \) or \( k = 6 \) and let \( g > 0 \) be of the form \( ak + b(k - 1) \) with \( a, b \in \mathbb{Z}_{\geq 0} \). Then \( \text{Mod}(S_g) \) is generated by three elements of order \( k \). If instead \( k = 7 \) and \( g \) is of the form \( 7 + 7a + 6b \) with \( a, b \in \mathbb{Z}_{\geq 0} \), then \( \text{Mod}(S_g) \) is generated by three elements of order 7.

Theorem 1.1 in the introduction follows directly from Theorem 4.1 along with Lemma 2.2 (for the case \( g = 5 \)) and the observation that any \( g \geq (k - 1)^2 + 1 \) may be written as a sum of \( k \)'s and \((k - 1)\)'s with at least one summand equal to \( k \).

In this section we prove the first part of Theorem 4.1 about generating with four elements of fixed finite order. In order to illustrate our construction, we depict the particular case \( k = 5 \) and \( g = 18 \) in Figure 4.3. In what follows, a chain of curves on a surface is a sequence of curves \( c_1, \ldots, c_t \) such that pairs of consecutive curves in the sequence intersect exactly once and each other pair of curves is disjoint.

Proof of Theorem 4.1, (1). We begin with the case where \( g = ak + b(k - 1) \) and treat the case where \( g = ak + 1 \) with a small modification at the end of the proof. Since \( g = ak + b(k - 1) \), we have a \( k \)-fold symmetric embedding of \( S_g \) in \( \mathbb{R}^3 \) as constructed in Lemma 2.1. Call this embedded surface \( \Sigma_g \) and let it be comprised of \( a \) surfaces of genus \( k \) followed by \( b \) surfaces of genus \( k - 1 \). Let \( \sigma_1 \) through \( \sigma_{a+b} \) denote these \( k \)-symmetric subsurfaces of \( \Sigma_g \). Let \( r \) be a rotation of \( \Sigma_g \) by \( 2\pi/k \) about its axis.
We will construct our desired elements by mapping $S_g$ to $\Sigma_g$, performing a rotation $r$, and then mapping back to $S_g$. In doing so we will specify how individual curves map over and back again, and so control how curves are permuted among themselves. In order to construct these maps, it is convenient to label curves on $S_g$ and $\Sigma_g$ as follows. Take on the one hand the usual embedding of the Humphries curves in $S_g$ as shown in Figure 1.1 and the upper-left of Figure 4.3. We will refer to the $\alpha_i$, $\beta_i$ and $\gamma_i$ curves as $\alpha$ curves, $\beta$ curves, and $\gamma$ curves, respectively. The Humphries curves consist of a chain of $2g - 1$ curves that alternate between $\beta$ curves and $\gamma$ curves as well as two additional $\alpha$ curves.

Similarly, take the $k$-fold symmetric embedded surface $\Sigma_g$ and embed in each $\sigma_i$ a chain of curves of length $2g_i - 1$, where $g_i$ is the genus of $\sigma_i$. See Figure 4.1 and the upper-left of Figure 4.3. We label the curves in these chains also as $\beta$ and $\gamma$ curves and note that they are embedded so that $r(\beta_i) = \beta_{i+1}$, $1 \leq i \leq g_i - 1$, and $r(\gamma_i) = \gamma_{i+1}$, $1 \leq i \leq g_i - 2$. We will use these labels as “local coordinates”—saying, for instance, “the $\beta_2$ curve in $\sigma_3$.” In $\sigma_1$ we additionally embed two $\alpha$ curves, $\alpha_1$ and $\alpha_2$, such that each respectively intersects $\beta_1$ and $\beta_2$ once, intersects no other curves, and $r(\alpha_1) = \alpha_2$.

We are now prepared to define three homeomorphisms $\hat{f}$, $\hat{g}$, and $\hat{h}$ from $S_g$ to $\Sigma_g$. We will use these maps to define three homeomorphisms of the form $\hat{f}^{-1}r\hat{f}$ and will show that the corresponding mapping classes (1) have order $k$, (2) satisfy Lemma 3.1, and (3) generate a subgroup that puts the Humphries curves into the same orbit.

We first construct a homeomorphism $\hat{f}$. The $\beta$ and $\gamma$ Humphries curves in $S_g$ form a chain of length $2g - 1$. By removing some of the $\gamma$ curves from this chain, we form $a + b$ smaller chains. The first $a$ chains will be $2k - 1$ curves long and the last $b$ chains will be $2k - 3$ curves long. We accomplish this by removing every $k$th $\gamma$ curve up to $\gamma_{ak}$, and then every $(k - 1)$st $\gamma$ curve thereafter. We call these the excluded $\gamma$ curves. Call the resulting chains $F_i$, keeping their sequential order. We add to $F_1$ the curves $\alpha_1$ and $\alpha_2$.

Note that the curves in each $F_i$ form a chain of simple closed curves in $S_g$ and the union of the $F_i$ is nonseparating. (Note that $F_1$ is not quite a chain because of the $\alpha_2$ curve.) By
the change of coordinates principle, there is a homeomorphism \( \hat{f} \) that takes curves in the \( F_i \) to the curves in the chains in \( \sigma_i \) as specified above, as these chains of curves have the same length. (Recall that in \( \sigma_1 \) we have two additional curves that correspond to the \( \alpha \) curves of \( S_g \).) Let \( f \) be the mapping class of \( \hat{f}^{-1}\tau\hat{f} \). Then \( f \) has order \( k \) and maps \( \gamma_1 \) to \( \gamma_2 \) as required by Lemma 3.1.

We now construct \( \hat{g} \). We form triples of curves \( G_i \), \( 2 \leq i \leq a + b \). To form each \( G_i \), we take the second-to-last \( \beta \) curve in \( F_{i-1} \), the excluded \( \gamma \) curve falling between \( F_{i-1} \) and \( F_i \), and the second \( \beta \) curve in \( F_i \).

Note that the curves in \( \bigcup_i G_i \) are in the complement of \( L \), that they are nonseparating simple closed curves, that they are disjoint, and that their union is nonseparating. By the change of coordinates principle, there is a homeomorphism \( \hat{g} \) that maps the curves in \( L \) and the curves \( \bigcup_i G_i \) to a collection of curves of the same topological type in \( \Sigma_g \) as follows:

\[
\hat{g}: S_g \rightarrow \Sigma_g
\]

\[
(x_3, x_1, \gamma_1, \gamma_2) \rightarrow (a, b, c, d) \text{ in } \sigma_1 \text{ as in Figure 4.2}
\]

\[
G_i \rightarrow (\beta_1, \beta_2, \beta_3) \text{ in } \sigma_i, \ 2 \leq i \leq a + b
\]

Note that the specified image curves are of the same topological type as the four curves in \( L \) and the curves in \( \bigcup_i G_i \). Note also that the embedding of the lantern curves depends on
whether the genus of $\sigma_1$ is $k$ or $k - 1$; see Figure 4.2. In both embeddings, the image of $L$ is again nonseparating. Let $g$ be the mapping class of $\hat{g}^{-1}r\hat{g}$. Then $g$ has order $k$ and maps the pair $(x_3, x_1)$ to the pair $(\gamma_1, \gamma_2)$ as required in Lemma 3.1.

Figure 4.2: The important curves of the subsurface $L$ as embedded by $\hat{g}$ and $\hat{h}$ in $\sigma_1$ when the genus of $\sigma_1$ is $k$ and when it is $k - 1$. The latter is depicted as seen from above.

Finally, we construct $\hat{h}$. We form pairs of curves $H_i$, $2 \leq i \leq a + b$. Each $H_i$ consists of the first $\beta$ curve and the second $\gamma$ curve in $F_i$. The map $\hat{h}$ also specifies the mapping of the Humphries curve $\beta_4$. Let $\hat{h}$ be a homeomorphism that maps curves as follows:

\[
\hat{h}: S_g \longrightarrow \Sigma_g
\]
\[
(\gamma_1, \gamma_2, x_2, \alpha_2) \mapsto (a, b, c, d) \text{ in } \sigma_1 \text{ as in Figure 4.2}
\]
\[
\beta_4 \mapsto r(d) \text{ in } \sigma_1
\]
\[
H_i \mapsto (\beta_1, \beta_2) \text{ in } \sigma_i, \ 2 \leq i \leq a + b
\]

Let $h$ be the mapping class of $\hat{h}^{-1}r\hat{h}$. Then $h$ has order $k$ and $h^{-1}$ maps the pair $(x_2, \alpha_2)$ to the pair $(\gamma_1, \gamma_2)$ as required by Lemma 3.1. (We use $h^{-1}$ here because we want all of our generators to be conjugate and because the lantern curves are in a fixed cyclic order.)

We now show that the Humphries curves are in the same orbit under $\langle f, g, h \rangle$. Refer to Figure 4.4. First, note that every $\beta$ and $\gamma$ Humphries curve in $S_g$ is in some $F_i$ or $G_i$. 

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Figure 4.3: The Humphries curves in $S_{18}$. $\Sigma_{18}$ with “local coordinate” curves in each $\sigma_i$. The curves in the $F_i$, $G_i$, $H_i$, and the subsurface $L$, along with their images under $\hat{f}$, $\hat{g}$, and $\hat{h}$. 
Figure 4.4: Each node is a collection of curves that are in the same orbit under the subgroup generated by a single element. Each arrow indicates when a power of an element maps a curve in one collection to a curve in another. Since every Humphries curve is in at least one of the collections, all Humphries curves are in the same orbit under the subgroup $\langle f, g, h \rangle$.

Additionally, powers of $f$ map any $\beta$ curve in $F_i$ to any other $\beta$ curve in the same $F_i$, and likewise for $\gamma$ curves. Call these orbits of curves $F_i\beta$ and $F_i\gamma$. In the same way, a power of $g$ maps any curve in $G_i$ to any other curve in the same $G_i$. Thus at most we have the following orbits of the Humphries curves under $\langle f, g, h \rangle$: the $F_i\beta$, the $F_i\gamma$, the $G_i$, $\alpha_1$, and $\alpha_2$. We will show that these are all in fact a single orbit under $\langle f, g, h \rangle$.

The element $f$ maps $\alpha_1$ to $\alpha_2$ when $\sigma_1$ has genus $k$ and maps $\alpha_1$ to $\gamma_1$ when $\sigma_1$ has genus $k - 1$. The element $g$ maps the lantern curve $\gamma_2$ to the lantern curve $\alpha_2$. Thus each of $\alpha_1$ and $\alpha_2$ is in the same orbit as some $\gamma$ curve.

A power of $g$ takes a curve in $F_i\beta$ to a curve in $F_{i-1}\beta$ as well as to a curve in $G_i$, $2 \leq i \leq a + b$. Additionally, a power of $h$ takes a curve in $F_i\beta$ to a curve in $F_i\gamma$, $1 \leq i \leq a + b$. (Note that in the case of $F_1$, we have $h^2(\gamma_2) = \beta_4$.) Thus all Humphries curves are in a single orbit under $\langle f, g, h \rangle$. By Lemma 3.1, the Dehn twist about $\alpha_1$ may be written as a product in $f$, $g$, and $h$, and $T_{\gamma_2}f^kT_{\gamma_2}^{-1}$. Thus all Dehn twists about the Humphries curves may be written as products in our four elements of order $k$, and so they generate $\text{Mod}(S_g)$.

In the case where $g = ak + 1$, we may modify the construction to show that $\text{Mod}(S_g)$ is again generated by four elements of order $k$. Take a connect sum of $a$ surfaces of genus $k$ and insert one further handle along the axis of rotation, as in Lemma 2.1. The element $r$ is a rotation of this embedded surface by $2\pi/k$ and $f$ is defined as above by ignoring
the final two Humphries curves $\beta_g$ and $\gamma_{g-1}$. We must modify our other elements of order $k$ so that they place these two additional Humphries curves into the same orbit as all of the other Humphries curves under the subgroup $\langle f, g, h \rangle$. Modify $\hat{g}$ so that it additionally maps $\beta_g$ to $r(d)$ in $\sigma_1$ and modify $\hat{h}$ so that it additionally maps $\gamma_{g-1}$ to $r^2(d)$ in $\sigma_1$. These modifications preserve the fact that the curves involved are disjoint and that their union is nonseparating. The elements $g$ and $h$ now put the curves $\beta_g$ and $\gamma_{g-1}$ into the same orbit as the other Humphries curves. Hence $\text{Mod}(S_g)$ is also generated by four elements of order $k$ when $g = ak + 1$. \qed
CHAPTER 5
SHARPENING TO THREE ELEMENTS

In this section we prove the second part of Theorem 4.1.

Proof of Theorem 4.1, (2). We first provide the construction for the cases \( k \geq 8 \) and then afterwards give the constructions for \( k = 7 \) and \( k = 6 \). Let \( k \geq 8 \). By assumption we may write \( g \) in the form \( ak + b(k - 1) \). We construct the homeomorphism \( \hat{f} : S_g \to \Sigma_g \) as in the proof of the first part of theorem, except with the modification that it additionally maps the \( \alpha \) curve that intersects the final \( \beta \) curve in \( F_1 \) (called \( \alpha_\ell \)) to the curve \( r^{-1}\hat{f}(\alpha_1) \). See Figure 5.1. We again let \( f \) be the mapping class of \( \hat{f}^{-1}r\hat{f} \), and \( f \) has order \( k \).

We now construct \( \hat{g} \). Let \( G_2 \) consist of \( \alpha_\ell \), the excluded \( \gamma \) curve falling between \( F_1 \) and \( F_2 \), the first \( \gamma \) curve in \( F_2 \), and the third \( \beta \) curve in \( F_2 \). For \( 2 < i \leq a + b \), let \( G_i \) be the last \( \gamma \) curve in \( F_{i-1} \), the excluded \( \gamma \) curve between \( F_{i-1} \) and \( F_i \), the first \( \gamma \) curve in \( F_i \), and the third \( \beta \) curve in \( F_i \). See Figure 5.2. Let \( \hat{g} \) be a homeomorphism that maps the specified curves as follows:

\[
\hat{g} : S_g \longrightarrow \Sigma_g \\
(x_3, x_1, \gamma_1, \gamma_2, x_2, \alpha_2) \longmapsto (a, b, c, d, e, f) \text{ as in Figure 5.3} \\
(\gamma_3, \gamma_4) \longmapsto (r^3(e), r^3(f)) \\
\beta_6 \longmapsto r^4(f) \text{ in } \sigma_1 \\
G_i \longmapsto (\beta_1, \beta_2, \beta_3, \beta_4) \text{ in } \sigma_i, \ 2 \leq i \leq a + b
\]

Let \( g \) be the mapping class of \( \hat{g}^{-1}r\hat{g} \). Then \( g \) has order \( k \) and maps the pair \( (x_3, x_1) \) to the pair \( (\gamma_1, \gamma_2) \) as required by Lemma 3.1. Additionally, \( g^3(x_2, \alpha_2) = (\gamma_3, \gamma_4) \) and \( f^{-2}(\gamma_3, \gamma_4) = (\gamma_1, \gamma_2) \). We may therefore define \( h = f^{-2}g^3 \) so that \( h \) satisfies the hypoth-
Figure 5.1: The images of \( \alpha \) curves embedded in \( \sigma_1 \) by \( \hat{f} \). With this embedding, the rotation \( r \) maps \( \alpha_\ell \) to \( \alpha_1 \).

Figure 5.2: The curves mapped by \( \hat{g} \) in the case \( k = 8, g = 21 \). This is a worst case example where \( k \) has the smallest possible value and all of the \( \sigma_i \) have genus \( k-1 \).

Figure 5.3: The embedding of the subsurface \( L \) in \( \sigma_1 \) when the genus of \( \sigma_1 \) is \( k \) and when it is \( k-1 \). Also, the embedding of \( \gamma_3, \gamma_4, \) and \( \beta_6 \). These diagrams depict the case \( k = 8 \).
esis of Lemma 3.1, as \( h(x_2, \alpha_2) = (\gamma_1, \gamma_2) \). Thus the Dehn twist about \( \alpha_1 \) may be written as a product in \( f, g, \) and \( T_{\gamma_2} f T_{\gamma_2}^{-1} \).

\[
\begin{array}{ccc}
G_2 & & G_3 \\
\uparrow & & \uparrow \\
F_1 \beta & F_2 \beta & F_3 \beta \\
\downarrow & \downarrow & \downarrow \\
F_1 \gamma & F_2 \gamma & F_3 \gamma \\
\end{array}
\]

\( h = f^{-2} g \)\( \beta \)\( g \)\( f \)\( a = 0 \)\( \alpha_1 \)\( \alpha_2 \)

Figure 5.4: Again, each node is a collection of curves that are in the same orbit under the subgroup generated by a single element. Each arrow indicates when a power of an element maps a curve in one collection to a curve in another. Since every Humphries curve is in at least one of the collections, all Humphries curves are in the same orbit under \( \langle f, g \rangle \).

Finally, we show that all of the Humphries curves are in the same orbit under \( \langle f, g \rangle \), as can be seen in Figure 5.4. Again, every \( \beta \) and \( \gamma \) Humphries curve in \( S_g \) is in some \( F_i \beta \), \( F_i \gamma \), or \( G_i \). Therefore we have at most the following orbits of the Humphries curves under \( \langle f, g \rangle \): the \( F_i \beta \), the \( F_i \gamma \), the \( G_i \), \( \alpha_1 \), and \( \alpha_2 \). We will show that these are all in fact a single orbit under \( \langle f, g \rangle \). For \( i > 2 \), powers of \( g \) put the curves in \( G_i, F_i \beta, F_i \gamma, \) and \( F_{i-1} \gamma \) in the same orbit. Note that powers of \( g \) map \( \beta_6 \) to \( \gamma_2 \) and a \( \gamma \) curve in \( F_2 \) to \( \alpha_\ell \), while the product \( h \) carries \( \alpha_2 \) to \( \gamma_2 \). Also, \( f \) maps \( \alpha_\ell \) to \( \alpha_1 \) and maps \( \alpha_1 \) either to \( \alpha_2 \) or \( \gamma_1 \), depending on the genus of \( \sigma_1 \). Considering this, all of the curves are in the same orbit under the subgroup \( \langle f, g \rangle \). Therefore the Dehn twist about each of the Humphries curves may be written as a product in the three elements \( f, g, \) and \( T_{\gamma_2} f T_{\gamma_2}^{-1} \), and so they generate \( \text{Mod}(S_g) \).

In the case where \( k = 7 \), the same construction as above goes through as long as the genus of \( \sigma_1 \) is 7. As illustrated in Figure 5.2, under this assumption there is enough room to configure all of the required curves in the construction of \( \hat{g} \). The hypotheses of the theorem in this case exactly demand that the genus of \( \sigma_1 \) be 7.

In the case where \( k = 6 \), we use the same construction as above for \( f \) and construct the
element $g$ as follows, exploiting the three-fold symmetry of a lantern. See Figure 5.5. Let $\hat{g}$ be a homeomorphism that maps the specified curves as follows:

$\hat{g} : S_g \rightarrow \Sigma_g$

$(x_3, x_1, \gamma_1, \gamma_2, x_2, \alpha_2) \mapsto (a, b, r^2(a), r^2(b), r^4(a), r^4(b))$ as in Figure 5.5

$\beta_4 \mapsto r(b)$ in $\sigma_1$

$G_i \mapsto (\beta_1, \beta_2, \beta_3, \beta_4)$ in $\sigma_i$, $2 \leq i \leq a + b$

In this case, $g^2$ and $g^4$ play the roles of $g$ and $h$ in Lemma 3.1, and all Humphries curves are again in the same orbit.
CHAPTER 6

UNIVERSAL BOUNDS FOR TORSION GENERATING SETS OF $\text{Mod}(S_g)$

6.1 Introduction

The author and Margalit proved the following theorem.

**Theorem 6.1.** [21, Theorem 1.1] For every $g \geq 3$, every nontrivial periodic mapping class that is not a hyperelliptic involution normally generates $\text{Mod}(S_g)$.

Based on this result and other corroborating evidence, the author and Margalit raised the following question, which was also recorded as Problem 4.3 by Margalit in his problems paper [22].

**Question 6.2.** [21, Question 3.4] Is there a number $N$, independent of $g$, so that if $f$ is a periodic normal generator of $\text{Mod}(S_g)$ then $\text{Mod}(S_g)$ is generated by $N$ conjugates of $f$?

It will sometimes be convenient for us to use the notation $N(f)$ for the number of conjugates of $f$ required to generate $\text{Mod}(S_g)$. Note that it is not difficult to give an upper bound on $N(f)$ for periodic normal generators that is linear in $g$. A soft bound is $N(f) \leq 24g + 12$, which follows from our Lemma 6.6 and the fact that $\text{Mod}(S_g)$ is generated by $2g + 1$ Dehn twists about non-separating curves.

The results in this paper resolve Question 6.2.

For the class of involutions—and therefore for the case of general periodic normal generators—we show that there is no universal bound on the number of conjugates of $f$ required to generate $\text{Mod}(S_g)$.

**Theorem 6.3.** There does not exist a number $N$, independent of $g$, so that if $f$ is a periodic normal generator of $\text{Mod}(S_g)$ with $|f| = 2$ then $\text{Mod}(S_g)$ is generated by $N$ conjugates of $f$.
Alternatively, this may be written as:

\[
\sup_{g \geq 3} \{ N(f) \mid f \in \text{Mod}(S_g) \text{ an involution normal generator} \} = \infty
\]

This result should be viewed in contrast with the results by Brendle–Farb, Kassabov, Korkmaz, and Yildiz giving involution generating sets for \( \text{Mod}(S_g) \) of universally bounded size [9, 10, 23, 24]. For these results, the involutions in a given generating set need not all be conjugate, and the conjugacy classes used are hand-picked, rather than arbitrary.

On the other hand, we show that involutions are the exceptional case: there does exist a universal upper bound \( N \) under the assumption that \( |f| \geq 3 \).

**Theorem 6.4.** There a number \( N \), independent of \( g \), so that if \( f \) is a periodic normal generator of \( \text{Mod}(S_g) \) with \( |f| \geq 3 \) then \( \text{Mod}(S_g) \) is generated by \( N \) conjugates of \( f \).

Our proof of this theorem shows that \( N \) may be taken to be 60. This can be rephrased as: for \( g \geq 3 \) and \( f \) periodic with \( |f| \geq 3 \), \( N(f) \leq 60 \). We can of course ask what the sharp universal upper bound may be under the hypothesis \( |f| \geq 3 \); we know of no obstacle to it being \( N = 2 \).

**Prior results**

There have been many results about generating mapping class groups with torsion; an overview is given in an earlier paper of the author [25]. In terms of giving upper bounds on the sizes of torsion generating sets for \( \text{Mod}(S_g) \) consisting of conjugate elements, the following results were previously known. Korkmaz showed that two conjugate elements of order \( 4g + 2 \) generate \( \text{Mod}(S_g) \) for \( g \geq 3 \). The author showed that three conjugate elements of order \( k \geq 6 \) generate \( \text{Mod}(S_g) \) for \( g \geq (k - 1) + 1 \) [25]; all of the elements used in the constructions of that paper can be realized as rotations of an embedding of \( S_g \) in \( \mathbb{R}^3 \).
In Section 2 we prove our results about generating sets for $\text{Mod}(S_g)$ comprised of conjugates of an involution. In Section 3 we describe our proof strategy for Theorem 6.4, prove several preliminary technical lemmas, and then apply these to prove Theorem 6.4.

### 6.2 Involution generating sets

Every pair $(r, s)$ of non-negative integers determines a conjugacy class of involution homeomorphisms on $S_g$ where $g = 2r + s$; we denote a representative of this class by $i_{r,s}$. The involution $i_{r,s}$ rotates $r$ pairs of handles to swap them and “skewers” $s$ handles, as illustrated in Figure 6.1. Pairs $(r, s)$ with $g = 2r + s$ in fact parametrize conjugacy classes of involutions in $\text{Mod}(S_g)$ for $g \geq 2$, a classification that goes back to the work of Klein; see, for instance, the survey by Dugger [26]. An involution $i_{r,s}$ induces an action on $H_1(S_g; \mathbb{Q})$ that preserves $2s$ subspaces of dimension 1 and induces involutions on $r$ pairs of subspaces of dimension 2.

The intersection of the preserved subspaces of the generators of a group action is also preserved by the group. Consequently, any generating set for $\text{Mod}(S_g)$ comprised of conjugates of $i_{r,s}$ must contain least $(2r + s)/2r$ generators, since $\text{Mod}(S_g)$ acts transitively on 1-dimensional subspaces of $H_1(S_g; \mathbb{Q})$. The quantity $(2r + s)/2r$ is arbitrarily large whenever we let $s$ be sufficiently large compared to $r$.

Note that $i_{0,g}$ is a hyperelliptic involution in $\text{Mod}(S_g)$ and it is not a normal generator of $\text{Mod}(S_g)$; for $i_{0,g}$, the quantity $(2r + s)/2r$ is undefined.

---

Figure 6.1: A schematic for the involution $i_{r,s}$, where an involution $i_{3,4}$ on $S_{10}$ is illustrated.
While (2r + s)/2r is a lower bound on the number of conjugates of \( i_{r,s} \) required to generate \( \text{Mod}(S_g) \), we can also give an easy upper bound. This will also show that with the constraint \((2r + s)/2r < n\), there exists a universal bound \( N(n) \) on the number of conjugates of \( i_{r,s} \) required to generate \( \text{Mod}(S_g) \). Our argument additionally serves as a warm-up for our proof of Theorem 6.4.

Let \( H_g \) be the set of curves \( \{a_1, \ldots, a_{2g}, b\} \) corresponding to the Humphries generating set, where the curves \( a_i \) form a chain of length \( 2g \); the curves \( H_g \) are depicted in Figure 6.2. Each generator that is a conjugate of \( i_{r,s} \) puts \( 2r \) disjoint non-separating curves into \( r \) 2-cycles, and these \( 2r \) curves are collectively non-separating. Therefore in a subgroup generated by \( k = \lceil g/2r \rceil = \lceil (2r + s)/2r \rceil \) generators, we can ensure that each of \( g \) disjoint curves is in a 2-cycle, and that these are collectively non-separating. In the subgroup generated by \( k \) additional generators, the same holds true for all \( 2g \) curves in \( H_g - \{b\} \). We may add a further \( 2k - 1 \) generators so that the curves in \( H_g - \{b\} \) are all in the same orbit under the generated subgroup, and adding one further generator ensures that \( b \) is also in the same orbit. So far this is a total of \( 4k \) generators. Since \( T_b \) can be written as a product in at most an additional 6 generators by the lantern relation trick (see Lemma 6.6), we have that all Dehn twists about the curves in \( H_g \) are in the generated subgroup, which therefore equals \( \text{Mod}(S_g) \). Summing up, we have that \( N(i_{r,s}) \leq 4k + 6 = 4 \cdot \lceil (2r + s)/2r \rceil + 6 \).

We record the results of this section in the following theorem.

**Theorem 6.5.** Let \( g \geq 3 \) and let \( i_{r,s} \) be an involution in \( \text{Mod}(S_g) \) that is not a hyperelliptic involution. Any generating set for \( \text{Mod}(S_g) \) consisting of conjugates of the involution \( i_{r,s} \) contains at least \( (2r + s)/2r \) generators. Further, \( 4 \cdot \lceil (2r + s)/2r \rceil + 6 \) conjugates of \( i_{r,s} \) suffice to generate \( \text{Mod}(S_g) \).

This result immediately implies Theorem 6.3, stated in the introduction.
6.3 A universal bound for non-involution torsion generating sets

In this section we prove our main result, Theorem 6.4. We begin by outlining our proof strategy, which breaks up into three steps. We then prove the lemmas that carry out the first two steps and then conclude with our proof of Theorem 6.4, which carries out the final step.

Proof strategy

The proof of Theorem 6.4 follows the same basic strategy taken by the author in his article [25], which gives a sharper version of Theorem 6.4 for a certain class of periodic elements (ones that can be represented by rotations of $S_g$ embedded in $\mathbb{R}^3$) under the further assumption that $|f| \geq 5$. It is also the strategy used in the proof of Theorem 6.5 above.

To prove Theorem 6.4, it suffices to show that each of the generators in the Humphries generating set of $2g+1$ Dehn twists about nonseparating curves lies in a subgroup generated by 60 conjugates of an arbitrary periodic element $f$ with $|f| \geq 3$. Again, call the set of Humphries curves $H_g$; it consists of a chain of curves $a_1, \ldots, a_{2g}$ and an additional curve $b$. See Figure 6.2. By passing to powers, it suffices to consider $f$ where $|f|$ is either 4 or an odd prime.

The proof then consists of three steps. First, using the lantern relation trick, we show that a single Dehn twist about a non-separating curve can be written as a product in at most 12 distinct conjugates of $f$. We show this as Lemma 6.6.

It remains to show that the curves in $H_g$ lie in a single orbit under the action of a subgroup generated by at most 48 conjugates of $f$. As the second step, we show that there
exists a maximal non-separating chain $C$ of $2g$ curves in $S_g$ such that $f$ acts on a subset $C^*$ of $C$ of size at least $g/4$ where every orbit under $\langle f \rangle$ that contains an element of $C^*$ in fact contains at least two elements of $C^*$. That is, we show that $f$ shuffles a nice collection of curves, of size a definite fraction of $g$, in a nice way. This is shown for irreducible elements in Lemma 6.8 and for reducible elements in Lemma 6.9. We conclude by proving the main result, which involves showing that at most 48 conjugates of $f$ generate a subgroup of $\text{Mod}(S_g)$ that acts transitively on $H_g$; along with the 12 conjugates of $f$ that generate a subgroup that contains $T_b$, this yields a generating set for $\text{Mod}(S_g)$ consisting of at most 60 conjugates of $f$.

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**Step 1**

Our first lemma is a straightforward consequence of the work in Sections 2 and 3 of the author’s paper with Margalit [21].

**Lemma 6.6.** Let $g \geq 3$ and let $f \in \text{Mod}(S_g)$ be a periodic mapping class with $|f| \geq 3$. Then the Dehn twist about any fixed non-separating curve in $S_g$ can be written as a product in at most 12 distinct conjugates of $f$.

**Proof.** By the proof of Theorem 1.1 of [21], there exists a non-trivial power $f^k$ of $f$ and a curve $c$ in $S_g$ such that

1. $c$ is non-separating and $c$ and $f^k(c)$ are disjoint and non-homologous,
2. $c$ is non-separating and $c$ and $f^k(c)$ intersect exactly once, or
3. $c$ is separating and $c$ and $f^k(c)$ are disjoint.

The third case implies the existence of a non-separating curve $d$ such that $d$ and $f^k(d)$ are disjoint and non-homologous, and so reduces to the first case. The second case implies that there is a conjugate $g$ of $f$ such that $c$ and $g^k f^k(c)$ are disjoint and non-homologous, and so reduces to the first case at the cost of doubling the number of conjugates of $f$. 

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required. Finally, using the lantern relation trick, the first case shows that a single Dehn twist about a non-separating curve can be written as a product of 6 conjugates of \( f \). (See for instance [20, Theorem 7.16].) So considering all cases, a single Dehn twist about a given non-separating curve can be written as a product in 12 conjugates of \( f \). \( \square \)

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**Step 2**

We now proceed to the main technical work of the paper, where we guarantee that a definite fraction, independent of \( g \), of the \( 2g + 1 \) curves in \( H_g \) can be shuffled around by a periodic element \( f \) with \( |f| \geq 3 \). We begin by classifying the irreducible mapping classes of odd prime order and of order 4.

**Lemma 6.7.** Let \( p \) be either an odd prime or 4. Then there exists a unique \( g(p) \geq 1 \) such that \( \text{Mod}(S_{g(p)}) \) contains an irreducible element of order \( p \). Up to conjugacy and powers, \( \text{Mod}(S_{g(p)}) \) contains exactly 1 such element when \( p = 4 \), which has signature \((4, 0; (1, 2), (1, 4), (1, 4))\); and finitely many such elements when \( p \) is an odd prime, each with signature of the form \((p, 0; (c_1, p), (c_2, p), (c_3, p))\), with \( 0 < c_i < p \).

**Proof.** Let \( f \) be an irreducible periodic mapping class of order \( p \), with \( p \) equaling either an odd prime or 4. By a result of Gilman [27], any irreducible periodic mapping class has as its quotient orbifold a sphere with three marked points. Since the index of each marked point is greater than 1 and must divide \( p \), each index must equal \( p \) when \( p \) is prime and must equal 2 or 4 when \( p \) is 4. By the Riemann–Hurwitz formula, we have when \( p \) is an odd prime

\[
2g - 2 = p(0 - 2) + 3(p - 1)
\]

And so \( g(p) = \frac{p-1}{2} \).

Similarly, when \( p = 4 \), the possible triples of indices are \( \{2, 2, 4\} \), \( \{2, 4, 4\} \), and \( \{4, 4, 4\} \), since the LCM of the indices must be 4. We have, respectively,
\[2g - 2 = 4(0 - 2) + 7\]

\[2g - 2 = 4(0 - 2) + 8\]

\[2g - 2 = 4(0 - 2) + 9\]

Only the second equation, corresponding to \(\{2, 4, 4\}\), yields a natural number for \(g\), so we have \(g(4) = 1\).

The conjugacy class of a periodic element (that is not a free action) is determined by its signature \((n, g_0; (c_1, n_1), \ldots, (c_{\ell}, n_{\ell}))\). For our irreducible element of order \(p\), we have either

\[(p, 0; (c_1, p), (c_2, p), (c_3, p))\]

when \(p\) is an odd prime and

\[(4, 0; (c_1, 2), (c_2, 2), (c_3, 4))\]

when \(p = 4\), with \(0 < c_i < p\). In each case, there is a requirement that \(\sum_i c_i = 0 \pmod{p}\). We see that there are finitely many combinations. For \(p = 4\), we have that there are two conjugacy classes: \((4, 0; (1, 2), (1, 4), (1, 4))\) and \((4, 0; (1, 2), (1, 4), (1, 4))\). These are powers of each other.

The next two lemmas form the main technical result of this paper. Under the hypothesis that \(|f|\) is an odd prime or 4, we give a topological decomposition of \(f\) and use this to show that \(f\) shuffles a definite fraction of the curves in a maximal non-separating chain in \(S_g\). Our decomposition applies work of Gilman [27] and follows the outline of a geometric decomposition for general periodic elements given by Parsad–Rajeevsarathy–Sanki [28];
see especially their Theorem 2.27. Our decomposition also resembles the construction of periodic elements of prime order in the thesis of Chrisman, although Chrisman’s goal is to produce periodic homeomorphisms that realize any given number of branch points [29, Chapter 3]. (That is, Chrisman gives a construction realizing each possible number of branch points for prime order elements, but he does not show that every prime order conjugacy class can be built in his manner.)

Lemma 6.8. Let $p$ be either an odd prime or $4$. Let $g = g(p)$ and let $f \in \text{Mod}(S_g)$ be an irreducible element of order $p$. Then there exists a set of curves $C^*$ in $S_g$ such that

- $C^*$ is a subset of a maximal non-separating chain of curves in $S_g$,
- $|C^*| \geq 2g/5$, (alternatively, $|C^*| \geq p/4 - 1$),
- every orbit of curves under the action of $\langle f \rangle$ that contains a curve in $C^*$ contains at least two curves in $C^*$, and
- each subset of curves in $C^*$ that lies in the same orbit under $\langle f \rangle$ forms either a chain of length at least 2 or else consists of disjoint curves.

In particular, when $p$ is 3, 4, 5, 7, or 11, we may take $C^*$ so that $|C^*|$ is 2, 2, 2, 3, or 3, respectively.

Proof. We first treat the case $p = 4$. Then $S_g$ is the torus $T$ and $f$ can be realized by a rotation of a square torus by $\pi/2$ or $3\pi/2$. In either case, $f$ exchanges the meridian and longitude of $T$, which form a chain of length 2. We take this chain to be $C^*$. We have that $|C^*| = 2 > 1 = g$.

Next we let $p \geq 13$ be an odd prime; we treat the cases $p \in \{3, 5, 7, 11\}$ afterwards. Since $f$ has a fixed point, by a result of Kulkarni, $f$ can be represented as a rotation of a polygon with an appropriate side pairing [30, Theorem 2]. Parsad–Rajeevsarathy–Sanki give an explicit construction for Kulkarni’s existence result [28, Theorem 2.10]. In particular, they show that $f$ can be realized by a rotation of a $2p$-gon $D$ by $2\pi m/p$, where
\((m, p) = 1\). The sides \(d_i\) of \(D\) are identified in pairs (in an orientation-preserving way) according to the formula \(d_i \sim d_{i+k}\), where indices are taken mod \(2p\) and where \(k\) is a constant that depends on the signature for \(f\). In other words, each pair of sides that is identified is at a fixed distance \(k\) apart.

Consider three cases, depending on whether \(k\) is 0, 1, or 2 (mod 3). In each case we will construct a set of curves \(C^*\) so that \(|C^*| \geq 2g/5\). In each case, we will select curves from an orbit of a curves \(c_i\) under \(\langle f \rangle\), \(1 \leq i \leq p\), where each curve is formed by the segments connecting midpoints of two pairs of edges: \(d_i\) and \(d_{i+2}\), and \(d_{i+k}\) and \(d_{i+k+2}\). Each of these curves is non-separating, and we will take a collection so that they form a subset of a chain. Further, these curves are all in the same orbit, and so satisfy the third and fourth properties of \(C^*\) in the statement.

When \(k = 0\) (mod 3), we may take \(\lfloor 2p/6 \rfloor\) of the \(c_i\) so that they are disjoint and together do not separate. This case is illustrated in Figure 6.3 in a case where \(p = 13\) and \(\lfloor 2p/6 \rfloor = 4\).

When \(k = 1\) (mod 3), we may take \(\lfloor 2p/7 \rfloor\) of the \(c_i\) so that they are disjoint and together do not separate.

When \(k = 2\) (mod 3), we may take \(\lfloor 2p/8 \rfloor\) of the \(c_i\) so that they are disjoint and together do not separate.

In each case, we have that \(|C^*| = \lfloor 2p/8 \rfloor = \lfloor (2g + 1)/4 \rfloor \geq 2g/5\) and also \(|C^*| \geq p/4 - 1\). For all odd primes \(p \geq 13\) we have that \(|C^*| \geq 2\) and that all of the curves in \(C^*\) belong to the same orbit, as desired.

For \(p \in \{3, 5, 7, 11\}\), a more careful analysis is required. In these cases the bound \(|C^*| \geq p/4 - 1\) is insufficient, since we require \(|C^*| \geq 2\). We give a modified construction for \(C^*\) in each case.

When \(p = 3\), \(g(p) = 1\). Up to conjugacy and powers there is a single element to consider, corresponding to the signature \((3, 0; (1, 3), (1, 3), (1, 3))\). We have that \(D\) is a hexagon with opposite sides identified, and \(f\) is a rotation by \(2\pi/3\). Then we may take for
Figure 6.3: The collections of curves $C^*$ for an irreducible periodic mapping classes of orders 13 with skip number 3 (\(=0 \pmod{3}\)).
$C^*$ a chain of 2 curves that are in the same orbit under $f$, namely, 2 curves corresponding to segments connecting midpoints of opposite sides of $D$.

When $p = 5$, $g(p) = 2$. Up to conjugacy and powers there is a single element to consider, corresponding to the signature $(5, 0; (1, 5), (1, 5), (3, 5))$. We may take $C^*$ to be a chain of length 2, as shown in Figure 6.4.

When $p = 7$, $g(p) = 3$. Up to conjugacy and powers there are two elements to consider, corresponding to the signatures $(7, 0; (1, 7), (1, 7), (5, 7))$ and $(7, 0; (1, 7), (2, 7), (4, 7))$. In each case we may take $C^*$ to be a chain of length 3, as shown in Figure 6.4.

When $p = 11$, $g(p) = 5$. Up to conjugacy and powers there are two elements to consider, corresponding to the signatures $(11, 0; (1, 11), (1, 11), (9, 11))$ and $(7, 0; (1, 11), (2, 11), (8, 11))$. In each case we may take $C^*$ to be a chain of length 3; the pictures are similar to those for $p = 7$.

Lemma 6.9. Let $p$ be either an odd prime or 4. Let $g \geq 3$ and let $f \in \text{Mod}(S_g)$ be a reducible element of order $p$. Then there exists a set of curves $C^*$ in $S_g$ such that

\begin{itemize}
  \item $C^*$ is a subset of a maximal non-separating chain of curves in $S_g$,
  \item $|C^*| \geq g/4$,
  \item every orbit of curves under the action of $\langle f \rangle$ that contains a curve in $C^*$ contains at least two curves in $C^*$, and
  \item each subset of curves in $C^*$ that lies in the same orbit under $\langle f \rangle$ forms either a chain of length at least 2 or else consists of disjoint curves.
\end{itemize}

Proof. Let $p$, $g$, and $f$ be as in the statement. We first observe that if two sets of curves $Y_1$ and $Y_2$ are each subsets of a chain of curves in $S_g$ and the $Y_i$ lie in disjoint subsurfaces of $S_g$ that are distinct in homology, then $Y_1 \cup Y_2$ is again a subset of a chain of curves in $S_g$. We will use this fact freely in forming our collection of curves $C^*$.
Figure 6.4: The collections of curves $C^n$ for the irreducible periodic mapping classes of orders 5 and 7.
We first treat the case when $p$ is an odd prime. Since $f$ is reducible, the quotient orbifold $Q = S_g/\langle f \rangle$ has either $g_0 > 0$ or $\ell > 3$ or both, by a result of Gilman [27]. (Here $g_0$ is the genus of $Q$ and $\ell$ is the number of orbifold points of $Q$.) If $g_0 > 0$, we may take preimages under $\langle f \rangle$ of $g_0$ disjoint separating curves in $Q$, each of which cuts off a single handle from $Q$. In the preimage, the curves corresponding to a single separating curve in $Q$ cut off an orbit of $p$ handles, distinct in homology. We may take a chain of two non-separating curves in each handle and produce all together $2p$ disjoint curves, which together form a subset of a chain of curves in $S_g$. These free orbits of handles, then, have a curve-to-genus ratio of $2/1$. Since we are only showing that we can form $C^*$ so that the curves-to-genus ratio is $1/4$, orbits of this type can only help the ratio.

We have therefore reduced to the case of elements $f$ such that $g_0 = 0$. Since $f$ is assumed reducible, again by a result of Gilman [27], $f$ has a reduction system of disjoint essential curves $C$ that are pairwise non-isotopic, which we may take to be maximal. Note that every orbit of curves in $C$ under $\langle f \rangle$ has size 1 or $p$, since $p$ is prime. Let $S_g(C)$ denote the (possibly disconnected) surface obtained from $S_g$ by cutting along $C$ and capping the resulting boundary components with disks containing a single marked point. Corresponding to $f$ there is an action $f'$ on $S_g(C)$.

We now consider the connected components $R_i$ of $S_g(C)$. We first claim that $f'$ induces the identity permutation on the $R_i$. Otherwise there would be an orbit of $R_i$ of size $p$. If these $R_i$ had positive genus, this would contradict the assumption of $g_0 = 0$. If they instead were spheres, they would each have at least 3 marked points in order for the curves of $C$ to be essential and non-isotopic, and this would again contradict the assumption that $g_0 = 0$. Therefore each $R_i$ is mapped to itself by $f'$.

We now analyze what the components $R_i$ may be and how $f'$ may act upon them. If $R_i$ is a sphere, then $f'$ acts on $R_i$ by a rotation of $2m\pi/p$, with $(m, p) = 1$. The action of $f'$ on $R_i$ has exactly two branch points, each of order $p$. If instead $R_i$ is not a sphere, $f'$ restricts to an irreducible self-map of order $p$, and we classified these in Lemma 6.7. In particular,
each non-sphere $R_i$ has genus $\frac{p-1}{2}$ and has exactly 3 branch points, each of order $p$.

We are now prepared to recover the action of $f$ on $S_g$ by reglueing annuli at pairs of marked points. For any $R_i$ that is a sphere, it cannot only have marked points at one or both of its branch points, since the curves of $C$ are essential and non-isotopic. Therefore each sphere $R_i$ has some $r_i > 0$ orbits of $p$ marked points, each having 0 local rotation number. Each of the $p$ marked points in each orbit is paired with $p$ other marked points that lie in some other $R_j$. (We have that $i \neq j$ by the assumption that $g_0 = 0$.) For each such orbit, we may add to $C^*$ a collection of $p - 1$ disjoint non-separating curves. The sphere $R_i$ may additionally have marked points at one or both of its branch points; each can either connect to a marked point on a different $R_j$, or else they may together form a pair. From these orbits we do not take any curves to add to $C^*$.

In the other case, where $R_i$ is a surface with positive genus, $R_i$ may have marked points at its branch points under the action of $f'$, or at non-branch points, or at both. The surface $R_i$ has some $r_i \geq 0$ orbits of $p$ marked points (possibly none), each having 0 local rotation number. Each of the $p$ marked points in each orbit is paired with $p$ other marked points that lie in some other $R_j$. (We have that $i \neq j$ by the assumption that $g_0 = 0$.) For each such orbit, we may add to $C^*$ a collection of $p - 1$ disjoint non-separating curves. The surface $R_i$ may additionally have marked points at its branch points; each can either connect to a marked point on a different $R_j$, or two of these marked points may together form a pair. In fact, this last possibility does not arise, since then $c_i + c_j = 0 \pmod{p}$ but we have that $c_1 + c_2 + c_3 = 0 \pmod{p}$ and that each $c_i$ is nonzero $\pmod{p}$. In any case, no further contribution to $C^*$ is made for these orbits. Also, we have by Lemma 6.8 that $R_i$ supports $p/4 - 1$ curves to contribute to $C^*$ whenever $g \geq 13$, and either 2 or 3 curves for smaller values of $p$.

We are now prepared to compute bounds for each of $g$ and $|C^*|$ in terms of the data described so far. Let $a$ be the number of $R_i$ that are spheres and let $b$ be the number of $R_i$ that are surfaces with positive genus. Then the number of branch points of the $R_i$ totals
$2a + 3b$. Therefore the maximal number of marked branch points is the same, and the number of fixed annuli is at most $\frac{2a + 3b}{2}$. Let $k = \frac{1}{2} \sum r_i$ be the number of orbits of annuli of size $p$. Since $(a + b - 1)$ of these annuli are required so that the resulting surface is connected—and therefore do not contribute to the genus—we have the following bound for $g$:

$$g \leq a \cdot 0 + b \cdot \frac{p - 1}{2} + kp + \frac{2a + 3b}{2} - (a + b - 1) = \frac{bp}{2} + kp + 1$$

Note that since $g \geq 3$ by assumption, at least one of $b$ or $k$ is positive.

Let $c$ be equal to the constant guaranteed by Lemma 6.8: $p/4 - 1$ when $g \geq 13$ and 2, 2, 3, or 3 when $g$ is 3, 5, 7, or 11, respectively. As we have a contribution of $c$ curves to $C^*$ for each $R_i$ of positive genus and a contribution of $p - 1$ curves from each pair of marked points with orbit of size $p$ (corresponding to an orbit of annuli of size $p$), we have the following equality for $|C^*|:

$$|C^*| = bc + k(p - 1)$$

We now apply the following three facts: (1) at least one of $b$ or $k$ is positive; (2) in general, $\frac{q + r}{s + t} \geq \min\{\frac{q}{s}, \frac{r}{t}\}$ for $q, r, s, t > 0$; (3) the individual inequalities

$$\frac{bc}{\frac{bp}{2} + 1} \geq \frac{1}{4} \quad \text{and} \quad \frac{k(p - 1)}{kp + 1} \geq \frac{1}{4}$$

hold for each $p \geq 3$ and its corresponding value of $c$ whenever $b$ is positive and whenever $k$ is positive, respectively.

Applying these facts yields the desired result:

$$\frac{|C^*|}{g} \geq \frac{bc + k(p - 1)}{\frac{bp}{2} + kp + 2} \geq \frac{1}{4}.$$ 

The case when $p = 4$ follows the same outline. 

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Figure 6.5: A schematic for the collections of curves $C'_1, \ldots, C'_{16}, C'_{17}, \ldots, C'_{32}$, and $C'_{33}, \ldots, C'_{48}$. Within a single surface, curves in different $C'_i$ are in different colors, and within a collection $C'_i$ the curves in the same orbit under the cyclic group $\langle f_i \rangle$ are in the same chain. Note that for the case illustrated, only four collections of curves are needed in each of the three groupings, and also that “overflow” curves are not depicted.

Figure 6.6: A schematic for the collections of curves $C_{\text{odd}}, C_{\text{even}}$, and the shifted versions of each. Within a single surface, curves in different $C'_i$ are in different colors, and within a collection $C'_i$ the curves in the same orbit under the cyclic group $\langle f_i \rangle$ have matching markings. Note that for the case illustrated, only four $C'_i$ are needed in each of the four groupings, and also that “overflow” curves are not depicted.
Proof of the main theorem

With all of our preliminaries in hand, we are prepared to prove Theorem 6.4.

**Proof of Theorem 6.4.** Let \( g \geq 3 \) and let \( H_g = \{a_1, \ldots, a_{2g}, b\} \) be a set of simple closed non-separating curves corresponding to the Humphries generating set for \( \text{Mod}(S_g) \). Let \( f \) be a periodic element of \( \text{Mod}(S_g) \) with \( |f| \geq 3 \). By passing to a power, we may assume without loss of generality that \( |f| \) is either 4 or an odd prime. By Lemma 6.6, a Dehn twist about a non-separating curve may be written as a product in at most 12 distinct conjugates of \( f \). Take 12 conjugates of \( f \) in which \( T_b \) is a product and let them be the start to our generating set for \( \text{Mod}(S_g) \).

By Lemmas 6.8 and 6.9, \( f \) acts on \( C^* \), a subset of a maximal non-separating chain of curves in \( S_g \), such that \( |C^*| \geq g/4 \) and every orbit of curves under the action of \( \langle f \rangle \) that contains a curve in \( C^* \) contains at least two curves in \( C^* \). The \( C^* \) curves that lie in a given orbit either form a chain of length at least 2, or they are all disjoint. We consider two cases, depending on whether at least half of the curves of \( C^* \) lie in orbits of the former type, or of the latter type. In either case, let this collection of curves be called \( C' \). We have \( |C'| \geq g/8 \). We treat the two cases in turn, in similar fashion.

First case: chains. When the curves of \( C' \) all lie in chains of length at least 2, we form at most \( 16 = \frac{2g}{g/8} \) subsets of \( H_g - \{b\}, C'_1, \ldots, C'_{16} \), that are each of the topological type of \( C' \) and that together cover \( H_g - \{b\} \) except for “gaps” of size 1. See Figure 6.5. Note that fewer than 16 subsets may be required, and that it does not matter how the “overflow” curves for the last \( C'_i \) are chosen. We may then take at most 16 conjugates of \( f, \{f_1, \ldots, f_{16}\} \), so that each individually acts on the corresponding \( C'_i \) in the way that \( f \) acts on \( C' \). In particular, each curve in each \( C'_i \) is in the same orbit as the other curves belonging to the same chain in \( C'_i \) under the action of the cyclic group \( \langle f_i \rangle \).

Similarly, we form \( C'_{17}, \ldots, C'_{32} \) and \( C'_{33}, \ldots, C'_{48} \) that are shifted one and two curves down the chain \( H_g - \{b\} \) from the corresponding curves in \( C'_1, \ldots, C'_{16} \). Note that it again
does not matter where the “overflow” curves are chosen, except that we ensure that $b$ is among them. Again, see Figure 6.5. We may take 32 conjugates of $f$, $\{f_{17}, \ldots, f_{48}\}$, so that each acts on $C'_i$ in the way that $f$ acts on $C'$, for $17 \leq i \leq 48$.

We now argue that all of the curves in $H_g$ are in the same orbit under the subgroup generated by the $48$ $f_i$. All but the “gap” curves are put into orbits by $\langle f_1, \ldots, f_{16}\rangle$. Each gap curve is put into the same orbit as the curve immediately “prior” to it by $\langle f_{17}, \ldots, f_{32}\rangle$ and also into the same orbit as the curve immediately “after” it by $\langle f_{33}, \ldots, f_{48}\rangle$. Thus all of the original orbits are collapsed into a single orbit “through” the gap curves. By construction we also have that $b$ is in this orbit.

Therefore the curves in $H_g$ are all in the same orbit under the action of a subgroup generated by 48 conjugates of $f$. With the additional 12 conjugates of $f$ we have a Dehn twist about $b$ in the subgroup, and so also the Dehn twists about all curves in $H_g$, and therefore the subgroup so generated is equal to $\text{Mod}(S_g)$. Thus $\text{Mod}(S_g)$ is generated by 60 conjugates of $f$, as required.

**Second case: disjoint curves.** When the curves of $C'$ are all disjoint, we form at most $8 = \frac{g}{g/8}$ subsets of $H_g - \{b\}$, $C'_{1'}, \ldots, C'_{8}$, that are each of the topological type of $C''$ and that together cover the $a_i$ curves in $H_g$ with odd indices. Call these subsets $C_{\text{odd}}$. See Figure 6.6. Note that fewer than 8 subsets may be required, and that it does not matter how the “overflow” curves for the last $C'_{i'}$ are chosen. We may then take at most 8 conjugates of $f$, $\{f_1, \ldots, f_8\}$, so that each individually acts on the corresponding $C'_{i'}$ in the way that $f$ acts on $C''$. Similarly, we form sets of curves $C'_{9'}, \ldots, C'_{16}$ that are each of the topological type of $C'$ and that together cover the $a_i$ curves in $H_g$ with even indices; we may do this by shifting all of the $C_{\text{odd}}$ curves down the $H_g - \{b\}$ chain by one curve. Call these subsets $C_{\text{even}}$. We take at most 8 conjugates of $f$, $\{f_9, \ldots, f_{16}\}$, so that each individually acts on the corresponding $C'_{i'}$ in the way that $f$ acts on $C''$.

We now take $C'_{17}, \ldots, C'_{24}$ and $C'_{25}, \ldots, C'_{32}$ that are shifted two and three three curves down the chain $H_g - \{b\}$ from the $C_{\text{odd}}$ curves. (So two down from $C_{\text{odd}}$ and two down
from $C_{\text{even}}$.) Note that fewer than 16 subsets may be required, and that it again does not matter where the “overflow” curves are chosen, except that we ensure that $b$ is among them, and also that an “overflow” curve from the shifted “odd” chain is one of the $a_i$ curves with an even index. Again, see Figure 6.6. We take at most 16 conjugates of $f$, $\{f_{17}, \ldots, f_{32}\}$, so that each acts on $C'_i$ in the way that $f$ acts on $C'$, for $17 \leq i \leq 32$.

We now argue that all of the curves in $H_g$ are in the same orbit under the subgroup generated by the 32 $f_i$. All of the $a_i$ curves with odd index are in the same orbit, since all of their orbits under $\langle f_1, \ldots, f_8 \rangle$ are collapsed by the “shifted” conjugates. Similarly, all of the $a_i$ curves with even index are in the same orbit under $\langle f_1, \ldots, f_{32} \rangle$. By construction, there exist an even $a_i$ curve and an odd $a_i$ curve that lie in the same orbit under $\langle f_1, \ldots, f_{32} \rangle$, and also $b$ is in the same orbit as some $a_i$ curve. Therefore the curves in $H_g$ are all in the same orbit under the action of a subgroup generated by 32 conjugates of $f$.

With an additional 12 conjugates of $f$ we have a Dehn twist about $b$ in the subgroup, and so also the Dehn twists about all curves in $H_g$, and therefore the subgroup so generated is equal to $\text{Mod}(S_g)$. Thus $\text{Mod}(S_g)$ is generated by 44 conjugates of $f$, as required. \qed
REFERENCES


VITA

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