APPROXIMATE SCHAUDER FRAMES FOR BANACH SEQUENCE SPACES

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APPROXIMATE SCHAUDER FRAMES FOR BANACH SEQUENCE SPACES

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For my parents.

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TABLE OF CONTENTS

Acknowledgments iv

List of Figures vii

Summary viii

Chapter 1: Background and Overview 1

1.1 Outline of the Thesis 1

1.2 Overview of Frames for Hilbert Spaces 5

1.3 Overview of Frames for Banach Spaces 9

1.4 Introduction to Finite Unit-Norm Tight Frames 12

1.5 Brief Discussion of Balanced Frames 18

Chapter 2: Finite Unit-Norm Tight Frames for Banach Spaces 22

2.1 Introduction 22

2.2 Existence of FUNTFs for Real ℓ_1^n 27

2.3 An Alternative FUNTF Construction for Real ℓ_1^n 37

2.4 The General Case and Open Questions 52

Chapter 3: Frame Inequalities for Banach Sequence Spaces 58

3.1 Introduction 58

3.2 Frame Inequality in Finite Dimensions 61

3.3 Motivational Example: Initial Construction 64

3.4 Motivational Example: Extension to a Schauder Frame 76

3.5 Frame Inequality in Infinite Dimensions 83

3.6 An Improved Frame Inequality Condition 90

3.7 Schur's Test and Frame Inequalities 96

3.8 Future Directions 101

Chapter 4: Introduction to ℓ_1-boundedness 103

4.1 Preliminaries and Definitions 103

4.2 Examples of ℓ_1-boundedness 105

4.3 Boundedness and ℓ_1-boundedness 108

4.4 Meagerness and ℓ_1-boundedness 110

4.5 Compactness and ℓ_1-boundedness 112

4.6 Further Properties of ℓ_1-bounded Sets 116

4.7 Connections between Riesz Bases and Frames 118

References 120

LIST OF FIGURES

3.1 Graphs of R_0, R_1 (top), and R_2, R_3 (bottom). 67

SUMMARY

The main topics of this thesis concern two types of approximate Schauder frames for the Banach sequence space ℓ_1. The first main topic pertains to finite-unit norm tight frames (FUNTFs) for the finite-dimensional real sequence space ℓ_1^n. We prove that for any $N \geq n$, FUNTFs of length N exist for real ℓ_1^n. This answers a fundamental existence question posed by Chavez-Dominguez, Freeman, and Kornelson in [1]. To show the existence of FUNTFs, specific examples are constructed for various lengths. These constructions involve repetitions of frame elements. However, a different method of frame constructions allows us to prove the existence of FUNTFs for real ℓ_1^n of lengths $2n-1$ and $2n-2$ that do not have repeated elements.

The second main topic of this thesis pertains to normalized unconditional Schauder frames for the sequence space ℓ_1. A Schauder frame provides a reconstruction formula for elements in the space, but need not be associated with a frame inequality. Our main theorem on this topic establishes a set of conditions under which an ℓ_1-type of frame inequality is applicable towards unconditional Schauder frames. On the basis of this theorem, we then consider various less restrictive conditions for such unconditional Schauder frames. A primary motivation for choosing this set of hypotheses involves appropriate modifications of the Rademacher system, a version of which we prove to be an unconditional Schauder frame that does not satisfy an ℓ_1-type of frame inequality.

The final topic of our thesis introduces the concept of ℓ_1-boundedness in the Hilbert sequence space ℓ_2. We prove various properties of ℓ_1-bounded sets of ℓ_2 and state some potentially mathematically significant open problems in this subject area.

CHAPTER 1

BACKGROUND AND OVERVIEW

1.1 Outline of the Thesis

The main topics of this thesis cover results on Schauder frames for Banach sequence spaces. Schauder frames are a generalization of frames in Hilbert spaces to the Banach space setting, a detailed definition and background discussion of which will be given later on in this chapter. The first main result of this manuscript considers a special subclass of approximate Schauder frames known as finite unit-norm tight frames, abbreviated as FUNTFs. Since FUNTFs are finite length frames, we work with the finite-dimensional Banach space real ℓ_1^n in our first set of main results, which are presented in Chapter 2. The second set of main results, presented in Chapter 3, covers normalized unconditional Schauder frames, this time in the infinite-dimensional setting. Specifically, the space that we work with in Chapter 3 is ℓ_1. Finally, Chapter 4 contains a few results on what we call ℓ_1-*bounded subsets* of Hilbert spaces.

The definitions and background information that underlie the main topics of this thesis are presented in Sections 1.2 to 1.5 of this thesis. In particular, we discuss the main definitions and theorems for frame theory in the Hilbert space setting in Section 1.2, and then introduce frames for Banach spaces in Section 1.3. In Hilbert spaces, a frame is defined by a certain norm equivalence, which then implies the existence of basis-like, though possibly non-unique, representations. However, this implication fails when we deal with Banach spaces that are not Hilbert spaces. Consequently, there are multiple ways to generalize frames to the Banach space setting. In this thesis, we deal with the generalization known as *Schauder frames*, which is based on the existence of representations rather than norm inequalities.

In the latter sections of this chapter, we discuss some existing results on FUNTFs for Hilbert spaces, and discuss the foundations that will be needed for our results in Chapter 2. The last section of Chapter 1 overviews some existing results and proposes some open questions on balanced frames.

Now we present an outline of this thesis. Chapter 2 contains the first main set of results. An extension of the concept of finite unit-norm tight frames (FUNTFs) from the Hilbert space setting to the Banach space setting was introduced by Chavez-Dominguez, Freeman, and Kornelson ([1]). That paper is foundational to our work, so we begin Chapter 2 with a detailed discussion of those results from [1] that are most relevant to our work.

Two of the main results of [1] that we build on are: (a) Proof of the existence of FUNTFs for complex finite-dimensional Banach spaces, and (b) necessary and sufficient conditions for Banach space sequence pairs to be a FUNTF. The existence of FUNTFs for real finite-dimensional Banach spaces is stated as an open question in [1]. Our main goal in Chapter 2 is to make significant progress towards answering this open question.

We will work in the finite-dimensional Banach space real ℓ_1^n, which is n-dimensional real Euclidean space but endowed with the ℓ_1-norm. In other words, this is the space consisting of all real sequences $x = (x_j)_{j=1}^n$ with norm $||x||_1 = \sum_{j=1}^{n} |x_j|$. One of the main theorems of Chapter 2 answers the aforementioned open question in [1] for real ℓ_1^n. Specifically, we prove that FUNTFs exist for real ℓ_1^n.

To summarize, the following facts proved in [1] are the foundations of our work in Chapter 2:

1. There exist FUNTFs of length $N \geq n$ for complex n-dimensional Banach spaces and real 2-dimensional Banach spaces.

2. There exists a FUNTF of length $n + 1$ for real ℓ_1^n with $n \geq 3$.

3. There exist FUNTFs of length $N \geq n$ for real ℓ_1^n with $3 \leq n \leq 6$.

We prove the following facts, which are the main theorems of the chapter:

1. There exists a FUNTF of length $N \geq n$ for real ℓ_1^n for any $n \geq 3$.

2. There exists FUNTFs of lengths $2n-1$ and $2n-2$ that do not contain repeated elements for real ℓ_1^n for any $n \geq 3$.

To prove the first statement, we provide specific examples of FUNTFs of lengths $n+2 \leq N \leq 2n-1$ for real ℓ_1^n for any $n \geq 3$. To obtain FUNTFs of longer lengths, we take appropriate unions of our shorter frames. All of the FUNTFs constructed by this method involve the repetition of frame elements.

To prove the second statement, we provide a method for constructing examples of FUNTFs which do not involve repeating frame elements. Using this method, we are able to construct FUNTFs of length $2n-1$ and $2n-2$ for real ℓ_1^n. This method is a generalization of the method used in [1] to construct FUNTFs of lengths $N \geq n$ for real ℓ_1^n with $n = 4, 5, 6$. Constructing this class of FUNTFs provides the foundations for finding a different class of FUNTFs for real ℓ_1^n from the ones we defined in the first main theorem of this chapter.

Since ℓ_2^n is a Hilbert space, we know that it contains FUNTFs. In this thesis, we extend the class of finite-dimensional Banach spaces in which FUNTFs are known to exist to include ℓ_1^n. We leave as an open question whether FUNTFs exist for general finite-dimensional Banach spaces. The ideas of [1] and our results for real ℓ_1^n provide a method leading to an open computational problem, which eventually addresses this question.

Chapter 3 of this thesis continues our study of Schauder frames for Banach sequence spaces. This chapter contains the second set of main results of this thesis, and is a joint collaboration project with Dr. Daniel Freeman and Dr. Christopher Heil. Instead of ℓ_1^n, we focus in this chapter on normalized unconditional Schauder frames for the infinite dimensional space ℓ_1. While our work in Chapter 2 is directly motivated by explicit open questions in [1], we draw from a wider array of motivations in this chapter. Since a set of conditions for Banach frames to satisfy the reconstruction formula were provided by Carando, Lassalle, and Schmidberg in [2], it is natural to consider conditions which imply that a Schauder frame satisfies a frame inequality. Both our work in Chapter 2 and that of

Lindenstrauss and Zippin in [3] motivate us to specifically choose ℓ_1 as the space we work with.

The main theorem of Chapter 3 provides a set of conditions on ℓ_1 normalized unconditional Schauder frames for which an ℓ_1-type of frame inequality holds. An initial motivating example involving a modification of the Rademacher sequence is described in detail beforehand. While this system does not satisfy an ℓ_1-type of frame inequality, it is not a Schauder frame.

In order to prove that a frame inequality will not hold in general, this initial example is further modified to be an unconditional Schauder frame. To define this new example, we still use the same finite-length sequence pairs of length $N \in \mathbb{N}$ for $\ell_1^{2^N+N}$ that are the building blocks of the initial non-Schauder frame construction. Instead of directly constructing an infinite sequence pair as in the first attempt, we construct a normalized unconditional Schauder frame for $\ell_1^{2^N+N}$ of length $\ell_1^{2^N+3N}$ which is shown to not satisfy an ℓ_1-type of frame inequality with an upper bound that is independent of N. The existence of a normalized unconditional Schauder frame for ℓ_1 not satisfying an ℓ_1-type of frame inequality can then be obtained by combining these individual finite systems appropriately to create an infinite system for ℓ_1. This discussion motivates the hypotheses we set for the main theorem of this chapter.

The latter sections of Chapter 3 discuss further possibilities of using different methods to widen the class of Schauder frames that can be shown to satisfy an ℓ_1-type of frame inequality. As corollaries of our main result on frame inequalities for Schauder frames, we prove a relaxed set of conditions for Schauder frames that satisfy an ℓ_1-type of frame inequality. Finally, we find a connection to Schur's Test that perhaps could be used to further widen the class of Schauder frames which satisfy a frame inequality.

The final chapter of this thesis, Chapter 4, is the only chapter whose results do not directly apply for the space ℓ_1 or a variation thereof, although the motivation for these results is still based on an ℓ_1-norm. In this chapter, we work with the Hilbert space ℓ_2, but we consider convergence under the ℓ_1-norm in ℓ_2. This ties in with the theme of the other main chapters. In order to do so, we formally introduce the concept of ℓ_1-boundedness. We prove some results about this class of ℓ_1-bounded sets. However, even some seemingly simple questions appear to be quite difficult. We believe that these are interesting and potentially significant mathematical problems. Therefore, a heavier emphasis is placed on conjectures and open questions in this chapter. These are dispersed in between various properties related to ℓ_1-boundedness that we state and prove.

1.2 Overview of Frames for Hilbert Spaces

In analysis, one type of sequence that recurrently appears is known as a *basis*. Even though bases can be too restrictive to work with, they have many useful properties. For example, an orthonormal basis for an infinite-dimensional Hilbert space satisfies Plancherel's Equality, which states that if $\{x_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for H, then

$$\sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 = ||x||^2 \tag{1.1}$$

for all $x \in H$. Some texts that discuss bases and their generalizations in detail, including Plancherel's Equality, are [4] and [5].

In the rest of this section, we present classes of sequences that are less restrictive than bases but still preserve many of the same properties. Namely, we will introduce and briefly describe *Bessel sequences*, *frames*, and *Riesz bases*. We begin our discussion with Bessel sequences, which are the least restrictive among the three categories of sequences just listed.

Definition 1. Let H be a Hilbert space. A sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq H$ is called a *Bessel sequence* if for all $x \in H$,

$$\sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 < \infty. \tag{1.2}$$

The Uniform Boundedness Principle implies that (Theorem 7.2 in [5]) if $\{x_n\}_{n\in\mathbb{N}}$ is a

Bessel sequence for H, then there exists a constant $B > 0$ such that

$$\sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \leq B||x||^2 \tag{1.3}$$

for all $x \in H$.

Given a Bessel sequence $\{x_n\}_{n\in\mathbb{N}}$ in a Hilbert space H, we define its associated *analysis operator* to be the map $L : H \to \ell_2$ defined by

$$L(x) = (\langle x, x_n \rangle)_{n\in\mathbb{N}} \tag{1.4}$$

for $x \in H$. The adjoint L^* of the analysis operator L is known as the *synthesis operator*. This is the map $L^* : \ell_2 \to H$ defined by

$$L^*((c_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} c_n x_n, \tag{1.5}$$

where $(c_n)_{n=1}^{\infty} \in \ell_2$, meaning that $\sum\limits_{n=1}^{\infty} |c_n|^2 < \infty$.

Since Bessel sequences satisfy the inequality (Equation 1.3) and orthonormal bases satisfy the Plancherel Equality, our next step is to discuss sequences whose properties lie between these two definitions. Consider Bessel sequences for which the quantity $\sum\limits_{n=1}^{\infty} |\langle x, x_n \rangle|^2$ is not only bounded above, but also bounded below. These sequences are known as *frames*.

Definition 2. Let H be a Hilbert space. A sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq H$ is called a *frame* if there exist constants $A, B > 0$, referred to as *frame constants*, such that for every $x \in H$,

$$A||x||^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \leq B||x||^2. \tag{1.6}$$

The inequality (Equation 1.6) is the *frame inequality*. A frame is a *tight frame* if we can take $A = B$, and is a *Parseval frame* if we can take $A = B = 1$.

While Bessel sequences must satisfy the upper bound of the frame inequality, they need not satisfy the lower bound. Hence, all frames are Bessel sequences, but not all Bessel sequences are frames. Also, notice that frame constants are not unique.

Frames were first introduced by Duffin and Schaefer in [6]. In contrast to our motivation, which is to find a class of sequences that lies between Bessel sequences and orthonormal bases, [6] defined frames as tools to understand nonharmonic Fourier series. This manuscript will not cover nonharmonic Fourier analysis; two references on this topic are [7] and [8]. A renewed interest in frame theory was sparked by the publication of the paper [9] by Daubechies, Grossmann, and Mayer, which constructed wavelet frames and Gabor frames for $L^2(\mathbb{R})$.

Since frames are a subset of Bessel sequences, the analysis and synthesis operators as defined for Bessel sequences are also applicable to frames. By composing the analysis and synthesis operators for frames, the *frame operator* and the *Gram operator* can be defined.

Definition 3. Let H be a Hilbert space. If $\{x_n\}_{n\in\mathbb{N}}$ is a Bessel sequence, then its associated *frame operator* is the map $S : H \to H$ with $S = L^*L$, while the associated *Gram operator* $G : \ell_2 \to \ell_2$ is defined by $G = LL^*$.

The frame operator is invertible and takes the explicit form

$$Sx = L^*Lx = \sum_{n=1}^{\infty} \langle x, x_n\rangle x_n \tag{1.7}$$

for $x \in H$. If $\{x_n\}_{n\in\mathbb{N}}$ is a tight frame with frame constant A, then its corresponding frame operator S is a multiple of the identity. Specifically, $S = AI$ in this situation.

As a consequence of the frame definition, if $\{x_n\}_{n\in\mathbb{N}}$ is a frame for a Hilbert space H, then for all $x \in H$,

$$x = \sum_{n=1}^{\infty} \langle x, S^{-1}x_n\rangle x_n = \sum_{n=1}^{\infty} \langle x, x_n\rangle S^{-1}x_n. \tag{1.8}$$

The series on the right hand side of the equation (Equation 1.8) converges unconditionally.

That is, the series $\sum_{n=1}^{\infty} \varepsilon_n \langle x, x_n \rangle S^{-1} x_n$ converges for any choice of $\epsilon_n = \pm 1$. The sequence $\{S^{-1}x_n\}_{n\in\mathbb{N}}$ is the *canonical dual frame* with respect to $\{x_n\}_{n\in\mathbb{N}}$. Any other sequence $\{y_n\}_{n\in\mathbb{N}}$ in H for which the equation

$$x = \sum_{n=1}^{\infty} \langle x, y_n \rangle x_n \tag{1.9}$$

holds for all $x \in H$ is an *alternative dual frame* for $\{x_n\}_{n\in\mathbb{N}}$.

Equation (Equation 1.9) is referred to as the *reconstruction formula*. Loosely speaking, the reconstruction formula and the frame inequality (Equation 1.6) are equivalent definitions for frames. For complete statements and proofs, and additional facts about frames, we refer to texts such as [4], [10], and [5].

Since G maps ℓ_2 into itself, it is given by an infinite matrix. Specifically, the matrix associated with the Gram operator G is the *Gram matrix*,

$$G = \Big[\langle x_n, x_m \rangle \Big]_{n,m}. \tag{1.10}$$

Gram matrices will only appear briefly in Chapter 3 of this manuscript.

Now we discuss the last class of sequences mentioned at the beginning of this section. A sequence $\{x_n\}_{n\in\mathbb{N}}$ for a Hilbert space H is a Riesz basis if it is complete in H and there exist constants $A, B > 0$ such that

$$A \sum_{n=1}^{N} |c_n|^2 \leq \left\| \sum_{n=1}^{N} c_n x_n \right\|^2 \leq B \sum_{n=1}^{N} |c_n|^2 \tag{1.11}$$

for any finite sequence of scalars $\{c_n\}_{n=1}^{N}$, $N \in \mathbb{N}$. An alternative definition of Riesz bases given in [5] is restated below.

Definition 4. A *Riesz basis* in a Hilbert space H is a sequence that is equivalent to an orthonormal basis for H. That is, a sequence is a Riesz basis if there exists a topological isomorphism (a bounded invertible linear bijection) that maps it to an orthonormal basis for H.

These two definitions are equivalent (Theorem 7.13 in [5]). Furthermore, all Riesz bases are frames (Theorem 8.32 in [5]). Therefore, we have the relations

$$\text{orthonormal bases} \subseteq \text{Riesz bases} \subseteq \text{frames} \subseteq \text{Bessel sequences}.$$

In this thesis, results involving Riesz bases appear only in Chapter 4.

1.3 Overview of Frames for Banach Spaces

Now we discuss generalizations of frames to the Banach space setting. The frame concept cannot be generalized naively. The problem is that given a Banach space X, the condition

$$A||x||^2 \leq \sum_{j\in\mathbb{N}} |x_j(x)|^2 \leq B||x||^2$$

does not in general imply the existence of a reconstruction formula.

We will provide a brief overview of the generalization of frames to Banach spaces. An in-depth background discussion can be found in [11] and the references therein.

While Hilbert spaces are self-dual, Banach spaces are not. In particular, canonical or alternative duals corresponding to frames for a Banach space X must lie in the dual space X^* rather than X itself. Also, there is the issue of what sequence space norm to work with when considering norm inequalities. Most often, this is an ℓ_p-type inequality with $p \neq 2$.

There are multiple ways to generalize frames to the Banach space setting. These include *Schauder frames* ([12] and [13]), *Banach frames* ([14], [2], and [15]), *framings* ([11]), and *atomic decompositions* ([16], [17], [18], and [19]). Since all of our results will pertain to Schauder frames, we focus on that generalization.

Definition 5. Let X be a Banach space with corresponding dual space X^*. A *Schauder* *frame*, or *quasibasis*, for X is a sequence $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ in $X \times X^*$ such that for all $x \in X$,

$$x = \sum_{j=1}^{\infty} f_j(x)x_j, \tag{1.12}$$

where this sequence converges in the norm of X.

A Schauder frame $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ is *normalized* if $||x_j||_X = ||f_j||_{X^*} = 1$ for all $j \in \mathbb{N}$. It is *unconditional* if the series $\sum_{j=1}^{\infty} f_j(x)x_j$ converges unconditionally. Furthermore, a constant $K > 0$ is said to be an *unconditionality constant* for a Schauder frame $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ if for all $x \in X$ and $\varepsilon_j = \pm 1$, we have

$$\left|\left|\sum_{j=1}^{\infty} \varepsilon_j\, f_j(x)x_j\right|\right| \leq K||x||.$$

In Chapter 3, we will define a normalized unconditional Schauder frames for ℓ_1 which does not satisfy an ℓ_1-type of frame inequality. This suggests that the reconstruction formula and the ℓ_1-type frame inequality are not equivalent definitions for Schauder frames.

As for Hilbert spaces, the *frame operator* can be defined for Schauder frames in the Banach space setting.

Definition 6. The *frame operator* $S : X \to X$ that is associated with a sequence pair $\{(x_j, f_j)\}_{j\in\mathbb{N}} \subseteq X \times X^*$ is the map

$$S(x) = \sum_{n=1}^{\infty} f_j(x)x_j \tag{1.13}$$

for $x \in X$.

For a Schauder frame, its frame operator is the identity operator on X. A more general notion is that of an *approximate Schauder frame*, where the frame operator need only be bounded and invertible.

Definition 7. Given a Banach space X, an *approximate Schauder frame* for X is a sequence pair $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ in $X \times X^*$ whose frame operator is bounded and invertible, and hence a topological isomorphism of X onto itself.

An in-depth discussion of frame operators in Banach space can be found in [20] and the references therein. Another characterization of (approximate) Schauder frames is to define them as compressions, or inner direct sums, of Schauder bases. We will not elaborate on these topics, but note that a detailed description of Schauder frames as compressions of Schauder bases can be found in [12]. Additional texts that elaborate on Schauder bases include [21] and [22].

We now introduce the notion of a *Banach frame*. Unlike a Schauder frame which is defined using the reconstruction formula as a sequence pair in a Banach space X and its dual X^*, more conditions need to be imposed when defining a Banach frame. Specifically, an associated Banach sequence space Z is also required. Here, a *Banach sequence space* is a Banach space consisting of scalar sequences whose coordinate functionals are continuous. While our results in this thesis do not pertain to Banach frames, some of our motivations for Chapter 3 involve these systems. We present the definition with an ℓ_1-type of frame inequality, but other types of frame inequalities are used in other settings.

Definition 8. Let X be a Banach space and let Z be a Banach sequence space. Also, let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence in X^* and let $S : Z \to X$ be a continuous linear operator. The pair $(\{f_j\}_{j\in\mathbb{N}}, S)$ is said to be *Banach frame* if for all $x \in X$, the following conditions hold.

(a) $(f_j(x))_{j\in\mathbb{N}} \in Z$.

(b) There exist positive constants $A, B > 0$ such that for all $x \in X$,

$$A||x|| \leq \sum_{n=1}^{\infty} |f_j(x)| \leq B||x||. \tag{1.14}$$

The inequality (Equation 1.14) is referred to as an ℓ_1-*type of frame inequality*.

(c) $x = S(f_j(x))$ for every $x \in X$.

The definition of a Banach frame involves both a frame inequality and a reconstruction condition in contrast to the definition of a Schauder frame. Furthermore, a Banach frame is defined as a pair consisting of a sequence and an operator while a Schauder frame is a sequence pair. Because of these differences in their definitions, a direct relationship between Banach and Schauder frames is not obvious.

We will only discuss Schauder frames for the Banach sequence space ℓ_p. However, Schauder frames can also be defined for continuous function spaces. For example, Berasategui and Carando in [23] cover Schauder frames for L^p. In addition, there are various sub-types of Schauder frames that we will not discuss, such as *weaving Schauder frames* ([24]) and *continuous Schauder frames* ([25]).

1.4 Introduction to Finite Unit-Norm Tight Frames

In this section, we give an overview of *finite unit-norm tight frames*, or FUNTFs. This type of frame will be the foundation of our discussion in Chapter 2.

Definition 9. A frame $\{x_n\}_{n=1}^{N}$ for a Hilbert space H is a *finite unit-norm tight frame*, or FUNTF, if $||x_n|| = 1$ for all $1 \leq n \leq N$ and its associated frame operator is a scalar multiple of the identity.

It can be shown that the frame constant of a FUNTF of length N for an n-dimensional Hilbert space is N/n (Theorem 3.1 in [26]).

Zimmermann in [27] proved that a FUNTF of length N exists for any n dimensional Hilbert space. In fact, explicit examples of FUNTFs with length $N \geq n$ were exhibited in both $\mathbb{C}^n$ and $\mathbb{R}^n$. We will discuss these in detail, since they motivate our results in Chapter 2.

Example ([27]). Fix $N \geq n$ and consider the complex Fourier matrix, which is the $N \times N$

matrix F_N defined by

$$F_N = [e^{2\pi ijk/N}]_{1\le j,k\le N} = \begin{bmatrix} e^{2\pi i/N} & e^{2\pi i\cdot 2/n} & e^{2\pi i\cdot 3/N} & \cdots & e^{2\pi i\cdot N/N} \\ e^{2\pi i\cdot 2/N} & e^{2\pi i\cdot 2\cdot 2/N} & e^{2\pi i 2\cdot 3/N} & \cdots & e^{2\pi i\cdot 2\cdot N/N} \\ e^{2\pi i\cdot 3/N} & e^{2\pi i\cdot 3\cdot 2/N} & e^{2\pi i 3\cdot 3/N} & \cdots & e^{2\pi i\cdot 3\cdot N/N} \\ e^{2\pi i\cdot 4/N} & e^{2\pi i\cdot 4\cdot 2/N} & e^{2\pi i 4\cdot 3/N} & \cdots & e^{2\pi i\cdot 4\cdot N/N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{2\pi i\cdot N/N} & e^{2\pi i\cdot N\cdot 2/N} & e^{2\pi i\cdot N\cdot 3/N} & \cdots & e^{2\pi i\cdot N\cdot N/N} \end{bmatrix}. \quad (1.15)$$

If we select any n rows from F_N, we obtain an $n \times N$ submatrix; the N columns of this resultant submatrix forms a set of N vectors for $\mathbb{C}^n$. Since $F_N\,\overline{F_N^T} = I$, the columns of this resultant submatrix defines a finite tight frame of length N for $\mathbb{C}^n$, which can then be normalized into a FUNTF.

The example above does not naively generalize from $\mathbb{C}^n$ to $\mathbb{R}^n$. Even so, Zimmermann in [27] defined FUNTFs for $\mathbb{R}^n$ via a real-valued variation on the complex construction.

Example ([27]). Our goal is to construct a FUNTF of length $N \ge n$ for $\mathbb{R}^n$. For N odd, let $N = 2k+1$ and define the $N \times N$ matrix

$$G_N = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & \cdots & 1/\sqrt{2} \\ 1 & \cos(2\pi/N) & \cos(2\pi\cdot 2/N) & \cdots & \cos(2\pi\cdot(N-1)/N) \\ 0 & \sin(2\pi/N) & \sin(2\pi\cdot 2/N) & \cdots & \sin(2\pi\cdot(N-1)/N) \\ 1 & \cos(2\pi\cdot 2/N) & \cos(2\pi\cdot 4/N) & \cdots & \cos(2\pi\cdot 2(N-1)/N) \\ 0 & \sin(2\pi\cdot 2/N) & \sin(2\pi\cdot 4/N) & \cdots & \sin(2\pi\cdot 2(N-1)/N) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \cos(2\pi\cdot k/N) & \cos(2\pi\cdot 2k/N) & \cdots & \cos(2\pi\cdot k(N-1)/N) \\ 0 & \sin(2\pi\cdot k/N) & \sin(2\pi\cdot 2k/N) & \cdots & \sin(2\pi\cdot k(N-1)/N) \end{bmatrix}. \quad (1.16)$$

For N even, let $N = 2k$ and define G_N by

$$G_N = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & \cdots & 1/\sqrt{2} \\ 1 & \cos(2\pi/N) & \cos(2\pi\cdot 2/N) & \cdots & \cos(2\pi\cdot(N-1)/N) \\ 0 & \sin(2\pi/N) & \sin(2\pi\cdot 2/N) & \cdots & \sin(2\pi\cdot(N-1)/N) \\ 1 & \cos(2\pi\cdot 2/N) & \cos(2\pi\cdot 4/N) & \cdots & \cos(2\pi\cdot 2(N-1)/N) \\ 0 & \sin(2\pi\cdot 2/N) & \sin(2\pi\cdot 4/N) & \cdots & \sin(2\pi\cdot 2(N-1)/N) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \cos(2\pi\cdot(k-1)/N) & \cos(2\pi\cdot 2(k-1)/N) & \cdots & \cos(2\pi\cdot(k-1)(N-1)/N) \\ 0 & \sin(2\pi\cdot(k-1)/N) & \sin(2\pi\cdot 2(k-1)/N) & \cdots & \sin(2\pi\cdot(k-1)(N-1)/N) \\ 1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} & \cdots & -1/\sqrt{2} \end{bmatrix}. \tag{1.17}$$

Choose an integer $N \in \mathbb{N}$. If n is odd, then extract rows 1 through n of G_N, using either equation (Equation 1.16) or (Equation 1.17) for the definition of G_N depending on the parity of N. Consider the sequence of N vectors whose columns are formed by this resultant $n \times N$ matrix.

If n is even, then extract rows 2 through $n+1$ of the appropriate G_N and consider the sequence of vectors formed by the columns of the resultant $n \times N$ matrix. Zimmermann shows that these sequences of vectors (for both n even and n odd) are finite tight frames, and hence can be subsequently normalized to become FUNTFs.

If n is even, then the entries of each element in this FUNTF alternate between pairs of cosines and sines for a given angle of the same frequency, with $n/2$ such pairs total. If n is odd, then the construction is slightly modified with $(n-1)/2$ pairs of alternating cosines and sines of the same angle in addition to a fixed value for the first entry of each element of our FUNTF.

Goyal, Kovavecic, and Kelner in [28] independently constructed FUNTFs of lengths $N \geq n$ for $\mathbb{C}^n$ motivated by issues in signal processing. As in [27], this was done by taking appropriate submatrices of the Fourier matrix.

After [27] proved the existence of FUNTFs for Hilbert spaces, the next step is to provide necessary and sufficient conditions for Hilbert space sequences to be a FUNTF. The first necessary and sufficient characterizations of FUNTFs were given by Benedetto and Fickus in [26]. Removing the unit-norm condition for FUNTFs, the results of [26] were further developed in [29]. In that paper, a characterization of finite tight frames (FTFs) were given with respect to fixed lengths of the frame elements.

One of the motivations behind [26] is that the collection of the vertices of a Platonic solid forms a FUNTF for $\mathbb{R}^3$. The Platonic solids consist of a tetrahedron (4 vertices), octahedron (6 vertices), octahedron (8 vertices), icosahedron (12 vertices), and dodecahedron (20 vertices). A generalization of this idea to higher order dimensions can be done by associating FUNTFs with equilibrium points under a force. This force is known as the *frame force*.

Definition 10. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$. The *frame force* is the map $FF : S^{n-1} \times S^{n-1} \to \mathbb{R}^n$ defined by

$$FF(a,b) = \langle a,b \rangle (a-b).$$

In order for a set of points to be under equilibrium with respect to the frame force, its elements need to be critical under the frame force. These elements are referred to as *FF-critical sequences*.

Definition 11. A finite sequence $\{x_j\}_{j=1}^N$ in a n-dimensional Hilbert space is *FF-critical* if each x_j is an eigenvector of the associated frame operator S. That is, for all $1 \leq j \leq N$, there must exist a scalar λ_j such that $Sx_j = \lambda_j x_j$.

Since all equilibrium points are critical but not vice versa, all FUNTFs are FF-critical frames but not all FF-critical frames are FUNTFs. This fact can also be checked using matrix theory: The frame operator S of a FUNTF is a scalar multiple of the identity and therefore there exists a scalar A such that $Sx_j = Ax_j$, showing that a FUNTF is FF-critical. However, the eigenvalues λ_j in the definition of a FF-critical frame need not be all equal. Hence, the matrix corresponding to the frame operator of an FF-critical frame is not necessarily a scalar multiple of the identity. Consequently, FF-critical frames need not necessarily be FUNTFs.

It is pointed out in [26] that finite sequence of unit-norm vectors is FF-critical if and only if it can be partitioned into mutually exclusive orthogonal sequences, each of which is a normalized tight frame for its span.

Finding the equilibrium under a force is equivalent to a constrained optimization problem, and therefore this problem can be solved using Lagrange multipliers. In this case, the local minimizers of the associated potential of the force are the equilibrium points of the force. The potential of the frame force is the *frame potential*.

Definition 12. Let $\{x_j\}_{j=1}^N$ be a finite unit-norm frame in $\mathbb{H}^n$, where $\mathbb{H}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. The *frame potential* corresponding to $\{x_j\}_{j=1}^N$ is the map $FP : (S^{n-1})^N \to [0, \infty)$ defined by

$$FP(\{x_j\}_{j=1}^N) = \sum_{j=1}^N \sum_{k=1}^N |\langle x_j, x_k \rangle|^2.$$

Here, S^{n-1} denotes the unit sphere in $\mathbb{H}$ and $(S^{n-1})^N = S^{n-1} \times \cdots \times S^{n-1}$ is the Cartesian product of N copies of S^{n-1}.

The main result of [26], which we state below, is that a sequence in a Hilbert space is a FUNTF if and only if it is a local minimizer of the frame potential. This is a consequence of proving that a sequence is a FUNTF in a Euclidean space if and only if it is in equilibrium under the frame force.

Theorem 1.4.1 ([26])**.** *Let $\{x_j\}_{j=1}^N$ be a finite sequence of unit vectors in $\mathbb{H}^n$. Then $FP(\{x_j\}_{j=1}^N) \geq N^2/n$. Moreover, $\{x_j\}_{j=1}^N$ is a FUNTF if and only if $FP(\{x_j\}_{j=1}^N) = N^2/n$.*

Theorem 1.4.1 provides a necessary and sufficient condition for a finite unit-norm sequence to be a FUNTF for n-dimensional Euclidean space. Since all norms on a finite-dimensional normed space are equivalent, the results in [27] and [26], which are for Euclidean spaces, can be applied to any finite-dimensional Hilbert spaces.

While FUNTFs have been characterized for Hilbert spaces as sequences that minimize the frame potential, explicit common structural properties of such sequences are yet unknown. Recall that vertices of Platonic solids are frames for $\mathbb{R}^3$, which provides a geometric visualization of FUNTFs for $\mathbb{R}^3$. On the other hand, there have been no known geometric visualizations of FUNTFs for $\mathbb{R}^n$ with $n > 3$, with the only known method to explicitly construct FUNTFs for $\mathbb{R}^n$ being Theorem 1.4.1.

A related result was provided by Abdollahi and Monfaredpour in [30], whose main theorem we state below. In that paper, a recursive method was obtained for constructing FUNTFs for $\mathbb{R}^{n+1}$ from FUNTFs whose elements sum to zero for $\mathbb{R}^n$.

Theorem 1.4.2 ([30])**.** *Let $\{x_j\}_{j=1}^N$ be a FUNTF for $\mathbb{R}^n$ such that $\sum_{j=1}^N x_j = 0$. Then there exists a FUNTF $\{y_j\}_{j=1}^{N+1}$ in $\mathbb{R}^{n+1}$ which is recursively defined from $\{x_j\}_{j=1}^N$.*

We briefly describe the method that [30] used to prove the theorem above. Let $\{x_j\}_{j=1}^N$ be a FUNTF for $\mathbb{R}^n$ satisfying the additional condition $\sum_{j=1}^N x_j = 0$, and let S be its corresponding $n \times N$ frame matrix. Next, [30] introduced the $(n+1) \times (N+1)$ block matrix

$$Y = \begin{bmatrix} \sqrt{\frac{n(N+1)}{N(n+1)}}S & H \\ \sqrt{\frac{N-n}{N(n+1)}}G & \pm 1 \end{bmatrix},$$

where $G = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$ and $H = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \pm 1 \end{bmatrix}$. They prove that the frame $\{y_j\}_{j=1}^{N+1}$ in $\mathbb{R}^{n+1}$, with the y_j's being the columns of Y, is a FUNTF of length $N+1$. This is directly constructed from the FUNTF $\{x_j\}_{j=1}^N$ of length N.

The FUNTF $\{x_j\}_{j=1}^N$ in the hypothesis of Theorem 1.4.2 satisfies the zero-sum condition. A finite sequence in a Hilbert space whose elements sum to zero is said to be *bal-*

anced. In other words a sequence $\{x_n\}_{n=1}^N$ in a Hilbert space H is balanced if $\sum_{n=1}^{N} x_n = 0$. We provide a brief overview of balanced frames in the next section.

1.5 Brief Discussion of Balanced Frames

As far as we are aware, the literature on balanced frames, even in the Hilbert space setting, is sparse. One of the few papers devoted to this topic is provided by Heineken, Morillas, and Tarazaga ([31]), where balanced FUNTFs are formally defined for $\mathbb{R}^n$ and various results pertaining to them are obtained, including the following examples.

Example. Consider the Fourier matrix

$$F_N = [e^{2\pi ijk/N}]_{1\leq j,k\leq N}. \tag{1.18}$$

Observe that for each $1 \leq j \leq N$,

$$\sum_{k=1}^{N} e^{2\pi ijk/N} = 0. \tag{1.19}$$

Any n rows of F_N forms a FTF (finite tight frame) of length N for $\mathbb{C}^n$. Equation (Equation 1.19) implies that the resultant FTF is also balanced. Since we can obtain a balanced FUNTF by normalizing this FTF, this shows that there exists a balanced FUNTF of any length $N \geq n$ for $\mathbb{C}^n$.

Moving on from $\mathbb{C}^n$, we illustrate an example of a balanced FUNTF in $\mathbb{R}^n$. The following example is from [31].

Example. Let $\{u_j\}_{j=1}^N$ be an orthonormal basis in $\mathbb{R}^n$ and define a sequence

$$\{v_j\}_{j=1}^{2N} = \{-u_N, -u_{N-1}, ..., -u_1, u_1, u_2, ..., u_N\}. \tag{1.20}$$

Since $\{u_j\}_{j=1}^N$ is an orthonormal basis, the Plancherel Equality implies that it is a

FUNTF. Similarly, since $\{v_j\}_{j=1}^{2N}$ is the union of two orthonormal bases, it is a FUNTF. Clearly, $\sum_{j=1}^{2N} v_j = 0$ and this defines a balanced FUNTF of length $2N$ for $\mathbb{R}^n$. This shows that there exist balanced FUNTFs of even length for $\mathbb{R}^n$.

The results of [31] provide a broad range of properties of balanced FUNTFs. These include multiple characterizations of balanced FUNTFs, a method for finding and completely characterizing the closest balanced frame to a given frame, and introducing a new concept of complements for balanced frames.

We leave the existence of a balanced FUNTF with odd length for $\mathbb{R}^n$ as an open question.

Question 1.5.1. *Let $n \in \mathbb{N}$ and $N \geq n$ be odd. Does there exist a balanced FUNTF in $\mathbb{R}^n$ of length N?*

For any $n \in \mathbb{N}$, there does exist an odd $N \geq n$ which does not yield a balanced FUNTF of length N for $\mathbb{R}^n$. To justify this claim, we need the characterization of balanced FUNTFs in terms of *spherical* 2*-designs*, obtained by Waldron in [32]. A spherical t-design is defined as follows.

Definition 13. Let $S = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}$ be the unit sphere in $\mathbb{R}^n$ and let $N \geq n$. The subset $\{z_j\}_{j=1}^{N} \subseteq S$ is a *spherical t-design* if and only if

$$\frac{1}{|S|} \int_S f(x)\, d\omega(x) = \frac{1}{N} \sum_{j=1}^{N} f(z_j). \tag{1.21}$$

for all polynomials $f : \mathbb{R}^n \to \mathbb{R}$ with degree $\leq t$.

Now, we state the characterization of balanced FUNTFs in $\mathbb{R}^n$ as given in [32].

Theorem 1.5.2 ([32])**.** *A finite sequence $\{x_j\}_{j=1}^{N}$ in $\mathbb{R}^n$ is a balanced FUNTF for $\mathbb{R}^n$ if and only if it is a spherical* 2*-design.*

In other words, determining the existence of balanced FUNTFs in $\mathbb{R}^n$ is equivalent to determining the existence of spherical 2-designs in $\mathbb{R}^n$. Mimura in [33] provided conditions under which spherical 2-designs for $\mathbb{R}^n$ exist.

Theorem 1.5.3 ([33])**.** *Let S be the unit sphere in $\mathbb{R}^n$. There does not exist a spherical 2-design with N points for S if and only if either $N = n+2$ with n odd, or $n = 1$ and N odd.*

Combining Theorems 1.5.2 and 1.5.3, we see that balanced FUNTFs do not always exist in $\mathbb{R}^n$. In contrast, there do exist FUNTFs of any length $N \geq n$ in $\mathbb{R}^n$.

Theorem 1.5.4. *If n is odd, then there does not exist a balanced FUNTF of length $n+2$ in $\mathbb{R}^n$.*

We extend the idea of balanced frames to Schauder frames for Banach spaces in the next definition.

Definition 14. A Schauder frame $\{(x_j, f_j)\}_{j=1}^{\infty}$ for a Banach space X is said to *balanced* if $\sum_{j=1}^{\infty} x_j = 0$ and $\sum_{j=1}^{\infty} f_j = 0$.

Recall that the existence and characterization of FUNTFs for all finite-dimensional Hilbert spaces are known: These results can be attributed to [27] and [26], respectively. We leave as open questions the existence and characterization of balanced Schauder frames for Banach spaces.

Question 1.5.5. *What are the necessary and sufficient conditions for a Schauder frame $\{(x_j, f_j)\}_{j=1}^{N}$ for a finite-dimensional Banach space X to be balanced?*

Question 1.5.6. *Let X be an n-dimensional Banach space and choose any $N \geq n$. Does there exist a balanced Schauder frame $\{(x_j, f_j)\}_{j=1}^{N}$ for X of length N?*

The simplest example of a finite-dimensional Banach space is ℓ_1^n. In general, proving results for a specific space can eventually lead to proofs for arbitrary spaces. It is thus appropriate to restate questions 1.5.5 and 1.5.6 for the space ℓ_1^n.

Question 1.5.7. *What are the necessary and sufficient conditions for a Schauder frame $\{(x_j, f_j)\}_{j=1}^N$ for ℓ_1^n to be balanced?*

Question 1.5.8. *Let X be an n-dimensional Banach space and choose any $N \geq n$. Does there exist a balanced Schauder frame $\{(x_j, f_j)\}_{j=1}^N$ for ℓ_1^n of length N?*

CHAPTER 2

FINITE UNIT-NORM TIGHT FRAMES FOR BANACH SPACES

2.1 Introduction

Schauder frames for Banach spaces were defined in such a way that the frame operator is the identity, and relaxing this restriction to just boundedness and invertibility yields the definition of an approximate Schauder frame. By letting the restrictions on the frame operator lie between these two notions, we can define another class of sequence pairs in a Banach space and its dual. This will be the focus for this chapter.

For a Hilbert space, FUNTFs are a specific subclass of unit-norm frames for which the frame operator is equal to a scalar multiple of the identity. Imposing this restriction on the frame operator in Banach spaces defines a class of sequence pairs that are more restrictive than approximate Schauder frames, but less restrictive than Schauder frames. By further imposing the unit-norm condition, the resultant class of sequence pairs is called a FUNTF, or finite unit-norm tight frame. We formally define a FUNTF for a Banach space below.

Definition 15. Let X be a finite-dimensional Banach space with dual X^* and let $\{(x_j, f_j)\}_{j=1}^N \subseteq X \times X^*$ satisfy $||x_j|| = f_j(x_j) = ||f_j|| = 1$ for all $1 \leq j \leq N$. Then $\{(x_j, f_j)\}_{j=1}^N$ is a *FUNTF* for X if its frame operator is a scalar multiple of the identity.

Motivated by previous work on the existence and characterization of FUNTFs in Hilbert spaces, Chavez-Dominguez, Kornelson, and Freeman in [1] derived results along similar lines for Banach spaces. While the *characterizations* of FUNTFs was successfully duplicated for all Banach spaces, they were not able to prove the *existence* of FUNTFs for a general finite-dimensional Banach space. What the authors of [1] were able to do in this direction was to provide examples of FUNTFs for select lower-dimensional Banach spaces. In particular, the existence of FUNTFs was proved for real ℓ_1^n with n in the range $2 \leq n \leq 6$

only.

One of the main theorems in [1] showed that FUNTFs for Banach spaces can be characterized as the finite sequences for which an appropriate associated frame potential is minimized. This is a similar condition to the one proved by Benedetto and Fickus in [26] for finite sequences to be FUNTFs for Hilbert spaces.

Theorem 2.1.1 ([1]). *Let X be an n-dimensional Banach space with dual X^* and fix a set of pairs $\{(x_j, f_j)\}_{j=1}^N \subseteq X \times X^*$. Then the frame potential of $\{(x_j, f_j)\}_{j=1}^N$ is at least N^2/n, and $\{(x_j, f_j)\}_{j=1}^N$ is a FUNTF for X if and only if its frame potential is equal to N^2/n.*

We have not yet specified how the frame potential is defined in a Banach space. Intuitively, two initial guesses to define the frame potential in Banach spaces are

$$FP(\{(x_j, f_j)\}_{j=1}^N) = \sum_{i=1}^N \sum_{j=1}^N |f_i(x_j)|^2 \tag{2.1}$$

and

$$FP(\{x_j, f_j\}_{j=1}^N) = \sum_{i=1}^N \sum_{j=1}^N |f_i(x_j) f_j(x_i)|. \tag{2.2}$$

However, it was shown in Proposition 2.5 of [1] that neither of these definitions are applicable to Banach spaces. Specifically, examples were given for FUNTFs of length 3 in ℓ_1^2 for which Theorem 2.1.1 does not hold using either of these two defintions of the frame operator. Consequently, an entirely different approach is needed when it comes to defining frame potentials in Banach spaces.

The frame potential is identical to the square of the Hilbert-Schmidt norm of the frame operator in the Hilbert space setting (see, e.g., [34] and [35] for discussions on Hilbert-Schmidt norms). Since Hilbert-Schmidt norms are only defined for operators on Hilbert spaces, it is necessary to find an alternative norm that is defined for both Banach and Hilbert spaces. Additionally, the new norm must coincide with the Hilbert-Schmidt norm for operators on Hilbert spaces.

One norm that satisfies all of these requirements is the *2-summing norm*. This allows us to alternatively define the frame potential as the square of the 2-summing norm.

Definition 16. Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be Banach spaces. The *2-summing norm* of an operator $T : X \to Y$ is the value $\pi_2(T)$ given by

$$\pi_2(T)^2 = \sup\left\{\sum_{j=1}^{n} ||Tx_j||_Y^2 \ : \ \sum_{j=1}^{n} |f(x_j)|^2 \leq ||f||_X^2, \ n \in \mathbb{N}, \ f \in X^*, \ (x_j)_{j=1}^n \subseteq X\right\}. \tag{2.3}$$

In other words, the square of the 2-summing norm is the supremum of the sum $\sum_{j=1}^{n} ||Tx_j||_Y^2$ taken over all $n \in \mathbb{N}$, functions $f \in X^*$, and sequences $(x_j)_{j=1}^n$ in X that satisfy the inequality

$$\sum_{j=1}^{n} |f(x_j)|^2 \leq ||f||_X^2. \tag{2.4}$$

Discussions on 2-summing norms and further information on its relationships with frame potentials and FUNTFs for Banach spaces can be found in texts such as [36], [37], [38], and [39].

Following [1], we define the frame potential in a Banach space as follows.

Definition 17. Let X be a finite-dimensional Banach space with dual X^*. The *frame potential* for a Schauder frame $\{(x_j, f_j)\}_{j=1}^N \subseteq X \times X^*$ is defined to be

$$FP(\{(x_j, f_j)\}_{j=1}^N) = \pi_2(S)^2,$$

where $S : X \to X$ is the frame operator of $\{(x_j, f_j)\}_{j=1}^N$.

From a previous discussion in Chapter 1, we know that [27] and [28] were the first to prove that FUNTFs exist in any finite-dimensional Hilbert space by finding and verifying FUNTFs explicitly. This occurred even before [26] gave a characterization of FUNTFs. More specifically, [27] and [28] independently computed FUNTFs in $\mathbb{C}^n$ explicitly using similar methods, with [27] going further to explicity compute FUNTFs of all lengths in $\mathbb{R}^n$.

For the Banach space setting the opposite ordering has taken place. While conditions for sequence pairs to be a FUNTF for any finite-dimensional Banach spaces that have a 1-unconditional basis were stated and proved in [1], the existence of FUNTFs of lengths $N \geq n$ has only been proved for complex n-dimensional and real 2-dimensional Banach spaces that have 1-unconditional bases.

Since 1-unconditional bases have not been defined previously, we do so formally as follows.

Definition 18. Given an n-dimensional Banach space X, a basis $\{x_j\}_{j=1}^n$ for X is 1-unconditional if given any sequences of scalars $\{\alpha_j\}_{j=1}^n$ and $\{\beta_j\}_{j=1}^n$ satisfying $|\alpha_j| \leq |\beta|_j$ for all $1 \leq j \leq n$, we have

$$\left|\left|\sum_{j=1}^{n} \alpha_j x_j\right|\right| \leq \left|\left|\sum_{j=1}^{n} \beta_j x_j\right|\right|.$$

To prove the existence of FUNTFs for Banach spaces, [1] used an operator theoretic approach without giving an explicit method to find examples of FUNTFs. That is, even though the proved that FUNTFs exist for complex finite-dimensional Banach spaces with 1-unconditional bases, they could not give explicit examples of such frames. Specifically, they gave necessary and sufficient conditions on scalar multiples of the identity operator that allow FUNTFs to be defined for Banach spaces. We state this below as a theorem.

In this theorem, the concept of a norm-one, rank-one projection is used as a part of the conditions. Given a finite-dimensional Banach space X whose dual space is X^*, norm-one rank-one projections is an operator $A : X \to X$ that satisfies $Ax = y^*(x)y$, for all $y \in X$ and $y \in X^*$ that satisfy $||y|| = ||y^*|| = y^*(y) = 1$.

Theorem 2.1.2 ([1])**.** *There exists a FUNTF of length N for a complex n-dimensional Banach space with a 1-unconditional basis if and only if a scalar multiple of the identity in X can be written as a sum of N norm-one, rank-one projections.*

When combined with Theorem 2.1.2, the next theorem, which is Proposition 4.5 in [1], concludes that there exists a FUNTF of length N for any complex n-dimensional Banach spaces with a 1-unconditional basis, for any $N \geq n$.

Theorem 2.1.3 ([1])**.** *Let X be a complex n-dimensional or a real 2-dimensional Banach space that has a* 1*-unconditional basis. Also, let $N \geq n$ be an integer and assume that the nonnegative numbers $\{\lambda_j\}_{j=1}^N$ satisfy $\sum_{j=1}^n \lambda_j = N$. Then the operator $T : X \to X$ defined by*

$$T = \sum_{j=1}^{n} \lambda_j \, e_j^* \otimes e_j$$

can be written as a sum of N norm-one, rank-one projections.

Here, the $\otimes$ symbol denotes the *tensor product*. In other words, we define the tensor product of the vectors $\mathbf{v}$ and $\mathbf{w}$ to be $\mathbf{v} \otimes \mathbf{w} = \mathbf{v}\mathbf{w}^T$.

To summarize, some of the major results that the authors in [1] were able to prove are listed here:

1. Given an n-dimensional Banach space X with an 1-unconditional basis, necessary and sufficient conditions were obtained for a sequence pair in $X \times X^*$ to be a FUNTF for X.

2. There exists a FUNTF of length $n \geq N$ for any complex n-dimensional Banach space that has a 1-unconditional basis, and for any real 2-dimensional Banach space that has a 1-unconditional basis.

What is left as an open problem in [1] is the existence FUNTFs of length $N \geq n$ in an n-dimensional real Banach space. Even though we are not able to give a complete answer to this question, we establish the following facts in this chapter:

1. There exists a FUNTF of length N for real ℓ_1^n for *any* $N \geq n$.

2. There exist FUNTFs without repeated elements of length $2n-2$ and $2n-1$ for real ℓ_1^n for any positive integer n.

The next section describes in detail the methods that we will use to prove the first statement above and the same will be done in the following section for the second statement. In the final section of this chapter, we list the open questions that need to be answered in order to draw conclusions about the existence of FUNTFs for any n-dimensional real Banach spaces containing a 1-unconditional basis.

2.2 Existence of FUNTFs for Real ℓ_1^n

Although the next lemma was stated and proved in [1], we give the modification of the proof that uses matrix theory to solve for the frame operator. We will then use this method multiple times throughout this section, including to obtain our main result where we show that there exists a FUNTF for real ℓ_1^n of any length $N \geq n$. In particular, we will explicitly compute a FUNTF of length $n+1$ for real ℓ_1^n and will later on use this as a starting point to compute FUNTFs of any length $N \geq n$.

Lemma 2.2.1. *For all $n \in \mathbb{N}$, there exists a FUNTF of length $n+1$ for real ℓ_1^n.*

Proof. Let $\{e_j\}_{j=1}^n$ be the standard basis in real ℓ_1^n and let $\{e_j^*\}_{j=1}^n$ be the standard basis in its dual, real ℓ_∞^n. We define a sequence $\{x_j\}_{j=1}^{n+1}$ in real ℓ_1^n as follows.

$$x_j = \begin{cases} be_j - \sum\limits_{i \neq j} ae_i, & \text{if } 1 \leq j \leq n, \\ \sum\limits_{i=1}^{n} \frac{1}{n} e_i, & \text{if } j = n+1, \end{cases} \tag{2.5}$$

where the constants $a, b > 0$ are yet to be computed. If the x_j's are to be a component of a FUNTF for real ℓ_1^n, they must be unit norm which in this case is achieved by setting

$$b + (n-1)a = 1. \tag{2.6}$$

The corresponding normalizing functionals of $\{x_j\}_{j=1}^{n+1}$ in real ℓ_∞^n, $\{f_j\}_{j=1}^{n+1}$, are given by

$$f_j = \begin{cases} e_j^* - \sum_{i \neq j} e_i^*, & \text{if } 1 \leq j \leq n, \\ \sum_{i=1}^{n} e_i, & \text{if } j = n+1. \end{cases} \tag{2.7}$$

Observe that $\{f_j\}_{j=1}^{n+1}$ is the sequence $\{x_j\}_{j=1}^{n+1}$ normalized with respect to the ℓ_∞-norm instead of the ℓ_1-norm. This is consistent with a given sequence's corresponding normalizing functionals being in the dual space.

Our goal now is to compute the constants $a, b > 0$ that were left undetermined in equation (Equation 2.5) so that $\{(x_j, f_j)\}_{j=1}^{n+1}$ does define a FUNTF for real ℓ_1^n. That is, we determine $a, b > 0$ so that the frame operator S associated with $\{(x_j, f_j)\}_{j=1}^{n+1}$ is given by $\frac{n+1}{n} I$. Using the definitions of the x_j's and f_j's given in equations (Equation 2.5) and (Equation 2.7), we compute that the matrix representation of S is

$$S = \sum_{j=1}^{n+1} f_j \otimes x_j = \begin{bmatrix} | & | & & | & | \\ f_1 & f_2 & \cdots & f_n & f_{n+1} \\ | & | & & | & | \end{bmatrix} \begin{bmatrix} — & x_1 & — \\ — & x_2 & — \\ & \vdots & \\ — & x_n & — \\ — & x_{n+1} & — \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 & 1 \\ -1 & 1 & -1 & \cdots & -1 & 1 \\ -1 & -1 & 1 & \cdots & -1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & 1 \\ -1 & -1 & -1 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} b & -a & -a & \cdots & -a \\ -a & b & -a & \cdots & -a \\ -a & -a & b & \cdots & -a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a & -a & -a & \cdots & b \\ 1/n & 1/n & 1/n & \cdots & 1/n \end{bmatrix}$$

$$= \begin{bmatrix} b+(n-1)a+1/n & -b+(n-3)a+1/n & \cdots & -b+(n-3)a+1/n \\ -b+(n-3)a+1/n & b+(n-1)a+1/n & \cdots & -b+(n-3)a+1/n \\ \vdots & \vdots & \ddots & \vdots \\ -b+(n-3)a+1/n & -b+(n-3)a+1/n & \cdots & b+(n-1)a+1/n \end{bmatrix}.$$

Each diagonal entry of S is $b + (n-1)a + 1/n$, while each non-diagonal entry is $-b + (n-3)a + 1/n$. If $\{(x_j, f_j)\}_{j=1}^{n+1}$ is to be a FUNTF for real ℓ_1^n, then S must be a diagonal matrix, so all nondiagonal entries must be zero. Finding the constants $a, b > 0$ for which $\{(x_j, f_j)\}_{j=1}^{n+1}$ defines a FUNTF now boils down to solving the following system of equations:

$$\begin{aligned} b + (n-1)a &= 1, \\ -b + (n-3)a + \frac{1}{n} &= 0. \end{aligned} \tag{2.8}$$

The first equation of the system (Equation 2.8) above represents the unit-norm condition for the x_j's, while the second equation represents the condition that S must be a diagonal matrix. The solution to this system is given by

$$a = \frac{1 - 1/n}{2n - 4} = \frac{n-1}{2n^2 - 4n}, \tag{2.9}$$

and

$$b = \frac{n^2 - 2n - 1}{2n^2 - 4n}. \tag{2.10}$$

These values of a and b do satisfy the relations given in the system of equations (Equation 2.8). Inserting these values into the frame matrix for S yields $S = (n+1)/nI$, and so the definition of a FUNTF is satisfied. Therefore, an explicit FUNTF of length $n+1$ for real ℓ_1^n is the pair $\{(x_j, f_j)\}_{j=1}^{n+1} \subseteq \ell_1^n \times \ell_\infty^n$, where

$$x_j = \begin{cases} \frac{n^2-2n-1}{2n^2-4n}\, e_j - \sum\limits_{i\neq j} \frac{1-1/n}{2n-4}\, e_i, & \text{if } 1 \leq j \leq n \\ \sum\limits_{i=1}^{n} \frac{1}{n}\, e_i, & \text{if } j = n+1, \end{cases} \tag{2.11}$$

and $\{f_j\}_{j=1}^{n+1}$ defined as in equation (Equation 2.7). □

Having explicitly defined a FUNTF of length $n+1$, the next step is to show whether there exists a FUNTF of length $n+2$ for real ℓ_1^n. Examining the construction of the FUNTF $\{(x_j, f_j)\}_{j=1}^{n+1}$ defined in Lemma 2.2.1, note that x_{n+1} differs from the other x_j's by being the only element which is independent of the constants $a, b > 0$. To define a sequence pair $\{(x_j, f_j)\}_{j=1}^{n+2}$ of length $n+2$ in real $\ell_1^n \times \ell_\infty^n$, we propose adding the terms $x_{n+2} = -x_{n+1}$ and $f_{n+2} = -f_{n+1}$, leaving the remaining $\{(x_j, f_j)\}_{j=1}^{n+1}$ unchanged. In the next theorem, we prove that this choice does indeed define a FUNTF of length $n+2$ for real ℓ_1^n.

Theorem 2.2.2. *For all $n \in \mathbb{N}$, there exists a FUNTF of length $n+2$ for real ℓ_1^n.*

Proof. Consider the sequence $\{x_j\}_{j=1}^{n+2}$ in real ℓ_1^n defined by:

$$x_j = \begin{cases} be_j - \sum\limits_{i\neq j} ae_i, & \text{if } 1 \leq j \leq n, \\ \sum\limits_{i=1}^{n} \frac{1}{n}\, e_i, & \text{if } j = n+1, \\ -\sum\limits_{i=1}^{n} \frac{1}{n}\, e_i, & \text{if } j = n+2, \end{cases} \tag{2.12}$$

with the constants $a, b > 0$ yet to be determined. Just as in Lemma 2.2.1, the x_j's must be normalized if they are to be components of a FUNTF sequence pair, so we set

$$b + (n-1)a = 1. \tag{2.13}$$

Just as in equation (Equation 2.7), it is readily seen that the normalizing functionals $\{f_j\}_{j=1}^{n+2}$

in real ℓ_∞^n corresponding to this sequence $\{x_j\}_{j=1}^{n+2}$ are given by

$$f_j = \begin{cases} e_j^* - \sum\limits_{i \neq j} e_i^*, & \text{if } 1 \leq j \leq n, \\ \sum\limits_{i=1}^{n} e_i, & \text{if } j = n+1, \\ -\sum\limits_{i=1}^{n} e_i, & \text{if } j = n+2. \end{cases} \tag{2.14}$$

A similar computation as in Lemma 2.2.1 then shows that the matrix of the frame operator S associated to the sequence pair $\{(x_j, f_j)\}_{j=1}^{n+2} \subseteq \ell_1^n \times \ell_\infty^n$ is

$$S = \begin{bmatrix} b+(n-1)a+2/n & -b+(n-3)a+2/n & \cdots & -b+(n-3)a+2/n \\ -b+(n-3)a+2/n & b+(n-1)a+2/n & \cdots & -b+(n-3)a+2/n \\ \vdots & \vdots & \ddots & \vdots \\ -b+(n-3)a+2/n & b+(n-1)a+2/n & \cdots & -b+(n-3)a+2/n \end{bmatrix}. \tag{2.15}$$

Since

$$S(m,m) = b + (n-1) + \frac{2}{n} \tag{2.16}$$

for all $1 \leq m \leq n$ and

$$S(k,m) = -b + (n-3)a + \frac{2}{n}, \tag{2.17}$$

for all $1 \leq k, m \leq n$ with $k \neq m$, we choose a and b so that the nondiagonal entries of S are all equal to 0 and the diagonal entries are given by $(n+2)/n$. In fact, we achieve this by setting

$$a = \frac{1 - 2/n}{2n-4} = \frac{1}{2n} \tag{2.18}$$

and

$$b = \frac{n+1}{2n}. \tag{2.19}$$

This shows that $\{(x_j, f_j)\}_{j=1}^{n+2} \subseteq \ell_1^n \times \ell_\infty^n$ with

$$x_j = \begin{cases} \frac{n+1}{2n} e_j - \sum\limits_{i \neq j} \frac{1}{2n} e_i, & \text{if } 1 \leq j \leq n, \\ \sum\limits_{i=1}^{n} \frac{1}{n}\, e_i, & \text{if } j = n+1, \\ -\sum\limits_{i=1}^{n} \frac{1}{n}\, e_i, & \text{if } j = n+2, \end{cases} \tag{2.20}$$

and

$$f_j = \begin{cases} e_j^* - \sum\limits_{i \neq j} e_i^*, & \text{if } 1 \leq j \leq n, \\ \sum\limits_{i=1}^{n} e_i, & \text{if } j = n+1, \\ -\sum\limits_{i=1}^{n} e_i, & \text{if } j = n+2. \end{cases} \tag{2.21}$$

defines a FUNTF of length $n+2$ for real ℓ_1^n. □

Now that we are working with higher dimensional spaces, one advantage of frames over bases is the ability to repeat items in the sequence. In particular, we can replace x_{n+2} in the proof above with $\sum\limits_{i=1}^{n} \frac{1}{n}\, e_i$ and make the corresponding appropriate change of sign to f_{n+2}. All other elements of the sequence pair defined by equations (Equation 2.12) and (Equation 2.14) remain unchanged. That is, we redefine the sequence pair $\{(x_j, f_j)\}_{j=1}^{n+2}$ by

$$x_j = \begin{cases} \frac{n+1}{2n} e_j - \sum\limits_{i \neq j} \frac{1}{2n} e_i, & \text{if } 1 \leq j \leq n, \\ \sum\limits_{i=1}^{n} \frac{1}{n}\, e_i, & \text{if } j = n+1 \text{ and } n+2. \end{cases} \tag{2.22}$$

and

$$f_j = \begin{cases} e_j^* - \sum\limits_{i \neq j} e_i^*, & \text{if } 1 \leq j \leq n, \\ \sum\limits_{i=1}^{n} e_i, & \text{if } j = n+1 \text{ and } n+2. \end{cases} \tag{2.23}$$

Notice here that $x_{n+1} = x_{n+2}$ and $f_{n+1} = f_{n+2}$. Unlike the FUNTF defined in Theorem 2.2.2, which did not repeat any frame elements, this new sequence pair now has one repeated element. Furthermore, since the negative signs cancel out when computing the frame matrix in Theorem 2.2.2, the frame matrix corresponding to the sequence pair $\{(x_j, f_j)\}_{j=1}^{n+2}$ defined by equations (Equation 2.22) and (Equation 2.23) is unchanged from the frame matrix corresponding to the FUNTF constructed in Theorem 2.2.2 above. Hence, the sequence pair $\{(x_j, f_j)\}_{j=1}^{n+2}$ given by equations (Equation 2.22) and (Equation 2.23) is still a FUNTF for real ℓ_1^n.

It was also shown in [1] that FUNTFs of lengths 4 and 5 exist for real ℓ_1^3, and that FUNTFs of lengths 5, 6, and 7 exist for real ℓ_1^4. Even though this is mentioned explicitly in [1], we provide an independent proof and new examples of the fact that FUNTFs of length M exist for real ℓ_1^3 for all $M \geq 3$, and those of length N exist for ℓ_1^4 for all $N \geq 4$.

Theorem 2.2.3. *There exists a FUNTF of length M for real ℓ_1^3 for all $M \geq 3$, and there exists a FUNTF of length N in real ℓ_1^4 for all $N \geq 4$.*

Proof. We prove this proposition for real ℓ_1^3 only, as a similar argument applies to real ℓ_1^4.

First, note that the sequence pair $\{(e_j, e_j^*\}_{j=1}^3$, with the e_j's in real ℓ_1^3 and the e_j^*'s in real ℓ_∞^3, provides a FUNTF of length 3 for real ℓ_1^3. In addition, Propositions 7.1 and 7.3 of [1] provide FUNTFs of lengths 4 and 5 for real ℓ_1^3, respectively. So, we need only consider the case $M \geq 6$.

Fix any M with $M \geq 6$. Write M as $M = 3(a+1)+b$, where $a \in \mathbb{N}$ and $0 \leq b \leq 2$. We can also write this as $M = 3a+(b+3)$, with $a \in \mathbb{N}$ and $3 \leq b+3 \leq 5$. Take the union of a copies of the standard bases in real ℓ_1^3 and real ℓ_∞^3 (i.e., a copies of $\{(e_j, e_j^*)\}_{j=1}^3 \subseteq \ell_1^3 \times \ell_\infty^3$) with a FUNTF of length $b+3$. Since the standard basis sequence pair is a FUNTF for real ℓ_1^3, this union provides a FUNTF of length $M = 3a + (b+3) \geq 6$ for ℓ_1^3. □

This argument can be generalized to any $n \in \mathbb{N}$ with $n \geq 3$. Therefore, Theorem 2.2.3 can be rewritten to be applicable for all real ℓ_1^n. We state this as a corollary, as follows.

Corollary 2.2.4. *Choose $n \in \mathbb{N}$ with $n \geq 3$ and $N \in \mathbb{N}$ with $N \geq n$. If there exists a FUNTF of length M for real ℓ_1^n for every $M \in \mathbb{N}$ with $n+1 \leq M \leq 2n-1$, then there* *exists a FUNTF of length N for real ℓ_1^n.*

Proof. By taking $M \in \mathbb{N}$ with $M \geq 2n$ and writing M as $M = 2n(a+1)+b$, with $a \in \mathbb{N}$ and $0 \leq b \leq n-1$, the proof follows by the same technique as used in Theorem 2.2.3. □

In order to prove that there exists a FUNTF of length $N \geq n$ for real ℓ_1^n, it suffices to construct FUNTFs of lengths $n+k$ for real ℓ_1^n for $1 \leq k \leq n$. Recall the discussion following Theorem 2.2.2, which demonstrated that a FUNTF of length $n+2$ can be recursively built from one with length $n+1$ by repeating an appropriate frame element. We prove that by repeating the said element not just once but $k-1$ times, the resultant sequence pair with $n+k$ elements is still a FUNTF for real ℓ_1^n.

Theorem 2.2.5. *For each $n \in \mathbb{N}$, there exists a FUNTF of length N for real ℓ_1^n for all $N \geq n$.*

Proof. Without loss of generality, we assume that $n \geq 3$. Recall that Lemmas 2.2.1 and 2.2.2 told us that there exist FUNTFs of length $N = n+1$ and $N = n+2$ for real ℓ_1^n. By Corollary 2.2.4, it suffices to prove the existence of a FUNTF of length $N = n+k$, where $3 \leq k \leq n-1$, since this will provide existence of FUNTFs of any length $N \geq n$.

Fix any $3 \leq k \leq n-1$ and let $N = n+k$. Define a sequence $\{x_j\}_{j=1}^N$ in real ℓ_1^n by

$$x_j = \begin{cases} be_j - \sum_{i \neq j} ae_i, & \text{if } 1 \leq j \leq n \\ \sum_{i=1}^{n} \frac{1}{n}\, e_i, & \text{if } n+1 \leq j \leq n+k, \end{cases} \tag{2.24}$$

with the constants a and b to be determined. Now, the normalizing functionals in real ℓ_∞^n corresponding to $\{x_j\}_{j=1}^N$, $\{f_j\}_{j=1}^N$, are given by

$$f_j = \begin{cases} e_i^* - \sum_{j \neq i} e_j^*, & \text{if } 1 \leq i \leq n, \\ \sum_{j=1}^{n} e_j, & \text{if } n+1 \leq i \leq n+k. \end{cases} \tag{2.25}$$

The frame matrix S corresponding to $\{(x_j, f_j)\}_{j=1}^N$ is computed below:

$$S = \sum_{j=1}^{N} f_j \otimes x_j = \begin{bmatrix} | & | & | & & | & & | \\ f_1 & f_2 & f_3 & \cdots & f_{n+1} & \cdots & f_{n+k} \\ | & | & | & & | & & | \end{bmatrix} \begin{bmatrix} — & x_1 & — \\ — & x_2 & — \\ — & x_3 & — \\ & \vdots & \\ — & x_{n+1} & — \\ & \vdots & \\ — & x_{n+k} & — \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 & 1 \\ -1 & 1 & -1 & \cdots & -1 & 1 \\ -1 & -1 & 1 & \cdots & -1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & 1 \\ -1 & -1 & -1 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} b & -a & -a & \cdots & -a \\ -a & b & -a & \cdots & -a \\ -a & -a & b & \cdots & -a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a & -a & -a & \cdots & b \\ 1/n & 1/n & 1/n & \cdots & 1/n \end{bmatrix}$$

$$= \begin{bmatrix} b+(n-1)a+k/n & -b+(n-3)a+k/n & \cdots & -b+(n-3)a+k/n \\ -b+(n-3)a+k/n & b+(n-1)a+k/n & \cdots & -b+(n-3)a+k/n \\ \vdots & \vdots & \ddots & \vdots \\ -b+(n-3)a+k/n & b+(n-1)a+k/n & \cdots & -b+(n-3)a+k/n \end{bmatrix}.$$

For all $1 \leq m \leq n$, we see that

$$S(m,m) = b + (n-1)a + \frac{k}{n}, \tag{2.26}$$

and whenever $1 \leq k, m \leq n$ with $k \neq m$, we have

$$S(k,m) = -b + (n-3)a + \frac{k}{n}. \tag{2.27}$$

Our goal now is to find constants $a, b > 0$ such that $\{(x_j, f_j)\}_{j=1}^N$ is a FUNTF for real ℓ_1^n. For this to hold, we must have $S = (N/n)I$. Equivalently, we must have $S(k, m) = 0$ whenever $k \neq m$ and $S(m, m) = N/n$ for all $1 \leq m \leq n$. Also taking into consideration that the $\{x_j\}_{j=1}^N$ must be normalized with respect to the ℓ_1-norm for $\{(x_j, f_j)\}_{j=1}^N$ if they are to be a FUNTF, this amounts to solving the following system of equations.

$$\begin{aligned} b + (n-1)a &= 1, \\ -b + (n-3)a + k/n &= 0. \end{aligned} \tag{2.28}$$

The first equation in (Equation 2.28) captures the ℓ_1 unit-norm condition that we impose on the x_j's, while the second equation in (Equation 2.28) encapsulates the $S(k, m) = 0$ condition for $k \neq m$. Solving for a and b, we obtain

$$a = \frac{n-k}{2n^2 - 4n} \tag{2.29}$$

and

$$b = \frac{n^2 + (k-3)n - k}{2n^2 - 4n}. \tag{2.30}$$

Inserting these values of a and b into the frame matrix S, we see that we indeed have $S(m, m) = (n + k)/n$ for all $1 \leq m \leq n$. This shows that the sequence pair $\{(x_j, f_j)\}_{j=1}^{n+k}$ in real $\ell_1^n \times \ell_\infty^n$ defined by

$$x_j = \begin{cases} \frac{n^2+(k-3)n-k}{2n^2-4n} e_j - \sum\limits_{i \neq j} \frac{n-k}{2n^2-4n} e_i, & \text{if } 1 \leq j \leq n, \\ \sum\limits_{i=1}^{n} \frac{1}{n} e_i, & \text{if } n+1 \leq j \leq n+k, \end{cases} \tag{2.31}$$

and

$$f_j = \begin{cases} e_i^* - \sum\limits_{j \neq i} e_j^*, & \text{if } 1 \leq i \leq n, \\ \sum\limits_{j=1}^{n} e_j, & \text{if } n+1 \leq i \leq n+k. \end{cases} \tag{2.32}$$

is a FUNTF for real ℓ_1^n.

Since k was an arbitrary integer in the range $3 \leq k \leq n-1$, this also shows that for all $M \in \mathbb{N}$ with $n+1 \leq M \leq 2n-1$, a FUNTF of length M exists for real ℓ_1^n. Therefore, by Corollary 2.2.4, we conclude that for all $N \in \mathbb{N}$ with $N \geq n$, a FUNTF of length N exists for real ℓ_1^n. More specifically, a FUNTF of length $N \geq 2n$ for real ℓ_1^n can be defined by taking a union of FUNTFs of lengths $n \leq M \leq 2n-1$ chosen in an appropriate manner. □

Even though we have been able to construct FUNTFs of length $N \geq n$ for any real ℓ_1^n, such an excessive repetition of frame elements may not be desirable. For example, all of the FUNTFs of length $n+1$ through $2n-1$ that we have discussed have the same geometric construction, with the differences between the FUNTFs being the number of times a selected frame element gets repeated. As a result, it is not possible to distinguish between the geometric structures for FUNTFs of different lengths that we have defined.

In the next section, we will give an alternative construction of FUNTFs for real ℓ_1^n that does not involve such frame repetitions. Specifically, we will construct FUNTFs that have a different geometric structure than the ones defined in Theorem 2.2.5. Eventually, this can pave the way for a better understanding of the properties of various FUNTFs for real ℓ_1^n, such as the potential possibility to geometrically characterize all FUNTFs of a given length.

Question 2.2.6. *Does there exist a geometric characterization of FUNTFs of each given length $N \geq n$ for real ℓ_1^n?*

2.3 An Alternative FUNTF Construction for Real ℓ_1^n

Before constructing FUNTFs without repeated elements in higher dimensions, we give a detailed interpretation of the FUNTF constructions in Proposition 7.3 of [1]. These are examples of FUNTFs for real ℓ_1 and ℓ_4 that do not have repeated elements. Additionally, they are not built upon the structure of the FUNTFs of length $n + 1$ for real ℓ_1^n that was defined earlier in [1].

The FUNTF of length 5 for real ℓ_1^3 defined in [1] is given by $\{(x_j, f_j)\}_{j=1}^5$, where

$$\{x_j\}_{j=1}^5 = \{(1,0,0), (-a,b,b), (-a,-b,b), (-a,b,-b), (-a,-b,-b)\}, \tag{2.33}$$

with $a = 1/6$ and $b = 5/12$, and the f_j's are the corresponding normalizing functionals of the x_j's in real ℓ_∞^3.

The sequence $\{x_j\}_{j=1}^5$ can be interpreted to be a pyramid-like construction for $\mathbb{R}^3$, albeit with respect to the ℓ_1-norm instead of the standard ℓ_2-norm for the Euclidean space $\mathbb{R}^3$. Specifically, the x_j's form the vertices of a pyramid in $\mathbb{R}^3$. In fact, in [1], the authors chose to use $-a$ instead of a in order to better visualize a pyramid in $\mathbb{R}^3$.

For real ℓ_1^4, the FUNTF defined in [1] is the sequence pair $\{(x_j, f_j)\}_{j=1}^6$, where

$$\{x_j\}_{j=1}^6 = \{(a,b,b,0), (a,-b,b,0), (a,b,-b,0), (a,-b,-b,0), (c,0,0,d), (c,0,0,-d)\}, \tag{2.34}$$

with a, b, c, and d defined by the condition that $a + 2b = c + d = 1$. As usual, the f_j's are the corresponding normalizing functionals of the x_j's in the dual space, which is real ℓ_∞^6.

The first four elements of $\{x_j\}_{j=1}^6$ can be interpreted as representing the vertices of a square with side-lengths $2b$, which can also be interpreted as the "base" of a pyramid-like construction generalized to real ℓ_1^4. In this setting, we view real ℓ_1^4 as $\mathbb{R}^4$ equipped with the ℓ_1-norm. Notice here that the fourth, or the last, entry is fixed to be 0 in the first four elements of the sequence (Equation 2.34) to account for the extra dimension.

Suppose that the "apex", which is $(1, 0, 0)$ for the real ℓ_1^3 example, remained a singleton standard basis element in real ℓ_1^4. In this case, we would have a sequence pair of length 5 in real $\ell_1^4 \times \ell_\infty^4$. One possible structure for the ℓ_1^4-component of such a sequence pair could be

$$\{x_j\}_{j=1}^{5} = \{(a,b,b,0),(a,-b,b,0),(a,b,-b,0),(a,-b,-b,0),(0,0,0,0,1)\}. \quad (2.35)$$

However, as we have shown in Lemma 2.2.1, there already exists a FUNTF without repeated elements of length $n+1$ for real ℓ_1^n. While this situation may allow us to define a different FUNTF of length $n+1$ with no repeated elements, it does not contribute to our goal of finding FUNTFs with no repeated elements of greater lengths.

On the other hand, the addition of the elements $(c,0,0,d)$ and $(c,0,0,-d)$, with $c = 1/4$ and $d = 3/4$, to the "base", which we recall consists of the elements $(a,b,b,0)$, $(a,-b,b,0)$, $(a,b,-b,0)$, and $(a,-b,-b,0)$, yields a FUNTF for real ℓ_1^4 with length 6. In other words, the "apex" of the pyramid-like FUNTF for real ℓ_1^4 is a line segment with endpoints $(c,0,0,d)$ and $(c,0,0,-d)$ in real ℓ_1^4.

Now that we have been able to deduce a pattern from the constructions of FUNTFs for real ℓ_1^3 of length 5 and for ℓ_1^4 of length 6 given in [1], we are able to create new constructions. Specifically, we generalize the pyramid-like constructions to derive an alternative construction of FUNTFs for real ℓ_1^5.

Taking into consideration the extra dimension as compared to real ℓ_1^4, instead of being a line segment, the "apex" should be a square in the last 2 coordinates with the first 3 coordinates fixed. As usual, the resulting sequence will be paired with normalizing functions in real ℓ_∞^5 to define a Schauder frame. In this case, the FUNTF for real ℓ_1^5 that we define using this method will be of length 8 (two "copies" of the square). The following lemma explains this construction in detail along with the derivations.

Lemma 2.3.1. *There exists a FUNTF of length 8 for real ℓ_1^5 that does not involve repetition of frame elements.*

Proof. Let a, b, c, and d be positive scalars, yet to be determined, that satisfy $a + 2b = 1 =$

$c + 2d$. Define $\{x_j\}_{j=1}^{8}$ in real ℓ_1^5 by

$$\{x_j\}_{j=1}^{8} = \{(a, b, b, 0, 0), (a, b, -b, 0, 0), (a, -b, b, 0, 0), (a, -b, -b, 0, 0), (c, 0, 0, d, d), (c, 0, 0, -d, d), (c, 0, 0, d, -d), (c, 0, 0, -d, -d)\}. \quad (2.36)$$

The corresponding normalizing functionals $\{f_j\}_{j=1}^{8}$ in real ℓ_∞^5 are

$$\{f_j\}_{j=1}^{8} = \{(1, 1, 1, 0, 0), (1, 1, -1, 0, 0), (1, -1, 1, 0, 0), (1, -1, -1, 0, 0), (1, 0, 0, 1, 1), (1, 0, 0, -1, 1), (1, 0, 0, 1, -1), (1, 0, 0, -1, -1)\}. \quad (2.37)$$

Our initial hypothesis for the constants a, b, c, and d ensures that the x_j's are normalized in real ℓ_1^5.

Our goal is to show that a, b, c, and d can be chosen so that $\{(x_j, f_j)\}_{j=1}^{8}$ defines a FUNTF for real ℓ_1^5. In order to do so, we compute the matrix of its associated frame operator S:

$$S = \sum_{j=1}^{8} f_j \otimes x_j = \begin{bmatrix} | & | & & | \\ f_1 & f_2 & \cdots & f_8 \\ | & | & & | \end{bmatrix} \begin{bmatrix} — & x_1 & — \\ — & x_2 & — \\ & \vdots & \\ — & x_8 & — \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a & b & b & 0 & 0 \\ a & b & -b & 0 & 0 \\ a & -b & b & 0 & 0 \\ a & -b & -b & 0 & 0 \\ c & 0 & 0 & d & d \\ c & 0 & 0 & -d & d \\ c & 0 & 0 & d & -d \\ c & 0 & 0 & -d & -d \end{bmatrix}$$

$$= \begin{bmatrix} 4a+4c & 0 & 0 & 0 & 0 \\ 0 & 4b & 0 & 0 & 0 \\ 0 & 0 & 4b & 0 & 0 \\ 0 & 0 & 0 & 4d & 0 \\ 0 & 0 & 0 & 0 & 4d \end{bmatrix}.$$

Setting $a = c = 1/5$ and $b = d = 2/5$, we see that $S = (8/5)I$. Therefore, $\{(x_j, f_j)\}_{j=1}^{8}$ with these values for a, b, c, and d provides a FUNTF of length 8 for real ℓ_1^5 without repeating frame elements. □

Just as [1] was able to obtain a FUNTF of length 7 for real ℓ_1^4 by adding a standard basis element to a FUNTF of length 6, a FUNTF of length 9 for real ℓ_1^5 can be obtained similarly from one of length 8. Also note that while the structure of the FUNTFs of lengths 6 and 7 for real ℓ_1^4 are kept the same other than the extra standard basis element, our FUNTF of length 7 has a different set of constants a, b, c, and d in equations (Equation 2.36) and (Equation 2.37). Likewise, we will do the same in order to define a FUNTF of length 9 for real ℓ_1^5 using this method.

Corollary 2.3.2. *There exists a FUNTF of length 9 for real ℓ_1^5 that does not involve repetition of frame elements.*

Proof. Consider the sequence pair $\{(x_j, f_j)\}_{j=1}^{8}$ defined by equations (Equation 2.36) and (Equation 2.37) from Theorem 2.3.1. As in the previous example, the constants a, b, c, and d are initially undetermined. To construct a sequence pair in real $\ell_1^5 \times \ell_\infty^5$, define $\{(x_j, f_j)\}_{j=1}^{9}$ by setting $x_9 = e_1$ and $f_9 = e_1^*$, while leaving the first eight pairs of this sequence unchanged from that of Theorem 2.3.1.

We will find constants a, b, c, and d so that $\{(x_j, f_j)\}_{j=1}^{9}$ defines a FUNTF for real ℓ_1^5. To do so, we compute the corresponding frame matrix S, and find that it is

$$S = \sum_{j=1}^{9} w_j \otimes v_j = \begin{bmatrix} 4a + 4c + 1 & 0 & 0 & 0 & 0 \\ 0 & 4b & 0 & 0 & 0 \\ 0 & 0 & 4b & 0 & 0 \\ 0 & 0 & 0 & 4d & 0 \\ 0 & 0 & 0 & 0 & 4d \end{bmatrix}.$$

Setting $a = c = 1/10$ and $b = d = 9/20$, we have $S = (9/5)I$ and therefore $\{(x_j, f_j)\}_{j=1}^{9}$ with those values of a, b, c, and d is a FUNTF for real ℓ_1^5 of length 9. $\square$

Notice here that we have not constructed a FUNTF of length 7 for real ℓ_1^5 without repeated elements. This is because the pyramid-like construction used in Lemma 2.3.1 and Corollary 2.3.2 is not applicable to sequences of length 7, or 7 vertices in a geometric interpretation. The main difficulty is that such a geometric visualization for higher dimensional ℓ_1^n cannot be directly generalized from those already known for lower dimensions. We state this explicitly as an open problem for FUNTFs of length 7 for real ℓ_1^5.

Question 2.3.3. *Does there exist a FUNTF of length 7 for real ℓ_1^5 without repeated elements?*

This method for constructing FUNTFs is applicable not only to real ℓ_1^5, but also to any real ℓ_1^n in higher dimensions. We prove below that by generalizing the construction of the FUNTF in Proposition 2.3.1 of length 8 for real ℓ_1^5 to n-dimensions, a FUNTF without repeating elements can be obtained of length $2n-2$ for real ℓ_1^n. Likewise, generalizing the construction of the FUNTF of length 9 in Corollary 2.3.2 will give us a FUNTF of length $2n-1$ for real ℓ_1^n, also without any repeated elements as desired.

Theorem 2.3.4. *There exists a FUNTF of length $2n-2$ for real ℓ_1^n that does not involve the repetition of frame elements.*

Proof. First, let n be even. Define the sequence $\{x_j\}_{j=1}^{2n-2}$ in real ℓ_1^n by

$$\begin{aligned} x_{4i-3} &= b_i e_1 + a_i e_{2i} + a_i e_{2i+1} \\ x_{4i-2} &= b_i e_1 - a_i e_{2i} + a_i e_{2i+1} \\ x_{4i-1} &= b_i e_1 + a_i e_{2i} - a_i e_{2i+1} \\ x_{4i} &= b_i e_1 - a_i e_{2i} - a_i e_{2i+1}, \end{aligned} \tag{2.38}$$

with $1 \leq i \leq n/2 - 1$. For the last 2 elements of this sequence, set

$$\begin{aligned} x_{2n-3} &= b_{n/2} e_1 + a_{n/2} e_n \\ x_{2n-2} &= b_{n/2} e_1 - a_{n/2} e_n. \end{aligned} \tag{2.39}$$

As in previous situations, the a_i's and the b_i's are yet to be computed. Now, the normalizing sequence of functionals for $\{x_j\}_{j=1}^{2n-2}$ in real ℓ_∞^n is the sequence $\{f_j\}_{j=1}^{2n-2}$ defined by

$$\begin{aligned} f_{4i-3} &= e_1^* + e_{2i}^* + e_{2i+1}^* \\ f_{4i-2} &= e_1^* - e_{2i}^* + e_{2i+1}^* \\ f_{4i-1} &= e_1^* + e_{2i}^* - e_{2i+1}^* \\ f_{4i} &= e_1^* - e_{2i}^* - e_{2i+1}^*, \end{aligned} \tag{2.40}$$

for $1 \leq i \leq n/2 - 1$, and

$$\begin{aligned} f_{2n-3} &= e_1^* + e_n^* \\ f_{2n-2} &= e_1^* - e_n^*. \end{aligned} \tag{2.41}$$

Our goal is to find constants a_i, b_i, for $1 \leq i \leq n/2$, such that $\{(x_j, f_j)\}_{j=1}^{2n-2}$ defines a FUNTF for real ℓ_1^n. First of all, the sequence $\{x_j\}_{j=1}^{2n-2}$ must be normalized in real ℓ_1^n. Therefore, one restriction we impose on the a_i's and b_i's is that $b_i + 2a_i = 1$ for all $1 \leq i \leq n/2 - 1$ and $b_{n/2} + a_{n/2} = 1$.

To verify that $\{(x_j, f_j)\}_{j=1}^{2n-2} \subseteq \ell_1^n \times \ell_\infty^n$ is a FUNTF for real ℓ_1^n, we need to find its corresponding frame operator. For $1 \leq k, m \leq n$, the (k, m)-th entry of the frame operator matrix S is given by

$$S(k, m) = \sum_{j=1}^{2n-2} f_j(e_m) e_k^*(x_j).$$

Set $m = k$. We have

$$\sum_{i=1}^{n/2-1} (e_1^* \pm e_{2i}^* \pm e_{2i+1}^*)(e_m) e_m^*(b_j e_1 \pm a_j e_{2i} \pm a_j e_{2i+1}) = \begin{cases} \sum_{i=1}^{n/2-1} b_i & \text{if } m = 1 \\ a_{m/2} & \text{if } m \text{ is even} \\ a_{(m-1)/2} & \text{if } m \text{ is odd} \\ 0 & \text{if } m = 2n-3 \\ 0 & \text{if } m = 2n-2, \end{cases}$$

where the appropriate plus or minus signs are chosen depending on the value of i, for each $1 \leq i \leq n/2 - 1$, corresponding to how the x_j's are defined in equation (Equation 2.38).

We compute that

$$\begin{aligned}\sum_{j=1}^{2n-4} f_j(e_m)e_m^*(x_j) &= \sum_{i=1}^{n/2-1} x_{4i-3}^*(e_m)e_m^*(x_{4i-3}) + \sum_{i=1}^{n/2-1} x_{4i-2}^*(e_m)e_m^*(x_{4i-2}) + \\ &\quad \sum_{i=1}^{n/2-1} x_{4i-1}^*(e_m)e_m^*(x_{4i-1}) + \sum_{i=1}^{n/2-1} x_{4i}^*(e_m)e_m^*(x_{4i}) \\ &= \begin{cases} 4\sum\limits_{i=1}^{n/2-1} b_i & \text{if } m = 1 \\ 4a_{m/2} & \text{if } m \text{ is even} \\ 4a_{(m-1)/2} & \text{if } m \text{ is odd.} \end{cases}\end{aligned} \tag{2.42}$$

Choosing the appropriate plus or minus signs corresponding to the definitions of x_{2n-2} and x_{2n-3} in equation (Equation 2.39), we have

$$(e_1^* \pm e_n^*)(e_m)e_m^*(b_{n/2}e_1 \pm a_{n/2}e_n) = \begin{cases} b_{n/2} & \text{if } m = 1 \\ a_{n/2} & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases}$$

This gives us

$$f_{2n-3}(e_m)e_m^*(x_{2n-3}) + f_{2n-2}(e_m)e_m^*(x_{2n-2}) = \begin{cases} 2b_{n/2} & \text{if } m = 1 \\ 2a_{n/2} & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases} \tag{2.43}$$

Combining equations (Equation 2.42) and (Equation 2.43), we conclude that

$$S(m,m) = \begin{cases} 4\sum_{i=1}^{n/2-1} b_i + 2b_{n/2} & \text{if } m = 1 \\ 4a_{m/2} & \text{if } 2 \leq m \leq n-1 \text{ is even} \\ 4a_{(m-1)/2} & \text{if } 2 \leq m \leq n-1 \text{ is odd} \\ 2a_{n/2} & \text{if } m = n. \end{cases}$$

Given appropriate choices of the a_i's and b_i's, all of the $S(m,m)$ entries will have the same value. In fact, this will be the case if we set

$$a_i = \begin{cases} \frac{n-1}{2n} & \text{if } 1 \leq i \leq \frac{n}{2} - 1 \\ \frac{n-1}{n} & \text{if } i = \frac{n}{2} \end{cases} \tag{2.44}$$

and

$$b_i = \frac{1}{n} \tag{2.45}$$

for $1 \leq i \leq n/2$. In this case, not only is the unit norm requirement for the x_j's satisfied, but we also have that $S(m,m) = (2n-2)/n$ for all $1 \leq m \leq n$.

To compute the nondiagonal entries of S, set $m \neq k$. We have

$$\sum_{i=1}^{n/2-1} (e_1^* \pm e_{2i}^* \pm e_{2i+1}^*)(e_m) e_k^*(b_i e_1 \pm a_i e_{2i} \pm a_i e_{2i+1}) = 0$$

and

$$(e_1^* \pm e_n^*)(e_m) e_k^*(b_{n/2} e_1 \pm a_{n/2} e_n) = 0.$$

Therefore, $S(k,m) = 0$. Again, the appropriate plus or minus signs are chosen here corresponding to the frame elements for which these sums are being computed.

This shows that for n even, the sequence pair $\{(x_j, f_j)\}_{j=1}^{2n-2}$, with the x_j's satisfying equations (Equation 2.38) and (Equation 2.39), the f_j's satisfying equations (Equation 2.40) and (Equation 2.41), and the a_i's, b_i's satisfying equations (Equation 2.44) and (Equation 2.45), respectively, is a FUNTF for real ℓ_1^n which does not have repeated frame elements.

Now, let n be odd. Define the sequence $\{y_j\}_{j=1}^{2n-2}$ by

$$\begin{aligned} y_{4i-3} &= b_i e_1 + a_i e_{2i} + a_i e_{2i+1} \\ y_{4i-2} &= b_i e_1 - a_i e_{2i} + a_i e_{2i+1} \\ y_{4i-1} &= b_i e_1 + a_i e_{2i} - a_i e_{2i+1} \\ y_{4i} &= b_i e_1 - a_i e_{2i} - a_i e_{2i+1}, \end{aligned} \tag{2.46}$$

where $1 \leq i \leq (n-1)/2$ and the a_i's and b_i's to be determined. The normalizing functionals corresponding to the y_j's are the sequence $\{z_j\}_{j=1}^{2n-2}$ defined by

$$\begin{aligned} z_{4i-3} &= e_1^* + e_{2i}^* + e_{2i+1}^* \\ z_{4i-2} &= e_1^* - e_{2i}^* + e_{2i+1}^* \\ z_{4i-1} &= e_1^* + e_{2i}^* - e_{2i+1}^* \\ z_{4i} &= e_1^* - e_{2i}^* - e_{2i+1}^*, \end{aligned} \tag{2.47}$$

also with $1 \leq i \leq (n-1)/2$.

For $\{(y_j, z_j)\}_{j=1}^{2n-2}$, a computation of the associated frame operator similar to the one performed in the even case gives us $S(k, m) = 0$ for $k \neq m$. Also, we have

$$S(m, m) = \begin{cases} 4 \sum\limits_{i=1}^{(n-1)/2} b_i & \text{if } m = 1 \\ 4a_{m/2} & \text{if } m \text{ is even} \\ 4a_{(m-1)/2} & \text{if } m \text{ is odd.} \end{cases}$$

Choose any $1 \le i \le (n-1)/2$. By setting

$$a_i = \frac{n-1}{2n} \tag{2.48}$$

and

$$b_i = \frac{1}{n}, \tag{2.49}$$

S becomes a scalar multiple of the identity with $S(m,m) = (2n-2)/n$ for all $1 \le m \le n$. Therefore, taking a_i and b_i satisfying equations (Equation 2.48) and (Equation 2.49), we conclude that for n odd, $\{(y_j, z_j)\}_{j=1}^{2n-2}$ defines a FUNTF of length $2n-2$ for real ℓ_1^n. $\square$

Adding an appropriate standard basis element, we can recursively obtain a FUNTF of length $2n-1$ for real ℓ_1^n with no repeating elements from one of length $2n-2$. Detailed computations of a specific example are given in the corollary below.

Corollary 2.3.5. *For all $n \in \mathbb{N}$, there exists a FUNTF of length $2n-1$ for real ℓ_1^n that does not involve the repetition of frame elements.*

Proof. First, let n be even. Our goal is to construct a FUNTF $\{(x_j, f_j)\}_{j=1}^{2n-1}$ of length $2n-1$ for real ℓ_1^n.

Consider the sequence pair $\{(x_j, f_j)\}_{j=1}^{2n-2}$ defined by equations (Equation 2.38) and (Equation 2.39) for the x_j's and equations (Equation 2.40) and (Equation 2.41) for the f_j's in Theorem 2.3.4. Just as in Theorem 2.3.4, we start by leaving the values for the a_i's and b_i's undetermined. Since we will construct a FUNTF of length $2n-1$ instead of $2n-2$, the scalars $\{a_i\}_{i=1}^{n/2-1}$ and $\{b_i\}_{i=1}^{n/2-1}$ to be determined will have different values from that computed in Theorem 2.3.4.

Set $x_{2n-1} = e_1$ and $f_{2n-1} = e_1^*$. The resulting sequence pair $\{(x_j, f_j)\}_{j=1}^{2n-1}$ is of length $2n-1$ in real $\ell_1^n \times \ell_\infty^n$. To show that $\{(x_j, f_j)\}_{j=1}^{2n-1}$ defines a FUNTF for real ℓ_1^n, we need to compute the a_i's and b_i's, for $1 \le i \le n/2-1$, so that the associated frame operator is $S = (2n-1)/nI$.

Similar to the computations given in the proof of Theorem 2.3.4, for $1 \leq k, m \leq n$, we have $S(k, m) = 0$ whenever $k \neq m$ and

$$S(m,m) = \begin{cases} 4\sum_{i=1}^{n/2-1} b_i + 2b_{n/2} + 1 & \text{if } m = 1 \\ 4a_{m/2} & \text{if } 2 \leq m \leq n-1 \text{ is even} \\ 4a_{(m-1)/2} & \text{if } 2 \leq m \leq n-1 \text{ is odd} \\ 2a_{n/2} & \text{if } m = n. \end{cases}$$

Define the constants $\{a_i\}_{i=1}^{n/2}$ and $\{b_i\}_{i=1}^{n/2}$ by

$$a_i = \begin{cases} \frac{2n-1}{4n} & \text{if } 1 \leq i \leq n/2 - 1 \\ \frac{2n-1}{2n} & \text{if } i = n/2 \end{cases} \tag{2.50}$$

and

$$b_i = \frac{1}{2n}. \tag{2.51}$$

Due to the extra frame element $(x_{2n-1}, f_{2n-1}) = (e_1, e_1^*)$, the constants $\{a_i\}_{i=1}^{n/2}$ and $\{b_i\}_{i=1}^{n/2}$ are different from those defined for $\{(x_j, f_j)\}_{j=1}^{2n-2}$ in Theorem 2.3.4. We need to do this in order to be able to define a FUNTF of length $2n - 1$.

Inserting these values of a_i, b_i, for $1 \leq i \leq n/2$ as computed in equations (Equation 2.50) and (Equation 2.51), into the definition of $\{(x_j, f_j)\}_{j=1}^{2n-1}$, we conclude that $S(m, m) = (2n - 1)/n$ for all $1 \leq m \leq n$. Since $S(k, m) = 0$ whenever $k \neq m$, this shows that for $1 \leq i \leq n/2 - 1$, $\{(x_j, f_j)\}_{j=1}^{2n-1}$ defined by

$$\begin{aligned}
x_{4i-3} &= \frac{1}{2n}e_1 + \frac{2n-1}{4n}e_{2i} + \frac{2n-1}{4n}e_{2i+1} \\
x_{4i-2} &= \frac{1}{2n}e_1 - \frac{2n-1}{4n}e_{2i} + \frac{2n-1}{4n}e_{2i+1} \\
x_{4i-1} &= \frac{1}{2n}e_1 + \frac{2n-1}{4n}e_{2i} - \frac{2n-1}{4n}e_{2i+1} \\
x_{4i} &= \frac{1}{2n}e_1 - \frac{2n-1}{4n}e_{2i} - \frac{2n-1}{4n}e_{2i+1} \qquad (2.52) \\
x_{2n-2} &= \frac{1}{2n}e_1 + \frac{2n-1}{2n}e_n \\
x_{2n-3} &= \frac{1}{2n}e_1 - \frac{2n-1}{2n}e_n \\
x_{2n-1} &= e_1
\end{aligned}$$

and

$$\begin{aligned}
f_{4i-3} &= e_1^* + e_{2i}^* + e_{2i+1}^* \\
f_{4i-2} &= e_1^* - e_{2i}^* + e_{2i+1}^* \\
f_{4i-1} &= e_1^* + e_{2i}^* - e_{2i+1}^* \\
f_{4i} &= e_1^* - e_{2i}^* - e_{2i+1}^* \qquad (2.53) \\
f_{2n-3} &= e_1^* + e_n^* \\
f_{2n-2} &= e_1^* - e_n^* \\
f_{2n-1} &= e_1^*
\end{aligned}$$

is a FUNTF of length $2n-1$ for real ℓ_1^n with no repeating frame elements.

Now, let n be odd. In this case, consider the sequence pair $\{(y_j, z_j)\}_{j=1}^{2n-1}$, where $y_{2n-1} = e_1 \in \ell_1^n$, $z_{2n-1} = e_1^* \in \ell_\infty^n$ and $\{(y_j, z_j)\}_{j=1}^{2n-2}$ has the same structure as equations (Equation 2.46) and (Equation 2.47), except with different values for the a_i's and b_i's that are yet to be determined. As with the case when n is even, the different values of the a_i's and b_i's are due to the presence of the extra frame element (y_{2n-1}, z_{2n-1}), requiring different normalizations.

Consider the frame operator S associated with $\{(y_j, z_j)\}_{j=1}^{2n-1}$. A computation similar to that of Theorem 2.3.4's proof shows that for $1 \leq m \leq n$,

$$S(m,m) = \begin{cases} 4\sum_{i=1}^{(n-1)/2} b_i + 1 & \text{if } m = 1 \\ 4a_{m/2} & \text{if } m \text{ is even} \\ 4a_{(m-1)/2} & \text{if } m \text{ is odd,} \end{cases}$$

and that $S(k,m) = 0$ for $1 \leq k, m \leq n$ with $k \neq m$. Setting

$$a_i = \frac{2n-1}{4n} \tag{2.54}$$

and

$$b_i = \frac{1}{2n}, \tag{2.55}$$

we have $S(m,m) = (2n-1)/n$ for all $1 \leq m \leq n$. For $1 \leq i \leq (n-1)/2$, this shows that $\{(y_j, z_j)\}_{j=1}^{2n-1}$ with

$$\begin{aligned} y_{4i-3} &= \frac{1}{2n}e_1 + \frac{2n-1}{4n}e_{2i} + \frac{2n-1}{4n}e_{2i+1} \\ y_{4i-2} &= \frac{1}{2n}e_1 - \frac{2n-1}{4n}e_{2i} + \frac{2n-1}{4n}e_{2i+1} \\ y_{4i-1} &= \frac{1}{2n}e_1 + \frac{2n-1}{4n}e_{2i} - \frac{2n-1}{4n}e_{2i+1} \\ y_{4i} &= \frac{1}{2n}e_1 - \frac{2n-1}{4n}e_{2i} - \frac{2n-1}{4n}e_{2i+1} \\ y_{2n-1} &= e_1 \end{aligned} \tag{2.56}$$

and

$$
\begin{aligned}
z_{4i-3} &= e_1^* + e_{2i}^* + e_{2i+1}^* \\
z_{4i-2} &= e_1^* - e_{2i}^* + e_{2i+1}^* \\
z_{4i-1} &= e_1^* + e_{2i}^* - e_{2i+1}^* \\
z_{4i} &= e_1^* - e_{2i}^* - e_{2i+1}^* \\
z_{2n-1} &= e_1^*,
\end{aligned} \tag{2.57}
$$

defines a FUNTF of length $2n-1$ without repeated frame elements for real ℓ_1^n. $\square$

Generalizing the geometric construction used in Proposition 7.3 of [1], we were able to obtain FUNTFs of lengths $2n-1$ and $2n-2$ for real ℓ_1^n without repeated elements. However, just as we were unable to provide a FUNTF of length 7 for real ℓ_1^5 without repeated elements, we do not know of other geometric constructions of FUNTFs that can be generalized for those of lengths $n-2 \leq N \leq 2n-3$ for real ℓ_1^n. We state this as an open question.

Question 2.3.6. *For $n-2 \leq N \leq 2n-3$, do there exist FUNTFs of length N for real ℓ_1^n without repeated elements?*

2.4 The General Case and Open Questions

Recall that FUNTFs of length $N \geq n$ exist for any *complex* n-dimensional Banach space with an 1-unconditional basis. Since we have proved that FUNTFs of length $N \geq n$ exist for real ℓ_1^n, the final step for proving the existence of FUNTFs in finite-dimensional Banach spaces will be to show whether FUNTFs exist for finite-dimensional *real* Banach spaces. As has been the case so far, we restrict our attention to Banach spaces with 1-unconditional bases.

The following lemma (see, e.g., [1], [40], and [41]), provides the foundation for generalizing methods to construct FUNTFs from real ℓ_1^n to general finite-dimensional real Banach spaces.

Lemma 2.4.1 ([1]). *Let X be an n-dimensional Banach space with a normalized 1-unconditional basis $\{e_j\}_{j=1}^N$, and let the corresponding biorthogonal functionals be $\{e_j^*\}_{j=1}^N$. For any sequence of nonnegative numbers $\{\lambda_j\}_{j=1}^N$ satisfying $\sum_{j=1}^n \lambda_j = 1$, there exist sequences of nonnegative numbers $\{\alpha_j\}_{j=1}^n$ and $\{\beta_j\}_{j=1}^N$ such that both $x = \sum_{j=1}^n \alpha_j e_j$ and $x^* = \sum_{j=1}^n \beta_i e_j^*$ have norm one, and, moreover, $\alpha_j \beta_j = \lambda_j$ for each $1 \leq j \leq n$.*

In other words, given any finite-dimensional Banach space and a sequence of nonnegative numbers summing to 1, it is possible to directly obtain unit norm elements in the aforementioned Banach space from that sequence.

If $\{(x_j, f_j)\}_{j=1}^N$ is a FUNTF for real ℓ_1^n, then

$$\sum_{i=1}^n |x_j(e_i)| = 1 \tag{2.58}$$

for all $1 \leq j \leq N$. As a consequence, we will prove below that for any finite-dimensional Banach space X with an 1-unconditional basis, Lemma 2.4.1 provides a finite sequence of unit-norm pairs in $X \times X^*$.

Lemma 2.4.2. *Given an n-dimensional Banach space X, there exists a pair of sequences $\{(x_j, f_j)\}_{j=1}^N$ in $X \times X^*$ such that $||x_j|| = ||f_j|| = 1$ and $f_j(x_j) = 1$ for all $1 \leq j \leq N$.*

Proof. Let $\{(y_i, z_i)\}_{i=1}^N$ be a FUNTF of length $N \geq n$ in real ℓ_1^n, which we know exists due to Theorem 2.2.5. Then for all $1 \leq i \leq N$,

$$\sum_{j=1}^n |y_i(e_j)| = 1.$$

Take any n-dimensional Banach space X. By Lemma 2.4.1, there exist nonnegative sequences of scalars $\{\alpha_{ij}\}_{j=1}^n$ and $\{\beta_{ij}\}_{j=1}^n$ such that the sequences $\{x_i\}_{i=1}^N$ and $\{f_i\}_{i=1}^N$ are finite unit-norm sequences in X and its dual X^* respectively, where

$$x_i = \sum_{j=1}^{n} \alpha_{ij} e_j, \qquad f_i = \sum_{j=1}^{n} \beta_{ij} e_j^*.$$

Moreover, note that for all $1 \leq i \leq N$ and $1 \leq j \leq n$, we have $\alpha_{ij}\beta_{ij} = |y_i(e_j)|$. Therefore for all $1 \leq i \leq N$, we have

$$x_i(f_i) = f_i(x_i) = \sum_{j=1}^{n} \alpha_{ij}\beta_{ij} = \sum_{j=1}^{n} |x_i(e_j)| = 1.$$

This completes the proof. □

As mentioned in the beginning of this section, our ultimate goal is to prove that there exists a FUNTF of length N for X, which will be stated as an open problem at the end of this chapter. In order to do so, we need to be able to compute the frame operator S of the sequence $\{(x_i, f_i)\}_{i=1}^N$ in $X \times X^*$ defined in Lemma 2.4.1. Just as in the case for real ℓ_1^n, the frame operator can also be expressed in matrix form with respect to X and X^*.

$$S = \sum_{i=1}^{N} z_i \otimes y_i = \begin{bmatrix} \beta_{11} & \beta_{21} & \beta_{31} & \cdots & \beta_{N1} \\ \beta_{12} & \beta_{22} & \beta_{32} & \cdots & \beta_{N2} \\ \beta_{13} & \beta_{23} & \beta_{33} & \cdots & \beta_{N3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{1n} & \beta_{2n} & \beta_{3n} & \cdots & \beta_{Nn} \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2n} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots & \alpha_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{N1} & \alpha_{N2} & \alpha_{N3} & \cdots & \alpha_{Nn} \end{bmatrix}.$$

Without loss of generality, let $N = n + k$, with $1 \leq k \leq n - 1$. As is the case for real ℓ_1^n, the union of FUNTFs is still a FUNTF for the n-dimensional Banach space X, so providing a FUNTF of length $n + k$ with $1 \leq k \leq n - 1$ will yield FUNTFs of any length $N \geq n$.

With $N = n+k$, let $\{(y_i, z_i)\}_{i=1}^N$ be the FUNTF defined in the proof of Theorem 2.2.5.

Then, for $1 \leq m \leq n$, we have

$$S(m,m) = \sum_{i=1}^{N} \beta_{im}\alpha_{im} = \sum_{i=1}^{n+k} |y_i(e_m)| = b + (n-1)a + \frac{k}{n} = \frac{N}{n}.$$

It remains to prove that $S(k,m) = 0$, where $1 \leq k, m \leq n$ and $k \neq m$. We have

$$S(k,m) = \sum_{i=1}^{N} \beta_{ik}\alpha_{im}.$$

Lemma 2.4.1 specifies that for all $1 \leq i \leq N$ and $1 \leq j \leq n$, the scalars α_{ij} and β_{ij} are nonnegative. With $\{(y_i, z_i)\}_{i=1}^{N}$ as defined in Lemma 2.4.2, $S(k,m) = 0$ only if $\beta_{ik}\alpha_{im} = 0$ for all $1 \leq i \leq N$. This contradicts our computation that $S(m,m) = N/n$ for all $1 \leq m \leq n$.

We start over by redefining the sequence pair $\{(x_i, f_i)\}_{i=1}^{N} \subseteq X \times X^*$ by

$$\begin{aligned} x_i &= \sum_{j=1}^{n} \varepsilon_{ij}\alpha_{ij}e_j \\ f_i &= \sum_{j=1}^{n} \sigma_{ij}\beta_{ij}e_j^* \end{aligned} \tag{2.59}$$

Here, we let the α_{ij}'s and β_{ij}'s for $1 \leq i \leq N$, $1 \leq j \leq n$ be as they were defined in the proof of Lemma 2.4.2. Also, choose signs $\varepsilon_{ij}, \sigma_{ij} \in \{-1, 1\}$ so that $\varepsilon_{ij}\sigma_{ij} = 1$ for $1 \leq i \leq N$ and $1 \leq j \leq n$.

From a cancellation of signs, the associated frame operator S of this new definition of $\{(x_i, f_i)\}_{i=1}^{N}$ still gives us

$$S(m,m) = \sum_{i=1}^{N} (\sigma_{im}\beta_{im})(\varepsilon_{im}\alpha_{im}) = \sum_{i=1}^{N} (\varepsilon_{im}\sigma_{im})(\beta_{im}\alpha_{im}) = \frac{N}{n}.$$

Now for all other entries of S, we have

$$S(k,m) = \sum_{i=1}^{N} (\varepsilon_{im}\sigma_{ik})(\beta_{ik}\alpha_{im}).$$

The quantity $\varepsilon_{im}\sigma_{ik}\beta_{im}\alpha_{ik}$ can either be positive or negative depending on the choices of k and m for each $1 \leq i \leq N$. Therefore, there is now a possibility for $S(k,m)$ to be equal to 0 for all $1 \leq k, m \leq n$ with $k \neq m$, and proving this will allow us to conclude the existence of FUNTFs of any length for real finite-dimensional Banach spaces.

We leave as an open ended computational problem of calculating the appropriate constants $\alpha_{ik}, \beta_{im} > 0$ and $\varepsilon_{im}, \sigma_{im} = \pm 1$ in order to obtain $S(k,m) = 0$. Doing so will yield a FUNTF $\{(x_j, f_j)\}_{j=1}^{n+k}$ in $X \times X^*$, with X being an arbitrary n-dimensional real Banach space with associated dual X^*. A general FUNTF of length $N \geq n$ can then be obtained by taking appropriate unions of FUNTFs of length $n + k$ with $1 \leq k \leq n - 1$.

Question 2.4.3. *Given a FUNTF of length $N \geq n$ for real ℓ_1^n, how can we deduce from it a FUNTF of length $N \geq n$ for an arbitrary n-dimensional Banach space?*

Since this is a computational problem, solving this for lower dimensions will provide insight into answering Question 2.4.3. Therefore, we leave as an open-ended computational problem the following question.

Question 2.4.4. *Do there exist FUNTFs of lengths 4 and 5 for every 3-dimensional real Banach space?*

To show whether there exist FUNTFs of lengths 4 and 5 for 3-dimensional real Banach spaces, one should start with the FUNTFs of lengths 4 and 5 for real ℓ_1^n as defined in Propositions 7.1 and 7.3 of [1] and apply Lemma 2.4.1 using the method that we have discussed.

If the answer to Question 2.4.4 is affirmative, then by taking the unions of appropriate FUNTFs of lengths 3, 4, and 5, one can obtain a FUNTF of any length $N \geq 3$ for 3-dimensional Banach spaces.

Likewise, the same can be done for 4-dimensional real Banach spaces. To show whether there exist a FUNTF of length $N \geq 4$ for any 4-dimensional real Banach space, it suffices to provide a method which yields FUNTFs of lengths 5, 6, and 7 from that for real ℓ_1^4.

Question 2.4.5. *Do there exist FUNTFs of lengths* 5, 6*, and* 7 *for every* 4*-dimensional real Banach space?*

After one obtains some intuition from answering Questions 2.4.4 and 2.4.5, Question 2.4.3 can potentially be answered by applying the method described in this section to the sequence defined in Theorem 2.2.5 of Section 2.2 to construct FUNTFs for arbitrary n-dimensional real Banach spaces.

As a consequence of the questions posed in the previous chapter, another open question that can be proposed deals with the existence of balanced FUNTFs for real ℓ_1^n. Balanced FUNTFs for Banach spaces can be defined by combining the definition of balanced Schauder frames and that of FUNTFs in the Banach space setting. That is, a FUNTF $\{(x_j, f_j)\}_{j=1}^N$ for a Banach space X is *balanced* if $\sum_{j=1}^{N} x_j = 0$ and $\sum_{j=1}^{N} f_j = 0$.

Question 2.4.6. *Fix $n \geq 3$ and let $N \geq n$. Does there exist a balanced FUNTF of length N for real ℓ_1^n?*

CHAPTER 3

FRAME INEQUALITIES FOR BANACH SEQUENCE SPACES

This chapter is joint work with Dr. Daniel Freeman and Dr. Christopher Heil.

3.1 Introduction

Other than Schauder frames, which were the focus in Chapter 2, an alternative generalization of frames to Banach spaces is the *Banach frame*, which were defined in Chapter 1. Unlike Schauder frames, which are defined in terms of the reconstruction formula, Banach frames are defined in terms of a frame inequality formulated with respect to an associated sequence space. Under certain circumstances, Banach frames may also satisfy the reconstruction formula which defines Schauder frames. Such conditions are given in [2], whose main theorem we state next.

Theorem 3.1.1 ([2]). *Any Banach frame for L_p and $L_{p,q}$, with $1 \leq p, q < \infty$, with respect to a solid sequence space (such as c_0 and ℓ_p) admits a reconstruction formula.*

Here, c_0 denotes the subspace of ℓ_∞ for which the sequence elements tend to zero. In other words, if $(x_j)_{j=1}^{\infty} \in c_0$, then $\lim_{j\to\infty} x_j = 0$.

The goal of this chapter is to state and prove a result in the same spirit for Schauder frames. With adherence to the reconstruction formula being the definition of a Schauder frame, it remains to find conditions for which the frame inequality holds.

Recall the definition of a frame $\{x_j\}_{j\in\mathbb{N}}$ for a Hilbert space H, which states that there exist constants $A, B > 0$ such that for all $x \in H$,

$$A||x||^2 \leq \sum_{j=1}^{\infty} |\langle x, x_j\rangle|^2 \leq B||x||^2.$$

A Schauder frame for a Banach space X with dual X^* is actually a sequence of pairs $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ in $X \times X^*$. To obtain an appropriate replacement of $|\langle x, x_j\rangle|^2$ for Schauder frames, we restate the definition of the frame operator S in the Banach space setting, which for a Schauder frame is the identity operator:

$$S(x) = \sum_{j=1}^{\infty} f_j(x)x_j = x.$$

This suggests that if a frame inequality is to be applied in the Banach space setting, the $|\langle x, x_j\rangle|$ term that appears in the frame inequality for the Hilbert space setting should be replaced by $|f_j(x)|$. There are many choices of appropriate sequence space norms that we could consider, but for our purposes here it will be most natural to use an ℓ_1-type of inequality. For a Schauder frame $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ for a Banach space X, such an ℓ_1-type of frame inequality has the form

$$A||x||_X \leq \sum_{j=1}^{\infty} |f_j(x)| \leq B||x||_X, \tag{3.1}$$

which needs to hold for all $x \in X$. That is, for a given Schauder frame, the goal is to determine whether inequality (Equation 3.1) holds for some constants $A, B > 0$. Section 3.4 of this chapter will provide an example which shows that this does not come readily for Schauder frames: There exists a normalized unconditional Schauder frame for which inequality (Equation 3.1) does not hold.

Taking these factors into consideration, we formally revise the goal of this chapter to be answering the next question.

Question 3.1.2. *Given an unconditional Schauder frame $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ for a Banach space X, under what conditions do there exist constants $A, B > 0$ such that*

$$A||x||_X \leq \sum_{j=1}^{\infty} |f_j(x)| \leq B||x||_X$$

for all $x \in X$?

Notice here that the Banach space X is not specified to be finite-dimensional, as was required in Chapter 2. On the other hand, Banach frames were defined in the infinite-dimensional continuous function spaces L_p and $L_{p,q}$ in Theorem 3.1.1. This suggests that working in infinite-dimensional Banach spaces will be more of interest than focusing exclusively on finite-dimensional spaces.

At this point, it is still unknown whether a frame inequality can even be obtained for infinite-dimensional Banach spaces, whether with extra conditions imposed or not. Banach frames and Schauder frames both generalize Hilbert space frames to Banach spaces, albeit from very different angles. Thus, Theorem 3.1.1's infinite dimensional setting also suggests that a frame inequality should indeed work in Banach spaces under certain conditions.

When proving the existence and characterizations of FUNTFs for Banach spaces, results from Hilbert spaces are typically not able to be naively generalized: Many properties in Hilbert spaces do not have analogs in Banach spaces. Therefore, we initially answer Question 3.1.2 for certain classes of infinite-dimensional Banach spaces.

To determine the infinite-dimensional Banach spaces that we first answer Question 3.1.2 for, the following classical theorem suggests the possibility that ℓ_1 and c_0 may share properties of ℓ_2 that other infinite-dimensional Banach spaces do not (see [42] and [3] for proof and further details).

Theorem 3.1.3 ([3]). *The only Banach spaces that have a unique normalized unconditional basis, which is equivalent to the standard basis, are c_0, ℓ_1, and ℓ_2.*

This motivates us to consider ℓ_1 and c_0 as the spaces to work with when trying to extend properties of ℓ_2, such as the frame inequality, to the Banach space setting, starting with ℓ_1.

The following lemma shows that an ℓ_1-type of lower bound in the frame inequality holds not only for ℓ_1, but also for all normalized Schauder frames for any Banach spaces. Here, a Schauder frame $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ for a Banach space X is said to be normalized if for all $j \in \mathbb{N}$, $||x_j||_X = ||f_j||_{X^*} = 1$, where X^* is the dual of X, $\{x_j\}_{j\in\mathbb{N}} \subseteq X$, and $\{f_j\}_{j\in\mathbb{N}} \subseteq X^*$.

Lemma 3.1.4. *If $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ is a normalized Schauder frame for X, then*

$$||x|| \leq \sum_{j=1}^{\infty} |f_j(x)|$$

for all $x \in X$.

Proof. Let $\{(x_j, f_j)\}_{j\in\mathbb{N}} \subseteq X \times X^*$ be a normalized Schauder frame for X and choose $x \in X$. By the Triangle Inequality and the definition of a Schauder frame, we have

$$||x|| = \left|\left|\sum_{j=1}^{\infty} f_j(x)x_j\right|\right| \leq \sum_{j=1}^{\infty} |f_j(x)|\,||x_j|| = \sum_{j=1}^{\infty} |f_j(x)|,$$

which completes the proof. □

Combining this with the lower bound in the frame inequality, we can now revise Question 3.1.2 as the main goal for the rest of the chapter.

Question 3.1.5. *Given a normalized unconditional Schauder frame $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ for ℓ_1, what are the most general properties that need to be imposed on the frame to ensure the existence of a constant $C > 0$ such that for all $x \in \ell_1$,*

$$\sum_{j=1}^{\infty} |f_j(x)| \leq C||x||_1? \tag{3.2}$$

Notice here that ℓ_∞ is viewed as the dual of ℓ_1.

3.2 Frame Inequality in Finite Dimensions

Before working with the infinite-dimensional situation, which will be the main focus of this chapter, we need to start with finite dimensions, where all frames considered are finite. In the context of our work so far, we turn our attention to ℓ_1^n.

Given any Schauder frame $\{(x_j, f_j)\}_{j=1}^N$ for ℓ_1^n, we must have $||x_j||_1, ||f_j||_\infty < \infty$ for all $1 \leq j \leq N$. Since there are only finitely many j's, we conclude that

$$\max_{1\leq j\leq N} ||x_j||_1 < \infty, \qquad \max_{1\leq j\leq N} ||f_j||_\infty < \infty.$$

In other words, $\{x_j\}_{j=1}^N$ and $\{f_j\}_{j=1}^N$ must be bounded in ℓ_1^n and ℓ_∞^n, respectively, by virtue of being finite sequences. Such boundedness allows us to prove (3.1.5) in finite dimensions.

Lemma 3.2.1. *If $\{(x_j, f_j)\}_{j=1}^N$ is a Schauder frame for ℓ_1^n, then there exists a constant $C > 0$ such that*

$$\sum_{j=1}^N |f_j(x)| \leq C||x||_1 \tag{3.3}$$

for all $x \in \ell_1^n$.

Proof. Let $\{(x_j, f_j)\}_{j=1}^N$ be a Schauder frame for ℓ_1^n. Since $\{f_j\}_{j=1}^N$ is a bounded sequence in ℓ_∞^n, there exists $K > 0$ such that

$$K = \max_{\substack{1\leq j\leq N\\ 1\leq k\leq n}} |f_j(e_k)|. \tag{3.4}$$

Also, we know that $x = \sum\limits_{i=1}^n e_i^*(x)e_i$ so therefore,

$$\begin{aligned}
\sum_{j=1}^N |f_j(x)| &= \sum_{j=1}^N \left|\sum_{i=1}^n e_i^*(f_j(x))e_i\right| \\
&= \sum_{j=1}^N \left|\sum_{i=1}^n e_i^*(x)f_j(e_i)\right| \\
&\leq \sum_{i=1}^n \left(|e_i^*(x)| \sum_{j=1}^N |f_j(e_i)|\right) \\
&\leq KN||x||_1
\end{aligned}$$

for all $x \in \ell_1^n$. Setting $C = KN$, we obtain inequality (Equation 3.3). $\square$

Combining this with the lower bound proved in Lemma 3.1.4, we draw the following conclusion for ℓ_1^n.

Theorem 3.2.2. *If $\{(x_j, f_j)\}_{j=1}^N$ is a normalized Schauder frame for ℓ_1^n, then there exist constants $A, B > 0$ such that*

$$A||x||_1 \leq \sum_{j=1}^{N} |f_j(x)| \leq B||x||_1$$

for all $x \in \ell_1^n$.

While Lemma 3.2.1 holds for any Schauder frame for ℓ_1^n, Lemma 3.1.4 was assumed to hold for only normalized Schauder frames. This restricts the hypothesis of Theorem 3.2.2 to normalized Schauder frames.

Other than normalized Schauder frames for ℓ_1^n, we prove in the corollary below that the frame inequality holds for FUNTFs for ℓ_1^n. Observe that proving the ℓ_1-type of frame inequality for FUNTFs provides a connection between the context of Chapter 2 and the topic of the current chapter. Also, recall that FUNTFs for ℓ_1^n are sequence pairs $\{(x_j, f_j)\}_{j=1}^N$ in $\ell_1^n \times \ell_\infty^n$ such that $||x_j||_1 = ||f_j||_\infty = |f_j(x_j)| = 1$ and that

$$Sx = \sum_{j=1}^{N} f_j(x)x_j = (N/n)x \tag{3.5}$$

for all $x \in \ell_1^n$.

Corollary 3.2.3. *If $\{(x_j, f_j)\}_{j=1}^N$ is a FUNTF for ℓ_1^n, then there exist constants $A, B > 0$ such that*

$$A||x||_1 \leq \sum_{j=1}^{N} |f_j(x)| \leq B||x||_1$$

for all $x \in \ell_1^n$.

Proof. If $\{(x_j, f_j)\}_{j=1}^N$ is a FUNTF for ℓ_1^n, then for all $1 \leq j \leq N$, we have $||x_j||_1 = ||f_j||_\infty = f_j(x_j) = 1$. Referring to equation (Equation 3.5) above, we have

$$||x||_1 = \frac{n}{N} \left|\left| \sum_{j=1}^N f_j(x) x_j \right|\right|_1 \leq \frac{n}{N} \sum_{j=1}^N |f_j(x)|$$

Setting $A = N/n$, we obtain the lower bound for the frame inequality.

We are left with proving the upper bound of the frame inequality. Since normalized Schauder frames are bounded in $\ell_1^n \times \ell_\infty^n$, the sequence $\{f_j\}_{j=1}^N$ must also be a bounded in ℓ_∞^n. Setting the constant $K > 0$ as in equation (Equation 3.4) in the proof of Lemma 3.2.1, the same computation yields

$$\sum_{j=1}^N |f_j(x)| \leq KN||x||_1.$$

Therefore, the conclusion of Lemma 3.2.1 also holds for FUNTFs. Setting $B = KN$ gives us the ℓ_1-type of frame inequality for FUNTFs. □

This completes our discussion of the ℓ_1-type of frame inequalities for ℓ_1^n. Next, we turn our attention to frame inequalities in infinite-dimensional Banach spaces, starting with ℓ_1. Before we discuss the conditions under which the ℓ_1-type of frame inequality holds for ℓ_1, we provide a motivational example which indicates that unlike for ℓ_1^n, not all Schauder frames for ℓ_1 satisfy the frame inequality. We initially provide a construction which does not yield a Schauder frame before undertaking appropriate modifications to obtain a normalized unconditional Schauder frame that does not satisfy an ℓ_1-type of frame inequality.

3.3 Motivational Example: Initial Construction

So far, we have considered frames only for finite-dimensional Banach spaces. In this section, we consider *infinite-dimensional* Banach spaces. Our goal is to find conditions under which a normalized unconditional Schauder frame for ℓ_1 will satisfy an ℓ_1-type of frame inequality. For motivation, we will discuss an example of a sequence pair in ℓ_1 whose analysis operator is not an embedding and which does *not* satisfy the frame inequality. Before we start, we discuss analysis and synthesis operators in the ℓ_1 setting.

For a frame in a Hilbert space H, its analysis operator maps H bijectively onto a closed subspace of ℓ_2 (Theorem 8.27 in [5]). Hence, a sequence $(x_j)_{j\in\mathbb{N}}$ in a Hilbert space H is a frame if and only if the analysis operator $T : \ell_2 \to \ell_2$ defined by $T(x) = (\langle x, x_j\rangle)_{j=1}^{\infty}$ is an *embedding*. Here, an operator is an embedding if it is injective, continuous, and closed. In this context, we view frames from the frame inequalities definition.

Now, consider a Schauder frame $\{(x_j, f_j)\}_{j\in\mathbb{N}} \subseteq \ell_1 \times \ell_\infty$ for ℓ_1. Since we do not know whether the analysis operator $T : \ell_1 \to \ell_1$ defined by $T(x) = (f_j(x))_{j=1}^{\infty}$ is an embedding, we leave as an open question the conditions for $T : \ell_1 \to \ell_1$ to be an embedding.

Question 3.3.1. *Let $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ be a sequence pair in $\ell_1 \times \ell_\infty$. What are the necessary and sufficient conditions for $T : \ell_1 \to \ell_1$ defined by $T(x) = (f_j(x))_{j=1}^{\infty}$ to be an embedding?*

Another important fact in the Hilbert space setting is that the synthesis operator of a frame is a surjective map of ℓ_2 onto H (Theorem 8.27 in [5]). We will define an analog of the synthesis operator for sequence pairs in $\ell_1 \times \ell_\infty$.

Given a normalized sequence pair $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ in $\ell_1 \times \ell_\infty$, define the synthesis operator $R : \ell_1 \to \ell_1$ by

$$R((a_j)_{j=1}^{\infty}) = \sum_{j=1}^{\infty} a_j x_j. \tag{3.6}$$

Since

$$\sum_{j=1}^{\infty} ||a_j\, x_j||_1 = \sum_{j=1}^{\infty} |a_j| < \infty, \tag{3.7}$$

the series in equation (Equation 3.6) converges absolutely, and the operator R is bounded.

Just as in the Hilbert space setting, we define the frame operator for ℓ_1 to be the composition of the analysis and synthesis operators:

$$S = (RT)(x) = R((f_j(x))_{j=1}^{\infty}) = \sum_{j=1}^{\infty} f_j(x)x_j. \tag{3.8}$$

Since R and T are bounded, $S : \ell_1 \to \ell_1$ is also bounded. What is unknown is whether it is invertible.

Question 3.3.2. *Let $T : \ell_1 \to \ell_1$ defined by $T(x) = (f_j(x))_{j=1}^{\infty}$ be an embedding and let $R : \ell_1 \to \ell_1$ be defined by equation (Equation 3.6). Is the operator $S = RT$ invertible?*

Since equivalent conditions for frames in ℓ_2 are that the synthesis operator is surjective and that the analysis operator is an embedding, we leave as an open question whether the surjectivity of the operator $R : \ell_1 \to \ell_1$ is a necessary and sufficient condition for $T : \ell_1 \to \ell_1$ to be an embedding.

Question 3.3.3. *Is R being onto a necessary and sufficient condition for T to be an embedding? If not, what are the necessary and sufficient conditions that need to be imposed on R in order for T to be an embedding?*

We are now ready to discuss the example mentioned in the beginning of this section. In order to demonstrate that the hypotheses for satisfying the ℓ_1-type of frame inequality for infinite-dimensional Banach spaces are more complicated than that for Hilbert spaces and finite-dimensional Banach spaces, we provide an example of a sequence pair for which the corresponding operator $T : \ell_1 \to \ell_1$ is not an embedding and most importantly, the ℓ_1-type of frame inequality does not hold. We begin by defining the Rademacher system.

Definition 19. The *Rademacher system* in $L^2[0,1]$ is the sequence $\{R_j\}_{j=0}^{\infty}$ defined by

$$R_j(t) = \operatorname{sign}(\sin(2^j \pi t)) = \begin{cases} 1, & t \in \bigcup_{k=0}^{2^{j-1}-1} \left(\frac{2k}{2^j}, \frac{2k+1}{2^j}\right) \\ 0, & t = \frac{k}{2^j}, \quad k = 0, ..., 2^j \\ -1, & t \in \bigcup_{k=0}^{2^{j-1}-1} \left(\frac{2k+1}{2^j}, \frac{2k+2}{2^j}\right). \end{cases} \tag{3.9}$$

The first four Rademacher functions are illustrated below.

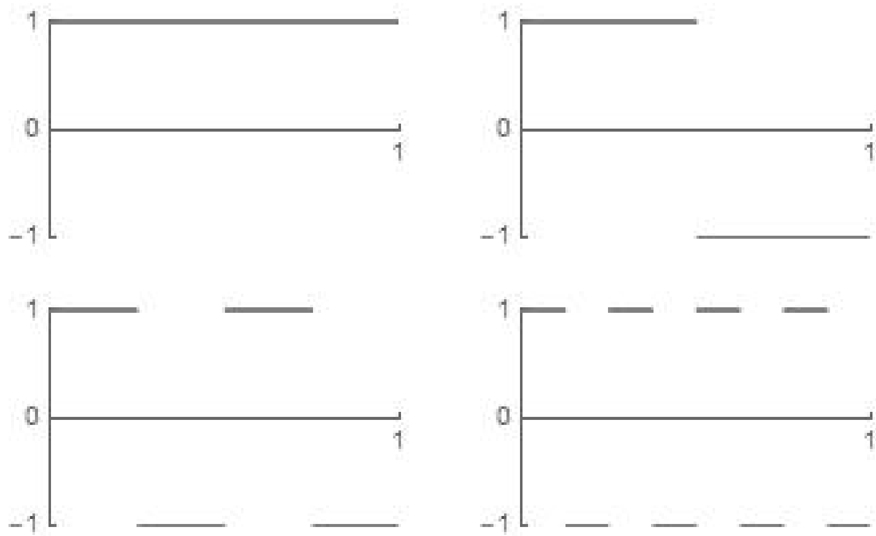

Figure 3.1: Graphs of R_0, R_1 (top), and R_2, R_3 (bottom).

We see that R_j has 2^j peaks and troughs, each being 2^{-j} in width. More reading on Rademacher functions in L^p can be found in texts such as [43] and [5].

Now we define a discrete version of the Rademacher system in $\ell_1^{2^N}$. By sampling the continuous Rademacher function $R_j(t)$ at $t = (2k-1)/2^{N+1}$ for integers $1 \leq k \leq 2^N$, we call the resultant sequence in $\ell_1^{2^N}$ the *Rademacher sequence*. The *normalized Rademacher system* can be obtained by normalizing the Rademacher sequence with respect to the ℓ_1-norm.

Definition 20. The *normalized Rademacher sequence* in $\ell_1^{2^N}$ is the sequence $\{r_j^N\}_{j=0}^{N+1}$ in ℓ_1^N with $N+1$ elements defined by

$$r_j^N(e_k) = \frac{1}{2^N} R_j((2k-1)2^{-(N+1)}), \tag{3.10}$$

for integers $1 \leq k \leq 2^N$ and $0 \leq j \leq N+1$.

Just as illustrations of the lower order Rademacher functions help visualize those of higher orders, we explicitly write out the Rademacher sequences in $\ell_1^{2^N}$ for $N = 1, 2$, and 3.

Example. First, let $N = 1$. In this case, the normalized Rademacher sequence is defined in ℓ_1^2 and consists of only two elements, r_0^1 and r_1^1, defined below.

$$\begin{aligned} r_0^1 &= \left(\frac{1}{2}, \frac{1}{2}\right) \\ r_1^1 &= \left(\frac{1}{2}, -\frac{1}{2}\right). \end{aligned} \tag{3.11}$$

Let $N = 2$. Now, the Rademacher sequence is defined in ℓ_1^4 and consists of the following 3 elements.

$$\begin{aligned} r_0^2 &= \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \\ r_1^2 &= \left(\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right) \\ r_2^2 &= \left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\right) \end{aligned} \tag{3.12}$$

Finally, let $N = 3$. In this case, the Rademacher sequence is defined in ℓ_1^8 and consists of the following 4 elements.

$$\begin{aligned} r_0^3 &= \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right) \\ r_1^3 &= \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, -\frac{1}{8}, -\frac{1}{8}, -\frac{1}{8}, -\frac{1}{8}\right) \\ r_2^3 &= \left(\frac{1}{8}, \frac{1}{8}, -\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, -\frac{1}{8}, -\frac{1}{8}\right) \\ r_3^3 &= \left(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, -\frac{1}{8}\right) \end{aligned} \tag{3.13}$$

Next, we define a sequence pair $\{(x_j^N, f_j^N)\}_{j=1}^N$ in $\ell_1^{2^N+N} \times \ell_\infty^{2^N+N}$ by

$$x_j^N(e_k) = \begin{cases} r_j^N(e_k), & \text{if } 1 \le k \le 2^N \\ 0, & \text{if } 2^N + 1 \le k \le 2^N + N \end{cases} \tag{3.14}$$

and

$$f_j^N(e_k) = \begin{cases} \frac{1}{j}, & \text{if } k = 1 \\ 1, & \text{if } k = 2^N + j \\ 0, & \text{elsewhere.} \end{cases} \tag{3.15}$$

While we started with $j = 0$ when defining the normalized Rademacher sequence, we now exclude the element r_0^N by starting our indexing at $j = 1$. We illustrate the relationship between $\{(x_j^N, f_j^N)\}_{j=1}^N$ and the normalized Rademacher sequence by explicitly defining $\{(x_j^N, f_j^N)\}_{j=1}^N$ for $1 \le N \le 3$.

Observe that for $N = 1$,

$$x_1^1 = \left(\frac{1}{2}, -\frac{1}{2}, 0\right) \tag{3.16}$$

and

$$f_1^1 = (1, 0, 1). \tag{3.17}$$

Let $N = 2$. Proceeding in the same way as above, we have

$$\begin{aligned} x_1^2 &= \left(\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 0, 0\right) \\ x_2^2 &= \left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, 0, 0\right) \end{aligned} \tag{3.18}$$

and

$$f_1^2 = \left(\frac{1}{2}, 0, 0, 0, 1, 0\right)$$
$$f_2^2 = \left(\frac{1}{2}, 0, 0, 0, 0, 1\right). \tag{3.19}$$

Likewise, for $N = 3$ we have

$$x_1^3 = \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, -\frac{1}{8}, -\frac{1}{8}, -\frac{1}{8}, -\frac{1}{8}, 0, 0, 0\right)$$
$$x_2^3 = \left(\frac{1}{8}, \frac{1}{8}, -\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, 0, 0, 0\right) \tag{3.20}$$
$$x_3^3 = \left(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, -\frac{1}{8}, 0, 0, 0\right)$$

and

$$f_1^3 = \left(\frac{1}{3}, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0\right)$$
$$f_2^3 = \left(\frac{1}{3}, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0\right) \tag{3.21}$$
$$f_3^3 = \left(\frac{1}{3}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1\right).$$

Our goal is to find a sequence pair $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ in $\ell_1 \times \ell_\infty$ which does not satisfy an ℓ_1-type of frame inequality. Even though we are not able to find an approximate Schauder frame which does not satisfy the ℓ_1-type of frame inequality, we are able to find a sequence pair $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ whose frame operator is bounded and whose analysis operator $T : \ell_1 \to \ell_1$ is not an embedding. In order to find such a sequence pair in $\ell_1 \times \ell_\infty$, we interleave the finite sequences $\{(x_j^N, f_j^N)\}_{j=1}^N$ with respect to N appropriately.

Before defining the sequence pair $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ in $\ell_1 \times \ell_\infty$, we define the sequence pair

$\{(X_j^N, F_j^N)\}_{j\in\mathbb{N}}$, also in $\ell_1 \times \ell_\infty$, for $N \geq 2$ below.

$$X_j^N(e_k) = \begin{cases} 0, & 1 \leq k \leq \sum\limits_{n=1}^{N-1}(2^n+n) \\ x_j^N\left(k - \sum\limits_{n=1}^{N-1}(2^n+n)\right), & 1 + \sum\limits_{n=1}^{N-1}(2^n+n) \leq k \leq \sum\limits_{n=1}^{N}(2^n+n) \\ 0, & \text{elsewhere} \end{cases} \tag{3.22}$$

$$F_j^N(e_k) = \begin{cases} 0, & 1 \leq k \leq \sum\limits_{n=1}^{N-1}(2^n+n) \\ f_j^N\left(k - \sum\limits_{n=1}^{N-1}(2^n+n)\right), & 1 + \sum\limits_{n=1}^{N-1}(2^n+n) \leq k \leq \sum\limits_{n=1}^{N}(2^n+n) \\ 0, & \text{elsewhere.} \end{cases} \tag{3.23}$$

Now for $N = 1$, we define

$$X_1^1(e_k) = \begin{cases} x_1^1(e_k), & 1 \leq k \leq 3 \\ 0, & \text{elsewhere} \end{cases} \tag{3.24}$$

and

$$F_1^1(e_k) = \begin{cases} f_1^1(e_k), & 1 \leq k \leq 3 \\ 0, & \text{elsewhere.} \end{cases} \tag{3.25}$$

Observe that each of the X_j^N's and F_j^N's were defined from the x_j^N's and f_j^N's by adding zeros appropriately to form infinite sequences. Just as we have done previously for Rademacher sequences, the definitions of the X_j^N's and F_j^N's can be better understood after explicitly constructing the sequence pairs for small values of N.

Example. Let $N = 1$. Then we have

$$X_1^1 = \left(\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0, ...\right)$$
$$F_1^1 = (1, 0, 1, 0, 0, 0, 0, ...)\,. \tag{3.26}$$

Let $N = 2$. Observe that

$$X_1^2 = \left(0, 0, 0,\ \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, ...\right)$$
$$F_1^2 = \left(0, 0, 0,\ \frac{1}{2}, 0, 0, 0, 1, 0, 0, 0, 0, 0, ...\right) \tag{3.27}$$

and

$$X_1^2 = \left(0, 0, 0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, ...\right)$$
$$F_1^2 = \left(0, 0, 0, \frac{1}{2}, 0, 0, 0, 0, 1, 0, 0, 0, 0, ...\right). \tag{3.28}$$

Notice here that for $N = 2$, both X_j^N and F_j^N are padded with 3 zeros in the front.

Finally, we illustrate our last example for $N = 3$ by explicitly constructing X_1^3 and F_1^3. For $N = 3$, X_j^N and F_j^N are padded with 9 zeros in the front:

$$X_1^3 = \left(0, 0, 0, 0, 0, 0, 0, 0, 0,\ \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, -\frac{1}{8}, -\frac{1}{8}, -\frac{1}{8}, -\frac{1}{8}, 0, 0, 0, 0, 0, 0, 0, ...\right)$$
$$F_1^3 = \left(0, 0, 0, 0, 0, 0, 0, 0, 0,\ \frac{1}{3}, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0...\right). \tag{3.29}$$

We define the sequence pair $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ by arranging and concatenating the sequences X_j^N and F_j^N appropriately:

$$\{x_j\}_{j\in\mathbb{N}} = \{X_1^1\} \cup \{X_j^2\}_{j=1}^2 \cup \{X_j^3\}_{j=1}^3 \cup \{X_j^4\}_{j=1}^4 \cup \cdots \{X_j^N\}_{j=1}^N \cup \cdots \tag{3.30}$$

and

$$\{f_j\}_{j\in\mathbb{N}} = \{F_1^1\} \cup \{F_j^2\}_{j=1}^2 \cup \{F_j^3\}_{j=1}^3 \cup \{F_j^4\}_{j=1}^4 \cup \cdots \{F_j^N\}_{j=1}^N \cup \cdots \tag{3.31}$$

That $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ is a normalized sequence pair in $\ell_1 \times \ell_\infty$ can be seen by observing that $||X_j^N||_1 = 1$ and $||F_j^N||_\infty = 1$ for all $1 \leq j \leq N$ and $N \in \mathbb{N}$.

We prove that the operator $T : \ell_1 \to \ell_1$ corresponding to $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ is not an embedding. Consider the operator $T_N : \ell_1^{2^N+N} \to \ell_1^{2^N+N}$ defined by $T_N(x) = ((f_j^N(x))_{j=1}^\infty$. Since $f_j^N(e_k) = 0$ whenever $2 \leq k \leq 2^N$, T_N is not injective and therefore, is not an embedding. Likewise, we conclude that $T : \ell_1 \to \ell_1$ defined by $T(x) = ((f_j(x))_{j=1}^\infty$ also cannot be an embedding.

The theorem that we state next is known as *Khintchine's Inequality*, a proof of which can be found in Theorem 3.25 in [5].

Theorem 3.3.4 (Khintchine's Inequality)**.** *Let $\{R_n\}_{n=0}^\infty$ be the Rademacher system in $L^2[0,1]$. For each $1 \leq p < \infty$, there exist constants $k_p, K_p > 0$ such that for every $N \in \mathbb{N}$ and real scalars $c_1, ..., c_N$,*

$$k_p \left(\sum_{j=1}^N c_j^2\right)^{1/2} \leq \left|\left|\sum_{j=1}^N c_j R_j\right|\right|_p \leq K_p \left(\sum_{j=1}^N c_j^2\right)^{1/2}. \tag{3.32}$$

In order to prove the boundedness of the frame operator of $\{(x_j, f_j)\}_{j\in\mathbb{N}}$, we will need the following discrete version of Khintchine's Inequality for the index $p = 1$.

Lemma 3.3.5. *Let $\{r_j^N\}_{j=1}^N$ in $\ell_1^{2^N}$ be the normalized Rademacher sequence in $\ell_1^{2^N}$. Then there exist constants $k_1, K_1 > 0$ such that*

$$k_1 \left(\sum_{j=1}^N c_j^2\right)^{1/2} \leq \left|\left|\sum_{j=1}^N c_j r_j^N\right|\right|_1 \leq K_1 \left(\sum_{j=1}^N c_j^2\right)^{1/2} \tag{3.33}$$

for every $N \in \mathbb{N}$ and real scalars $c_1, ..., c_N$.

Proof. For a given Rademacher function $R_j(t)$, all of its peaks and troughs have lengths 2^{-N}, are of height 1, and occur at $t = 2^{-N}k$, for integers $1 \leq k \leq 2^N$. Hence,

$$\int_0^1 |R_j(t)|\ dt = 2^{-N} \sum_{k=1}^{2^N} \left| R_j((2k-1)2^{-(N+1)}) \right|. \tag{3.34}$$

This gives us

$$\left\| \sum_{j=1}^{N} c_j R_j \right\|_1 = \int_0^1 \left| \sum_{j=1}^{N} c_j\, R_j(t) \right|\ dt = 2^{-N} \sum_{k=1}^{2^N} \left| \sum_{j=1}^{N} c_j\, R_j((2k-1)2^{-(N+1)}) \right| \tag{3.35}$$

By equation (Equation 3.10),

$$\begin{aligned} 2^{-N} \sum_{k=1}^{2^N} \left| \sum_{j=1}^{N} c_j\, R_j((2k-1)2^{-(N+1)}) \right| &= 2^{-N} \sum_{k=1}^{2^N} \left| \sum_{j=1}^{N} c_j\, 2^N r_j^N(k) \right| \\ &= \sum_{k=1}^{2^N} \left| \sum_{j=1}^{N} c_j\, r_j^N(k) \right| \\ &= \left\| \sum_{j=1}^{N} c_j\, r_j^N \right\|_1, \end{aligned} \tag{3.36}$$

where the final equality above holds because $r_j^N \in \ell_1^{2^N}$ for all $1 \leq j \leq N$.

This shows that

$$\left\| \sum_{j=1}^{N} c_j R_j \right\|_1 = \left\| \sum_{j=1}^{N} c_j\, r_j^N \right\|_1. \tag{3.37}$$

Since the Rademacher system in $L^2[0,1]$ satisfies Khintchine's Inequalities, we conclude that there exist constants $k_1, K_1 > 0$ such that

$$k_1 \left(\sum_{j=1}^{N} c_j^2 \right)^{1/2} \leq \left\| \sum_{j=1}^{N} c_j r_j^N \right\|_1 \leq K_1 \left(\sum_{j=1}^{N} c_j^2 \right)^{1/2}.$$

for every $N \in \mathbb{N}$ and real scalars $c_1, ..., c_N$. $\square$

Lemma 3.3.5 allows us to compute

$$\begin{aligned}\left|\left|\sum_{j=1}^{N} f_j^N(e_1)x_j^N\right|\right|_1 = \left|\left|\sum_{j=1}^{N} \frac{1}{j}\, x_j^N\right|\right|_1 &= \left|\left|\sum_{j=1}^{N} \frac{1}{j}\, r_j^N\right|\right|_1 \\ &\leq K_1\left(\sum_{j=1}^{N}\frac{1}{j^2}\right)^{1/2} \\ &\leq K_1\left(\sum_{j=1}^{\infty}\frac{1}{j^2}\right)^{1/2} \\ &= \frac{\pi K_1}{\sqrt{6}}.\end{aligned} \tag{3.38}$$

Now, suppose that $2 \leq k \leq 2^N$. In this case, $f_j^N(e_k) = 0$ for all $1 \leq j \leq N$, so

$$\left|\left|\sum_{j=1}^{N} f_j^N(e_k)x_j^N\right|\right|_1 = 0. \tag{3.39}$$

Finally, let $2^N + 1 \leq k \leq 2^N + N$. In this case, $f_j^N(e_k) = 1$ when $j = k - 2^N$, which gives us

$$\left|\left|\sum_{j=1}^{N} f_j^N(e_k)x_j^N\right|\right|_1 = \left|\left|x_{k-2^N}^N\right|\right|_1 = \left|\left|r_{k-2^N}^N\right|\right|_1 = 1. \tag{3.40}$$

Let $k_N = 1 + \sum_{n=1}^{N-1}(2^n + n)$, with $N \in \mathbb{N}$. Referring to the definition of the f_j's in terms of the F_j^N's and to equation (Equation 3.38), we compute

$$||S(e_{k_N})|| = \left|\left|\sum_{j=1}^{\infty} f_j(e_{k_N})x_j\right|\right|_1 \leq \frac{\pi K_1}{\sqrt{6}}. \tag{3.41}$$

Likewise, it follows from equation (Equation 3.39) and (Equation 3.40) that $||S(e_k)|| \leq 1$ for all $k \in \mathbb{N}$ whenever $k \neq k_N$ and $N \in \mathbb{N}$. Since S is bounded with respect to all of the e_k's, we conclude that it is a bounded operator.

Even though S is bounded, it must also be invertible in order for $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ to be an approximate Schauder frame. Since there exists $k \in \mathbb{N}$ such that $S(e_k) = 0$, S cannot be invertible, and hence $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ cannot be an approximate Schauder frame.

Finally, we need to show that $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ does not satisfy the ℓ_1-type of frame inequality. First, we know that

$$\sum_{j=1}^{N} |f_j^N(e_1)| = \sum_{j=1}^{N} \frac{1}{j}.$$

For all $N \in \mathbb{N}$, we have

$$\sum_{j=1}^{\infty} \left| f_j \left(1 + \sum_{n=1}^{N-1} (2^n + n) \right) \right| = \sum_{j=1}^{N} \frac{1}{j}. \tag{3.42}$$

Therefore, for large values of N, the sum in equation (Equation 3.42) becomes arbitrarily large.

If $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ were to satisfy the ℓ_1-type of frame inequality, then there must exist a constant $C > 0$ such that

$$\sum_{j=1}^{\infty} |f_j(x)| \leq C||x||_1 \tag{3.43}$$

for all $x \in \mathbb{N}$. In other words, the sum $\sum\limits_{j=1}^{\infty} |f_j(x)|$ must be finite. Referring to equation (Equation 3.42), we conclude that the frame inequality does not hold for $\{(x_j, f_j)\}_{j\in\mathbb{N}}$.

In the next section, this example is a foundation that allows us to define another modified discrete Rademacher system that is actually a normalized unconditional Schauder frames for ℓ_1 not satisfying an ℓ_1-type of frame inequality.

3.4 Motivational Example: Extension to a Schauder Frame

While our example in the previous section did not satisfy an ℓ_1-type of frame inequality, it failed to be a Schauder frame. In this section, we modify the example constructed in the previous section to obtain an unconditional Schauder frame which also does not satisfy an ℓ_1-type of frame inequality. This new construction keeps the core structure of the preceding example intact. That is, the sequence pair $\{(x_j^N, f_j^N)\}_{j=1}^{N} \subseteq \ell_1^{2^N+N}$ defined in the previous section by equations (Equation 3.14) and (Equation 3.15) remains a building block for our new example.

First, we construct a modified discrete Rademacher system for ℓ_1 in finite dimensions. Our initial goal is to show that this finite-dimensional sequence pair is an unconditional Schauder frame for $\ell_1^{2^N+N}$ for which a uniform ℓ_1-type of frame inequality does not hold. In other words, we show that any such frame inequality for this system depends on N. We start by proving that the series $\sum\limits_{j=1}^{N} f_j^N(x) x_j^N$ converges unconditionally for any choice of $x \in \ell_1^{2^N+N}$.

Lemma 3.4.1. *There exists a constant $K > 0$ such that for all $N \in \mathbb{N}$ and $\varepsilon_j = \pm 1$,*

$$\left|\left|\sum_{j=1}^{N} \varepsilon_j \, f_j^N(x) \, x_j^N\right|\right| \leq K||x|| \tag{3.44}$$

for all $x \in \ell_1^{2^N+N}$.

Proof. In the previous section, we proved that

$$\left|\left|\sum_{j=1}^{N} f_j^N(e_k) \, x_j^N\right|\right| \leq \max\left\{\frac{\pi K_1}{\sqrt{6}}, 1\right\}$$

for all $1 \leq k \leq 2^N + N$. Here, K_1 is the constant associated with the upper Khintchine's Inequality. Setting $K = \max\{\pi K_1/\sqrt{6}, 1\}$, we can rewrite the inequality above as

$$\left|\left|\sum_{j=1}^{N} f_j^N(e_k) \, x_j^N\right|\right| \leq K. \tag{3.45}$$

Now, for any $\varepsilon_j = \pm 1$ for $1 \leq j \leq N$, Lemma 3.3.5 gives us

$$\begin{aligned}
\left|\left|\sum_{j=1}^{N} \varepsilon_j f_j^N(e_1) x_j^N\right|\right| &= \left|\left|\sum_{j=1}^{N} \frac{\varepsilon_j}{j}\, r_j^N\right|\right|_1 \\
&\leq K_1 \left(\sum_{j=1}^{N} \frac{\varepsilon_j^2}{j^2}\right)^{1/2} \\
&= K_1 \left(\sum_{j=1}^{N} \frac{1}{j^2}\right)^{1/2} \\
&= \frac{\pi K_1}{\sqrt{6}}.
\end{aligned}$$

For $2 \leq k \leq 2^N$, $f_j^N(e_k) = 0$ gives us

$$\left|\left|\sum_{j=1}^{N} \varepsilon_j f_j^N(e_k) x_j^N\right|\right| = 0,$$

and finally for $2^N + 1 \leq k \leq 2^N + N$, we have

$$\left|\left|\sum_{j=1}^{N} \varepsilon_j f_j^N(e_k) x_j^N\right|\right| = ||\varepsilon_{k-2^N} x_{k-2^N}^N|| = ||x_{k-2^N}^N|| = 1.$$

Therefore, for all $1 \leq k \leq 2^N + N$, we have

$$\left|\left|\sum_{j=1}^{N} \varepsilon_j f_j^N(e_k) x_j^N\right|\right| \leq \max\left\{\frac{\pi K_1}{\sqrt{6}}, 1\right\} = K. \tag{3.46}$$

We conclude that

$$\left|\left|\sum_{j=1}^{N} \varepsilon_j\, f_j^N(x)\, x_j^N\right|\right| \leq K||x|| \tag{3.47}$$

for all $x \in \ell_1^{2^N+N}$. □

Also, recall from the previous section that

$$\sum_{j=1}^{N} |f_j^N(e_1)| = \sum_{j=1}^{N} \frac{1}{j},$$

which becomes arbitrarily large for large values of $N \in \mathbb{N}$. Thus, there does *not* exist a constant $B > 0$ such that

$$\sum_{j=1}^{N} |f_j^N(x)| \leq B||x||$$

holds simultaneously for all $N \in \mathbb{N}$ and all $x \in \ell_1^{2^N+N}$.

Next, define the following sequence pair in $\ell_1^{2^N+N} \times \ell_\infty^{2^N+N}$:

$$\{(x_j, f_j)\}_{j=1}^{2^N+3N} = \{(e_j, e_j^*)\}_{j=1}^{2^N+N} \cup \{(x_j^N, f_j^N)\}_{j=1}^{N} \cup \{(x_j^N, -f_j^N)\}_{j=1}^{N}. \tag{3.48}$$

Choose any $x \in \ell_1^{2^N+N}$. We have

$$\begin{aligned}\sum_{j=1}^{2^N+3N} f_j(x)x_j &= \sum_{j=1}^{2^N+N} e_j^*(x)e_j + \sum_{j=1}^{N} f_j^N(x)x_j^N - \sum_{j=1}^{N} f_j^N(x)x_j^N \\ &= x.\end{aligned}$$

This shows that $\{(x_j, f_j)\}_{j=1}^{2^N+3N}$ is a Schauder frame for $\ell_1^{2^N+N}$.

In addition, for all $\varepsilon_j = \pm 1$ with $1 \leq j \leq N$ and any $x \in \ell_1^{2^N+N}$, we have

$$\begin{aligned}\left|\left|\sum_{j=1}^{2^N+3N} \varepsilon_j f_j(x)x_j\right|\right| &\leq \left|\left|\sum_{j=1}^{2^N+N} \varepsilon_j\, e_j^*(x)e_j\right|\right| + 2\left|\left|\sum_{j=1}^{N} \varepsilon_j\, f_j^N(x)x_j^N\right|\right| \\ &\leq (2K+1)||x||.\end{aligned}$$

This shows that $\{(x_j, f_j)\}_{j=1}^{2^N+3N}$ is an unconditional Schauder frame with unconditionality constant $2K+1$, which is a constant independent of N.

Now, we note that

$$\sum_{j=1}^{2^N+3N} |f_j(e_1)| = \sum_{j=1}^{2^N+N} |e_j^*(e_1)| + 2\sum_{j=1}^{N} |f_j(e_1)| = 1 + 2\sum_{j=1}^{N} \frac{1}{j},$$

which becomes arbitrarily large for large values of $N \in \mathbb{N}$. In other words, there does not exist a constant $B > 0$ such that

$$\sum_{j=1}^{2^N+3N} |f_j(x)| \leq B||x||$$

holds simultaneously for all $N \in \mathbb{N}$ and all $x \in \ell_1^{2^N+N}$.

We have shown that for all $N \in \mathbb{N}$, there exists a normalized unconditional Schauder frame for $\ell_1^{2^N+N}$ for which an ℓ_1-type of frame inequality is not applicable. Therefore, it remains to do so in the infinite-dimensional space ℓ_1.

Recall the sequences $\{X_j^N\}_{j=1}^N$ and $\{F_j^N\}_{j=1}^N$ in ℓ_1 and ℓ_∞, respectively, defined in the previous section, where

$$X_j^N(e_k) = \begin{cases} 0, & 1 \leq k \leq \sum_{n=1}^{N-1}(2^n+n) \\ x_j^N\left(k - \sum_{n=1}^{N}(2^n+n)\right), & 1 + \sum_{n=1}^{N-1}(2^n+n) \leq k \leq \sum_{n=1}^{N}(2^n+n) \\ 0, & \text{elsewhere} \end{cases}$$

and

$$F_j^N(e_k) = \begin{cases} 0, & 1 \leq k \leq \sum_{n=1}^{N-1}(2^n+n) \\ f_j^N\left(k - \sum_{n=1}^{N}(2^n+n)\right), & 1 + \sum_{n=1}^{N-1}(2^n+n) \leq k \leq \sum_{n=1}^{N}(2^n+n) \\ 0, & \text{elsewhere} \end{cases}$$

for $N \geq 2$, and

$$X_j^1(e_k) = \begin{cases} x_j^1(e_k), & 1 \leq k \leq 2^N + N \\ 0, & \text{elsewhere} \end{cases}$$

and

$$F_j^1(e_k) = \begin{cases} f_j^1(e_k), & 1 \leq k \leq 2^N + N \\ 0, & \text{elsewhere} \end{cases}$$

for $N = 1$.

Now, define the sequence pair $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ in $\ell_1 \times \ell_\infty$ by

$$\{(x_j, f_j)\}_{j\in\mathbb{N}} = \{(e_j, e_j^*)\}_{j\in\mathbb{N}} \cup \bigcup_{N=1}^{\infty} \{(X_j^N, F_j^N)\}_{j=1}^N \cup \bigcup_{N=1}^{\infty} \{(X_j^N, -F_j^N)\}_{j=1}^N. \quad (3.49)$$

Choose any $x \in \ell_1$. We have

$$\sum_{j=1}^{\infty} f_j(x)x_j = \sum_{j=1}^{\infty} e_j^*(x)e_j + \sum_{N=1}^{\infty}\sum_{j=1}^{N} F_j^N(x)X_j^N - \sum_{N=1}^{\infty}\sum_{j=1}^{N} F_j^N(x)X_j^N$$

$$= x.$$

This shows that $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ is a Schauder frame for ℓ_1.

Next, we prove the unconditionality of $\{(x_j, f_j)\}_{j\in\mathbb{N}}$. Before doing so, we write $x \in \ell_1$ as follows.

$$x = \bigcup_{N=1}^{\infty} x^N = (x^1, x^2, x^3, \ldots), \quad (3.50)$$

where $x^N \in \ell_1^{2^N+N}$ is given by $x^N(e_k) = x(e_k)$ for all $N \in \mathbb{N}$ and $1 \leq k \leq 2^N + N$. In this case, we have $||x||_1 = \sum_{N=1}^{\infty} ||x^N||_1$ and $F_j^N(x) = f_j^N(x^N)$ for any $x \in \ell_1$.

For any $\varepsilon_j = \pm 1$, we compute

$$\begin{aligned}
\left\|\sum_{N=1}^{\infty}\sum_{j=1}^{N}\varepsilon_j F_j^N(x)X_j^N\right\| &= \left\|\sum_{N=1}^{\infty}\sum_{j=1}^{N}\varepsilon_j\, f_j^N(x^N)X_j^N\right\| \\
&= \left\|\sum_{N=1}^{\infty}\sum_{j=1}^{N}\varepsilon_j\, f_j^N(x^N)x_j^N\right\| \\
&\leq \sum_{N=1}^{\infty}\left\|\sum_{j=1}^{N}\varepsilon_j\, f_j^N(x^N)x_j^N\right\| \\
&\leq \sum_{N=1}^{\infty} K||x^N|| \\
&= K||x||.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\left\|\sum_{j=1}^{\infty}\varepsilon_j f_j(x)x_j\right\| &\leq \left\|\sum_{j=1}^{\infty}\varepsilon_j\, e_j^*(x)e_j\right\| + 2\left\|\sum_{N=1}^{\infty}\sum_{j=1}^{N}\varepsilon_j F_j^N(x)X_j^N\right\| \\
&\leq (2K+1)||x||
\end{aligned}$$

This shows that $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ is an unconditional Schauder frame with unconditionality constant $2K+1$.

The normalization of $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ follows from the fact that $\{(x_j^N, f_j^N)\}_{j=1}^N$ is normalized, which implies that $\{(X_j^N, F_j^N)\}_{j=1}^N$ is normalized for all $N \in \mathbb{N}$.

Finally, fix $x \in \ell_1$ to be the sequence defined by

$$x(e_k) = \begin{cases} 1, & k = 1 + \sum\limits_{n=1}^{N-1}(2^n + n) \\ 0, & \text{elsewhere.} \end{cases} \tag{3.51}$$

In this case, we have

$$F_j^N(x) = f_j^N(e_1) = \frac{1}{j},$$

and therefore

$$\begin{aligned}\sum_{j=1}^{\infty} |f_j(x)| &= \sum_{j=1}^{\infty} |e_j^*(x)| + 2\sum_{N=1}^{\infty}\sum_{j=1}^{N} |F_j^N(x)| \\ &= 1 + 2\sum_{N=1}^{\infty}\sum_{j=1}^{N} \frac{1}{j} \qquad (3.52) \\ &= \infty.\end{aligned}$$

This proves that $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ is a normalized unconditional Schauder frame for ℓ_1 which does not satisfy an ℓ_1-type of frame inequality. We state this formally as a theorem as follows.

Theorem 3.4.2. *There exists a normalized unconditional Schauder frame $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ for ℓ_1 such that there does not exist a constant $B > 0$ such that*

$$\sum_{j=1}^{\infty} |f_j(x)| \leq B||x||$$

for all $x \in \ell_1$.

In other words, an ℓ_1-type of frame inequality is not applicable for $\{(x_j, f_j)\}_{j\in\mathbb{N}}$.

3.5 Frame Inequality in Infinite Dimensions

From the modified Rademacher systems discussed in the previous sections, we conclude that unlike for the finite-dimensional case, additional restrictions need to be imposed on unconditional Schauder frames for ℓ_1 for the frame inequality to hold. Continuing with the spirit that the Rademacher system is a normalized system, we initially focus on normalized unconditional Schauder frames $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ for ℓ_1, in which case $||f_j||_\infty = 1$ for all $j \in \mathbb{N}$. In other words,

$$\sup_{k\in\mathbb{N}} |f_j(e_k)| = 1$$

for all $j \in \mathbb{N}$. One of the simplest, albeit restrictive, such conditions for the f_j's is to let all $|f_j(e_k)|$'s be equal to 0 or 1 for all $k \in \mathbb{N}$. That is, assume that $f_j(e_k) \in \{-1, 0, 1\}$ for all j, k. We propose this condition as a property that will yield an affirmative answer to Question 3.1.5.

Since the ℓ_2-type of frame inequality holds for all frames for ℓ_2, it may be helpful to introduce the notion of the distance between ℓ_1 and ℓ_2. This notion will be useful for deducing a version of the frame inequality for ℓ_1 from that of ℓ_2. One definition that provides a link between a Banach space and its distance to a Hilbert space are the *type* and *cotype* of a Banach space. The type and cotype properties of Banach spaces is a deep subject in its own right: For further reading, we refer the reader to [44] and references therein.

Since only the cotype property of a Banach space will be relevant our discussion, we give the formal definition below. We will not explicitly state the definition of the type property.

Definition 21. A Banach space X is of *cotype* p, with $1 \le p < \infty$, if there exists a constant $C > 0$ such that for any finite sequence $\{x_j\}_{j=1}^n \in X$,

$$\frac{1}{C}\sum_{j=1}^{n} ||x_j||^p \le \operatorname*{avg}_{\varepsilon_j=\pm 1} \left|\left|\sum_{j=1}^{n} \varepsilon_j x_j\right|\right|^p,$$

where the average is taken over all possible choices of $\varepsilon_j = \pm 1$.

It is a well known fact that ℓ_p is of cotype 2 for $1 \le p \le 2$ and of cotype p for $2 \le p < \infty$ (see, e.g., [45]). Since ℓ_1 is of cotype 2, there exists a constant $C > 0$ such that for every finite sequence $\{x_j\}_{j=1}^n$ in ℓ_1,

$$\sum_{j=1}^{n} ||x_j||_1^2 \le C \operatorname*{avg}_{\varepsilon_j=\pm 1} \left|\left| \sum_{j=1}^{n} \varepsilon_j x_j \right|\right|_1^2,$$

where the average is taken over all possible choices of $\varepsilon_j = \pm 1$. Also, note that $n \in \mathbb{N}$ is chosen arbitrarily, and hence can be arbitrarily large. We prove in the next lemma that the inequality defining the cotype 2 property of ℓ_1 enables us to obtain a preliminary version of the frame inequality.

Lemma 3.5.1. *Let $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ be a normalized sequence pair in $\ell_1 \times \ell_\infty$. If the series $\sum_{j=1}^{\infty} f_j(x)x_j$ converges unconditionally for all $x \in \ell_1$, then there exists a constant $C > 0$ such that*

$$\sum_{j=1}^{\infty} |f_j(x)|^2 \le C||x||_1^2. \tag{3.53}$$

for all $x \in \ell_1$.

Proof. Fix an arbitrary $x \in \ell_1$. Since ℓ_1 is of cotype 2, there exists some $K > 0$ such that for every $x \in \ell_1$ and every $n \in \mathbb{N}$,

$$\sum_{j=1}^{n} ||f_j(x)x_j||_1^2 \le K \operatorname*{avg}_{\varepsilon_j=\pm 1} \left|\left| \sum_{j=1}^{n} \varepsilon_j f_j(x)x_j \right|\right|_1^2. \tag{3.54}$$

Suppose that the series $\sum_{j=1}^{\infty} f_j(x)x_j$ converges unconditionally. Keeping $x \in \ell_1$ fixed, then (see, e.g., Theorem 3.15 in [5]) there exists a finite scalar $R_\varepsilon > 0$ such that

$$R_{\mathcal{E}} = \sup_{\substack{n\in\mathbb{N} \\ \varepsilon_i=\pm 1}} \left|\left| \sum_{j=1}^{n} \varepsilon_j f_j(x)x_j \right|\right|_1.$$

Fix $n \in \mathbb{N}$. This implies that

$$\operatorname*{avg}_{\varepsilon_j=\pm 1} \left|\left| \sum_{j=1}^{n} \varepsilon_j f_j(x)x_j \right|\right|_1^2 \le R_{\mathcal{E}}^2.$$

Therefore, by combining equations (Equation 3.54) and the definition of $R_{\mathcal{E}}$, we see that

$$\sum_{j=1}^{n} ||f_j(x)x_j||_1^2 \leq KR_{\mathcal{E}}^2. \tag{3.55}$$

By our hypothesis, we have $||x_j||_1 = 1$ for every $j \in \mathbb{N}$, and so

$$\sum_{j=1}^{n} |f_j(x)|^2 = \sum_{j=1}^{n} ||f_j(x)x_j||_1^2 \leq KR_{\mathcal{E}}^2 < \infty. \tag{3.56}$$

Since inequality (Equation 3.56) holds for any $n \in \mathbb{N}$, letting $n \to \infty$ gives us

$$\sum_{j=1}^{\infty} |f_j(x)|^2 \leq KR_{\mathcal{E}}^2. \tag{3.57}$$

While this shows that the left hand side of equation (Equation 3.53) is finite, it does not actually prove equation (Equation 3.53). In order to do so, consider for $N \in \mathbb{N}$ the map $A_N : \ell_1 \to \ell_2$ defined by

$$(A_N x)(e_j) = \begin{cases} f_j(x), & 1 \leq j \leq N, \\ 0, & j > N. \end{cases}$$

For all $x \in \ell_1$, we have

$$\sup_{N\in\mathbb{N}} ||A_N x||_2^2 = \sup_{N\in\mathbb{N}} \sum_{j=1}^{N} |f_j(x)|^2 = \sum_{j=1}^{\infty} |f_j(x)|^2 \leq KR_{\mathcal{E}}^2 < \infty. \tag{3.58}$$

Before applying the Uniform Boundedness Principle to the A_N's, we need to show that $A_N : \ell_1 \to \ell_2$ is a bounded linear map for all $N \in \mathbb{N}$. Since f_j is bounded for all $j \in \mathbb{N}$ and

$$||A_N||_2^2 = \sup_{||x||_1=1} ||A_N x||_2^2 = \sup_{||x||_1=1} \sum_{j=1}^{N} |f_j(x)|^2,$$

with $N \in \mathbb{N}$ being finite, the boundedness of A_N follows for each $N \in \mathbb{N}$.

We are left with proving the linearity of A_N. First, fix $1 \leq j \leq N$. For any $x, y \in \ell_1$ and constants $a, b > 0$, we have

$$(A_N(ax+by))(e_j) = f_j(ax+by) = af_j(x) + bf_j(y) = (aA_N(x) + bA_N(y))(e_j).$$

The result is obvious for $j > N$, since $(A_N x)(e_j) = 0$ for all $x \in \ell_1$ in that situation. This proves the linearity of A_N. Therefore, the Uniform Boundedness Principle gives us $\sup_{N \in \mathbb{N}} ||A_N|| < \infty$.

Now, consider the operator $A : \ell_1 \to \ell_2$ defined by $Ax = (f_j(x))_{j \in \mathbb{N}}$. Notice that $Ax = \lim_{N \to \infty} A_N x$ for all $x \in \ell_1$. Thus,

$$||A|| = \sup_{||x||=1} ||Ax||_2 = \sup_{||x||=1} \lim_{N \to \infty} ||A_N x||_2 \leq \sup_{||x||=1} \sup_{N \in \mathbb{N}} ||A_N x|| = \sup_{N \in \mathbb{N}} ||A_N|| < \infty.$$

Set $C = ||A||_2^2$. Then for all $x \in \ell_1$,

$$\sum_{j=1}^{\infty} |f_j(x)|^2 = ||Ax||_2^2 \leq ||A||_2^2 \, ||x||_1^2 \leq C||x||_1^2,$$

which completes the proof. □

Lemma 3.5.1 shows that normalized unconditional Schauder frames satisfy inequality (Equation 3.59). While this brings us a step closer to the ℓ_1-type of frame inequality, inequality (Equation 3.53) does not immediately yield the upper bound of the frame inequality, given by

$$\sum_{j=1}^{\infty} |f_j(x)| \leq C||x||_1.$$

In other words, there is still a nontrivial gap between Lemma 3.5.1 and finding conditions for which the frame inequality holds.

The next theorem, which contains two parts, shows that under the hypothesis of letting

$f_j(e_k) \in \{-1, 0, 1\}$ for all $j, k \in \mathbb{N}$, this gap can be filled. The first part of this theorem uses Lemma 3.5.1 to show that under this condition and for each $k \in \mathbb{N}$, only finitely many of of the $f_j(e_k)$'s are nonzero indexed with respect to j. The second part of this theorem applies the conclusion of the previous part to prove the frame inequality.

Theorem 3.5.2. *Let $\{(x_j, f_j)\}_{j\in\mathbb{N}} \subseteq \ell_1 \times \ell_\infty$ be a normalized unconditional Schauder frame for ℓ_1 with $f_j(e_k) \in \{-1, 0, 1\}$ for all $j, k \in \mathbb{N}$. Then there exists some $K > 0$ such that*

$$|\{j \in \mathbb{N} : f_j(e_k) \neq 0,\ k \in \mathbb{N}\}| \leq K \tag{3.59}$$

and

$$||x||_1 \leq \sum_{j=1}^{\infty} |f_j(x)| \leq K||x||_1 \tag{3.60}$$

for all $x \in \ell_1$.

Proof. Suppose that $f_j(e_k) \in \{-1, 0, 1\}$ for all $j, k \in \mathbb{N}$. Also, suppose that there does not exist any $K > 0$ such that equation (Equation 3.59) holds.

Then for each constant $C > 0$ there exists some $n_k \in \mathbb{N}$ such that

$$|\{j \in \mathbb{N} : f_j(e_{n_k}) \neq 0\}| > C. \tag{3.61}$$

Since $|f_j(e_{n_k})| = 1$ whenever $f_j(e_{n_k}) \neq 0$, this gives us

$$\begin{aligned}\sum_{j=1}^{\infty} |f_j(e_{n_k})|^2 &= \sum_{j\in\mathbb{N},\, f_j(e_{n_k})\neq 0} |f_j(e_{n_k})|^2 \\ &= |\{j \in \mathbb{N} : f_j(e_{n_k}) \neq 0\}| \\ &> C \\ &= C||e_{n_k}||_1^2.\end{aligned}$$

By Lemma 3.5.1, we conclude that the series $\sum_{j=1}^{\infty} f_j(e_{n_k})x_j$ cannot converge unconditionally. This contradicts our hypothesis of $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ being an unconditional Schauder frame. Therefore, there must exist some $K > 0$ such that

$$|\{j \in \mathbb{N} : f_j(e_k) \neq 0,\ k \in \mathbb{N}\}| \leq K.$$

This proves the first part of the theorem.

Since $|f_j(e_k)| = 1$ whenever $f_j(e_k) \neq 0$, for any $k \in \mathbb{N}$, we have

$$\sum_{j=1}^{\infty} |f_j(e_k)| \leq K.$$

Choose any $x \in \ell_1$. Note that $x = \sum_{i=1}^{\infty} e_i^*(x)e_i$. Proceeding similarly to the calculation in the proof of Lemma 3.2.1, we have

$$\begin{aligned}\sum_{j=1}^{\infty} |f_j(x)| &= \sum_{j=1}^{\infty} \left| \sum_{i=1}^{\infty} e_i^*(f_j(x))e_i \right| \\ &= \sum_{j=1}^{\infty} \left| \sum_{i=1}^{\infty} e_i^*(x) f_j(e_i) \right| \\ &\leq \sum_{i=1}^{\infty} \left(|e_i^*(x)| \sum_{j=1}^{\infty} |f_j(e_i)| \right) \\ &\leq K||x||_1.\end{aligned}$$

Therefore, applying Lemma 3.1.4 proves inequality (Equation 3.60). □

In other words, Theorem 3.5.2 provides a class of Schauder frames for ℓ_1 where the frame inequality holds.

Our next goal is to tie together the discussions in the previous section with Theorem 3.5.2. Recall that for a sequence pair $\{(x_j, f_j)\}_{j \in \mathbb{N}}$ in $\ell_1 \times \ell_\infty$, the operators $T : \ell_1 \to \ell_1$ and $R : \ell_1 \to \ell_1$ are defined by

$$T(x) = (f_j(x))_{j=1}^{\infty}$$

and

$$R((a_j)_{j=1}^{\infty}) = \sum_{j=1}^{\infty} a_j \, x_j.$$

One open question that we left in the previous section asked the necessary and sufficient conditions that need to be imposed for $T : \ell_1 \to \ell_1$ to be an embedding. Referring back to our discussions in the previous section, we claim that necessary and sufficient conditions for $T : \ell_1 \to \ell_1$ to be an embedding under the hypothesis that $f_j(e_k) \in \{-1, 0, 1\}$ for all $j, k \in \mathbb{N}$ as in Theorem 3.5.2 is for $\{(x_j, f_j)\}_{j \in \mathbb{N}}$ to be an *approximate* Schauder frame and $R : \ell_1 \to \ell_1$ to be onto.

Conjecture 3.5.3. *Let $\{(x_j, f_j)\}_{j \in \mathbb{N}}$ be a sequence pair in ℓ_∞ such that $f_j(e_k) \in \{-1, 0, 1\}$ for all $j, k \in \mathbb{N}$. Then the following are equivalent:*

(a) T is an embedding.

(b) $\{(x_j, f_j)\}_{j=1}^{\infty}$ is an approximate Schauder frame.

(c) R is onto.

The main focus of the rest of this chapter is on finding the largest possible class of Schauder frames for which the ℓ_1-type of frame inequality still holds.

3.6 An Improved Frame Inequality Condition

Requiring that $f_j(e_k) \in \{-1, 0, 1\}$ for all $j, k \in \mathbb{N}$ is a restrictive condition that can be relaxed. Observe that the proof of Theorem 3.5.2 used the hypothesis that $f_j(e_k) \in \{-1, 0, 1\}$ twice, once in each part of the theorem.

In the first instance, this was used to prove that only finitely many of the $f_j(e_k)$'s can be nonzero for a fixed $k \in \mathbb{N}$. Recall in the proof of Theorem 3.5.2 that if for infinitely many $j \in \mathbb{N}$, we assume $|f_j(e_k)| = 1$ for all $k \in \mathbb{N}$, then there exists $n_k \in \mathbb{N}$ where

$$\sum_{j=1}^{\infty} |f_j(e_{n_k})|^2 = |\{j \in \mathbb{N} : f_j(e_{n_k}) \neq 0\}| > C$$

for any constant $C > 0$. If we let K be a constant with $0 < K < 1$ such that $|f_j(e_k)| > K$ for all $k \in \mathbb{N}$, then there exists $n_k \in \mathbb{N}$ such that for any given constant $C > 0$,

$$\sum_{j=1}^{\infty} |f_j(e_{n_k})|^2 > K^2 \, |\{j \in \mathbb{N} : f_j(e_{n_k}) \neq 0\}| > CK^2.$$

For this situation, the same method used in the proof of Theorem 3.5.2 yields

$$\sum_{j=1}^{\infty} |f_j(e_{n_k})|^2 > CK^2 ||e_{n_k}||_1^2. \tag{3.62}$$

This change to the hypothesis on the f_j's only alters the lower bound constant in the first part of Theorem 3.5.2, which does not change its conclusion. Requiring that the nonzero $f_j(e_k)$'s satisfy the inequality $K < |f_j(e_k)| \leq 1$ for some constant $0 < K < 1$ still allows us to prove that only finitely many of the $f_j(e_k)$'s can be nonzero for all $k \in \mathbb{N}$.

In the second instance where the hypothesis of Theorem 3.5.2 is used, this hypothesis is combined with the first part of the theorem to obtain the frame inequality. Specifically, recall that $|f_j(e_k)| = 1$ for all $f_j(e_k) \neq 0$ gave us

$$\sum_{j=1}^{\infty} |f_j(x)| \leq \sum_{i=1}^{\infty} \left(|e_i^*(x)| \sum_{j=1}^{\infty} |f_j(e_i)| \right) \leq K||x||_1.$$

For this case, it is not necessary that the nonzero $f_j(e_k)$'s satisfy $|f_j(e_k)| = 1$: Just requiring $|f_j(e_k)| \leq 1$, or having $\{f_j\}_{j \in \mathbb{N}}$ be normalized in ℓ_∞, still gives us the above calculation. Since this is an even more generalized condition than that of the first instance, where the nonzero $|f_j(e_k)|$'s required a nonzero lower bound, applying the bound $K < |f_j(e_k)| \leq 1$ will allow us to arrive at the same conclusion as that of Theorem 3.5.2.

Therefore, we can relax the condition on $\{f_j\}_{j\in\mathbb{N}}$ in Theorem 3.5.2. Instead of requiring that $|f_j(e_k)| = 1$ whenever $f_j(e_k) \neq 0$, it suffices to define a scalar K, with $0 < K < 1$, such that $|f_j(e_k)| \geq K$ whenever $f_j(e_k) \neq 0$. We state the results of this discussion as the following corollary.

Corollary 3.6.1. *Let $\{(x_j, f_j)\}_{j\in\mathbb{N}} \subseteq \ell_1 \times \ell_\infty$ be a normalized Schauder frame for ℓ_1. Assume that there exists a constant K, with $0 < K < 1$, such that $|f_j(e_k)| > K$ whenever $f_j(e_k) \neq 0$. Then inequality (Equation 3.59) holds and thus there exists $C > 0$ such that inequality (Equation 3.3) also holds for all $x \in \ell_1$.*

By the process of elimination, the only normalized Schauder frames $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ in $\ell_1\times\ell_\infty$ that have not been considered are those for which the nonzero entries of $\{|f_j(e_k)|\}_{j\in\mathbb{N}}$ do not have a lower bound for each $k \in \mathbb{N}$. That is, the Schauder frames for which the $f_j(e_k)$'s tend to 0. Therefore, we impose the requirement that $\liminf_{j\to\infty} |f_j(e_k)| = 0$ and prove in the remainder of this section that this type of Schauder frame also satisfies the frame inequality.

Now, notice that another unnecessarily strict initial condition we imposed was requiring that $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ be normalized. This forces $||x_j||_1 = ||f_j||_\infty = 1$ for all $j \in \mathbb{N}$. The most relaxed condition that can possibly be applied is letting $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ to be just bounded. In this situation, there exists $K_1, K_2 > 0$ such that $||x_j||_1 < K_1$ and $||f_j||_\infty < K_2$.

Retracing the proofs of Lemma 3.5.1 and Theorem 3.5.2, boundedness on its own is an insufficient condition. Specifically, in the proof of Lemma 3.5.1, we obtained the inequality

$$\sum_{j=1}^{\infty} |f_j(x)|^2 \leq KR_{\mathcal{E}}^2 \tag{3.63}$$

because $||x_j||_1 = 1$ for all $j \in \mathbb{N}$ and

$$\sum_{j=1}^{\infty} |f_j(x)|^2 = \sum_{j=1}^{\infty} ||f_j(x)x_j||^2 \frac{1}{||x_j||_1^2} = \sum_{j=1}^{\infty} ||f_j(x)x_j||^2 \leq KR_{\mathcal{E}}^2. \tag{3.64}$$

If we let $\lim_{j\to\infty} ||x_j||_1 = 0$, then the left hand side of inequality (Equation 3.63) will tend to infinity. In this case, there will be no upper bound so the ℓ_1-type of frame inequality no longer holds. In fact, the condition that $||x_j||_1 \leq K_1$, for some constant $K_1 > 0$, was not used in inequality (Equation 3.64).

On the other hand, if there exists a constant $C > 0$ for which $||x_j||_1 \geq C$, then inequality (Equation 3.63) can be rewritten as

$$\sum_{j=1}^{\infty} |f_j(x)|^2 \leq (K/C)R_{\mathcal{E}}^2. \tag{3.65}$$

Therefore, we impose the extra condition of having our Schauder frame $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ be *bounded below* by a nonzero constant. We formally define sequences whose norms are bounded above and below by nonzero constants as being *bounded above and below*.

Definition 22. A sequence $\{x_j\}_{j\in\mathbb{N}}$ in a Banach space X is said to be *bounded above and below* if there exist constants $C_1, C_2 > 0$ such that $C_1 \leq ||x_j||_X \leq C_2$ for all $j \in \mathbb{N}$.

Since a Schauder frame for a Banach space X is a sequence pair in $X \times X^*$ instead of a sequence in X itself, we modify the notion of being bounded above and below for a Schauder frame to take into account of it being a sequence pair.

Definition 23. A Schauder frame $\{(x_j, f_j)\}_{j\in\mathbb{N}} \subseteq X \times X^*$ for a Banach space X, with X^* being the dual of X, is said to be *bounded and below* if there exist constants $C_1, C_2, C_3, C_4 > 0$ such that

$$C_1 \leq ||x_j||_1 \leq C_2, \qquad C_3 \leq ||f_j||_\infty \leq C_4 \tag{3.66}$$

for all $j \in \mathbb{N}$.

This allows us to further generalize Lemma 3.5.1.

Lemma 3.6.2. *Let $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ be a bounded above and below sequence pair in $\ell_1 \times \ell_\infty$. If the series $\sum_{j\in\mathbb{N}} f_j(x)x_j$ converges unconditionally for all $x \in \ell_1$, then there exists $C > 0$*

such that

$$\sum_{j=1}^{\infty} |f_j(x)|^2 \leq C||x||_1^2.$$

Proof. Since $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ is a bounded above and below sequence in $\ell_1 \times \ell_\infty$, there exists a constant $C > 0$ such that $||f_j||_\infty > C$ for all $j \in \mathbb{N}$. With this new assumption instead of $||x_j||_1 = ||f_j||_\infty = 1$, the proof of Lemma 3.5.1 remains unchanged, except inequality (Equation 3.55) becomes

$$\sum_{j=1}^{\infty} |f_j(x)|^2 = \sum_{j=1}^{\infty} ||f_j(x)x_j||_1^2 \leq (K/C)R_{\mathcal{E}}^2 < \infty.$$

Here, the constant $K > 0$ comes from the cotype 2 property of ℓ_1, which tells us that

$$\sum_{j=1}^{n} ||f_j(x)x_j||_1^2 \leq K \underset{\varepsilon=\pm 1}{\operatorname{avg}} \left|\left|\sum_{j=1}^{n} \varepsilon_j f_j(x)x_j\right|\right| \tag{3.67}$$

for all $x \in \ell_1$ and every $n \in \mathbb{N}$.

Such a change in upper bound constants does not affect any other parts of the proof of Lemma 3.5.1. □

Finally, we give the furthest possible naive generalization of the properties of $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ such that the ℓ_1-type of frame inequality still holds. The conditions imposed in both Theorem 3.5.2 and Corollary 3.6.1 yielded

$$|\{j \in \mathbb{N} : f_j(e_k) \neq 0\}| < \infty$$

for all $k \in \mathbb{N}$. Thus, there exists $N \in \mathbb{N}$ such that $f_j(e_k) = 0$ for all $j > N$. In other words, $\lim_{j\to\infty} f_j(e_k) = 0$ for all $k \in \mathbb{N}$. This limit holds for all Schauder frames $\{(x_j, f_j)\}_{j\in\mathbb{N}} \subseteq \ell_1 \times \ell_\infty$ that are bounded above and below.

Letting $\lim_{j\to\infty} f_j(e_k) = 0$ and referring to the method of proof in Theorem 3.5.2, we are ready to further generalize Theorem 3.5.2. Expanding upon the conditions of Corollary 3.6.1, the frame inequality still holds if we let the bounded above and below Schauder frame $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ for ℓ_1 satisfy

$$\sup_{k\in\mathbb{N}} \sum_{j=1}^{\infty} |f_j(e_k)| < \infty.$$

Theorem 3.6.3. *Let $\{(x_j, f_j)\}_{j\in\mathbb{N}} \subseteq \ell_1 \times \ell_\infty$ be a bounded above and below unconditional Schauder frame for ℓ_1 such that for each $k \in \mathbb{N}$, we have that $(f_j(e_k))_{j\in\mathbb{N}} \in \ell_1$, and furthermore,*

$$\sup_{k\in\mathbb{N}} \sum_{j=1}^{\infty} |f_j(e_k)| < \infty. \tag{3.68}$$

Then the frame inequality holds. That is, there exist constants $A, B > 0$ such that

$$A||x||_1 \leq \sum_{j=1}^{\infty} |f_j(x)| \leq B||x||_1$$

for all $x \in \ell_1$.

Proof. Let $\{(x_j, f_j)\}_{j\in\mathbb{N}} \subseteq \ell_1 \times \ell_\infty$ be a bounded above and below Schauder frame for ℓ_1. Since the condition (Equation 3.68) holds, there exists a constant $K > 0$ such that

$$K = \sup_{k\in\mathbb{N}} \sum_{j=1}^{\infty} |f_j(e_k)| < \infty.$$

Applying Lemma 3.6.2 in the same manner how we applied Lemma 3.5.1 in the proof of Theorem 3.5.2, we have

$$\begin{aligned}\sum_{j=1}^{\infty} |f_j(x)| &= \sum_{j=1}^{\infty} \left| \sum_{i=1}^{\infty} e_i^* f_j(x) e_i \right| \\ &= \sum_{j=1}^{\infty} \left| \sum_{i=1}^{\infty} e_i^*(x) f_j(e_i) \right|\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{\infty}\left(|e_i^*(x)|\sum_{j=1}^{\infty}|f_j(e_i)|\right)\\
&\leq K\sum_{i=1}^{\infty}|e_i^*(x)|\\
&= K||x||_1.
\end{aligned}
$$

This gives us the upper bound of the frame inequality. Combining this with the lower bound of the frame inequality given in Lemma 3.1.4, we obtain the frame inequality. □

By naively generalizing the methods used in Section 3.4, the hypotheses of Theorem 3.6.3 are the least restrictive conditions that we are able to impose on a Schauder frame that satisfies the ℓ_1-type of frame inequality. However, it is possible this condition can be relaxed further using a different method, which we leave leave as an open question.

Question 3.6.4. *What are the least restrictive conditions for a Schauder frame* $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ *for* ℓ_1 *to satisfy an* ℓ_1*-type of frame inequality?*

In the next section, we lay the foundations of a potentially different method for finding necessary and sufficient conditions for Schauder frames satisfying an ℓ_1-type of inequality. This will be accomplished using Schur's Test.

3.7 Schur's Test and Frame Inequalities

Recall from Chapter 1 the definition of a Gram matrix for frames in Hilbert spaces: If $\{x_n\}_{n\in\mathbb{N}}$ is a frame in a Hilbert space H, its (infinite-dimensional) Gram matrix G is given by

$$G = [\langle x_i, x_j\rangle]_{i,j\in\mathbb{N}}$$

By comparing the reconstruction formula for Schauder frames to that of Hilbert space frames, we are able to generalize the Gram matrix to the Banach space setting. Let X be a

Banach space with dual X^*. Define the Gram matrix G for a Schauder frame $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ in $X \times X^*$ as

$$G = [f_j(x_k)]_{j,k\in\mathbb{N}}\,. \tag{3.69}$$

The hypothesis in Theorem 3.6.3 can be reworded as imposing the condition

$$\sup_{k\in\mathbb{N}} \sum_{j=1}^{\infty} |f_j(e_k)| < \infty. \tag{3.70}$$

on the matrix G defined in equation (Equation 3.69). In other words, whenever a Schauder frame $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ satisfies inequality (Equation 3.70), then the frame inequality holds.

Our goal now is to check whether Theorem 3.6.3 can be sharpened using a different method from that of the previous section. Also, notice here that the given conditions have been redefined as a condition on the infinite Gram matrix of a Schauder frame. One theorem that may help accomplish our goal is *Schur's Test*. While we refer the reader to [46] for a detailed discussion and proof, we formally state Schur's Test below.

Theorem 3.7.1. *Let $A = [a_{ij}]_{i,j\in\mathbb{N}}$ be an infinite matrix such that*

$$C_1 = \sup_{i\in\mathbb{N}} \sum_{j=1}^{\infty} |a_{ij}| < \infty, \qquad C_2 = \sup_{j\in\mathbb{N}} \sum_{i=1}^{\infty} |a_{ij}| < \infty. \tag{3.71}$$

Then the following statements hold for each $1 \le p \le \infty$.

(a) If $x \in \ell_p$, then the series $\sum_{j=1}^{\infty} a_{ij}x_j$ converges for each $i \in \mathbb{N}$, and the vector $y = Ax = (y_i)_{i\in\mathbb{N}}$ belongs to ℓ_p.

(b) A is a bounded linear mapping of ℓ_p into ℓ_p.

(c) The operator norm of $A : \ell_p \to \ell_p$ satisfies

$$||A|| \le C_1^{1/p'} C_2^{1/p}. \tag{3.72}$$

The conditions in Theorem 3.6.3 only provides the latter inequality in (Equation 3.71). While the full results from Schur's Test for $1 \leq p \leq \infty$ cannot be obtained in this situation, these conditions are sufficient for Schur's Test to hold with $p = 1$ and $p = \infty$. The following lemma shows that the right-hand side inequality in (Equation 3.71) yields a partial version of Schur's Test for ℓ_∞ while the left-hand side yields that for ℓ_1.

Lemma 3.7.2. *Let $A = [a_{ij}]_{i,j\in\mathbb{N}}$ be an infinite matrix.*

(a) Suppose that

$$\sup_{j\in\mathbb{N}} \sum_{i=1}^{\infty} |a_{ij}| < \infty. \tag{3.73}$$

Then the conclusions of Schur's Test hold for ℓ_1, with $||A|| \leq C_2$.

(b) Suppose that

$$\sup_{i\in\mathbb{N}} \sum_{j=1}^{\infty} |a_{ij}| < \infty. \tag{3.74}$$

Then the conclusions of Schur's Test hold for ℓ_∞, with $||A|| \leq C_1$.

Proof. (a) Suppose that the infinite matrix $A = [a_{ij}]_{i,j\in\mathbb{N}}$ satisfies

$$C_2 = \sup_{j\in\mathbb{N}} \sum_{i=1}^{\infty} |a_{ij}| < \infty. \tag{3.75}$$

Choose any $x \in \ell_1$. For all $i \in \mathbb{N}$, we have

$$(Ax)(e_i) = \sum_{j=1}^{\infty} a_{ij}x_j. \tag{3.76}$$

First, we need to show that the series $\sum_{j=1}^{\infty} a_{ij}x_j$ converges for each $i \in \mathbb{N}$. Note that

$$\begin{aligned}
\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}x_j| &= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |a_{ij}||x_j| \\
&\leq C_2 \sum_{j=1}^{\infty} |x_j| \\
&\leq C_2 ||x||_1
\end{aligned}$$

and therefore, $\sum_{j=1}^{\infty} a_{ij}x_j$ is absolutely convergent and therefore, converges.

To complete the proof, we compute the following:

$$\begin{aligned}
||Ax||_1 &= \sum_{i=1}^{\infty} |(Ax)(e_i)| \\
&= \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} a_{ij}x_j \right| \\
&\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}x_j| \\
&\leq C_2 ||x||_1.
\end{aligned}$$

The rest of Schur's Test for $p = 1$ can be readily seen from the calculations above. That is, $Ax \in \ell_1$ and $A : \ell_1 \to \ell_1$ is a bounded mapping with $||A|| \leq C_2$.

(b) In this situation, we let

$$C_1 = \sup_{i \in \mathbb{N}} \sum_{j=1}^{\infty} |a_{ij}| < \infty. \tag{3.77}$$

Choose any $z \in \ell_\infty$, and we compute the following.

$$\begin{aligned} ||Az||_\infty &= \sup_{i\in\mathbb{N}} |(Az)(e_i)| \\ &= \sup_{i\in\mathbb{N}} \left| \sum_{j=1}^{\infty} a_{ij} x_j \right| \\ &\le \left(\sup_{i\in\mathbb{N}} \sum_{j=1}^{\infty} |a_{ij}| \right) \left(\sup_{j\in\mathbb{N}} |x_j| \right) \\ &\le C_1 ||x||_\infty. \end{aligned}$$

This proves the conclusions of Schur's Test for $p = \infty$.

□

In the context of Schauder frames for ℓ_1, part (a) of Lemma 3.7.2 can be written as follows.

Corollary 3.7.3. *If $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ is a bounded Schauder frame in $\ell_1 \times \ell_\infty$ such that*

$$\sup_{k\in\mathbb{N}} \sum_{j=1}^{\infty} |f_j(e_k)| < \infty.$$

Then the Gram matrix G associated with $\{(x_j, f_j)\}_{j=1}^{\infty}$ is a bounded linear mapping of ℓ_1 into ℓ_1.

Invoking Schur's Test does not provide information on whether conditions for the frame inequality can be relaxed. The only information it does provide is the boundedness of the Gram matrix of the Schauder frame under the hypotheses of Theorem 3.6.3. We leave as an open question whether the ℓ_1-type of frame inequality can be deduced from the boundedness of the Gram matrix of our given Schauder frame.

Question 3.7.4. *Does the boundedness of the Gram matrix of a normalized unconditional Schauder frame imply an ℓ_1-type of frame inequality?*

3.8 Future Directions

Besides ℓ_1, recall from Theorem 3.1.3 that c_0 is the other Banach space which has a normalized unconditional basis equivalent to the standard basis. Since c_0 is a closed subspace of ℓ_∞, the dual space corresponding to c_0 is ℓ_1. Therefore, if $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ is a Schauder frame for c_0, then $\{f_j\}_{j\in\mathbb{N}} \subseteq \ell_1$. Additionally, in the c_0 setting, the frame inequality is an ℓ_∞-type frame inequality. That is, if a Schauder frame $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ for c_0 were to satisfy the frame inequality, then there exist scalars $A, B > 0$ such that

$$A||x||_\infty \leq \sup_{j\in\mathbb{N}} |f_j(x)| \leq B||x||_\infty \tag{3.78}$$

for all $x \in c_0$.

Since ℓ_∞ is *not* of cotype p for any $2 \leq p < \infty$, Lemma 3.5.1 no longer holds. In this situation, proving the conditions, if any, for the frame inequality to hold for c_0 requires using a different method from that used in Section 3.4. Hence, we leave as an open question the conditions under which Schauder frames for c_0 satisfy the ℓ_∞-type of frame inequality.

Question 3.8.1. *If $\{(x_j, f_j)\}_{j\in\mathbb{N}}$ is a Schauder frame for c_0, then under what conditions does the ℓ_∞-type frame inequality hold?*

When working with FUNTFs in Chapter 2, difficulties arose in our attempts to generalize results for ℓ_1^n to other finite-dimensional Banach spaces, which includes ℓ_p^n for $1 < p < \infty$. Even though we are working with normalized Schauder frames instead of FUNTFs, and dealing with infinite instead of finite dimensions, a naive generalization of our results in this chapter from ℓ_1 to ℓ_p for $1 < p < \infty$ is still not to be expected. This observation is further emphasized by the fact that Theorem 3.1.3 does not apply to ℓ_p for $1 < p < \infty$.

Since the frame inequality for Hilbert spaces was of ℓ_2-type and that for ℓ_1 was of ℓ_1-type, we propose that the frame inequality for ℓ_p, if it were to exist, should be of ℓ_p-type.

Observe that since the dual of ℓ_p is $\ell_{p'}$, where $1/p + 1/p' = 1$, a Schauder frame for ℓ_p is of the form $\{(x_j, f_j)\}_{j\in\mathbb{N}} \subseteq \ell_p \times \ell_{p'}$.

We leave as an open question the conditions under which the the ℓ_p-type of frame inequality holds for Schauder frames in $\ell_p \times \ell_{p'}$ for ℓ_p.

Question 3.8.2. *Let $\{(x_j, f_j)\}_{j\in\mathbb{N}} \subseteq \ell_p \times \ell_{p'}$ be a Schauder frame for ℓ_p, where $\ell_{p'}$ is the dual of ℓ_p and so $1/p + 1/p' = 1$. Under what conditions does the ℓ_p-type of frame inequality hold? In other words, under what conditions do there exist constants $A, B > 0$ such that*

$$A||x||^p \leq \sum_{j=1}^{\infty} |f_j(x)|^p \leq B||x||^p \tag{3.79}$$

for all $x \in \ell_p$?

CHAPTER 4

INTRODUCTION TO ℓ_1-BOUNDEDNESS

This chapter is joint work with Dr. Christopher Heil.

4.1 Preliminaries and Definitions

In previous chapters, the primary focus was on frames in the Banach space setting. Specifically, those results and discussions were largely focused on frames for ℓ_1. Recall that in the Hilbert space setting, frames are a special case of Bessel sequences. One topic that can bridge these two statements is the formal definition and properties of Bessel sequences in the Banach space setting. Examples of work done in this direction include [14] and [47].

Alternatively, we can continue to remain in Hilbert spaces and instead of the inequality $\sum_{j\in\mathbb{N}} |\langle x, x_j\rangle|^2 < \infty$ for all $x \in H$, consider sequences $\{x_j\}_{j\in\mathbb{N}} \subseteq H$ which satisfy

$$\sum_{j=1}^{\infty} |\langle x, x_j\rangle| < \infty \tag{4.1}$$

for all $x \in H$. While one option is to formally characterize such sequences themselves, we will determine the properties of $x \in H$ for which inequality (Equation 4.1) holds given a sequence $\{x_j\}_{j\in\mathbb{N}}$ in H.

To begin, consider the most restrictive type of Bessel sequence in Hilbert spaces, namely, Riesz bases. Let $\mathcal{E} = \{x_j\}_{j\in\mathbb{N}}$ be an arbitrary Riesz basis in a Hilbert space H. An element $x \in H$ for which inequality (Equation 4.1) holds is said to be ℓ_1-*bounded* with respect $\mathcal{E}$. Before defining ℓ_1-boundedness in detail, we define the ℓ_1-*norm* with respect to $\mathcal{E}$.

Definition 24. Let $\mathcal{E} = \{x_n\}_{n\in\mathbb{N}}$ be a Riesz basis in a Hilbert space H. For each $x \in H$, the ℓ_1*-norm of* x *with respect to* $\mathcal{E}$ is

$$||x||_{1,\mathcal{E}} = \sum_{j=1}^{\infty} |\langle x, x_j \rangle| \,. \tag{4.2}$$

Now, we can define what it means for a subset H to be ℓ_1-bounded with respect to a Riesz basis $\mathcal{E}$.

Definition 25. A subset S of a Hilbert space H is said to be ℓ_1*-bounded* if there exists a Riesz basis $\mathcal{E} = \{x_n\}_{n\in\mathbb{N}}$ for H such that

$$\sup_{x\in S} ||x||_{1,\mathcal{E}} < \infty. \tag{4.3}$$

Since Riesz bases are a special case of frames, we can generalize the concept of the ℓ_1-norm to the setting of frames instead of just Riesz bases.

Definition 26. Let $\mathcal{F} = \{f_j\}_{j\in\mathbb{N}}$ be a frame for a Hilbert space H. For all $x \in H$, we define the ℓ_1*-norm of* x *with respect to* $\mathcal{F}$ by

$$||x||_{1,\mathcal{F}} = \sum_{n=1}^{\infty} |\langle x, f_n \rangle| \,. \tag{4.4}$$

Similarly, the concept of ℓ_1-boundedness of a set with respect to a Riesz basis can also be defined for frames. We will refer to this as ℓ_1*-frame boundedness*.

Definition 27. A subset S of a Hilbert space H is said to be ℓ_1*-frame bounded* if there exists a frame $\{f_j\}_{j\in\mathbb{N}}$ for H such that

$$\sup_{x\in S} ||x||_F = \sup_{x\in S} \sum_{n=1}^{\infty} |\langle x, f_j \rangle| < \infty.$$

As far as we are aware, the concept of ℓ_1-boundedness has not been studied in detail in the literature. In this chapter, we will attempt to determine the basic properties of ℓ_1-boundedness as the basis for a future buildup of a complete theoretical development of this subject. In particular, we will give a heavy emphasis to formulating open questions on this topic.

4.2 Examples of ℓ_1-boundedness

Before discussing and proving properties related to ℓ_1-boundedness, we give examples of sets that are ℓ_1-bounded and those that are not. The following example demonstrates a set that is not ℓ_1-bounded with respect to a given Riesz basis.

Example. Consider a singleton set $M = \{x\}$ in the Hilbert space ℓ_2, where

$$x = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}. \tag{4.5}$$

Let $\mathcal{E} = \{e_j\}_{j\in\mathbb{N}}$ be the standard basis for ℓ_2. Then $\mathcal{E}$ is a Riesz basis with respect to which M is not ℓ_1-bounded, since

$$||x||_{1,\mathcal{E}} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty. \tag{4.6}$$

However, this does *not* necessarily imply that the set M is not ℓ_1-bounded. In order for a set S to be ℓ_1-bounded, there only needs to be *at least one* Riesz basis $\mathcal{E}$ with respect to which $\sup_{x\in S} ||x||_{1,\mathcal{E}} < \infty$.

To demonstrate a set that is *not* ℓ_1-bounded, we need to show that it cannot be ℓ_1-bounded with respect to *any* Riesz basis for the given Hilbert space. One example of this is the closed unit disk for an infinite-dimensional Hilbert space.

Example. Let H be an arbitrary infinite-dimensional Hilbert space and let D be the closed unit disk in H. Explicitly,

$$D = \{x \in H : ||x|| \leq 1\}. \tag{4.7}$$

We will show that D is *not* an ℓ_1-bounded subset of H.

Let $\mathcal{E} = \{x_j\}_{j\in\mathbb{N}}$ be any Riesz basis in H and let $\{\tilde{x}_j\}_{j\in\mathbb{N}}$ be its corresponding biorthogonal system. Since $\{\tilde{x}_j\}_{j\in\mathbb{N}}$ is a complete Bessel sequence (see, e.g., [5]), there exists a constant $B > 0$ such that

$$\left|\left|\sum_{j=1}^{\infty} c_j \tilde{x}_j\right|\right|^2 \leq B \sum_{j=1}^{\infty} |c_j|^2 \tag{4.8}$$

for all sequences $\{c_j\}_{j\in\mathbb{N}} \in \ell_2$.

Now, set

$$x = \sum_{j=1}^{\infty} \frac{1}{j}\, \tilde{x}_j. \tag{4.9}$$

Clearly $\{1/j\}_{j\in\mathbb{N}} \in \ell_2$ and in fact,

$$\sum_{j=1}^{\infty} \left|\frac{1}{j}\right|^2 = \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}. \tag{4.10}$$

Therefore, we have

$$||x|| = \left|\left|\sum_{j=1}^{\infty} \frac{1}{j}\, \tilde{x}_j\right|\right| \leq \sqrt{\frac{B}{6}}\, \pi. \tag{4.11}$$

Set $C = ||x||$ and let $y = (1/C)x$. Note that $||y|| \leq 1$ and therefore, $y \in D$. This gives us

$$\sum_{j=1}^{\infty} |\langle y, x_j\rangle| = \sum_{j=1}^{\infty} \left|\left\langle \sum_{k=1}^{\infty} \frac{1}{k}\, \tilde{x}_k, x_j \right\rangle\right| = \sum_{j=1}^{\infty} \frac{1}{j} = \infty. \tag{4.12}$$

This shows that $||y||_{1,\mathcal{E}} = \infty$ with respect to the Riesz basis $\mathcal{E}$. Since $\mathcal{E}$ was an arbitrary Riesz basis, this shows that the closed unit disk D is not an ℓ_1-bounded set in H.

Now that examples of sets that are not ℓ_1-bounded have been provided, we consider sets that are ℓ_1-bounded. One such type of ℓ_1-bounded set is an appropriate subset of ℓ_1, viewed as a subspace of the Hilbert space ℓ_2. The following example gives a detailed description.

Example. Consider the Hilbert space ℓ_2 and let $\mathcal{E} = \{e_j\}_{j\in\mathbb{N}}$ be the standard basis with respect to ℓ_2. If $x \in \ell_2$ also satisfies $x \in \ell_1$, then

$$||x||_{1,\mathcal{E}} = \sum_{j=1}^{\infty} |\langle x, e_j\rangle| = \sum_{j=1}^{\infty} |x_j| = ||x||_1. \tag{4.13}$$

Recall from the definition of ℓ_1-boundedness with respect to $\mathcal{E}$ that if a set $M \subseteq \ell_2$ is to be ℓ_1-bounded, then $\sup_{x\in M} ||x||_{1,\mathcal{E}} < \infty$. To provide an example of an ℓ_1-bounded set, we fix an arbitrary scalar $C > 0$ and define the set

$$M = \{x \in \ell_2 \,:\, ||x||_1 < C\}. \tag{4.14}$$

In this case, if we take $\mathcal{E} = \{e_j\}_{j\in\mathbb{N}}$, then

$$\sup_{x\in M} ||x||_{1,\mathcal{E}} = \sup_{x\in M} ||x||_{1,\mathcal{E}} = \sup_{x\in M} ||x||_1 < C. \tag{4.15}$$

This shows that M is ℓ_1-bounded.

An ℓ_1-bounded set need not be ℓ_1-bounded with respect to every Riesz basis. We demonstrate this by showing that there does exist a Riesz basis with respect to which the set considered in Example section 4.2 is ℓ_1-bounded, which shows that that the set in Example section 4.2 is an ℓ_1-bounded set.

Example. Returning to Example section 4.2, we consider the same singleton set $M = \{x\}$ with $x = (1/n)_{n=1}^{\infty}$ in the Hilbert space ℓ_2 but consider a different Riesz basis $\mathcal{F} = \{f_j\}_{j\in\mathbb{N}}$ for ℓ_2. Let $\mathcal{F}$ be an orthogonal basis for ℓ_2 such that $f_1 = x$, and then choosing an orthonormal basis $\{f_j\}_{j\geq 2}$ for $\{x\}^{\perp}$. By the orthogonality of $\mathcal{F}$, we have $x \perp f_n$ for all $n \geq 2$. This gives us

$$||x||_{1,\mathcal{F}} = \sum_{j=1}^{\infty} |\langle x, f_j\rangle| = |\langle x, f_1\rangle| = |\langle x, x\rangle| = \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}. \tag{4.16}$$

Therefore, even though M is not ℓ_1-bounded with respect to the standard basis $\mathcal{E} = \{e_j\}_{j\in\mathbb{N}}$, it is still an ℓ_1-bounded set since it is ℓ_1-bounded with respect to our orthogonal basis $\mathcal{F} = \{f_j\}_{j\in\mathbb{N}}$.

Even though we have exhibited examples of sets that are ℓ_1-bounded and those that are not, we have not established any patterns which allow us to characterize ℓ_1-bounded sets with respect to a given Riesz basis. In order to gain more insight on this topic, we return to the example of the closed unit disk in a Hilbert space, which is not ℓ_1-bounded.

We start by considering subsets of the closed unit disk D that are not closed. One example of a subset of D that is not closed is $D \cap c_{00}$. Here, the sequence space c_{00} consists of all sequences which only have finitely many nonzero elements. Therefore, for any $x \in c_{00}$, there exists $N \in \mathbb{N}$ such that $x(e_k) = 0$ for all $k \geq N$. In addition, c_{00} is *not* closed in ℓ_1 since it is a dense but proper subspace of ℓ_1.

We leave as an open question whether the set $D \cap c_{00}$ is ℓ_1-bounded. That is, we ask whether there exists a Riesz basis $\mathcal{E}$ for ℓ_2 such that $\sup\limits_{x \in D \cap c_{00}} ||x||_{1,\mathcal{E}} < \infty$.

Question 4.2.1. *Let D be the closed unit disk in ℓ_2. Is the set $D \cap c_{00}$ ℓ_1-bounded?*

Additional foundations for an eventual characterization of ℓ_1-bounded sets can be established by discussing the properties that ℓ_1-bounded sets do, or may, satisfy. This will be our goal for the rest of this chapter.

4.3 Boundedness and ℓ_1-boundedness

As the first topic on our discussion of the properties of ℓ_1-bounded sets, we explore the connection between boundedness and ℓ_1-boundedness. Here, *boundedness* without qualification means boundedness in the usual sense in a Hilbert space. Specifically, a set S in a Hilbert space H is bounded if $\sup\limits_{x \in S} ||x||_H < \infty$.

The proposition below shows that all finite sets are ℓ_1-bounded.

Proposition 4.3.1. All finite subsets of a Hilbert space are ℓ_1-bounded.

Proof. Without loss of generality, let H be an infinite-dimensional separable Hilbert space and let S be a finite subset with N elements. Then there exists a subspace $M \subseteq H$ of dimension $n \leq N$ such that $S \subseteq M$.

Let $\{a_j\}_{j=1}^n$ be an orthonormal basis for M and consider its orthogonal complement, $M^\perp$. Since M is finite dimensional and H is infinite dimensional, $M^\perp$ is infinite dimensional. In this case, let $\{b_j\}_{j \in \mathbb{N}}$ be an orthonormal basis for $M^\perp$. Since $S = M \oplus M^\perp$, the sequence $\{x_k\}_{k=1}^{\infty} = \{a_k\}_{k=1}^{n} \cup \{b_j\}_{j=1}^{\infty}$ is an orthonormal basis for H, and hence is a Riesz basis for H.

Choose any $x \in S$. Applying the Cauchy-Bunyakovsky-Schwarz Inequality, we obtain

$$\begin{aligned}\sum_{k=1}^{\infty} |\langle x, x_k\rangle| &= \sum_{k=1}^{n} |\langle x, a_k\rangle| + \sum_{j=1}^{\infty} |\langle x, b_j\rangle| \\ &= \sum_{k=1}^{n} |\langle x, a_k\rangle| \\ &\leq n^{1/2} \left(\sum_{k=1}^{n} |\langle x, a_k\rangle|^2\right)^{1/2} \\ &= n^{1/2}||x|| \\ &\leq N^{1/2}||x||.\end{aligned}$$

Since S is bounded, $C = \sup\limits_{x\in S} ||x|| < \infty$. Therefore

$$\sup_{x\in S} \sum_{n=1}^{\infty} |\langle x, x_n\rangle| \leq CN^{1/2} < \infty.$$

This shows that S is ℓ_1-bounded. □

Observe that the bound given in Proposition 4.3.1 grows with the number of points in the set S.

The next proposition shows that all ℓ_1-bounded sets are bounded with respect to the norm of the given Hilbert space H.

Proposition 4.3.2. All ℓ_1-bounded sets in a Hilbert space H are bounded.

Proof. Let S be an ℓ_1-bounded set in a Hilbert space H and let $\mathcal{E} = \{x_n\}_{n\in\mathbb{N}}$ be the corresponding Riesz basis for which equation (Equation 4.3) holds. In this case, $C = \sup\limits_{x\in S} ||x||_{1,\mathcal{E}} < \infty$.

Choose any $x \in S$. Since Riesz bases are frames, there exist frame bounds $A, B > 0$

such that for every $x \in H$,

$$A||x||^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \leq B||x||^2$$

Using the fact that the ℓ_2-norm is dominated by the ℓ_1-norm, we therefore obtain

$$||x|| \leq \left(\frac{1}{A} \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \right)^{1/2} \leq \frac{1}{A^{1/2}} \sum_{n=1}^{\infty} |\langle x, x_n \rangle| \leq A^{-1/2} C < \infty.$$

Since $x \in S$ was arbitrary, this proves that S is a bounded set in H. □

Even though we have established that ℓ_1-boundedness implies boundedness, the converse does not hold. The closed unit disk D in Example Equation 4.2 is an example of a bounded set in a Hilbert space H that is not ℓ_1-bounded.

4.4 Meagerness and ℓ_1-boundedness

One well-known result in functional analysis is that ℓ_1 is a meager subset of ℓ_2. In other words, ℓ_1 is a countable union of nowhere dense subsets of ℓ_2 (see, e.g., [5]). The concept of meagerness can be connected with that of ℓ_1-boundedness. By modifying the proof of the meagerness of ℓ_1, we can show that ℓ_1-bounded sets of a Hilbert space H are meager subsets of H.

First, we need the following lemma.

Lemma 4.4.1. *Let H be a Hilbert space and $\mathcal{E} = \{x_n\}_{n \in \mathbb{N}}$ be a Riesz basis in H. Then*

$$S = \{x \in H : ||x||_{1,\mathcal{E}} < \infty\}$$

is a meager subset of H.

Proof. Fix $N \in \mathbb{N}$ and define the set

$$S_N = \{x \in H : ||x||_{1,\mathcal{E}} \leq N\}\,. \tag{4.17}$$

Observe that $S = \cup_{N\in\mathbb{N}} S_N$, and therefore it suffices to show that S_N is a nowhere dense subset of H for each $N \in \mathbb{N}$. In other words, we prove that the closure of S_N has empty interior for all $N \in \mathbb{N}$.

If $x \in H$ is in the closure of S_N, then there exists a sequence $\{y_k\}_{k\in\mathbb{N}} \subseteq S_N$ such that $\lim_{k\to\infty} ||x - y_k|| = 0$. From the continuity of the norm, this implies that for all $z \in H$,

$$\lim_{k\to\infty} \langle y_k, z\rangle = \langle x, z\rangle.$$

Therefore, by applying Fatou's Lemma for series, we have

$$\begin{aligned}
||x||_{1,\mathcal{E}} &= \sum_{n=1}^{\infty} |\langle x, x_n\rangle| \\
&= \sum_{n=1}^{\infty} \lim_{k\to\infty} |\langle y_k, x_n\rangle| \\
&\leq \liminf_{k\to\infty} \sum_{n=1}^{\infty} |\langle y_k, x_n\rangle| \\
&= \liminf_{k\to\infty} ||y_k||_{1,\mathcal{E}} \\
&\leq N.
\end{aligned}$$

This shows that $x \in S_N$. We conclude that the closure of S_N is itself, and therefore S_N is a closed subset of H for each $N \in \mathbb{N}$.

To complete the proof that S_N is nowhere dense, we need to show that it has empty interior for all $N \in \mathbb{N}$. In order to do so, we prove that S is a *proper* subspace of H. That S is a subspace of H can be clearly seen so it suffices to prove that S is proper.

Let $\{\tilde{x}_n\}_{n\in\mathbb{N}}$ be the biorthogonal sequence corresponding to $\mathcal{E}$. Notice that $\{\tilde{x}_n\}_{n\in\mathbb{N}}$ is still a Riesz basis for our Hilbert space H.

Therefore, if $\sum_{n=1}^{\infty} |c_n|^2 < \infty$, then the series $\sum_{n=1}^{\infty} c_n \tilde{x}_n$ is convergent in H (Theorem 7.13 in [5]). In particular,

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \right|^2 = \frac{\pi^2}{6} < \infty, \tag{4.18}$$

so the series $x = \sum_{n=1}^{\infty} (1/n)\tilde{x}_n$ converges in H. However,

$$||x||_{1,\mathcal{E}} = \sum_{n=1}^{\infty} |\langle x, x_n \rangle| = \sum_{n=1}^{\infty} \left| \left\langle \sum_{j=1}^{\infty} \frac{1}{j}\, \tilde{x}_j, x_n \right\rangle \right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty, \tag{4.19}$$

so x does not belong to S. Therefore, S is a proper subspace of H.

Since $S_N \subseteq S$, it is also a proper subspace of H. Consequently, $S_N^{\circ} = \emptyset$ since every subset of a proper subspace of a normed space has empty interior.

This shows that S_N is closed and has empty interior for all $N \in \mathbb{N}$. Thus, $S = \cup_{N \in \mathbb{N}} S_N$ is the union of countably many nowhere dense sets, so S is a meager subset of H. □

Applying Lemma 4.4.1, we obtain the meagerness of ℓ_1-bounded sets.

Theorem 4.4.2. *All ℓ_1-bounded subsets of a Hilbert space are meager.*

Proof. Let S be an ℓ_1-bounded subset of a Hilbert space H. Then there exists a Riesz basis $\mathcal{E}$ in H such that

$$C = \sup_{x \in S} ||x||_{1,\mathcal{E}} < \infty.$$

Notice that $S \subseteq M$, where $M = \{x \in H : ||x||_{1,\mathcal{E}} < \infty\}$.

By Lemma 4.4.1, M is meager, and therefore S is a meager subset of H. □

4.5 Compactness and ℓ_1-boundedness

We have only described some of the properties that ℓ_1-bounded sets satisfy, but have not yet given properties that they do *not* satisfy. We show in this section that ℓ_1-bounded sets need not be compact. Specifically, we show that unit-length segments in ℓ_2 are non-compact ℓ_1-bounded sets.

Example. Consider the set S that consists of the union of unit vector line segments in ℓ_2 that lie on the coordinate axes. Specifically, define S by

$$S = \bigcup_{k\in\mathbb{N}} \{c_k e_k : 0 \le c_k \le 1\} = \{x \in \ell_2 : x = c_k e_k,\ 0 \le c_k \le 1,\ k \in \mathbb{N}\}, \tag{4.20}$$

where e_k is the kth element of the standard basis for ℓ_2 with $k \in \mathbb{N}$. Consider the sequence $\mathcal{E} = \{e_j\}_{j\in\mathbb{N}}$, the standard basis in ℓ_2, which is a Riesz basis by definition. If $x \in S$, then $x = c_k e_k$ for some $k \in \mathbb{N}$ and some $c_k \in [0, 1]$. We compute

$$\sup_{x\in S} ||x||_{1,\mathcal{E}} = \sup_{x\in S} \sum_{j=1}^{\infty} |\langle x, e_j\rangle| = \sup_{0\le c_k\le 1} \sum_{j=1}^{\infty} |\langle c_k e_k, e_j\rangle| = \sup_{0\le c_k\le 1} |c_k| \le 1.$$

This shows that S is ℓ_1-bounded.

Since H is a Hilbert space, it is also a metric space, and thus sequential compactness is equivalent to compactness. The sequence $\{e_k\}_{k\in\mathbb{N}}$ does not contain convergent subsequences, and thus is not sequentially compact. Hence, we conclude that S is not compact. In fact, a similar argument shows that the countable set $\{e_k\}_{k\in\mathbb{N}}$ is ℓ_1-bounded but not compact.

We leave as an open question the sufficient conditions for ℓ_1-bounded sets to be compact.

Question 4.5.1. *What extra conditions should be imposed on ℓ_1-bounded sets to ensure their compactness?*

In order to have a better understanding of the relationship between compactness and ℓ_1-boundedness, we provide an example of a compact set that is also ℓ_1-bounded. This can be done by considering an appropriate version of the *Hilbert cube*, viewed as a subset of ℓ_2. Before we start, we define the *standard* Hilbert cube.

Let the standard Hilbert cube H be the product of intervals

$$\prod_{n=1}^{\infty}\left[0,\frac{1}{n}\right]=[0,1]\times\left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{3}\right]\times\cdots. \tag{4.21}$$

By Tychonoff's Theorem, the standard Hilbert cube H is compact with respect to the product topology. In addition, the standard Hilbert cube H is compact with respect to the ℓ_2-norm (see, e.g., [46]).

An element $x\in H$ is the sequence $x=(x_n)_{n=1}^{\infty}$, where $0\le x_n\le 1/n$ for all $n\in\mathbb{N}$. Observe that $H\subseteq\ell_2$ since for $x=(x_n)_{n=1}^{\infty}\in H$, we have

$$\sum_{n=1}^{\infty}|x_n|^2\le\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty.$$

Therefore, H is a bounded subset of ℓ_2.

Now, we will show that the standard Hilbert cube H is not ℓ_1-bounded respect to the standard basis $\mathcal{E}=\{e_n\}_{n\in\mathbb{N}}$ for ℓ_2.

Let $x=(1/n)_{n=1}^{\infty}\in H$. Then we compute

$$\sum_{n=1}^{\infty}|\langle x,e_n\rangle|=\sum_{n=1}^{\infty}\frac{1}{n}=\infty.$$

This shows that the Hilbert cube is not ℓ_1-bounded with respect to the standard basis for ℓ_2.

However, the definition of ℓ_1-boundendess states that a set S is ℓ_1-bounded provided that $\sup_{x\in S}||x||_{1,\mathcal{E}}<\infty$ for *some* Riesz basis $\mathcal{E}$. Hence, we *cannot* conclude that the Hilbert cube is *not an ℓ_1-bounded set* even if it is not ℓ_1-bounded with respect to the standard basis for ℓ_2. We leave as an open question whether the standard Hilbert cube can be ℓ_1-bounded.

Question 4.5.2. *Is the standard Hilbert cube ℓ_1-bounded? That is, does there exist a Riesz basis $\mathcal{E}$ for ℓ_2 such that the Hilbert cube H satisfies*

$$\sup_{x\in H}||x||_{1,\mathcal{E}}<\infty? \tag{4.22}$$

Next, we define the set K by modifying the Hilbert cube appropriately as follows:

$$K = \prod_{n=1}^{\infty} \left[0, \frac{1}{2^{n-1}}\right] = [0,1] \times \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{4}\right] \times \left[0, \frac{1}{8}\right] \cdots . \tag{4.23}$$

Similar to the standard Hilbert cube H, this set K is compact with respect to both the product topology (via Tychonoff's Theorem) and the ℓ_2-norm (also see [46]).

The set K can be viewed as a subset of ℓ_2 because each element $y \in K$ can be written as $y = (y_n)_{n=1}^{\infty}$, with $0 \leq y_n \leq 1/2^{n-1}$ and $n \in \mathbb{N}$. We show that the modified Hilbert cube K is compact with respect to the ℓ_2-norm and ℓ_1-bounded.

Theorem 4.5.3. *Let K be the modified Hilbert cube defined in equation (Equation 4.23). Then K is compact and ℓ_1-bounded in ℓ_2.*

Proof. Since we already know that K is compact with respect to the ℓ_2-norm, it suffices to show that K is an ℓ_1-bounded set in ℓ_2. Choose any $y = (y_n)_{n=1}^{\infty} \in K$ and let $\mathcal{E} = \{e_n\}_{n \in \mathbb{N}}$ be the standard basis for ℓ_2. In this case, notice that $|y_n| \leq 1/2^{n-1}$ for all $n \in \mathbb{N}$. This gives us

$$\sum_{n=1}^{\infty} |\langle y, e_n \rangle| = \sum_{n=1}^{\infty} |\langle y_n, e_n \rangle| \leq \sum_{n=1}^{\infty} \left| \frac{1}{2^{n-1}} \right| = 1 + \sum_{n=1}^{\infty} \frac{1}{2^n} = 2.$$

Therefore,

$$\sup_{y \in K} \sum_{n=1}^{\infty} |\langle y, e_n \rangle| = 2 < \infty. \tag{4.24}$$

This shows that the modified Hilbert cube K is also ℓ_1-bounded in addition to being compact. □

In-depth discussions on Hilbert cubes, especially from a topological viewpoint, can be found in texts such as [48].

We leave as open questions whether we can find a Riesz basis in ℓ_2 to ensure the ℓ_1-boundedness of the Hilbert cube, and as to what further properties compact sets are required to have in order to ensure ℓ_1-boundedness with respect to a given Riesz basis.

Question 4.5.4. *Let H be a Hilbert space and $\mathcal{E}$ be a Riesz basis for H. If $K \subseteq H$ is compact, under what conditions is it ℓ_1-bounded with respect to $\mathcal{E}$?*

We also leave as an open question whether a compact subset can be ℓ_1-bounded.

Question 4.5.5. *Let K be a compact subset of a Hilbert space H and let $M \subseteq K$ be bounded. Is M ℓ_1-bounded in H?*

4.6 Further Properties of ℓ_1-bounded Sets

Given a Hilbert space and a topological property associated with an element of the space, a *topological isomorphism* preserves those properties after mapping into a different Hilbert space. We prove below that ℓ_1-boundedness is preserved by topological isomorphisms, a definition of which can be found in [5].

Proposition 4.6.1. Let H and K be Hilbert spaces. If $S \subseteq H$ is ℓ_1-bounded and $A : H \to K$ is a topological isomorphism, then $A(S)$ is ℓ_1-bounded in K.

Proof. Let S be a ℓ_1-bounded set in the Hilbert space H. Since S is ℓ_1-bounded in H, there exists a Riesz basis $\{x_j\}_{j\in\mathbb{N}}$ of H such that

$$\sup_{x\in S} \sum_{j=1}^{\infty} |\langle x, x_j\rangle| < \infty.$$

Now, let $A : H \to K$ be a topological isomorphism. Then for each $x \in S$, we have

$$\langle x, x_j\rangle = \langle A^{-1}Ax, x_j\rangle = \langle Ax, (A^{-1})^* x_j\rangle. \tag{4.25}$$

Therefore,

$$\sum_{j=1}^{\infty} \left|\langle Ax, (A^{-1})^* x_j\rangle\right| = \sum_{j=1}^{\infty} |\langle x, x_j\rangle| \,. \tag{4.26}$$

We conclude that

$$\sup_{Ax \in A(S)} \sum_{j=1}^{\infty} \left|\langle Ax, (A^{-1})^* x_j \rangle\right| = \sup_{x \in S} \sum_{j=1}^{\infty} |\langle x, x_j \rangle| < \infty. \tag{4.27}$$

In addition, topological isomorphisms preserve Riesz bases, so $\{(A^{-1})^* x_j\}_{j\in\mathbb{N}}$ is a Riesz basis for K. This shows that $A(S)$ is ℓ_1-bounded in K. □

Even though topological isomorphisms preserve ℓ_1-boundedness, they are a highly restrictive class of operators. Therefore, one direction where further results can be obtained is finding the least restrictive class of operators that preserve ℓ_1-boundedness. In particular, we leave as an open question whether compact operators preserve ℓ_1-boundedness.

Question 4.6.2. *Let H and K be Hilbert spaces. If $S \subseteq H$ is ℓ_1-bounded and $A : H \to K$ is compact, is $A(S)$ ℓ_1-bounded in K?*

Since compact operators are bounded, a natural follow-up question would be to determine whether bounded operators preserve ℓ_1-boundedness.

Question 4.6.3. *Let H and K be Hilbert spaces. If $S \subseteq H$ is ℓ_1-bounded and $A : H \to K$ is bounded, is $A(S)$ ℓ_1-bounded in K?*

Another topic on ℓ_1-boundedness that can be further explored is whether the union of ℓ_1-bounded sets is still ℓ_1-bounded. Even though this appears to be a straightforward question on the surface, serious difficulties arise because two Riesz bases are involved, which makes the analysis quite delicate.

To illustrate this, let X and Y be ℓ_1-bounded sets in a Hilbert space H with respect to the Riesz bases $\mathcal{E}_1 = \{x_j\}_{j\in\mathbb{N}}$ and $\mathcal{E}_2 = \{y_j\}_{j\in\mathbb{N}}$, respectively. Our goal is to show whether $Z = X \cup Y$ is still ℓ_1-bounded. That is, we want to show whether there exists a Riesz basis $\mathcal{E} = \{z_j\}_{j\in\mathbb{N}}$ such that

$$\sup_{z \in X \cup Y} \sum_{j=1}^{\infty} |\langle z, z_j \rangle| < \infty. \tag{4.28}$$

In other words, showing whether $Z = X \cup Y$ is ℓ_1-bounded is equivalent to showing whether there exists a Riesz basis $\mathcal{E} = \{z_j\}_{j \in \mathbb{N}}$ such that

$$\sup_{x \in X} \sum_{j=1}^{\infty} |\langle x, z_j \rangle| < \infty \tag{4.29}$$

and

$$\sup_{y \in Y} \sum_{j=1}^{\infty} |\langle y, z_j \rangle| < \infty. \tag{4.30}$$

We leave as an open question whether the union of ℓ_1-bounded sets are ℓ_1-bounded.

Question 4.6.4. *Is the union of ℓ_1-bounded sets also ℓ_1-bounded?*

4.7 Connections between Riesz Bases and Frames

In our discussions so far, we have not made explicit use of the properties of Riesz bases which are not applicable to frames. Consequently, *all results in the previous sections on ℓ_1-boundedness apply for ℓ_1-frame boundedness.*

Since Riesz bases of a Hilbert space are also frames, this suggests a connection between ℓ_1-bounded sets and ℓ_1-frame bounded sets. We prove below that ℓ_1-bounded sets in a given Hilbert space can be ℓ_1-frame bounded in a larger Hilbert space for which the given space is a subspace of.

Proposition 4.7.1. Let M be a ℓ_1-frame bounded set in a Hilbert space H. Then there exists a Hilbert space K that contains H such that M is ℓ_1-bounded in K.

Proof. Since M is ℓ_1-frame bounded in H, there exists a frame $\mathcal{F} = \{f_j\}_{j \in \mathbb{N}}$ such that

$$\sup_{x \in M} \sum_{j=1}^{\infty} |\langle x, f_j \rangle| < \infty.$$

As a consequence of the Naimark Dilation Theorem (see, e.g., [12]), there exists a Hilbert space K with $K \supseteq H$ and a Riesz basis $\mathcal{E} = \{x_j\}_{j \in \mathbb{N}}$ for K such that $f_j = P x_j$ for all $j \in \mathbb{N}$, where P is the orthogonal projection of K onto H.

Notice that $x = Px$ for any $x \in M$. Therefore, using the fact that P is self-adjoint, we compute

$$\begin{aligned}\sup_{x\in M} ||x||_{1,\mathcal{E}} &= \sup_{x\in M} \sum_{j=1}^{\infty} |\langle x, x_j\rangle| = \sup_{x\in M} \sum_{j\in\mathbb{N}} |\langle Px, x_j\rangle| \\ &= \sup_{x\in M} \sum_{j=1}^{\infty} |\langle x, Px_j\rangle| \\ &= \sup_{x\in M} \sum_{j=1}^{\infty} |\langle x, f_j\rangle| \\ &< \infty.\end{aligned} \tag{4.31}$$

This shows that

$$\sup_{x\in M} ||x||_{1,\mathcal{E}} < \infty, \tag{4.32}$$

and hence M is ℓ_1-bounded in K using the Riesz basis $\mathcal{E}$. □

Proposition 4.7.1 shows us how an ℓ_1-frame bounded set in a Hilbert space can be an ℓ_1-bounded set in a larger Hilbert space. On the other hand, we do not know whether the converse holds. Let $M \subseteq H$ be a ℓ_1-bounded set in K with H being a subset of K. We leave as an open question whether M is an ℓ_1-frame bounded set in H.

Question 4.7.2. *Let H be a Hilbert space and let $K \subseteq H$ be a subspace of H. If M is an ℓ_1-bounded set with respect to a Riesz basis for K, is it ℓ_1-frame bounded with respect to a frame for H?*

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