THE EXTREMAL FUNCTION FOR $K_{10}$ MINORS

A Dissertation
Presented to
The Academic Faculty

By

Dantong Zhu

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in
Algorithms, Combinatorics, and Optimization

School of Mathematics
Georgia Institute of Technology

December 2021

© Dantong Zhu 2021
THE EXTREMAL FUNCTION FOR $K_{10}$ MINORS

Thesis committee:

Dr. Anton Bernshteyn
School of Mathematics
*Georgia Institute of Technology*

Dr. Grigoriy Blekherman
School of Mathematics
*Georgia Institute of Technology*

Dr. Zi-Xia Song
Department of Mathematics
*University of Central Florida*

Dr. Zhiyu Wang
School of Mathematics
*Georgia Institute of Technology*

Dr. Xingxing Yu
School of Mathematics
*Georgia Institute of Technology*

Date approved: October 15, 2021
To my parents.
ACKNOWLEDGMENTS

I first would like to thank my advisor Robin Thomas for guiding me into the fascinating world of structural graph theory, always encouraging me to follow my ideas, giving me just the right amount of help on research, and training me to become a more independent and knowledgeable young mathematician over years. I would also like to thank him for supporting me to travel around the world for both academic and non-academic opportunities, including conferences, summer schools, and even the NSF-funded internship at Microsoft. I am extremely grateful to have been a student of him.

I would like to thank my co-advisor Xingxing Yu for helping me finish up the $K_{10}$ minor work after Robin passed away. I am grateful for his patience and energy going through the lengthy proofs with me. I would also like to thank Dr. Anton Bernshteyn, Dr. Grigoriy Blekherman, Dr. Zi-Xia Song, and Dr. Zhiyu Wang for their service on my thesis committee.

I would like to thank Professor Derek Smith for introducing me to pure math research when I was a student of him at Lafayette College, as well as providing professional advice and always encouraging and trusting me this while time after I graduated from Lafayette. I would also like to thank Professors Gary Gordon, Thomas Hill, Lorenzo Traldi, and several other professors at Lafayette and the Budapest Semesters in Mathematics program for inspiring me to pursue graduate school in math.

I would like to thank Minjia Zhang and Yuxiong He for hosting me at Microsoft during my internship there. I would like to thank all my friends at Georgia Tech who has made my time here more enjoyable. I would like to thank Atlanta Ballet for developing my love and passion for ballet and being the best place to get me recharged outside work.

Finally, I would like to thank my parents for their unwavering love and constant support and belief in me.
TABLE OF CONTENTS

Acknowledgments ........................................................ iv

List of Tables .......................................................... viii

Summary ................................................................. ix

Chapter 1: Introduction ............................................... 1

  1.1 Motivation ......................................................... 1

  1.2 Related Work and Impact of the Main Theorem ................. 3

    1.2.1 The extremal function for $K_t$ minors .................. 3

    1.2.2 Relating to Hadwiger's Conjecture ....................... 4

    1.2.3 Variants of the extremal function for $K_t$ minors .... 6

Chapter 2: Preliminaries ............................................. 7

  2.1 Graph Basics ..................................................... 7

  2.2 Rooted $K_3$ and Rooted $K_4$ Minors ....................... 9

    2.2.1 Rooted $K_3$ minors .................................... 10

    2.2.2 Rooted $K_4$ minors .................................... 11

  2.3 Disjoint Paths .................................................. 11

    2.3.1 Two Disjoint Paths ..................................... 12
2.3.2 Three Disjoint Paths .............................................. 13
2.3.3 Menger’s Theorem and Perfect’s Theorem ....................... 13
2.4 Bridges and Tripods ................................................. 15
  2.4.1 Bridges ........................................................... 15
  2.4.2 Tripods ........................................................... 17
2.5 Exceptional graphs .................................................... 24
  2.5.1 Deletion Lemma .................................................. 28
  2.5.2 Contraction Lemmas ............................................. 30

Chapter 3: Structure of Possible Minimal Counter-Examples ............ 51
  3.1 Basic Properties ..................................................... 51
  3.2 Separations and Connectivity ....................................... 53
  3.3 Bounding Minimum Degree ......................................... 71

Chapter 4: Main Technical Lemma ........................................... 79
  4.1 Statements and Proof Outline ....................................... 79
  4.2 Problem Graphs ..................................................... 80
  4.3 $K_{2,3,3,3}$, $K_{3,3}$ + $C_5$, and $K_{4,4,4}$ ............................. 84
     4.3.1 Proof of Lemma 4.3.4 ........................................... 90
     4.3.2 Proof of Lemma 4.3.5 ........................................... 102
     4.3.3 Proof of Lemma 4.3.6 ........................................... 120
     4.3.4 Proof of Lemma 4.3.7 ........................................... 126
  4.4 Other Problem Graphs .............................................. 135
LIST OF TABLES

2.1 Case discussion in Lemma 2.5.9 ........................................ 34
2.2 Case discussion in Case 2.1 in Lemma 2.5.10 .......................... 39
2.3 Case discussion in Case 2.2 in Lemma 2.5.10 .......................... 40

3.1 Case discussion in Claim 1 in Lemma 3.2.5 ............................ 62
3.2 Case discussion in Lemma 3.2.5 ........................................ 63
3.3 Case discussion in Claim 1 in Lemma 3.2.6 ............................ 64
3.4 Case discussion in Lemma 3.2.6 ........................................ 65
For two graphs $G$ and $H$, $G$ has $H$ as a minor if a graph isomorphic to $H$ can be obtained from a subgraph of $G$ by repeatedly contracting edges. The Four Color Theorem (4CT) says that every planar graph is 4-colorable. Due to the Kuratowski-Wagner theorem, the 4CT can be restated that every graph with no $K_5$ minor and $K_{3,3}$ minor is 4-colorable. The famous Hadwiger Conjecture is a generalization of the 4CT, which says that every graph with no $K_{t+1}$ minor for integers $t \geq 1$ is $t$-colorable. The Hadwiger’s Conjecture is true for all $t \leq 5$ and remains widely open for $t \geq 6$.

To make progress on Hadwiger’s Conjecture for $t \geq 6$, one major line of work has focused on giving an upper bound on the number of edges for graphs without a $K_t$ minor. The maximum number of edges of an $n$-vertex graph with no $K_t$ minor is known as the extremal function for $K_t$ minors. This dissertation focuses on the extremal function for $K_{10}$ minors.

Our main theorem says that every graph on $n \geq 8$ vertices and at least $8n - 35$ edges either has a $K_{10}$ minor or falls into a few families of exceptional graphs. In Chapter 1, we discuss more into the motivation and related results on the $K_{10}$ minor work. In Chapter 2, we present necessary graph theoretical background and a series of observations of the exceptional graphs in the main theorem. We study structural properties of possible minimal counter-examples to the main theorem in Chapter 3 and later dive into proving a main technical lemma in Chapter 4. Finally, we conclude our main theorem in Chapter 5.

We note that the proof for the main technical lemma (Lemma 4.1.1) in our proof for the main theorem is computer-assisted. We do not yet have a computer-free proof for Lemma 4.1.1.
CHAPTER 1
INTRODUCTION

We discuss motivation, related results, and further impact of my work in this chapter.

1.1 Motivation

My work is motivated by Hadwiger’s conjecture, which is a longstanding open problem that generalizes the Four Color Theorem. The Four Color Theorem (4CT) states that every planar graph is 4-colorable. By the Kuratowski-Wagner theorem [20, 39], a graph is planar if and only if it has no $K_5$ or $K_{3,3}$ minor, which allows us to restate the 4CT that every graph with no $K_5$ minor and no $K_{3,3}$ minor is 4-colorable. As $K_{3,3}$ can in fact be colored by four colors, one might wonder if every graph with no $K_5$ minor is 4-colorable. This leads us to Hadwiger’s famous conjecture [7].

**Conjecture 1.1.1** (Hagwidger’s Conjecture). *For every integer $t \geq 0$, every graph with no $K_{t+1}$ minor is $t$-colorable.*

For $t \leq 3$, Hadwiger’s conjecture is reasonably easy, as shown by Hadwiger [7] and Dirac [5]. Wagner [39] proved the case $t = 4$ is equivalent to the 4CT in 1937, so the case $t = 4$ was eventually proved in 1976 when Appel and Haken [2, 3] proved the 4CT. In 1993, Robertson, Seymour, and Thomas [23] showed the case $t = 5$ is also equivalent to the 4CT. Hadwiger’s conjecture remains open for $t \geq 6$.

One major line of work on Hadwiger’s conjecture has focused on giving an upper bound on the number of edges for graphs that lack a $K_t$ minor. For positive integers $t$ and $n$, the maximum number of edges that an $n$-vertex graph with no $K_t$ minor can have is known as the *extremal function for $K_t$ minors*. My work in this dissertation focuses on the case $t = 10$. Mader [21] proved the following theorem in 1968.
Theorem 1.1.2. For every integer \( t = 1, 2, \ldots, 7 \), a graph on \( n \geq t \) vertices and at least \((t - 2)n - \binom{t - 1}{2} + 1\) edges has a \( K_t \) minor.

Mader also pointed out \( K_{2,2,2,2,2} \) is a counter-example for the case \( t = 8 \). One can construct further counter-examples by repeatedly identifying cliques of size 5.

In general, for graphs \( H_1 \) and \( H_2 \) and an integer \( k \), we define an \((H_1, H_2, k)\)-cockade recursively as follows: Every graph isomorphic to \( H_1 \) or \( H_2 \) is an \((H_1, H_2, k)\)-cockade; If \( G_1, G_2 \) are both \((H_1, H_2, k)\)-cockades, then the graph obtained from the disjoint union of \( G_1 \) and \( G_2 \) by identifying a clique of size \( k \) in \( G_1 \) with a clique of the same size in \( G_2 \) is also an \((H_1, H_2, k)\)-cockade; Every \((H_1, H_2, k)\)-cockade can be constructed this way. If \( H_1 = H_2 = H \), then an \((H_1, H_2, k)\)-cockade is also called an \((H, k)\)-cockade. A graph \( G \) is a trivial \((H, k)\)-cockade if \( G \cong H \), and otherwise a non-trivial \((H, k)\)-cockade. For a \((H, k)\)-cockade \( G \), the multiplicity of \( G \) is defined recursively as follows: \( G \) has multiplicity 1 if it is a trivial \((H, k)\)-cockade; \( G \) has multiplicity \( m = m_1 + m_2 \) for some \( m_1, m_2 \geq 1 \) if there exist induced subgraphs \( G_1, G_2 \) of \( G \) such that \( G = G_1 \cup G_2, G_1 \cap G_2 \cong K_k \), and \( G_i \) is an \((H, k)\)-cockade of multiplicity \( m_i \) for \( i = 1, 2 \).

Jørgensen [10] and Song and Thomas [32] generalized Theorem 1.1.2 for \( K_8 \) minors and \( K_9 \) minors, respectively, as follows.

Theorem 1.1.3. Every graph on \( n \geq 8 \) vertices and at least \( 6n - 20 \) edges either has a \( K_8 \) minor or is a \((K_{2,2,2,2,2}, 5)\)-cockade.

Theorem 1.1.4. Every graph on \( n \geq 9 \) vertices and at least \( 7n - 27 \) edges either has a \( K_9 \) minor or is a \((K_{1,2,2,2,2,2}, 6)\)-cockade, or is isomorphic to \( K_{2,2,2,3,3} \).

It is then natural to ask if every graph on \( n \geq 10 \) vertices and \( 8n - 35 \) edges with no \( K_{10} \) minor falls into a few families of graphs. We prove Theorem 1.1.5 for \( K_{10} \) minors, which is the main theorem of this dissertation.

For graphs \( H \) and \( G \), let \( H + G \) denote the graph obtained from the disjoint union of \( H \) and \( G \) by adding edges \( xy \) for all \( x \in V(H) \) and \( y \in V(G) \).
**Theorem 1.1.5.** Every graph on \( n \geq 8 \) vertices and at least \( 8n - 35 \) edges either has a \( K_{10} \) minor or is isomorphic to one of the following graphs:

1. a \((K_{1,1,2,2,2,2,2}, 7)\)-cockade;
2. \(K_{1,2,2,2,3,3}\);
3. \(K_{2,2,2,2} + C_5\);
4. \(K_{2,2,3,3,4}\);
5. \(K_{3,3,3} + C_5\);
6. \(K_{2,2,2,2,2,3}\);
7. \(G_1 = K_{2,2,2,2,2,3} - e \) where \( e \in E(K_{2,2,2,2,2,3}) \) and both ends of \( e \) have degree 11;
8. \(G_2 = K_{2,2,2,2,2,3} - e \) where \( e \in E(K_{2,2,2,2,2,3}) \) and the ends of \( e \) have degree 10 and 11;
9. \(K_{2,3,3,3,3}\);
10. \(G_3 = K_{2,3,3,3,3} - e \) where \( e \in E(K_{2,3,3,3,3}) \) and both ends of \( e \) have degree 11;
11. \(G_4 = K_{2,3,3,3,3} - e \) where \( e \in E(K_{2,3,3,3,3}) \) and the ends of \( e \) have degree 11 and 12;
12. a \((K_{2,2,2,2,2,3}, 6)\)-cockade of multiplicity 2.

We note that in the proof for Theorem 1.1.5 that we will present later in this dissertation, the proof for the main technical lemma (Lemma 4.1.1) is computer-assisted, and we do not yet have a computer-free proof for it.

**1.2 Related Work and Impact of the Main Theorem**

We now discuss related work to our problem and impact of Theorem 1.1.5 to give a more general context.

**1.2.1 The extremal function for \( K_t \) minors**

We first point out that the linear edge bound given by Mader in Theorem 1.1.2 actually is incorrect for large \( t \). Kostochka [17, 18] and de la Vega [38] proved that for large \( t \), a graph on \( n \) vertices must have at least \( \Omega(t\sqrt{\log tn}) \) edges to guarantee a \( K_t \) minor by showing that a random graph with no \( K_t \) minor may have average degree of order \( t\sqrt{\log t} \). Kostochaka
and Thomason [35] proved the order of $t\sqrt{\log tn}$ is also an upper bound, and later Thomason [36] was able to determine the constant of proportionality exactly. Although it may now seem unnecessary to study the extremal function for specific small values of $t$, the random graph examples only provide finitely many counter-examples. Of course, for any given value of $t$, more counter-examples can be made by taking disjoint unions or gluing counter-examples along small cut sets, but we know of no construction of highly connected infinite families of counter-examples. More specifically, Seymour and Thomas conjecture the following.

**Conjecture 1.2.1.** For every integer $t \geq 1$, there exists a constant $N = N(t)$ such that every $(t - 2)$-connected graph on $n \geq N$ vertices and at least $(t - 2)n - \binom{t-1}{2} + 1$ edges has a $K_t$ minor.

Note that Theorems 1.1.2, 1.1.3, and 1.1.4 imply that Conjecture 1.2.1 is true for $t \leq 9$. Since every $K_{10}$-minor-free graph in Theorem 1.1.5 is not 8-connected, Theorem 1.1.5 implies that Conjecture 1.2.1 is also true for $t = 10$. In particular, we have the following corollary of Theorem 1.1.5.

**Corollary 1.2.2** (Corollary of Theorem 1.1.5). Every 8-connected graph on $n \geq 8$ vertices and at least $8n - 35$ edges has a $K_{10}$ minor.

1.2.2 Relating to Hadwiger’s Conjecture

The proof of the case $t = 5$ of Hadwiger’s conjecture [23] uses the case $t = 6$ of Theorem 1.1.2 to get an upper bound on the number of edges for $K_6$ minor-free graphs. As the case $t = 6$ of Hadwiger’s conjecture remains open, Kawarabayashi and Toft [15] proved that every graph with no $K_7$ minor is either 6-colorable or has a $K_{4,4}$ minor. It is not known yet if every $K_7$ minor-free graph is 7-colorable. Albar and Gonçalves [1] and Rolek and Song [28] proved that for $t = 7, 8, 9$, a graph with no $K_t$ minor is $(2t - 6)$-colorable. Their proofs use the extremal function results for $t \leq 9$ to find a vertex of degree of at most $2t - 5$ in
every graph with no \( K_t \) minor. Now, Corollary 1.2.2 and Theorem 5.2 in [28] immediately imply the following corollary that every graph with no \( K_{10} \) minor is 14-colorable.

**Corollary 1.2.3** (Corollary of Theorem 1.1.5). *Every graph with no \( K_{10} \) minor is 14-colorable.*

Another weaker version of Hadwiger’s conjecture is the doubly-critical conjecture by Kawarabayashi, Pedersen, and Toft [13]. A connected \( t \)-chromatic graph \( G \) is called *doubly-critical* if \( G - \{u, v\} \) is \((t - 2)\)-colorable for every edge \( uv \in E(G) \). The doubly-critical conjecture states that every doubly-critical \( t \)-chromatic graph contains a \( K_t \) minor. Rolek and Song showed in [27] that the doubly-critical conjecture is true for all \( t \leq 9 \), and their proof again uses the extremal function for \( t \leq 9 \). According to Song (private communication), by following the ideas in [27] and the ideas proving Theorem 1.1.5 in this dissertation, one can prove, with effort, that every double-critical 10-chromatic graph has a \( K_{10} \) minor, which then resolves the the doubly-critical conjecture for \( t = 10 \).

Another way of weakening Hadwiger’s conjecture is to only consider \( t \)-chromatic graphs with a unique \( t \)-coloring. A recent work by Kriesell [19] shows that for \( t \leq 10 \), every graph of chromatic number \( t \) with a unique \( t \)-coloring has a \( K_{10} \) minor. Following the ideas in [19] and the ideas proving Theorem 1.1.5 in this dissertation, we can then extend to obtain the following corollary.

**Corollary 1.2.4** (Corollary of Theorem 1.1.5). *Every 11-chromatic graph with a unique 11-coloring has a \( K_{11} \) minor.*

The last line of work related to Hadwiger’s conjecture we want to mention here is the Erdős-Lovász Tihany Conjecture.

**Conjecture 1.2.5** (Erdős-Lovász Tihany Conjecture). *For any pair of integers \( s, t \geq 2 \) and any graph \( G \) with \( \omega(w) < \chi(G) = s + t - 1 \), there are two disjoint subgraphs \( G_1 \) and \( G_2 \) of \( G \) such that \( \chi(G_1) \geq s \) and \( \chi(G_2) \geq t \).*
For integers $s, t \geq 2$ with $s \leq t$, say a graph an $(s, t)$-graph if it is a connected $(s + t - 1)$-chromatic graph and does not contain two disjoint subgraphs with chromatic numbers $s$ and $t$, respectively. Then following the ideas in [16] by Kawarabayashi, Pedersen, and Toft and the ideas of proving Theorem 1.1.5 in this dissertation, we can conclude the following corollary, settling Conjecture 1.4 in the same paper [16] for $s = 2$ and $2 \leq t \leq 9$.

**Corollary 1.2.6** (Corollary of Theorem 1.1.5). *For $t = 2, 3, \ldots, 9$, every $(2, t)$-graph with clique number at most $t$ has a $K_2 \cup K_t$ minor.*

### 1.2.3 Variants of the extremal function for $K_t$ minors

Thomas and Yoo [34] studied the extremal function for $K_t$ minors for triangle-free graphs. They proved a theorem that for $t = 2, 3, \ldots, 9$, a triangle-free graph on $n \geq 2t - 5$ vertices and at least $(t - 2)n - (t - 2)^2 + 1$ edges has a $K_t$ minor. Now by Theorem 1.1.5 in this dissertation and Theorem 3.2 in [34], we can extend the triangle-free theorem to the case $t = 10$ and conclude the following corollary immediately.

**Corollary 1.2.7** (Corollary of Theorem 1.1.5). *Every triangle-free graph on $n \geq 15$ vertices and at least $8n - 63$ edges has a $K_{10}$ minor.*

The extremal functions for $K_t^-$ minors have also been studied, where $K_t^-$ denotes the graph obtained from $K_t$ by deleting one edge. Jakobsen [8, 9] proved that for $t \leq 7$, every graph on $n \geq t$ vertices and at least $(t - \frac{5}{2})n - \frac{1}{2}(t - 3)(t - 1)$ edges has a $K_t^-$ minor, or is a $(K_{t-1}, t - 3)$-cockade, or $G$ is a $(K_{2,2,2,2}, K_6, 4)$-cockade in the case $t = 7$. Song [31] later showed that every graph on $n \geq 8$ vertices and at least $\frac{4n - 35}{2}$ edges either has a $K_8^-$ minor or is a $(K_{1,2,2,2,2}, K_7, 5)$-cockade. Moreover, Song pointed out (private communication) it is promising that the way of using the 3-linkage theorem by Thomas and Wollan [33] in our proof for Theorem 1.1.5 in this dissertation can be applied to prove an analogous theorem for $K_9^-$ minor-free graphs.
CHAPTER 2
PRELIMINARIES

2.1 Graph Basics

All graphs are simple in this dissertation. For a graph \( G \), \( V(G) \) and \( E(G) \) denote the set of vertices and the set of edges of \( G \), respectively. If two vertices \( x, y \) are adjacent in \( G \), we say they are neighbors in \( G \) and write \( xy \) or \( yx \) to denote the edge between them. \( x \) and \( y \) are called the end vertices, or simply ends, of the edge \( xy \). We use \( |G| = |V(G)| \) to denote the number of vertices in \( G \).

If a graph \( G' \) satisfies that \( V(G') \subseteq V(G) \) and \( E(G') \subseteq E(G) \), then \( G' \) is a subgraph of \( G \), denoted by \( G' \subseteq G \). For a subset \( A \subseteq V(G) \) of vertices, \( G[A] \) denotes the subgraph of \( G \) that has set of vertices equal to \( A \) and set of edges containing every edge of \( G \) with both end vertices in \( A \). Say \( G[A] \) is induced by the subset \( A \). An induced subgraph of \( G \) is a subgraph of \( G \) equal to \( G[A] \) for some \( A \subseteq V(G) \).

For a vertex \( x \in V(G) \), \( N(x) \) is the set of neighbors of \( x \) in \( G \), and \( \overline{N}[x] = N(x) \cup \{x\} \). We also use \( N(x) \) and \( \overline{N}[x] \) denote the induced subgraphs of \( G \) on the subsets of vertices \( N(x) \) and \( \overline{N}[x] \), respectively. The degree of \( x \) is the size of \( N(x) \), denoted by \( d(x) \). The minimum degree over all vertices in \( G \) is denoted by \( \delta(G) \). A subset \( A \subseteq V(G) \) of vertices is a clique if every pair of vertices in \( A \) is an edge in \( G \). The clique number of \( G \) is maximum size of a clique in \( G \), denoted by \( \omega(G) \). For a subset \( A \subseteq V(G) \) of vertices and a subset \( F \subseteq E(G) \) of edges, \( G - A \) denotes the graph obtained from \( G \) by deleting all vertices in \( A \), and \( G - F \) denotes the graph obtained from \( G \) by deleting all edges in \( F \). In the case \( A = \{v\} \), \( G - v = G - A \). Let \( e \in E(G) \), \( G - e \) is the graph obtained from \( G \) by deleting \( e \).

We now define edge contraction and a more general notion of graph containment called
minor. Let \( e = xy \in E(G) \). The graph obtained from \( G \) by contracting \( e \), denoted as \( G \setminus e \), is the graph obtained from \( G \) by deleting both \( x \) and \( y \) and adding a new vertex whose neighborhood in the new graph is equal to \( N_G(x) \cup N_G(y) - \{x, y\} \). For some graph \( H \), say \( G \) contains \( H \) as a minor, or simply \( G \) has an \( H \) minor, if a graph isomorphic to \( H \) can be obtained from a subgraph of \( G \) by repeatedly contracting edges, denoted as \( G \succ H \). Equivalently, \( G \) has an \( H \) minor if there exist pairwise disjoint subsets \( S_1, \ldots, S_{|H|} \subseteq V(G) \) of vertices such that and a bijective function \( \phi : V(H) \to \{S_1, \ldots, S_{|H|}\} \) such that \( G[S_i] \) is a connected subgraph of \( G \) for all \( i = 1, \ldots, |H| \), and that for every \( xy \in E(H) \), there is an edge in \( G \) with one end in \( \phi(x) \) and one end in \( \phi(y) \). \( H \) is called a proper minor of \( G \) if \( G \succ H \) and \( G \not\cong H \).

A graph \( P \) is a path if we can label the vertices of \( P \) as \( v_1, \ldots, v_k \) such that \( E(P) = \{v_1v_2, \ldots, v_{k-1}v_k\} \). We denote \( P = v_1v_2\ldots v_k \). The length of a path is the number of edges on it. \( v_0 \) and \( v_k \) are the end vertices or simply ends of \( P \). Say \( P \) links \( v_0 \) and \( v_k \) or joins \( v_0 \) and \( v_k \), and say \( P \) is a \( v_0-v_k \) path. Say \( P \) is an \( A-B \) path if \( A, B \subseteq V(G) \) such that \( v_0 \in A \) and \( v_k \in B \). The vertices in \( V(P) - \{v_0, v_k\} \) are the internal vertices of \( P \).

We also consider the graph on a single vertex as a path. A path is trivial if it has length zero, and otherwise non-trivial. A subgraph \( P' \) of \( P \) is a subpath if it is also a path. For \( v_i, v_j \in V(P) \), \( v_iPv_j \) denotes the subpath of \( P \) linking \( v_i \) and \( v_j \). A graph \( C \) is a cycle if we can label the vertices of \( C \) as \( v_1, \ldots, v_k \) such that \( E(C) = \{v_1v_2, \ldots, v_{k-1}v_k, v_kv_1\} \). We write \( C = v_1v_2\ldots v_kv_1 \). The length of a cycle is the number of edges on it. A cycle of length \( k \) is called a \( k \)-cycle, denoted \( C_k \).

In a graph \( G \), two paths \( P, Q \subseteq G \) are disjoint if \( V(P) \cap V(Q) = \emptyset \), and they are internally disjoint if the sets of internal vertices of \( P, Q \) are disjoint. For \( A, B \subseteq V(G) \), say a set of paths \( P_1, \ldots, P_t \subseteq G \) links or joins \( A, B \), if every \( P_i \) for all \( i = 1, \ldots, t \) has one end in \( A \), one end in \( B \), and is otherwise disjoint from \( A \cup B \).

A graph \( G \) is connected if for every \( x, y \in V(G) \), there exists some path in \( G \) linking \( x \) and \( y \); it is disconnected otherwise. A connected component or simply a component of
$G$ is a maximal connected subgraph of $G$. $G$ is \textit{k-connected} for some integer \(k\) if for every \(X \subseteq V(G)\) such that \(|X| < k\), $G - X$ is connected. A subset $X \subseteq V(G)$ is a \textit{separating set} of $G$ if $G - X$ is disconnected. For vertices $a, b \in V(G) - X$, say $X$ separates $a$ from $b$ if there is no path linking $a, b$ in $G - X$. For $a \in V(G) - X$ and $B \subseteq V(G)$, say $X$ \textit{separates} $a$ from $B$, or $X$ \textit{separates} $B$ from $a$, if there is no path linking $a$ and some vertex in $B$ in $G - X$.

A \textit{separation} of $G$ is a pair $(A, B)$ of subsets of vertices of $G$ such that $A \cup B = V(G)$ and $ab \notin E(G)$ for all $a \in A - B$ and $b \in B - A$. For $X \subseteq V(G)$, the pair $(A, B)$ is a \textit{separation of $(G, X)$} if $(A, B)$ is a separation of $G$ such that $X \subseteq A$. For a separation $(A, B)$ of $G$, or $(G, X)$ for some $X \subseteq V(G)$, the \textit{order} of the separation is the size of $A \cap B$. It is called a \textit{k-separation} (or \(\leq k\)-separation), if $|A \cap B| = k$ (or $|A \cap B| \leq k$, respectively). A separation $(A, B)$ is \textit{trivial} if $A \subseteq B$ or $B \subseteq A$; it is \textit{non-trivial} otherwise. It is an easy exercise that a graph is \textit{k-connected} if and only if it has no non-trivial separation of order at most $k - 1$.

### 2.2 Rooted $K_3$ and Rooted $K_4$ Minors

Rooted minor is a special type of minor. Let $G, H$ be graphs, and let $X \subseteq V(G)$ such that $|X| = |H|$. Say $G$ has an $H$ \textit{minor rooted at} $X$ if there is a function $\phi$ mapping vertices in $H$ to disjoint connected subgraphs of $G$ such that $|V(\phi(u)) \cap X| = 1$ for all $u \in V(H)$, and that if $uv \in E(H)$ then there exists an edge in $G$ joining a vertex in $\phi(u)$ to a vertex in $\phi(v)$.

In this section, we will present our own result on rooted $K_3$ minors, which is relatively straight-forward, followed by a result on rooted $K_4$ minors due to Robertson, Seymour, and Thomas.
2.2.1 Rooted $K_3$ minors

Lemma 2.2.1. Let $G$ be a connected graph, and let $X = \{x_1, x_2, x_3\}$ be a subset of three distinct vertices in $G$. Then, $G$ has a $K_3$ minor rooted at $X$ if and only if there does not exist a cut vertex $w$ of $G$ such that every component of $G - \{w\}$ contains at most one vertex from $X$.

Proof. Since $G$ connected, we assume without loss of generality that there is a path $P$ in $G$ linking $x_1, x_2$ such that $x_3 \notin V(P)$.

First assume that $G$ has a $K_3$ minor rooted at $X$. Without loss of generality, we can then assume that there exist two paths $Q_1, Q_2$ such that $Q_1, Q_2$ each link $x_3$ and some vertex on $P$ and are otherwise disjoint from $P$, and that $V(Q_1) \cap V(Q_2) = \{x_3\}$. It follows that there is no cut vertex separating $x_3$ from $P$ in $G$, and therefore there is no cut vertex $w$ of $G$ such that every component of $G - \{w\}$ contains at most one vertex in $X$.

Now, assume that $G$ does not have a $K_3$ minor rooted at $X$. We will prove that there exists a cut vertex $w$ of $G$ such that every component of $G - \{w\}$ contains at most one vertex in $X$. Since $G$ is connected, there exists a path $Q$ linking $x_3$ and some vertex $w \in V(P)$ that is otherwise disjoint from $P$. Since $G$ does not have a $K_3$ minor rooted at $X$, there do not exist two paths $Q_1, Q_2$ each joining $x_3$ and some vertex on $P$ such that $Q_1, Q_2$ are disjoint except for $x_3$. Therefore, $w$ is a cut vertex of $G$, and there is a component $J_3$ of $G - \{w\}$ such that $x_3 \in V(J_3)$ and $V(P) \cap V(J_3) = \emptyset$.

If $w \in \{x_1, x_2\}$, without loss of generality, assume that $w = x_2$. Then, the component of $G - \{w\}$ that contains $x_1$, say $J_1$, satisfies that $V(J_1) \cap X = \{x_1\}$. It follows that $w$ satisfies that every component of $G - \{w\}$ contains at most one vertex in $X$. If $w \notin \{x_1, x_2\}$, we consider the block decomposition of $G - V(J_3)$. Notice that $P \subseteq G - V(J_3)$. Choose blocks $B_1, \ldots, B_k$ of $G - V(J_3)$ and vertices $v_1, v_2, \ldots, v_{k+1} \in V(P)$ such that $v_1 = x_1, v_{k+1} = x_2$, and $v_i, v_{i+1} \in V(B_i)$ for all $i = 1, \ldots, k$. Assume for a moment that $w \in V(B_i) - \{v_i, v_{i+1}\}$ for some $i \in \{1, \ldots, k\}$. Since $B_i$ is 2-connected, it follows that $B_i$
has $K_3$ minor rooted at $\{w, v_i, v_{i+1}\}$. It follows that $B_i \cup P \cup Q = B_i \cup v_1 P v_i \cup v_{i+1} P v_{k+1} \cup Q$ has a $K_3$ minor rooted at $X$, a contradiction. Hence, $w = v_i$ for some $i \in \{2, ..., k\}$. Let $J_1, J_2$ be the components of $G - \{w\} \cup V(J_3)$ such that $V(v_1 P v_i) - \{v_i\} \subseteq V(J_1)$ and $V(v_i P v_{k+1}) - \{v_i\} \subseteq V(J_2)$. It follows that $J_1, J_2, J_3$ are distinct components of $G - \{w\}$ such that $x_i \in V(J_i)$ for $i = 1, 2, 3$. 

2.2.2 Rooted $K_4$ minors

The following theorem on rooted $K_4$-minors was proven by Robertson, Seymour, and Thomas in their proof of Hadwiger's Conjecture for graphs with no $K_6$-minor [23], in which a $K_4$-minor rooted at $x_1, x_2, x_3, x_4$ is called a cluster traversing $\{x_1, x_2, x_3, x_4\}$. A trisection of a graph $G$ is a triple $(A, B, C)$ of subsets of $V(G)$ such that $A \cap B = A \cap C = B \cap C$ and $G[A] \cup G[B] \cup G[C] = G$; the order of the trisection $(A, B, C)$ is $|A \cap B \cap C|$.

**Theorem 2.2.2** (Rooted $K_4$-minor Theorem). Let $G$ be a graph and let $Z \subseteq V(G)$ with $|Z| = 4$. Then

(i) $G$ has a $K_4$ minor rooted at $X$, or

(ii) there is a trisection $(A_1, A_2, B)$ of order 2 such that $|Z \cap (A_i - B)| = 1$ for $i \in \{1, 2\}$, or

(iii) there is a $(\leq 3)$-separation $(A, B)$ with $Z \subseteq A$ and $|B - A| \geq 2$ and $|Z \cap B| \leq 2$, or

(iv) $G$ can be drawn in the plane so that every vertex in $Z$ is incident with the infinite region.

We note that Fabila-Monroy and Wood [6] proved a stronger theorem than Theorem 2.2.2 by giving a complete characterization of graphs that have a $K_4$ minor rooted at four nominated vertices.

2.3 Disjoint Paths

Let $G$ be a graph. For a path $P$ in $G$ and vertices $s, t \in V(G)$, say $P$ links $s, t$ if $s, t$ are the end vertices of $P$. For an integer $k$ and distinct vertices $s_1, ..., s_k, t_1, ..., t_k \in V(G)$, the $k$: 
**disjoint paths problem** asks whether there exist \( k \) disjoint paths \( P_1, ..., P_k \) in \( G \) such that \( P_i \) link \( s_i \) and \( t_i \) for all \( i = 1, ..., k \). Robertson and Seymour [26] showed that there is a polynomial time algorithm for deciding whether such disjoint paths exist. We now discuss results on the \( k \) disjoint paths problem for \( k = 2, 3 \).

### 2.3.1 Two Disjoint Paths

In fact, the two disjoint path problem is a special case for another. Let \( C \) be a cycle in a graph \( G \). A **\( C \)-cross** is a pair of disjoint paths \( P_1, P_2 \) with ends \( x_1, y_1 \) and \( x_2, y_2 \), respectively, such that \( x_1, x_2, y_1, y_2 \) occur on \( C \) in the order listed, and the paths are otherwise disjoint from \( C \). The feasibility problem for a \( C \)-cross generalizes the feasibility problem for the two disjoint paths problem: Notice that for distinct vertices \( s_1, s_2, t_1, t_2 \in V(G) \) in a graph \( G \), there exist two disjoint paths \( P_1, P_2 \) in \( G \) such that \( P_i \) for \( i = 1, 2 \) links \( s_i \) and \( t_i \) if and only if the graph \( G' = G \cup \{ s_1s_2, s_2t_1, t_1t_2, t_2s_1 \} \) has a \( C \)-cross, where \( C = s_1s_2t_1t_2s_1 \) is the cycle that goes through the four vertices \( s_1, s_2, t_1, t_2 \) in order. It follows that, to study the feasibility of the two disjoint path problem, it suffices to study the characterization of \( C \)-crosses in a graph.

Theorem 2.3.1 gives a characterization of graphs containing \( C \)-crosses. This exact version of the theorem is Theorem 1.3 in [14], obtained in various forms by Jung [11], Robertson and Seymour [24], Seymour [29], Shiloach [30], and Thomassen [37]. Let \( G \) be a graph, and let \( X \subseteq V(G) \). Let \( (A, B) \) be a \( \leq 3 \)-separation of \( (G, X) \) such that there exist \( |A \cap B| \) paths from some vertex \( v \in B - A \) to \( X \) that are disjoint except for \( v \). Let \( H \) be the graph obtained from \( G[A] \) by adding an edge joining every pair of distinct vertices in \( A \cap B \). We say that \( H \) is an elementary \( X \)-reduction of \( G \) (determined by \( (A, B) \)). We say that a graph \( J \) is an \( X \)-reduction of \( G \) if it can be obtained from \( G \) by a series of elementary \( X \)-reductions. If \( C \) is a subgraph of \( G \), then by an (elementary) \( C \)-reduction we mean an (elementary) \( V(C) \)-reduction.

**Theorem 2.3.1** (Jung; Robertson and Seymour; Seymour; Shiloach; Thomassen). *Let \( G \) be
a graph, and let $C$ be a cycle in $G$. Then, $G$ has no $C$-cross if and only if some $C$-reduction of $G$ can be drawn in the plane with $C$ bounding a face.

### 2.3.2 Three Disjoint Paths

We introduce a result on 3-linkage in Theorem 2.3.2 due to Thomas and Wollan [33]. Theorem 2.3.2 is used later in this dissertation to find three disjoint paths. Let $G$ be a graph.

For $X \subseteq V(G)$ and an integer $t$, the pair $(G, X)$ is $t$-linked if for all $k \leq t$ and distinct vertices $s_1, ..., s_k, t_1, ..., t_k \in X$, there exist $k$ disjoint paths $P_1, ..., P_k$ in $G$ such that $P_i$ links $s_i, t_i$ for all $i = 1, ..., k$. The pair $(G, X)$ is linked if it is $\lceil |X|/2 \rceil$-linked. A separation $(A, B)$ of $G$ is $t$-linked if $(G[B], A \cap B)$ is $t$-linked. Use $\rho_G(X)$ to denote the number of edges of $G$ that have at least one end in $V(G) - X$, i.e. $\rho_G(X) = |E(G)| - |E(G[X])|$.

**Theorem 2.3.2** (Wollan and Thomas). Let $G$ be a graph. Let $X \subseteq V(G)$ be a subset of vertices such that $|X| = 6$. Then, $(G, X)$ is linked if $\rho(V(G) - X) \geq 5|V(G) - X| + 4$ and $\rho(B - A) \leq 5|B - A|$ for every $\leq 5$-separation $(A, B)$ of $(G, X)$.

### 2.3.3 Menger’s Theorem and Perfect’s Theorem

A classic theorem of Menger states that if a graph $G$ is $k$-connected, then for any two disjoint subsets $A, B \subseteq V(G)$ such that $|A| = |B| = k$, there exist $k$ disjoint paths $P_1, ..., P_k$ such that each $P_i$ links one vertex in $A$ and one vertex in $B$.

Theorem 2.3.3 is a stronger version of Menger’s theorem due to Perfect [22], which is also stated in Section 3.3 in Diestel’s text [4].

**Theorem 2.3.3** (Perfect). Let $G$ be a graph with $A, B \subseteq V(G)$. Let $k$ be the minimum number of vertices separating $A$ from $B$ in $G$. If $\mathcal{P}$ is any set of fewer than $k$ disjoint $A - B$ paths in $G$, then $G$ contains a set $\mathcal{Q}$ of disjoint $A - B$ paths such that $|\mathcal{Q}| = |\mathcal{P}| + 1$, and that the set of vertices in $A$ that lie on a path in $\mathcal{P}$ is a proper subset of the vertices in $A$ that lie on a path in $\mathcal{Q}$, and likewise for $B$.  

13
Corollary 2.3.4 (Corollary of Theorem 2.3.3). Let $G$ be a graph with $a \in V(G)$ and $B \subseteq V(G) - \{a\}$. Let $k$ be the minimum number of vertices separating $a$ from $B$ in $G$. If $\mathcal{P}$ is any set of fewer than $k$ paths from $a$ to $B$ that are disjoint except for $a$, then $G$ contains a set $Q$ of paths from $a$ to $B$ that are disjoint except for $a$ such that $|Q| = |\mathcal{P}| + 1$, and that the set of vertices in $B$ that lie on a path in $\mathcal{P}$ is a proper subset of the vertices in $B$ that lie on a path in $Q$.

Let $S_k$ denote the permutation group on $k$ elements.

Corollary 2.3.5 (Corollary of Theorem 2.3.3). Let $G$ be a graph. Let $A, B \subseteq V(G)$ with a vertex $a_1 \in A - B$. Let $k = |A|$. Suppose there exist $k$ disjoint paths $P_1, \ldots, P_k$ such that $a_i \in A$ and $b_i \in B$ are the ends of $P_i$ for $i = 1, \ldots, k$. If there does not exist $a \leq k$-separation $(X, Y)$ of $G$ such that $A \subseteq X$, $B \subseteq Y$, and $X - Y \neq \emptyset$, then there exists some $b_{k+1} \in B - \{b_1, \ldots, b_k\}$, a permutation $\theta \in S_{k+1}$, and $k + 1$ internally disjoint paths $Q_1, \ldots, Q_{k+1}$ linking $A$ and $B$ such that $Q_i$ links $a_i$ and $b_{\theta(i)}$ for all $i = 1, \ldots, k$, and that $Q_{k+1}$ links $a_1$ and $b_{\theta(k+1)}$.

Proof. Let $G'$ be the graph obtained from $G$ by adding a new vertex $a'_1$ and an edge between $a'_1$ and every neighbor of $a_1$ in $G$. Let $A' = A \cup \{a'_1\}$. Note that $P_1, \ldots, P_k$ are $k$ disjoint paths linking $A'$ and $B$ in $G'$.

Assume for a moment that there exists a $k$-separation $(X', Y')$ such that $A' \subseteq X'$ and $B \subseteq Y'$. Since there is no $\leq k$-separation $(X, Y)$ of $G$ such that $A \subseteq X$, $B \subseteq Y$, and $X - Y \neq \emptyset$, it follows that $(X' - Y') \cap V(G) = \emptyset$, and therefore $X' - Y' = \{a'_1\}$ and $X' \cap Y' = \{a_1, \ldots, a_k\}$. Since $a_1 \notin B$, $P_1$ contains at least 2 vertices. Let $u$ be the unique vertex in $P_1$ that is adjacent to $a_1$, and note that $u \in Y' - X'$. However, since $a_1 u \in E(G)$, we know $a'_1 u \in E(G')$, which is a contradiction to that $(X', Y')$ is a separation of $G'$ and there is no edge between $X' - Y'$ and $Y' - X'$.

Hence, there is no $k$-cut separating $A'$ and $B$ in $G'$. By Theorem 2.3.3, there exist some $b_{k+1} \in B - \{b_1, \ldots, b_k\}$, a permutation $\theta \in S_{k+1}$, and $k + 1$ disjoint paths $Q'_1, \ldots, Q'_{k+1}$
linking $A'$ and $B$ in $G'$ such that $Q_i'$ links $a_i$ and $b_{\theta(i)}$ for $i = 1, ..., k$ and $Q_{k+1}'$ links $a_1'$ and $b_{\theta(k+1)}$. Back in the graph $G$, let $Q_i = Q_i'$ for all $i = 1, ..., k$, and let $Q_{k+1}$ be the path obtained from $Q_{k+1}'$ by replacing $a_1'$ with $a_1$. It follows that the vertex $b_{k+1}$, the permutation $\theta \in S_{k+1}$, and the paths $Q_1, ..., Q_{k+1}$ are as desired. \hfill \Box

2.4 Bridges and Tripods

2.4.1 Bridges

Let $G$ be a graph, and let $S$ be a subgraph of $G$. An $S$-bridge in $G$ is a connected subgraph $B$ of $G$ such that $E(B) \cap E(S) = \emptyset$ and either $E(B)$ consists of a unique edge with both ends in $S$, or for some component $C$ of $G \setminus V(S)$ the set $E(B)$ consists of all edges of $G$ with at least one end in $V(C)$. The vertices in $V(B) \cap V(S)$ are called the attachments of $B$ on $S$. We say an $S$-bridge $B$ attaches to a subgraph $H$ of $S$ if $V(H) \cap V(B) \neq \emptyset$, and in the case $H = \{v\}$ for some $v \in V(S)$, we say $B$ attaches to $v$. An $S$-bridge $B$ is called trivial if it consists of a unique edge with both ends in $S$, and it is called non-trivial otherwise.

Let $S$ be a subgraph of $G$, and let $W \subseteq V(S)$. A $W$-segment of $S$ is a subpath $P$ of $S$ of length at least one such that both end vertices of $P$ are contained in $W$, and that every internal vertex $v$ of $P$ is not in $W$ and has degree two in $S$. Say $W$ is a segmenting set of $S$ if $S$ is equal to the union of all $W$-segments of $S$. It is an easy exercise to check that a segmenting set of $S$ includes all vertices of degree not equal to two in $S$. Note that if $W$ is a segmenting set of $S$, every edge of $S$ is contained in a unique $W$-segment of $S$. Say an $S$-bridge $B$ is $W$-unstable if all attachments of $B$ on $S$ belong to some $W$-segment of $S$, and otherwise $B$ is $W$-stable.

The next lemma, Lemma 2.4.1, says that it is possible to make every $S$-bridge $W$-stable by making the following “local” changes. For a segmenting set $W \subseteq V(S)$ of $S$, let $P$ be a $W$-segment of $S$ of length at least two, and let $Q$ be a path in $G$ linking $x, y$ that is otherwise disjoint from $S$. Let $S'$ be obtained from $S$ by replacing the path $xPy$ by $Q$. Then we say
that $S'$ is obtained from $S$ by rerouting $P$ along $Q$, or simply that $S'$ is obtained from $S$ by rerouting. Please note that $P$ is required to have length at least two, and hence this relation is not symmetric. Also note that $W$ is a segmenting set of $S'$, as $S'$ is equal to the union of all $W$-segments of $S'$. We say the rerouting is proper with respect to $W$ if all attachments of the $S$-bridge that contains $Q$ belong to $P$. Lemma 2.4.1 is a generalization of Lemma 2.1 in [12] and is essentially due to Tutte.

**Lemma 2.4.1 (Rerouting Lemma).** Let $G$ be a graph. Let $S$ be a subgraph of $G$, and $W \subseteq V(S)$ be a segmenting set of $S$. Then, there exists a subgraph $S'$ of $G$ obtained from $S$ by a sequence of proper reroutings with respect to $W$ such that if all attachments of an $S'$-bridge $B$ belong to some $W$-segment $P$ of $S'$, then there exist vertices $x, y \in V(P)$ such that some component of $G \setminus \{x, y\}$ includes a vertex of $B$ and is disjoint from $S' \setminus V(P)$.

**Proof.** Choose a subgraph $S' \subseteq G$ that can be obtained from $S$ by a sequence of proper reroutings with respect to $W$ such that the number of vertices in $G - V(S')$ belonging to $W$-stable $S'$-bridges is maximum, and subject to this, $|V(S')|$ is minimum. We will prove that $S'$ is as desired. Let $B$ be a $W$-unstable $S'$-bridge, and say all attachments of $B$ on $S'$ belong to a $W$-segment $P$ of $S'$.

Let $v_0, v_1, ..., v_k$ be distinct vertices of $P$, listed in order of occurrence on $P$ such that $v_0$ and $v_k$ are the ends of $P$ and $\{v_1, ..., v_{k-1}\}$ is the set of all internal vertices of $P$ that are attachments of $W$-stable $S'$-bridges.

Assume for a moment that for some $i \in \{1, ..., k - 1\}$, there exist two attachments $u, v$ of $B$ such that $v_i$ is an internal vertex of $uPv$. Let $S''$ be a subgraph of $G$ obtained from $G$ by replacing $uPv$ by an induced subpath of $B$ linking $u, v$ that is otherwise disjoint from $S'$. It follows that every vertex in $V(G) - V(S')$ belonging to a $W$-stable $S'$-bridge is in $V(G) - V(S'')$ and belongs to a $W$-stable $S''$-bridge, and that $v_i \in V(S')$ is in $V(G) - V(S'')$ and belongs to a $W$-stable $S''$-bridge, a contradiction to the choice of $S'$. Therefore, all attachments of $B$ are on $v_iPv_{i+1}$ for some $i \in \{0, 1, ..., k - 1\}$.

Since $B$ is arbitrary, it follows that for every $W$-unstable $S'$-bridge $B'$ that attaches to
some vertex in the interior of \(v_iPv_{i+1}\), all attachments of \(B'\) are on \(v_iPv_{i+1}\). Hence, there is a component \(K\) of \(G\setminus\{v_i, v_{i+1}\}\) such that \(V(B) - \{v_i, v_{i+1}\} \subseteq V(K)\) and \(V(K) \cap (V(S') \setminus V(P)) = \emptyset\). This means that if \(V(B) - \{v_i, v_{i+1}\} \neq \emptyset\), then \(v_i\) and \(v_{i+1}\) give the desired \(x\) and \(y\). Therefore, we may assume that \(B\) is simply an edge joining \(v_i\) and \(v_{i+1}\).

Note this implies that \(v_iPv_{i+1}\) has length at least two. Let \(S''\) be the subgraph of \(G\) obtained from \(G\) by replacing \(v_iPv_{i+1}\) by the edge \(v_iv_{i+1}\). It follows that \(|V(S'')| < |V(S')|\) and every vertex in \(V(G) - V(S')\) belonging to a \(W\)-stable \(S'\)-bridge is in \(V(G) - V(S'')\) and belongs to a \(W\)-stable \(S''\)-bridge, a contradiction to the choice of \(S'\).

2.4.2 Tripods

We now introduce the tripod structure, which is originally due to Robertson and Seymour [25].

**Definition 2.4.2.** In a graph \(G\) with a subset of three distinct vertices \(X = \{x_1, x_2, x_3\} \subseteq V(G)\), a subgraph \(T\) of \(G\) is called an \(X\)- tripod if \(T\) can be written as a union of internally disjoint subpaths \(P_1, P_2, P_3, Q_1, Q_2, Q_3, L_1, L_2, L_3\) of \(G\) satisfying the following: For some distinct vertices \(z_1, z_2, z_3 \in V(G)\), \(L_i\) links \(z_i\) and \(x_i\) for \(i = 1, 2, 3\); and for distinct vertices \(p, q \in V(G) - \bigcup_{i=1}^3 V(L_i)\), \(P_i\) links \(p, z_i\) and \(Q_i\) links \(q, z_i\) for \(i = 1, 2, 3\). Each path \(L_i\) for \(i = 1, 2, 3\) is called a leg of \(T\). A leg is trivial if it is a single vertex; it is non-trivial otherwise.

**Lemma 2.4.3.** Let \(G\) be a graph and \(X = \{x_1, x_2, x_3\} \subseteq V(G)\) be a subset of three distinct vertices such that \(G\) cannot be drawn in the plane with \(x_1, x_2, x_3\) incident to the infinite face. If there is no non-trivial \(\leq 2\)-separation of \((G, X)\), then \(G\) has an \(X\)- tripod.

**Proof.** We will prove the lemma by inducting on \(|V(G)|\).

First consider the case that \(|V(G)| \leq 5\). Notice that the lemma is true if \(G \cong K_5\), as the complete graph \(K_5\) is non-planar, has no non-trivial \(\leq 2\)-separation, and contains an \(X'\)-tripod as a subgraph for every subset \(X'\) of three distinct vertices in it. Also notice that the
Lemma is trivially true if $|V(G)| \leq 4$, since the complete graph $K_4$ is planar, meaning that, if $|V(G)| \leq 4$, $G$ can always be drawn in the plane with $x_1, x_2, x_3$ incident to the infinite face. We may then assume $|V(G)| = 5$. If every vertex in $X$ and every vertex in $G - X$ are adjacent, then $G$ has an $X$-tripod. So assume that there is some non-edge between a vertex in $X$ and a vertex in $V(G) - X$. This then implies that $G \cup \{x_1x_2, x_1x_3, x_2x_3\}$ is isomorphic to some proper subgraph of $K_5$ and therefore is planar. It follows that $G \cup \{x_1x_2, x_1x_3, x_2x_3\}$ can be drawn in the plane with $x_1, x_2, x_3$ incident to the infinite face, and therefore $G$ can be drawn in the plane with $x_1, x_2, x_3$ incident to the infinite face.

From now on, we can assume $|V(G)| \geq 6$ and the assertion holds for all graphs on fewer than $|V(G)|$ vertices.

Observe that $x_2$ does not separate $x_1$ from $x_3$ in $G$: For the sake of a contradiction, assume that $x_2$ is a cut vertex of $G$, and there exist distinct components $K_1, K_3$ of $G - \{x_2\}$ such that $x_i \in V(K_i)$ for $i = 1, 3$. Note that if $K_3 - \{x_3\}$ is non-empty, then $\{x_2, x_3\}$ separates the non-trivial subgraphs $K_1$ and $K_3 - \{x_3\}$, a contradiction to the fact that $(G, X)$ does not have a non-trivial $\leq 2$-separation. By symmetry, it follows that $V(K_i) = \{x_i\}$ for $i = 1, 3$, meaning that $x_2$ is the only neighbor for $x_1$ and $x_3$ in $G$. Since $|V(G)| \geq 6$, we know that $|V(G) - X| > 0$. It follows that $x_2$ separates $\{x_1, x_3\}$ from $G - X$, again a contradiction to the fact that $(G, X)$ does not have a non-trivial $\leq 2$-separation.

Next, observe that we may assume $E(G[X]) = \{x_1x_2, x_2x_3\}$: Let $G' = (G \cup \{x_1x_2, x_2x_3\}) - \{x_1x_3\}$. Note that since $G$ cannot be drawn in the plane with $x_1, x_2, x_3$ incident to the infinite plane, neither can $G'$; and since $(G, X)$ does not have a non-trivial $\leq 2$-separation, neither does $(G', X)$. Also note that $G$ has an $X$-tripod if and only if $G'$ has one. Hence, it suffices to consider $G'$ instead of $G$, so we may assume that $E(G[X]) = \{x_1x_2, x_2x_3\}$.

Let $P_X$ be the path $x_1x_2x_3$ in $G$. We next prove that there exists some cycle $C$ such that $P_X \subseteq C$ and every $C$-bridge attaches to $x_2$.

**Claim 1.** There exists a cycle $C$ such that $P_X \subseteq C$ and every $C$-bridge attaches to $x_2$.

**Proof of Claim 1.** First observe that $X$ is a segmenting set for every cycle in $G$ that includes
all three vertices in $X$. Since $x_2$ does not separate $x_1$ from $x_3$ in $G$, there exists some cycle in $G$ that includes all three vertices in $X$. By Lemma 2.4.1, since $(G, X)$ does not have a non-trivial $\leq 2$-separation, there exists a cycle $C' \subseteq G$ such that $X \subseteq V(C')$ and every $C'$-bridge in $G$ is $X$-stable. For $i = 1, 2$, let $P_i$ be the subpath of $C'$ linking $x_i$ and $x_{i+1}$ such that $V(P_i) \cap X = \{x_i, x_{i+1}\}$.

Let $C$ be the cycle obtained from $C'$ by replacing $P_1$ with the edge $x_1x_2$ and replacing $P_2$ with the edge $x_2x_3$. Let $B \subseteq G$ be any $C$-bridge. Since $V(C) \subseteq V(C')$, $B$ is contained in a $C'$-bridge $B'$. Since every $C'$-bridge is $X$-stable, we know that $B'$ is an $X$-stable $C'$-bridge and therefore attaches to some vertex on $P_1 \cup P_2 - \{x_1, x_3\}$. By the construction of $C$, it follows that $B$ attaches to $x_2$. Since $B$ is arbitrary, it follows that the cycle $C$ is as desired. $\dashv$

By Claim 1, let $C \subseteq G$ be a cycle such that $P_X \subseteq C$ and every $C$-bridge attaches to $x_2$. We next show that we may assume $G$ has a $C$-cross.

Claim 2. If $G$ has no $C$-cross, then it has an $X$-tripod.

Proof of Claim 2. Assume that $G$ has no $C$-cross. Recall that $G$ cannot be drawn in the plane with $x_1, x_2, x_3$ incident to the infinite face. Since $x_1, x_2, x_3$ are all contained in $C$, it follows that $G$ cannot be drawn in the plane with $C$ bounding a face. By Theorem 2.3.1, some non-trivial $C$-reduction of $G$ can be drawn in the plane with $C$ bounding a face. This means that there exists a non-trivial $\leq 3$-separation $(A, B)$ of $(G, V(C))$ such that $G_1 = G[A] \cup \{uv : u, v \in A \cap B\}$ can be drawn in the plane with $C$ bounding a face. Since there is no non-trivial $\leq 2$-separation of $(G, X)$, every non-trivial separation of $(G, V(C))$ has order at least three. It follows that $|A \cap B| = 3$, and there exist three disjoint paths $L_1, L_2, L_3$ linking $X$ and $A \cap B$ in $G[A]$. It also follows that there is no non-trivial $\leq 2$-separation of $(G[B], A \cap B)$. Due to the disjoint paths $L_1, L_2, L_3$, since $G$ does not have an $X$-tripod, we know that $G[B]$ does not have an $(A \cap B)$-tripod. Since $|B| < |V(G)|$, by induction, $G[B]$ can be drawn in the plane with every vertex in $A \cap B$ incident to the infinite face. This means that the graph $G_2 = G[B] \cup \{uv : u, v \in A \cap B\}$ can be drawn.
in the plane with the cycle on $A \cap B$ bounding the infinite face. Now, the drawings of $G_1$ and $G_2$ can be combined to form a planar drawing of $G_1 \cup G_2 = G \cup \{uv : u, v \in A \cap B\}$ with $C$ bounding a face, a contradiction.

By Claim 2, we may assume that $G$ has a $C$-cross. This means that there exist four distinct vertices $s_1, s_2, t_1, t_2$ in order on $C$ and two disjoint paths $R_1, R_2$ in $G$ such that $R_i$ links $s_i, t_i$ for $i = 1, 2$ and is otherwise disjoint from $C$. Let $P = C \setminus \{x_2\}$.

First observe that we may assume $x_2 \in \{s_1, t_1, s_2, t_2\}$. To see this is true, assume that $x_2 \notin \{s_1, t_1, s_2, t_2\}$, which means that $\{s_1, s_2, t_1, t_2\} \subseteq V(P)$. Without loss of generality, assume that $P$ goes through $x_1, s_1, s_2, t_1, t_2, x_3$ in order, where $\{x_1, x_3\}$ may or may not be disjoint from $\{s_1, t_2\}$. Since every $C$-bridge attaches to $x_2$, it follows that there is a path $R_0$ linking $x_2$ and some $r \in V(R_1 \cup R_2) - V(P)$ such that $R_0$ is otherwise disjoint from $C \cup R_1 \cup R_2$. Without loss of generality, assume that $r$ is an internal vertex of $R_2$. By replacing $R_2$ with the path $R_0 \cup rR_2s_2$, we would then have $x_2$ as an end of $R_2$, as desired.

Now, without loss of generality, say $s_1, t_1$ are distinct vertices on $P$ such that $V(x_1Ps_1) \cap V(x_3Pt_1) = \emptyset$, $s_2$ is an internal vertex of $s_1Pt_1$, and that $x_2 = t_2$. Since every $C$-bridge attaches to $x_2$, the path $R_1$ has length at least two and is contained in some $C$-bridge attaching to $x_2$. It follows that there exists a path $W$ linking an internal vertex of $R_1$ and some vertex on $R_2 \setminus \{s_2\}$ such that $W$ is otherwise disjoint from $C\cup R_1 \cup R_2$. Let $T = P \cup R_1 \cup R_2 \cup W$. It follows that $T$ is an $X$-tripod as desired.

Here we introduce more notations and definitions related to tripods.

Let $G$ be a graph and $X = \{x_1, x_2, x_3\} \subseteq V(G)$ be a subset of three distinct vertices in $G$. Let $T \subseteq G$ be an $X$-tripod. Let vertices $z_1, z_2, z_3, p, q \in V(T)$ and paths $L_1, L_2, L_3, P_1, P_2, P_3, Q_1, Q_2, Q_3$ be labeled as in Definition 2.4.2 for the $X$-tripod $T$.

Let $L(T) = V(L_1 \cup L_2 \cup L_3)$, $Z(T) = \{z_1, z_2, z_3\}$, and $R(T) = V(P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup Q_3)$. For two subsets of vertices $A, B \subseteq V(G)$, say the ordered pair $(A, B)$ splits $T$ if $(A, B)$ is a 3-separation of $G$ such that $L(T) \subseteq A$, $R(T) \subseteq B$, $Z(T) = A \cap B$. 

20
Equivalently, we also say $T$ is split by $(A, B)$ or simply that $T$ is split in $G$ if some ordered pair $(A, B)$ splits $T$ in $G$. Let $\mathcal{W}(T) = \mathcal{Z}(T) \cup X \cup \{p, q\}$. Observe that $\mathcal{W}(T)$ is a segmenting set of $T$. For convenience, we say a $T$-bridge is stable if it is $\mathcal{W}(T)$-stable and otherwise unstable, and we say a subpath of $T$ is a segment of $T$ if it is a $\mathcal{W}(T)$-segment of $T$. Also, we call a (proper) rerouting with respect to $\mathcal{W}(T)$ simply a (proper) rerouting.

We next define three types of “local” changes, called tripod-transformations, that could be made on some $X$-tripods.

Let $R$ be a path in $G$ linking some $r_1 \in \mathcal{L}(T) - \mathcal{Z}(T)$ and $r_2 \in \mathcal{R}(T) - \mathcal{Z}(T)$. Without loss of generality (up to relabeling the vertices and segments in $T$), assume that $r_1 \in V(L_1) - \{z_1\}$ and either $r_2 \in V(P_1) - \{z_1\}$ or $r_2 \in V(P_2) - \{z_2\}$. If $r_2 \in V(P_1) - \{z_1\}$, let $T'$ be obtained from $T$ by including $R$ and deleting internal vertices of $z_1P_1r_2$. If $r_2 \in V(P_2) - \{z_2\}$, let $T'$ be obtained from $T$ by including $R$ and deleting internal vertices of $P_1$. Then, $T'$ is an $X$-tripod in both cases. We say $T'$ is obtained from $T$ by a tripod-transformation of Type I, or simply that $T'$ is obtained from $T$ by a tripod-transformation.

For some distinct indices $i, j \in \{1, 2, 3\}$, let $S_1, S_2 \subseteq G$ be two disjoint paths such that $S_1$ links $x_i$ and a vertex on $L_j - \{x_j\}$, $S_2$ links $x_j$ and a vertex on $L_i - \{x_i\}$, and that $S_1, S_2$ each are internally disjoint from $T$. Without loss of generality, say $i = 1$ and $j = 2$. Let $T'$ be obtained from $T$ by including $S_1 \cup S_2$ and deleting the internal vertices of $P_2$ and $Q_1$. Then, $T'$ is an $X$-tripod. We say $T'$ is obtained from $T$ by a tripod-transformation of Type II, or simply that $T'$ is obtained from $T$ by a tripod-transformation.

Let $r \in V(G) - V(T)$ and $R_1, R_2, R_3$ be three paths in $G$ such that $R_i$ for $i = 1, 2, 3$ links $r$ and some $u_i \in V(L_i)$ and is otherwise disjoint from $T$, $u_i \neq z_i$ for some $i \in \{1, 2, 3\}$, and that $R_1, R_2, R_3$ are pairwise disjoint except for $r$. Let $T'$ be obtained from $T$ by including $R_1 \cup R_2 \cup R_3$ and deleting $V(P_1 \cup P_2 \cup P_3) \setminus \mathcal{Z}(T)$, and let $T''$ be obtained from $T$ by including $R_1 \cup R_2 \cup R_3$ and deleting $V(Q_1 \cup Q_2 \cup Q_3) \setminus \mathcal{Z}(T)$. Then, we say $T'$ and $T''$ are obtained from $T$ by a tripod-transformation of Type III, or simply that $T'$ is
obtained from $T$ by a tripod-transformation.

In the next lemma, we observe a series of properties of tripod-transformations and reroutings on tripods.

**Lemma 2.4.4.** Let $G$ be a graph and $X = \{x_1, x_2, x_3\} \subseteq V(G)$ be a subset of three distinct vertices in $G$. Let $T \subseteq G$ be an $X$-tripod. Then, the following statements are true.

1. There is a 3-separation of $G$ splitting $T$ if and only if there is no $X$-tripod in $G$ that can be obtained from $T$ by a tripod-transformation of Type I.

2. If at least two legs of $T$ are trivial, then there is no $X$-tripod in $G$ that can be obtained from $T$ by a tripod-transformation of Type II.

3. If $T'$ is an $X$-tripod in $G$ obtained from $T$ by a tripod-transformation of Type II, then $T'$ has at least two trivial legs.

4. There exists an $X$-tripod $T' \subseteq G$ that can be obtained from $T$ by a sequence of tripod-transformations of Type I such that some 3-separation of $G$ splits $T'$.

5. If there is no non-trivial $\leq 2$-separation of $(G, X)$, then there exists an $X$-tripod $T' \subseteq G$ obtained from $T$ by a sequence of proper reroutings such that every $T'$-bridge is stable. Furthermore, if some 3-separation $(A, B)$ of $G$ splits $T$, then $(A, B)$ also splits $T'$.

**Proof.** (1)-(3) are simply true due to the definitions of tripod-transformations of Type I and Type II.

To prove (4), we may assume that $T$ is not split by any 3-separation of $G$, since otherwise we could just let $T' = T$. Let $T_0 = T$. By (1), we can recursively find a sequence of $X$-tripods $T_1, T_2, \ldots$ such that for every $i \geq 0$, if $T_i$ is not split by a 3-separation of $G$, then $T_{i+1}$ is an $X$-tripod obtained by a tripod-transformation of Type I. Notice that $3 \leq |\mathcal{L}(T_{i+1})| < |\mathcal{L}(T_i)|$ for all $i$ by the definition of tripod-transformations of Type I. It follows that the sequence of $X$-tripods $T_0, T_1, T_2, \ldots$ must have finite length. Let $T'$ be the last $X$-tripod in this sequence. By (1), it follows that $T'$ is split by some 3-separation of $G$.

It remains to prove (5). Let $W = \mathcal{W}(T)$, and recall that $W$ is a segmenting set of $T$. By Lemma 2.4.1, there exists an $X$-tripod $T' \subseteq G$ obtained from $T$ by a sequence of
proper reroutings such that if all attachments of an unstable $T'$-bridge $B_0$ belong to some segment $P$ of $T'$, then there exist vertices $x, y \in V(P)$ such that some component $K$ of $G \setminus \{x, y\}$ includes a vertex of $B_0$ and is disjoint from $T' \setminus P$. Note that if such an unstable $T$-bridge $B_0$ exists, let the segment $P$ of $T'$, vertices $x, y \in V(P)$, and the component $K$ of $G \setminus \{x, y\}$ be labeled as in the description above. Then, $V(K) \cap V(T)$ is a subset of the set of internal vertices of $P$, and therefore $\{x, y\}$ separates $K$ from $X$ in $G$, a contradiction to the fact that there is no non-trivial $\leq 2$-separation of $(G, X)$. Therefore, every $T'$-bridge is stable. Now, assume that $(A, B)$ is a 3-separation of $G$ that splits $T$. Then, since $T'$ is obtained from $T$ by a sequence of proper reroutings, which contains $A \cap B$ as a subset, every proper rerouting in the sequence is completely included in either $A$ or $B$. It follows that $(A, B)$ also splits $T'$ as well.

Lemma 2.4.5. Let $G$ be a graph and $X = \{x_1, x_2, x_3\} \subseteq V(G)$ be a subset of three distinct vertices such that there exists some $X$-tripod in $G - N[x]$. If there is no non-trivial $\leq 2$-separation of $(G, X)$, there exist an $X$-tripod $T$ satisfying the following properties:

(i) Some 3-separation of $G$ splits $T$.

(ii) Every $T$-bridge in $G$ is stable.

(iii) There is no $X$-tripod in $G$ that can be obtained from $T$ by a tripod-transformation.

Proof. By (4) and (5) of Lemma 2.4.4, there exists an $X$-tripod $T_1$ in $G$ such that every $T_1$-bridge in $G$ is stable and some 3-separation of $G$ splits $T_1$. By (1) of Lemma 2.4.4, there is no $X$-tripod in $G$ that can be obtained from $T_1$ by a tripod-transformation of Type I. If there is no $X$-tripod in $G$ that can be obtained from $T_1$ by a tripod-transformation of Type II, then $T_1$ would be as desired. So we may assume that there exists an $X$-tripod $T_2$ in $G$ that can be obtained from $T_1$ by a tripod-transformation of Type II. By (3) of Lemma 2.4.4, at least two legs of $T_2$ are trivial.

By (4) and (5) of Lemma 2.4.4 again, we can obtain an $X$-tripod $T_3$ from $T_2$ by a sequence of tripod-transformations of Type I, followed by a sequence of proper reroutings, such that every $T_3$-bridge is stable and some 3-separation of $G$ splits $T_3$. Note that $T_2$
has at least two trivial legs, and so does $T_3$, due to the construction of $T_3$. By (1) and (2) of Lemma 2.4.4, there is no $X$-tripod in $G$ that can be obtained from $T$ by a tripod-transformation. Hence, $T_3$ is an $X$-tripod in $G$ satisfying the desired properties.

### 2.5 Exceptional graphs

Say a graph is an exceptional graph if it is isomorphic to one of the $K_{10}$ minor-free graphs stated in Theorem 1.1.5. In a graph $G$, a subset $U \subseteq V(G)$ of vertices is called an island of $G$ if it is a minimal subset of vertices such that $G = G[U] + G[U']$ where $U' = V(G) \setminus U$. An island of size $k$ is called a $k$-island. A partition $P = (V_1, ..., V_t)$ of $V(G)$ is called an island partition of $G$ if every $V_i$ is an island of $G$.

**Lemma 2.5.1.** Every graph $G$ has a unique island partition.

**Proof.** If there exist two non-empty graphs $K$ and $L$ such that $G \cong K + L$, then $G$ has an island partition of size at least 2; otherwise $V(G)$ itself is an island in $G$. This shows the existence of an island partition of $G$.

For the sake of a contradiction, suppose $P = (V_1, ..., V_t)$ and $Q = (U_1, ..., U_r)$ are two distinct island partitions of $G$. Without loss of generality, assume $V_1 \cap U_1 \neq \emptyset$ and $V_1 - U_1 \neq \emptyset$. Since $V_1 - U_1$ is not included in $U_1$ in the partition $Q$, for every $a \in V_1 - U_1$ and every $b \in V_1 \cap U_1$, $ab \in E(G)$. This shows $G[V_1] = G[V_1 - U_1] + G[V_1 \cap U_1]$ where both $V_1 - U_1$ and $V_1 \cap U_1$ are non-empty, a contradiction to the fact that $V_1$ is an island.

Let $G$ be a graph. For any edge $xy \in E(G)$, the vertex $w \in V(G/xy)$ obtained from the contraction of $xy$ in $G$ is called the new vertex of $G/xy$. For a subset $U \subseteq V(G)$ of vertices, denote $q_G(U) = |U| - \omega(G[U])$, i.e. the minimum number of vertices to delete from $U$ such that the remaining vertices induce a complete subgraph. For a partition $P = (V_1, ..., V_t)$ of $V(G)$, denote $q_G(P) = \sum_{V_i \in P} q_G(V_i)$ and $l_G(P) = \lfloor q_G(P)/2 \rfloor$. 

24
Lemma 2.5.2. Let $G$ be a graph on $n$ vertices. Let $\mathcal{P} = (V_1, ..., V_t)$ be a partition of $V(G)$ such that $G = G[V_1] + ... + G[V_t]$. If $\max\{q_G(V_i) : 1 \leq i \leq t\} \leq \frac{1}{2}q_G(\mathcal{P})$, then $G > K_{n-l}$ where $l = l_G(\mathcal{P})$.

Proof. For convenience, let $q_i = q_G(V_i)$ for all $i$. If $q_i = 0$ for some $i$, then every vertex in $V_i$ is adjacent to all other vertices in $G$. We can then just delete $V_i$ from $G$ since $G > K_{n-l}$ if and only if $G - V_i > K_{n-|V_i|-l}$. This means we may assume $q_i \geq 1$ for all $i$. Since $q_i \leq |V_i| - 1$ for every $i$, it follows that $|V_i| \geq 2$ for all $i$. Also, since $\max\{q_G(V_i) : 1 \leq i \leq t\} \leq \frac{1}{2}q_G(\mathcal{P})$ and $q_i \geq 1$ for all $i$, it follows that $t \geq 2$. By relabeling the subsets in $\mathcal{P}$, we may assume $V_i$ are sorted in the decreasing order of $q_i$, meaning that $q_1 \leq \frac{1}{2} \sum_{i=1}^t q_i$.

We are going to prove the lemma by inducting on $\sum_{i=1}^t q_i$. The base case is $\sum_{i=1}^t q_i = 2$, which happens precisely when $t = 2$ and $q_1 = q_2 = 1$. In this case, $l = \lceil(1+1)/2\rceil = 1$.

Let $u_i \in V_i$ for each $i \in \{1, 2\}$ such that $G[V_i - \{u_i\}]$ is complete. By contracting the edge $u_1u_2$ we can get a $K_{n-1}$ minor.

So assume $\sum_{i=1}^t q_i \geq 3$. Choose $u_i \in V_i$ for each $i \in \{1, 2\}$ such that $\omega(G[V_i - \{u_i\}]) = \omega(G[V_i])$. Let $w$ be the new vertex of $G/u_1u_2$. Let $H = G/u_1u_2 - \{w\}$. Note $|H| = n-2$.

To show $G > K_{n-l}$, we will show $H > K_{n-l-1} = K_{|H|-l'}$ where $l' = l - 1$ by proving that a complete minor can be obtained within $l' = l - 1$ contractions from $H$.

Let $W_i = V_i - \{u_i\}$ for $i \in \{1, 2\}$ and let $W_i = V_i$ for all $3 \leq i \leq t$. For each $i$, since $|V_i| \geq 2$, $|W_i| \geq 1$, and therefore $\mathcal{P}' = (W_1, ..., W_t)$ is a partition of $V(H)$ where each $W_i$ is non-empty. Let $q'_{i} = q_H(W_i)$. Note $q'_{i} = q_i$ for $i \geq 3$ and $q'_{1} = q_1 - 1$ for $i = 1, 2$. This implies

$$l' = l-1 = \lfloor(\sum_{i=1}^t q_i)/2\rfloor - 1 = \lfloor\frac{1}{2}((q_1-1)+(q_2-1)) + \frac{1}{2}\sum_{i=3}^t q_i\rfloor = \lfloor\sum_{i=1}^t q'_i/2\rfloor = l_H(\mathcal{P}')$$.

By induction, it suffices to prove $\max\{q'_i : 1 \leq i \leq t\} \leq \frac{1}{2} \sum_{i=1}^t q'_i$. Since $q_i$ are in the decreasing order, either $\max\{q'_i : 1 \leq i \leq t\} = q'_1$ or $\max\{q'_i : 1 \leq i \leq t\} = q'_3$ in the case
If $t \geq 3$. If $\max\{q_i' : 1 \leq i \leq t\} = q_1'$, then

$$q_1' = q_1 - 1 \leq \sum_{i=2}^{t} q_i - 1 = \sum_{i=2}^{t} q_i', $$

meaning that $q_1' \leq \frac{1}{2} \sum_{i=1}^{t} q_i'$. We may then assume $t \geq 3$ and $q_3' > q_1' \geq q_2'$. Since $q_1 \geq q_2 \geq q_3$, it follows that $q_1 = q_2 = q_3$, and we let this value be $q$ for convenience. If $q \geq 2$, then

$$q_3' = q \leq 2q - 2 = (q - 1) + (q - 1) = q_1' + q_2' \leq \sum_{i=1}^{t} q_i' - q_3', $$

meaning $q_3' \leq \frac{1}{2} \sum_{i=1}^{t} q_i'$. We may then assume $q = 1$, meaning $q_i = 1$ for all $i$. If there exists $q_4 \geq 1$, then we get $q_3' \leq \frac{1}{2} \sum_{i=1}^{t} q_i'$ again. If $t = 3$, then $l' = \lceil (0 + 0 + 1)/2 \rceil = 1$. Recall $|W_1| \geq 1$. Let $u_1' \in W_1$ be arbitrary and let $u_3 \in W_3$ such that $H[W_3 - \{u_3\}]$ is complete. By contracting the edge $u_1'u_3$ in $H$, we can get a complete minor. \qed

**Lemma 2.5.3.** Let $G$ be an exceptional graph that is not isomorphic to a non-trivial $(K_{1,1,2,2,2,2,2}, 7)$-cockade or a non-trivial $(K_{2,2,2,2,2,3}, 6)$-cockade. Let $\mathcal{P}$ be the island partition of $G$. Then, $q_G(\mathcal{P}) = 2(|G| - 10) + 1$ and $q_G(C) \leq |G| - 10$ for every $C \in \mathcal{P}$.

**Proof.** Suppose $G \not\cong G_i$ for any $i$. Let $C$ be an island of $G$. Note that $C$ is either an independent set or induces some 5-cycle. If $C$ is an independent set, then $q_G(C) = |C| - 1$; and if $G[C] \cong C_5$, then $q_G(C) = 3$. One can then check the proposition holds by simply counting.

Suppose $G = G_i$ for some $i \in \{1, 2, 3, 4\}$ and $e \not\in E(G)$ such that $G + e \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$. Let $C$ be an island of $G$. Note if $C$ is an independent set, then again $q_G(C) = |C| - 1 = q_{G+e}(C)$. Otherwise, we can write $C = C_1 \cup C_2$ where $C_1, C_2$ are two distinct islands of $G + e$, and $e$ has one end in $C_1$ and the other one in $C_2$. Observe that, in any case, we always have $q_G(C) = q_G(C_1) + q_G(C_2) = q_{G+e}(C_1) + q_{G+e}(C_1)$. It follows that $q_G(\mathcal{P}) = q_{G+e}(\mathcal{P}')$ where $\mathcal{P}'$ is the island partition for $G + e$. If $G + e \cong K_{2,2,2,2,2,3}$,
|G| = 13 and max\{q_G(C) : C \in \mathcal{P}\} = 3; and if G + e \cong K_{2,3,3,3,3}, |G| = 14 and max\{q_G(C) : C \in \mathcal{P}\} = 4. Therefore, the proposition holds for G \cong G_i where i \in \{1, 2, 3, 4\}, too. 

**Lemma 2.5.4.** Let G be an exceptional graph. If G \not\cong K_{2,2,2,2,2,2,2} or K_{2,3,3,3,3,3}, then e(G) = 8|G| - 35. If G \cong K_{2,2,2,2,2,2,2,2} or K_{2,3,3,3,3,3,3}, then e(G) = 8|G| - 34.

**Proof.** If G is not a non-trivial (K_{1,1,2,2,2,2,2,7})-cockade or a (K_{2,2,2,2,2,2,2,2,6,2})-cockade of multiplicity 2, it suffices to check the following:

\[
\begin{align*}
e(K_{1,1,2,2,2,2,2}) &= 61 = 8 \cdot 12 - 35, \\
e(K_{1,2,2,2,3,3}) &= 69 = 8 \cdot 13 - 35, \\
e(K_{2,2,2,2} + C_5) &= 69 = 8 \cdot 13 - 35, \\
e(K_{2,2,3,3,4}) &= 77 = 8 \cdot 14 - 35, \\
e(K_{3,3,3} + C_5) &= 77 = 8 \cdot 14 - 35, \\
e(K_{2,2,2,2,2,2,2}) &= 70 = (8 \cdot 13 - 35) + 1, \text{ and} \\
e(K_{2,3,3,3,3}) &= 78 = (8 \cdot 14 - 35) + 1.
\end{align*}
\]

If G is a (K_{2,2,2,2,2,2,2,6})-cockade of multiplicity 2, then e(G) = 2e(K_{2,2,2,2,2,2,2}) - \binom{6}{2} = 8 \cdot 20 - 35.

If G is a non-trivial (K_{1,1,2,2,2,2,2,7})-cockade, we prove e(G) = 8|G| - 35 by inducting on |G|. The base case is |G| = 12 and was just shown above. Write G = G_1 \cup G_2 such that G_1, G_2 are both (K_{1,1,2,2,2,2,2,7})-cockades, G_1 \cap G_2 \cong K_7, and there is no edge between G_1 - G_2 and G_2 - G_1. By induction, e(G_i) = 8|G_i| - 35 for i \in \{1, 2\}. It follows that

\[
e(G) = e(G_1) + e(G_2) - e(K_7) = 8(|G_1| + |G_2|) - 70 - 21 = 8(|G| + 7) - 91 = 8|G| - 35.
\]

**Lemma 2.5.5.** Let H be a t-connected graph for integer t > 0. Let G be an (H, t)-cockade. Then, G is t-connected.
Proof. We prove this lemma by inducting on $|G|$. The base case $G \cong H$ is trivially true, and we may assume that $G = G_1 \cup G_2$ such that $G_1, G_2$ are both $(H, t)$-cockades, $G_1 \cap G_2 \cong K_t$, and there is no edge between $G_1 - G_2$ and $G_2 - G_1$. By induction, both $G_1, G_2$ are $t$-connected. Let $A \subseteq V(G)$ such that $|A| = t - 1$. Let $A_i = A \cap V(G_i)$ for $i = 1, 2$. Note that $|A_i| \leq t - 1$ for $i = 1, 2$ and $V(G_1 \cap G_2) - A \neq \emptyset$. For $i = 1, 2$, since $G_i$ is $t$-connected, $G_i - A_i$ is connected. Since $V(G_1 \cap G_2) - A \neq \emptyset$, we know that $G - A = (G_1 - A_1) \cup (G_2 - A_2)$ is connected. It follows that $G$ is $t$-connected, as $A$ was chosen arbitrarily.

Lemma 2.5.6. Let $H$ be a graph, and let $\omega(H) = t$. Let $G$ be an $(H, t)$-cockade. Then, $G$ does not contain a subgraph isomorphic to $K_{t+1}$.

Proof. We prove this lemma by inducting on $|G|$. The base case $G \cong H$ is trivially true, as $\omega(H) = t$. We may then assume that $G = G_1 \cup G_2$ such that $G_1, G_2$ are both $(H, t)$-cockades, $G_1 \cap G_2 \cong K_t$, and there is no edge between $G_1 - G_2$ and $G_2 - G_1$. By induction, neither $G_1$ nor $G_2$ contains a subgraph isomorphic to $K_{t+1}$. Since there is no edge between $V(G_1 - G_2)$ and $V(G_2 - G_1)$, every subset $A \subseteq V(G)$ of vertices such that $G[A]$ is a clique is completely contained in either $G_1$ or $G_2$. It follows that $G$ does not have a subgraph isomorphic to $K_{t+1}$.

2.5.1 Deletion Lemma

We will next prove that adding an edge to an exceptional graph would make it have a $K_{10}$ minor, unless the new graph is isomorphic to another exceptional graph. We call this lemma the deletion lemma, which will later be used to show that a minimum counter-example graph to our main theorem on $n$ vertices has exactly $8n - 35$ edges.

Lemma 2.5.7 (Deletion Lemma). Let $G$ be an exceptional graph. Let $x$ and $y$ be two non-adjacent vertices in $G$. Then, either $G + xy$ is an exceptional graph, or $G + xy > K_{10}$.
Proof. We will consider the two cases whether or not $G$ is a non-trivial $(H, t)$-cockade, where $(H, t) = (K_{2,2,2,2,2,3}, 2)$ or $(K_{1,1,2,2,2,2,2}, 7)$, separately.

Case 1: $G$ is NOT a non-trivial $(H, t)$-cockade, where $(H, t) = (K_{2,2,2,2,2,3}, 2)$ or $(K_{1,1,2,2,2,2,2}, 7)$.

Let $\mathcal{P}$ be the island partition of $G$. By Proposition 2.5.3, $q_G(\mathcal{P}) = 2(|G| - 10) + 1$. Note that $x, y$ are in the same island $C \in \mathcal{P}$ as $x, y$ are non-adjacent. We will first show $q_{G+xy}(\mathcal{P}) = 2(|G| - 10)$.

If $G \not\cong G_i$ for some $i \in \{1, 2, 3, 4\}$, then every island of $G$ is either an independent set or induces a 5-cycle. It follows that $q_{G+xy}(C) = q_G(C) - 1$ and therefore $q_{G+xy}(\mathcal{P}) = q_G(\mathcal{P}) - 1 = 2(|G| - 10)$.

If $G \cong G_i$ for some $i \in \{1, 2, 3, 4\}$, let $e \not\in E(G)$ such that $G + e \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$. Notice that since $e \not\in E(G)$, $C$ includes either zero or two ends of $e$. If $C$ does not include any end of $e$, then $C$ is just an independent set and we have $q_{G+xy}(C) = q_G(C) - 1$ again. It follows that $q_{G+xy}(\mathcal{P}) = 2(|G| - 10)$. We may then assume $C$ includes both ends of $e$. This means that $C$ is the disjoint union of islands $C_1, C_2$ of $G + e$, where $(G + e)[C_i] \cong \overline{K}_2$ or $\overline{K}_3$ for both $i = 1, 2$. Note that if $e = xy$, then $G + xy \cong G_i$ for some $i \in \{1, 2, 3, 4\}$, and we would then be done with the proof. So we may assume that $e \neq xy$, meaning that $x, y$ are both contained in $C_i$ for some $i \in \{1, 2\}$. One can then check that we have $q_{G+xy}(C) = q_G(C) - 1$ in all cases. It follows that $q_{G+xy}(\mathcal{P}) = q_G(\mathcal{P}) - 1 = 2(|G| - 10)$.

Now in both cases, we have $q_{G+xy}(\mathcal{P}) = 2(|G| - 10)$. By Lemma 2.5.3, $q_G(C') \leq |G| - 10 = \frac{1}{2}q_{G+xy}(\mathcal{P})$ for every island $C'$ of $G$. It follows that $q_{G+xy}(C') \leq q_G(C') \leq \frac{1}{2}q_{G+xy}(\mathcal{P})$. By Lemma 2.5.2, we conclude that $G + xy > K_{10}$.

Case 2: $G$ is a non-trivial $(H, t)$-cockade, where $(H, t) = (K_{2,2,2,2,2,3}, 2)$ or $(K_{1,1,2,2,2,2,2}, 7)$.

Note that in the case $(H, t) = (K_{2,2,2,2,2,3}, 2)$, $G$ is exactly a $(K_{2,2,2,2,2,3}, 2)$-cockade of multiplicity 2. Write $G = G_1 \cup G_2$ such that $G_1, G_2$ are both $(H, t)$-cockades, $G_1 \cap G_2 \cong K_t$, and there is no edge between $G_1 - G_2$ and $G_2 - G_1$. Due to Case 1 and Lemma 2.5.4,
we may assume that for both \( i = 1, 2 \), \( G_i + zw > K_{10} \) for every pair of non-adjacent vertices \( z, w \in V(G_i) \). This means that we may assume that \( x, y \) not both contained in one of \( G_1 \) and \( G_2 \).

Without loss of generality, say \( x \in V(G_1) - V(G_2) \) and \( y \in V(G_2) - V(G_1) \). Since \( G_1, G_2 \) are both \((H,t)\)-cockades, where \((H,t) = (K_{2,2,2,2,2,3}, 2)\) or \((K_{1,1,2,2,2,2}, 7)\), we know that there exists some \( x' \in V(G_1 \cap G_2) \) such that \( xx' \notin E(G) \), and that \( G[V(G_2 - G_1) \cup \{x'\}] \) is connected due to Lemma 2.5.5. It follows that there exists a path \( Q \subseteq G[V(G_2 - G_1) \cup \{x'\}] \) linking \( x' \) and \( y \). Notice that \( G_1 + xx' > K_{10} \), as \( xx' \notin E(G_1) \). It follows that by contracting edges on \( Q \) to one single vertex in \( G + xy \), we would then obtain resulting graph that contains a subgraph isomorphic to \( G_1 + xx' \), meaning that \( G + xy > K_{10} \).

### 2.5.2 Contraction Lemmas

The goal of this subsection is to prove that if a graph \( G \) has \( \delta(G) \geq 8 \) and two adjacent vertices \( x, y \) such that \( G/xy \) is isomorphic to an exceptional graph, and that \( x, y \) share exactly \( 8|G| - e(G/xy) - 36 \), then either \( G > K_{10} \) or \( G \) is isomorphic to some other exceptional graphs. We call lemmas in this form contraction lemmas. We will prove 5 contraction lemmas in this subsection, namely Lemma 2.5.10, Lemma 2.5.11, Lemma 2.5.12, Lemma 2.5.13, and Lemma 2.5.14. These contraction lemmas will be later used to show that in a minimum counter-example graph to our main theorem, every edge is contained in at least eight triangles.

**Lemma 2.5.8.** Let \( G \) be a graph, and let \( x, y \in V(G) \) be two distinct vertices. Let \( N = N_G(x) \cup N_G(y) - \{x, y\} \). Let \( N' \subseteq N \) and \( \alpha, \beta \in \mathbb{Z}^+ \) be such that \( x, y \) each have at least \( \alpha \) neighbors in \( N' \), and that they have at least \( \beta \) common neighbors in \( N' \). Let \( \mathcal{P} = (C_1, ..., C_t) \) be the island partition of \( G[N'] \), and let \( d \) be the number of 1-islands of \( G[N'] \).

Suppose the triple \( (G[N'], \alpha, \beta) \) satisfies the following properties: (i) \( t \geq 2 \), (ii) \( \alpha \geq \beta + 1 \),
(iii) $\beta \geq \sum_{i=1}^{t} \omega(G[C_i])$, and (iv) $\beta \geq \max\{|C_i|\} + d + 1$. Then, there exist two distinct islands $C_i, C_j$ of $G[N']$ such that $C_i$ contains non-adjacent vertices $w_1, w_2 \in N(x)$, and $C_j$ contains non-adjacent vertices $w_3, w_4 \in N(y)$.

Proof. Let $Z = N(x) \cap N(y) \cap N'$, and let $Z_i = Z \cap C_i$ for all $i = 1, ..., t$. For convenience, let $\omega_i = \omega(G[C_i])$ for all $i = 1, ..., t$. By (iii), $\sum_{i=1}^{t} \omega_i \leq \beta = \sum_{i=1}^{t} |Z_i|$. Note that $\omega_i \geq 1$ for all $i$.

Note that $|Z| = \beta \geq \alpha + 1 \geq 2$. We now observe that we may assume $G[Z]$ is a clique. To see this is true, assume that there exist non-adjacent vertices $w_1, w_2 \in Z_i \subseteq C_i \in \mathcal{P}$ for some island $C_i$. By (iv), there exists some $C_j$ in $\mathcal{P} - \{C_i\}$ such that $|C_j| \geq 2$ and $Z_j \neq \emptyset$. Let $w_3 \in Z_j$ and $w_4 \in C_j - \{w_3\}$. Note that $w_4$ is adjacent to at least one of $x$ and $y$. It follows that $C_i, C_j$ and $w_1, w_2, w_3, w_4$ are as desired.

Therefore, we now assume that $G[Z_i]$ is a clique for all $i = 1, ..., t$, and hence $\omega_i \geq |Z_i|$. Since $\sum_{i=1}^{t} \omega_i \leq \sum_{i=1}^{t} |Z_i|$, we know that $|Z_i| = \omega_i$ and that $G[Z_i]$ is a maximum clique in $G[C_i]$ for all $i = 1, ..., t$. By (ii), there exists an island $C_i \in \mathcal{P}$ such that $x$ has at least $\omega_i + 1$ neighbors in $C_i$, meaning that there exists some vertex of $C_i$ that is adjacent to $x$ but not to $y$. For the sake of a contradiction, we may assume that for every island $C_j \in \mathcal{P} - \{C_i\}$, every vertex in $C_j - Z_j$ is adjacent to $x$ but not to $y$.

Assume for a moment that $C_j - Z_j = \emptyset$ for all $j \neq i$. Since $G[Z_j]$ is a clique and $C_j = Z_j$ is an island of $G[N']$, it follows that $|Z_j| = |C_j| = 1$ for all $j \neq i$. Since $|C_i| \geq 2$, we have $d = t - 1$ and thus $\beta = |Z_i| + t - 1 = |Z_i| + d < |C_i| + d$, a contradiction to (iv). It follows that there exists some island $C_j \in \mathcal{P} - \{C_i\}$ such that $C_j - Z_j \neq \emptyset$. Since every vertex in $C_j - Z_j$ is adjacent to $x$ but not to $y$, we may then assume every vertex in $C_i - Z_i$ is adjacent to $x$ but not to $y$. It follows that every vertex in $N' - Z$ is adjacent to $x$ but not to $y$, meaning that $y$ has exactly $|Z| = \beta$ neighbors in $N'$, a contradiction to (ii). \hfill \square

Lemma 2.5.9. Let $G$ be a graph with $\delta(G) \geq 8$. Suppose there is an edge $xy \in E(G)$ such that $x$ and $y$ share at least 6 common neighbors in $G$. Suppose $G/xy$ is isomorphic to an exceptional graph that is neither a non-trivial $(K_{1,1,2,2,2,2,7})$-cockade nor non-trivial
(K_{2,2,2,2,3}, 6)-cockade. If \( G/xy \cong G_i \) for some \( i \in \{1, 2, 3, 4\} \), let \( e \not\in E(G/xy) \) be the unique non-edge of \( G/xy \) such that \( G/xy + e \cong K_{2,2,2,2,3} \) or \( K_{2,3,3,3,3} \), and let \( H = G/xy + e \); otherwise, let \( H = G/xy \). Let \( w \) be the new vertex of \( G/xy \). Let \( C_w \) be the island of \( H \) that contains \( w \). Then, there are two distinct islands \( C_1, C_2 \) of \( H - C_w \) such that

1. \( C_1 \) contains two non-adjacent vertices \( w_1, w_2 \in N_G(x) \),
2. \( C_2 \) contains two non-adjacent vertices \( w_3, w_4 \in N_G(y) \),
3. for every \( u \in V(G/xy) - N_{G/xy}[w] \), \( u \) is adjacent to \( w_i \) for every \( i \in \{1, 2, 3, 4\} \) in \( G/xy \),
4. if \( G/xy \cong K_{2,2,2,2} + C_5 \) and \( G/xy[C_w] \cong K_2 \), then \( G/xy[C_i] \) is isomorphic to a 5-cycle for some \( i \in \{1, 2\} \).

**Proof.** Let \( N = N_{G/xy}(w) = N_G(x) \cup N_G(y) - \{x, y\} \subseteq V(G) \). Note that \( H = G/xy \) if and only if \( H \not\cong G_i \) for any \( i \in \{1, 2, 3, 4\} \). We will prove this lemma by considering the following two cases: \( C_w \) is an island of \( G/xy \), or \( G/xy \cong G_i \) for some \( i \in \{1, 2, 3, 4\} \), and \( C_w \) is not an island of \( G/xy \).

**Case 1:** \( C_w \) is an island of \( H \) as well as \( G/xy \).

Note now it is not the case that \( G/xy \cong G_i \) for some \( i \in \{1, 2, 3, 4\} \) and an end of \( e \) is contained in \( C_w \). Since \( C_w \) is an island of \( G/xy \), every vertex \( u \in V(G/xy) - N_{G/xy}[w] \) must be in \( C_w \), and thus is adjacent to every vertex in \( V(G/xy) - C_w \) in the graph \( G/xy \). This means that it suffices to find two distinct islands satisfying (1), (2), and (4).

First assume that \( C_w \) is an independent set of \( H \) first, namely \( H \cong K_{2,2,3,3,4} \), \( K_{2,2,2,2,2,3} \), \( K_{2,3,3,3,3} \), \( K_{1,2,2,2,3,3} \), \( K_{1,1,2,2,2,2} \), or \( H \cong K_{3,3,3} + C_5 \) or \( K_{2,2,2,2} + C_5 \) with \( H[C_w] \) not isomorphic to a 5-cycle. In this case, it suffices to find desired islands satisfying (1) and (2), and we will use Lemma 2.5.8 to find them. Note that \( H[N] = H - C_w \), and every island of \( H[N] \) is equal to a unique island of \( H - C_w \). Also note that \( x \) and \( y \) have at least \( \beta = 6 \) common neighbors in \( N \), and since \( \delta(G) \geq 8 \) they each have at least \( \alpha = 7 \) neighbors in \( N \). Let \( P = (C'_1, \ldots, C'_t) \) be the island partition of \( G[N] \), and let \( d \) be the number of
1-islands in it. Notice that for every \( C_i' \in \mathcal{P}, \omega(C_i') = 1 \) if \( H[C_i'] \) is an independent set, and \( \omega(C_i') = 2 \) if \( H[C_i'] \) is isomorphic to a 5-cycle. Since \( H[N] \) contains at most one island isomorphic to a 5-cycle, \( \sum_{i=1}^{t} \omega(H[C_i']) = t + 1 \) if \( H[N] \) contains a 5-cycle, and \( \sum_{i=1}^{t} \omega(H[C_i']) = t \) otherwise. One can then check that the triple \((H[N], \alpha, \beta)\) satisfies all (i)-(iv) in Lemma 2.5.8, and it follows that the desired islands satisfying (1) and (2) can be found.

We may then assume that \( H \cong K_{3,3,3} + C_5 \) or \( K_{2,2,2,2} + C_5 \), and \( H[C_w] \) is isomorphic to a 5-cycle. We will again use Lemma 2.5.8 to find islands \( C_1, C_2 \) that satisfy (1) and (2) first. Let \( N' = V(H) - C_w \subseteq N \), so every island of \( H[N'] \) is equal to a unique island of \( H - C_w \). Since in either case \( w \) has exactly two neighbors in \( C_w \) in \( G/xy \), we know \(|N - N'| = 2\). It follows that \( x \) and \( y \) each have at least \( \alpha' = 7 - 2 = 5 \) neighbors in \( N' \), and they have at least \( \beta' = 6 - 2 = 4 \) common neighbors in \( N' \). One can then check the triple \((H[N'], \alpha', \beta')\) satisfies all (i)-(iv) in Lemma 2.5.8, and therefore there exist distinct islands \( C_1, C_2 \) satisfying (1) and (2).

It remains to show (4) in the case \( H = G/xy \cong K_{2,2,2,2} + C_5 \) and \( G/xy[C_w] \cong K_2 \). Note that we may assume the two islands \( C_1, C_2 \) of \( H - C_w \) that we just found using Lemma 2.5.8 are both 2-islands. Let \( C \) denote the 5-island of \( H \). Since every vertex in \( C \) is adjacent to \( w \) in \( G/xy \), it must be adjacent to at least one of \( x \) and \( y \) in \( G \). Since \( G/xy[C] \) is a 5-cycle, there exist two non-adjacent vertices \( w_1', w_2' \in C \) such that they are both adjacent to \( x \) or both adjacent to \( y \) in the graph \( G \). Without loss of generality, assume \( w_1' \) and \( w_2' \) are both adjacent to \( x \). We can then use \( C \) to replace \( C_1 \), use \( w_1' \) and \( w_2' \) to replace \( w_1 \) and \( w_2 \), and keep \( C_2 = \{w_3, w_4\} \) the same. The modified islands \( C_1 \) and \( C_2 \) are as desired.

**Case 2:** \( C_w \) is not an island of \( H \) as well as \( G/xy \).

In this case, \( G/xy \cong G_i \) for some \( i \in \{1, 2, 3, 4\} \), and \( C_w \) is not an island of \( G/xy \). It suffices to find distinct islands \( C_1, C_2 \) that satisfy (1)-(3) now. Recall that \( e \) is the unique edge in \( E(H) - E(G/xy) \), and note in this case \( C_w \) must contain an end of \( e \). Let \( N' = \{v \in N : v \text{ is adjacent to } u \text{ for every } u \in V(G/xy) - N_{G/xy}[w]\} \subseteq N \). Note that since
every island of $H$ is an independent set, every island $C'_i$ of $H[N']$ is a subset of a unique island of $H - C_w$, say $h(C'_i)$. Assume that there exist distinct islands $C'_i, C'_j$ of $H[N']$ such that $C'_i$ contains two non-adjacent vertices both adjacent to $x$ in $G$, and that $C'_j$ contains two non-adjacent vertices both adjacent to $y$ in $G$. Then by the definition of $N'$, $C_1 = h(C'_i)$ and $C_2 = h(C'_j)$ are distinct islands of $H - C_w$ satisfying (1), (2), and (3). Note that $x$ and $y$ each have at least $\alpha' = 7 - |N - N'|$ neighbors in $N'$, and they have at least $\beta' = 6 - |N - N'|$ common neighbors in $N'$. It follows that to find the desired islands, it suffices to show $(H[N'], \alpha', \beta')$ satisfies (i)-(iv) in Lemma 2.5.8. Again let $\mathcal{P} = (C'_1, \ldots, C'_i)$ be the island partition of $H[N']$, and let $d$ be the number of 1-islands in it. We consider all cases in the following table.

Table 2.1

| $G/xy$ | $[C_w]$ | if $w$ is an end of $e$ | $G[N]$ | $H[N'] = G[N']$ | $|N - N'|$ | $(\alpha', \beta', t, d)$ |
|--------|---------|----------------------|--------|-----------------|---------|-------------------|
| $G_1$  | 2       | yes                  | $K_{1,2,2,2,3}$ | $K_{2,2,2,3}$   | 1       | (6, 5, 4, 0)      |
|        | 2       | no                   | $K_{2,2,2,3}$   | $K_{1,2,2,2,3}$ | 1       | (6, 5, 1)        |
| $G_2$  | 2       | yes                  | $K_{2,2,2,2,2}$ | $K_{2,2,2,2}$   | 2       | (5, 4, 4, 0)     |
|        | 3       | yes                  | $K_{1,2,2,2,2}$ | $K_{2,2,2,2}$   | 1       | (6, 5, 4, 0)     |
|        | 2       | no                   | $K_{2,2,2,2,3}$ | $K_{2,2,2,2,2}$ | 1       | (6, 5, 5, 0)     |
|        | 3       | no                   | $K_{2,2,2,2,2}$ | $K_{1,2,2,2,2}$ | 1       | (6, 5, 5, 1)     |
| $G_3$  | 3       | yes                  | $K_{2,2,3,3}$   | $K_{2,3,3}$     | 2       | (5, 4, 3, 0)     |
|        | 3       | no                   | $K_{2,2,3,3}$   | $K_{2,2,3,3}$   | 1       | (6, 5, 4, 0)     |
| $G_4$  | 2       | yes                  | $K_{2,3,3,3}$   | $K_{3,3,3}$     | 2       | (5, 4, 3, 0)     |
|        | 3       | yes                  | $K_{1,3,3,3}$   | $K_{3,3,3}$     | 1       | (6, 5, 3, 0)     |
|        | 2       | no                   | $K_{3,3,3,3}$   | $K_{2,3,3,3}$   | 1       | (6, 5, 4, 0)     |
|        | 3       | no                   | $K_{2,3,3,3}$   | $K_{1,3,3,3}$   | 1       | (6, 5, 4, 1)     |

For each case in this table, one can check that the triple $(H[N'], \alpha', \beta')$ satisfies (i)-(iv) in Lemma 2.5.8. Therefore, the desired islands can be found.

**Lemma 2.5.10** (Contraction Lemma 1). Let $G$ be a graph with $\delta(G) \geq 8$. Suppose there is an edge $xy \in E(G)$ such that $x$ and $y$ share at least 6 common neighbors in $G$. Suppose $G/xy$ is isomorphic to an exceptional graph that is neither a non-trivial $(K_{1,1,2,2,2,2,2,2}, 7)$-cockade nor a $(K_{2,2,2,2,2,3}, 6)$-cockade of multiplicity 2. If the new vertex of $G/xy$ is not adjacent to all other vertices, then $G > K_{10}$.
Proof. We continue using the same definitions and notations used in Lemma 2.5.9: If $G/xy \cong G_i$ for some $i \in \{1,2,3,4\}$, let $e \notin E(G/xy)$ be the unique non-edge of $E(G/xy)$ such that $G/xy + e \cong K_{2,2,2,2,3}$ or $K_{2,3,3,3,3}$, and let $H = G/xy + e$; otherwise, let $H = G/xy$. Let $w$ be the new vertex of $G/xy$ and $C_w$ be the island of $H$ containing $w$.

By Lemma 2.5.9, we can choose distinct islands $C_1$, $C_2$ of $H - C_w$ and vertices $w_1, w_2, w_3, w_4$ such that (1) $w_1w_2 \notin E(G)$ and $\{w_1, w_2\} \subseteq C_1 \cap N_G(x)$, (2) $w_3w_4 \notin E(G)$ and $\{w_3, w_4\} \subseteq C_2 \cap N_G(y)$, (3) for every $u \in V(G/xy) - N_{G/xy}[w]$, $u$ is adjacent to $w_i$ for every $i \in \{1,2,3,4\}$ in $G/xy$, and (4) if $G/xy \cong K_{2,2,2,2} + C_5$ and $G/xy[C_w] \cong K_2$, then $G/xy[C_i]$ is isomorphic to a 5-cycle for some $i \in \{1,2\}$.

Define $H' = (H - \{w\}) + w_1w_2 + w_3w_4$. We first prove the following claim that $H' > K_{10}$.

Claim 1. $H' > K_{10}$.

Proof of Claim 1. Let $P$ be the island partition of $H$. Note $P$ contains $C_w, C_1$, and $C_2$. Let $P_1 = \{C_w, C_1, C_2\}$ and $P_2 = P - P_1$. Let $P'$ be the partition of $V(H')$ obtained from $P$ by replacing $C_w$ with $C_w - \{w\}$. Let $P'_1 = \{C_w - \{w\}, C_1, C_2\}$ and $P'_2 = P' - P'_1$. Observe that for every $C \in P_2$, $H[C] = H'[C]$ and thus $q_{H'}(C) = q_H(C)$. Also observe that every island in $P$ either is an independent set of size at least 2 or induces a 5-cycle. This implies that for $i \in \{1,2\}$, $q_{H'}(C_1) = q_{H+w_1w_2}(C_1) = q_H(C_1) - 1$ and $q_{H'}(C_2) = q_{H+w_3w_4}(C_2) = q_H(C_2) - 1$, and furthermore that $q_{H'}(C_w - \{w\}) = q_H(C_w) - 1$. Hence, we can write

\[
q_{H'}(P') = \sum_{C' \in P'_1} q_{H'}(C') + \sum_{C' \in P'_2} q_{H'}(C') = (q_H(C_w) - 1) + \sum_{i=1,2} (q_H(C_i) - 1) + \sum_{C \in P_2} q_H(C)
\]

\[
= \sum_{C \in P_1} q_H(C) - 3 \sum_{C \in P_2} q_H(C)
\]

35
\[ = q_H(\mathcal{P}) - 3. \]

Since \( H \) is an exceptional graph which is neither a non-trivial \((K_{1,1,2,2,2,2,2,2}, 7)\)-cockade nor a \((K_{2,2,2,2,2,3}, 6)\)-cockade of multiplicity 2, by Lemma 2.5.3, \( q_H(\mathcal{P}) = 2(|H| - 10) + 1. \) Since \(|H'| = |H| - 1\), it follows that

\[ l_{H'}(\mathcal{P}') = \left\lceil \frac{1}{2} q_{H'}(\mathcal{P}') \right\rceil = \left\lceil \frac{1}{2} \cdot (|H| - 10 + 1 - 3) \right\rceil = |H| - 11 = |H'| - 10. \]

By Lemma 2.5.2, to show \( H' > K_{10} \) it now suffices to prove \( q_{H'}(C') \leq \frac{1}{2} q_{H'}(\mathcal{P}') \) for every island \( C' \in \mathcal{P}' \). Note that \( \frac{1}{2} q_{H'}(\mathcal{P}') = |H| - 11 \) as shown above. Let \( C' \in \mathcal{P}' \) be arbitrary. If \( C' = C_w - \{w\} \), let \( C = C_w \); otherwise let \( C = C' \). Recall that if \( C \in \mathcal{P}_1 \), then \( q_{H'}(C') = q_H(C) - 1 \); and if \( C \in \mathcal{P}_2 \), then \( q_{H'}(C') = q_H(C) \). Note \( H \) is isomorphic to one of the following graphs: \( K_{1,1,2,2,2,2,2,2,2}, K_{1,2,2,2,3,3}, K_{2,2,2,2} + C_5, K_{2,2,3,4}, K_{3,3,3} + C_5, K_{2,2,2,2,2,2,2,2}, \) and \( K_{2,3,3,3,3} \). By checking through every one of these exceptional graphs, we observe that \( q_H(C) \leq |H| - 11 \) unless \( H \cong K_{2,2,2,2,2} + C_5 \) and \( H[C] \cong C_5 \). If \( q_H(C) \leq |H| - 11 \), then we have \( q_{H'}(C') \leq q_H(C) \leq |H| - 11 \). If \( H \cong K_{2,2,2,2,2} + C_5 \) and \( H[C] \cong C_5, q_H(C) = 3 = |K_{2,2,2,2,2} + C_5| - 11 + 1 \). We may then assume \( q_{H'}(C') = q_H(C) \), meaning that \( C \in \mathcal{P}_2 \) and thus \( |C_w| = |C_1| = |C_2| = 2 \), a contradiction property (4) of our choice of \( C_1 \) and \( C_2 \).

Let \( L \) be the graph obtained from \( G \) by contracting the edges \( xw_1 \) and \( yw_3 \). To prove \( G > K_{10} \), we just need to show \( L > K_{10} \). Note if \( G/xy = H \), then \( H' = (H - \{w\}) + w_1w_2 + w_3w_4 \subseteq L \) by properties (1)-(3). By Claim 1, it follows that \( L > K_{10} \) if \( G/xy = H \). We can then assume \( G/xy \neq H \), meaning that \( G/xy \cong G_i \) for some \( i \in \{1, 2, 3, 4\} \). Recall that in this case \( H \cong K_{2,2,2,2,2,2} \) or \( K_{2,3,3,3,3,3} \), and \( e \) is the unique edge in \( H - G/xy \). Let \( a, b \) be the ends of \( e \), and note \( a \) and \( b \) are in two distinct islands of \( H \). Let \( C_a \) and \( C_b \) be the islands of \( H \) containing \( a \) and \( b \), respectively. Observe that if \( w \) is an end of \( e \) or \( e = w_1w_3 \), then \( L \supseteq H' > K_{10} \). This means we may assume \( w \) is not an end of \( e \) and \( e \neq w_1w_3 \). More generally, we can assume \( e \) is not an edge between \( \{w_1, w_2\} \) and \( \{w_3, w_4\} \), since otherwise
we can relabel the vertices $w_1, w_2, w_3, w_4$ and use the previous argument to show $L > K_{10}$.

Let $G' = (G/xy - \{w\}) + w_1w_2 + w_3w_4$. Observe that $G' \subseteq L$, so it suffices to prove $G' > K_{10}$. The rest of the proof now falls into two cases: $\{C_a, C_b\} \subseteq \{C_w, C_1, C_2\}$ or $\{C_a, C_b\} \not\subseteq \{C_w, C_1, C_2\}$.

**Case 1:** $\{C_a, C_b\} \subseteq \{C_w, C_1, C_2\}$.

Note $\{C_a, C_b\} \subseteq \{C_w, C_1, C_2\}$ means that $e = ab$ crosses two distinct islands of $H$ among $C_w, C_1$, and $C_2$. Let $Q$ be the island partition of $G/xy$. Let $D_w$ be the island in $Q$ that contains $w$. Let $Q'$ be the partition of $V(G')$ obtained from $Q$ by replacing $D_w$ with $D_w' = D_w - \{w\}$. We make the following claim.

**Claim 1.** (i) $q_{G'}(Q') = q_{G/xy}(Q) - 3$, and (ii) $q_{G'}(C') \leq |G'| - 10$ for every $C' \in Q'$.

Before proving Claim 1, we first show it implies $G' > K_{10}$. Assume Claim 1 is true. By Lemma 2.5.3, since $G/xy$ is an exceptional graph that neither a non-trivial $(K_{1,1,2,2,2,2,2}, 7)$-cockade nor a $(K_{2,2,2,2,2,3}, 6)$-cockade of multiplicity 2, $q_{G/xy}(Q) = 2(|G/xy| - 10) + 1$. Since $|G'| = |G/xy| - 1$, by (i) in Claim 1,

$$q_{G'}(Q') = q_{G/xy}(Q) - 3 = 2(|G/xy| - 10) + 1 - 3 = 2(|G'| - 10),$$

meaning that $l_{G'}(Q') = \lceil \frac{1}{2}q_{G'}(Q') \rceil = |G'| - 10$. By (ii) and Lemma 2.5.2, it follows that $G' > K_{10}$.

**Proof of Claim 1.** Note that $D_e = C_a \cup C_b$ is one single island in $Q$. $D_e$ and $D_w$ may or may not be distinct islands, but it does not matter. Let $Q_1$ be the minimal subset of $Q$ that covers vertices in $C_1, C_2$, and $C_w$, and let $Q'_1$ be the minimal subset of $Q'$ that covers vertices in $C_1, C_2$, and $C_w - \{w\}$. Observe that $D_w \in Q_1$ and $D_w' \in Q'_1$. Let $Q_2 = Q - Q_1$ and $Q'_2 = Q' - Q'_1$. Since $\{C_a, C_b\} \subseteq \{C_w, C_1, C_2\}$, we know $Q_2 = Q'_2$ and $E_{G/xy}(C) = E_{G'}(C)$ for every $C \in Q_2 = Q'_2$. It follows that $q_{G/xy}(C') = q_{G'}(C)$ for every $C \in Q_2 = Q'_2$.

We now prove (i) first. If $\{C_a, C_b\} = \{C_1, C_2\}$, then $D_e = C_a \cup C_b = C_1 \cup C_2 \in Q$.
and $D_e \neq D_w$. It follows that $Q_1 = \{D_e, D_w\}$ and $Q'_1 = \{D_e, D'_w\}$. Since $e = ab$ is not between $\{w_1, w_2\}$ and $\{w_3, w_4\}$, one of $C_1$ and $C_2$ contains at least 3 vertices. Without loss of generality, assume $|C_1| = 3$, $C_1 = \{w_1, w_2, a\}$, and $b \in C_2$. Note $E(G'[D_e]) - E(G/xy[D_e]) = \{w_1 w_2, w_3 w_4\}$. One can check that if $|C_2| = 2$, $q_{G/xy}(D_e) = 3$ and $q_{G'}(D_e) = 1$; and if $|C_2| = 3$, $q_{G/xy}(D_e) = 4$ and $q_{G'}(D_e) = 2$. Therefore, in any case, we have $q_{G'}(D_e) = q_{G/xy}(D_e) - 2$ and $q_{G'}(D_e) \leq 2$. Since $D_w$ is an independent set in $G/xy$ and $|D'_w| = |D_w| - 1$, it follows that $q_{G'}(D'_w) = q_{G/xy}(D_w) - 1$. Therefore,

$$q_{G'}(Q') = \sum_{C' \in Q'_1} q_{G'}(C') + \sum_{C' \in Q'_2} q_{G'}(C')$$

$$= q_{G'}(D_e) + q_{G'}(D'_w) + \sum_{C' \in Q'_2} q_{G'}(C')$$

$$= (q_{G/xy}(D_e) - 2) + (q_{G/xy}(D_w) - 1) + \sum_{C \in Q_2} q_{G/xy}(C)$$

$$= q_{G/xy}(Q) - 3.$$  

This proves (i) for the case $\{C_a, C_b\} = \{C_1, C_2\}$. To finish proving (i), we may assume $\{C_a, C_b\} = \{C_w, C_i\}$ for some $i \in \{1, 2\}$. Without loss of generality, assume $i = 1$, $C_w = C_a$, and $C_1 = C_b$. It follows that $D_e = D_w \cup C_1 = D_w$, $Q_1 = \{D_w, C_2\}$, and $Q'_1 = \{D'_w, C_2\}$. It is easy to see that $q_{G'}(C_2) = q_{G/xy}(C_2) - 1$ since $G/xy[C_2] - G'[C_2] = \{w_3 w_4\}$. Since $w$ is not an end of $e$, $w \in C_a - \{a\}$ and is not adjacent to $a$ in $G/xy$. By property (3), both $w_1$ and $w_2$ are adjacent to every vertex that is not adjacent to $w$ in $G/xy$. It follows that $\{w_1, w_2\} \subseteq N_{G/xy}(a)$, and thus $b \neq w_1$ or $w_2$ and $C_1 = \{b, w_1, w_2\}$ is a 3-island. One can then check that if $|C_w| = 2$, $q_{G/xy}(D_w) = 3$ and $q_{G'}(D'_w) = 1$; and if $|C_w| = 3$, $q_{G/xy}(D_w) = 4$ and $q_{G'}(D'_w) = 2$. In either case, we have $q_{G'}(D_w) = q_{G/xy}(D'_w) - 2$ and $q_{G'}(D'_w) \leq 2$. Using the same argument as above, we can show $q_{G'}(Q') = q_{G/xy}(Q) - 3$ again. This finishes proving (i) in Claim 1.

To prove (ii), note when we were proving (i) we also showed that $q_{G'}(D_e) \leq 2$ if
\(\{C_a, C_b\} = \{C_1, C_2\}\), and that \(q_{G'}(D_w) = q_{G'}(D_e - \{w\}) \leq 2\) if \(\{C_a, C_b\} = \{C_w, C_i\}\) for some \(i \in \{1, 2\}\). For every island \(C' \in Q'\) that is not \(D_e\) or \(D_e - \{w\}\), \(|C'| \leq 3\) and therefore \(q_{G'}(C') \leq 2\). It follows that for every \(C' \in Q', q_{G'}(C') \leq 2\). Since \(G/xy \cong K_{2,2,2,2,2,3}\) or \(K_{2,2,2,3,3,3}\), \(|G'| - 10 = |G/xy| - 1 - 10 \geq 13 - 11 = 2\). It follows that \(q_{G'}(C') \leq 2 \leq |G'| - 10\) for every \(C' \in Q'\), which proves (ii).

\(\square\)

**Case 2:** \(\{C_a, C_b\} \not\subseteq \{C_w, C_1, C_2\}\).

Without loss of generality, assume \(C_a \not\subseteq \{C_w, C_1, C_2\}\). In the case \(\{C_a, C_b\} \cap \{C_w, C_1, C_2\} = \emptyset\), we choose \(C_a\) to be a 3-island if possible. For the rest of the proof, the goal is to find a vertex \(a' \in N_{G'}(a)\) such that \(G'/aa' > K_{10}\). Note this then implies \(G' > K_{10}\).

**Case 2.1:** \(H \cong K_{2,2,2,2,2,3}\)

Note \(H \cong K_{2,2,2,2,2,3}\) now has exactly 6 islands, and recall that \(H' = (H - \{w\}) + w_1w_2 + w_3w_4\). Let \(H'' = H' - C_1 \cup C_2 \cup C_w\), and note \(H''\) contains exactly 3 islands of \(H\) and that \(C_a\) is one of them. Also note this implies there exists an island \(C_{a'}\) of \(H''\) such that \(C_{a'} \not\subseteq \{C_a, C_b\}\). Let \(a' \in C_{a'}\). Since \(a'\) and \(b\) are adjacent in \(G'\), it follows that the new vertex of \(G'/aa'\) is adjacent to \(b\) in \(G'/aa'\), and therefore \(G'/aa' = H'/aa'\). This means that to prove \(G' > K_{10}\), we just need to choose an island \(C_{a'}\) of \(H''\) such that \(C_{a'} \not\subseteq \{C_a, C_b\}\) and \(H'/aa' > K_{10}\) for some \(a' \in C_{a'}\). In the following table, we list all possible cases; and for each case, we show our choice of \(C_{a'}\) by giving the size of it as well as the graph \(H/aa'\) where \(a'\) is any vertex in the chosen \(C_{a'}\).

| \(C_w\) | \(|C_1| = |C_2| = 2\) | \(H'' = H' - C_1 \cup C_2 \cup C_w\) | \(|C_a|\) | \(|C_{a'}|\) | \(H'/aa'\) |
|---|---|---|---|---|---|
| 2 | \(|C_1| = |C_2| = 2\) | \(K_{2,2,3}\) | 3 | 2 | \(K_7 + K_{2,2}\) |
| 2 | \(|C_1| = |C_2| = 2\) | \(K_{2,2,3}\) | 2 | 3 | \(K_7 + K_{2,2}\) |
| 2 | \(|C_1| \neq |C_2|\) | \(K_{2,2,2}\) | 2 | 2 | \(K_6 + P_5 + P_2\) |
| 3 | \(|C_1| = |C_2| = 2\) | \(K_{2,2,2}\) | 2 | 2 | \(K_7 + K_{2,2}\) |

We note that in the case \(|C_w| = |C_1| = |C_2| = 2\) and \(|C_a| = 2\) (second row in the table), \(C_b\) must not be the 3-island of \(H''\) by the choice of \(C_a\), which allows us to choose the 3-island of \(H''\) to be \(C_{a'}\). In other cases, it is easy to see that it is possible to choose the
island $C_{a'}$ of $H''$ of the size listed in the table such that $C_{a'} \not\in \{C_a, C_b\}$. Since each case in the table has $H'/aa' \cong K_7 + K_{2,2}$ or $K_6 + \overline{P}_3 + \overline{P}_2$, and both of these two graphs have a $K_{10}$ minor, it follows that $H'/aa' > K_{10}$. It follows that $G'/aa' = H'/aa' > K_{10}$.

**Case 2.2**: $H \cong K_{2,3,3,3}$ In this case, there are fewer islands in $H$, and we will have to choose the vertex $a'$ more carefully. Observe that at least one of $C_1$ and $C_2$ is not equal to $C_b$, so without loss of generality we can assume $C_1 \neq C_b$. In the table below, we list all possible cases with the range to choose $a'$ from and the corresponding resulting graph $H'/aa'$ in each case.

**Table 2.3**

| $|C_{w}|$ | $C_1$ and $C_2$ | $|C_a|$ | choose $a'$ in | $H'/aa'$ |
|---|---|---|---|---|
| 2 | $|C_1| = |C_2| = 3$ | 3 | $C_1 - \{w_1, w_2\}$ | $K_4 + K_{2,3} + \overline{P}_3$ |
| 3 | $|C_1| = |C_2| = 3$ | 2 | $C_1 - \{w_1, w_2\}$ | $K_4 + K_{2,3} + \overline{P}_3$ |
| 3 | $|C_1| = |C_2| = 3$ | 3 | $C_1 - \{w_1, w_2\}$ | $K_3 + K_{2,2,2} + \overline{P}_3$ |
| 3 | $|C_1| = 3, |C_2| = 2$ | 3 | $C_1 - \{w_1, w_2\}$ | $K_5 + K_{2,2,3}$ |
| 3 | $|C_1| = 3, |C_2| = 3, C_2 \neq C_b$ | 3 | $C_2 - \{w_3, w_4\}$ | $K_5 + K_{2,2,3}$ |
| 3 | $|C_1| = 2, |C_2| = 3, C_2 = C_b$ | 3 | $H - C_{w} \cup C_1 \cup C_2 \cup C_a$ | $K_3 + K_{2,2,2} + \overline{P}_3$ |

Observe that for every case in this table, the graph $H'/aa'$ always has a $K_{10}$ minor.

Since $a'$ is always chosen from an island of $H$ that is not $C_a$ or $C_b$, $a'$ is adjacent to $b$ in $G'$, and therefore the new vertex of $G'/aa'$ is adjacent to $b$ in $G'/aa'$. It follows that $G'/aa' = H'/aa' > K_{10}$.

\[ \square \]

**Lemma 2.5.11** (Contraction Lemma 2). Let $G$ be a graph with $\delta(G) \geq 8$. Suppose there is an edge $xy \in E(G)$ such that $x$ and $y$ share exactly 7 common neighbors in $G$. If $G/xy \cong K_{1,2,2,2,3,3}$, then either $G > K_{10}$ or $G$ is isomorphic to $K_{2,2,3,3,4}$, $G_3$, or $G_4$.

In particular, (1) if $G \cong K_{2,2,3,3,4}$, then one of $x$ and $y$ has degree 10 and the other one has degree 11, and (2) if $G \cong G_3$ or $G_4$, let $e$ be the unique non-edge such that $G + e \cong K_{2,3,3,3,3}$: if $G \cong G_3$, one of $x$ and $y$ is an end of $e$, and the other one is not an end of $e$ in a 3-island of $G + e$; if $G \cong G_4$, one of $x$ and $y$ is an end of $e$ in a 3-island, and the other one is not an end of $e$ in a 3-island of $G + e$.  

40
Proof. Let \( w \) be the new vertex of \( G/xy \). By Lemma 2.5.10, we may assume \( w \) is the vertex in \( G/xy \) that is adjacent to all other vertices. Let \( N = N_{G/xy}(w) = N_G(x) \cup N_G(y) - \{x, y\} \). Note \( G[N] \cong K_{2,2,2,3,3} \). It follows that

\[(d_G(x) - 1) + (d_G(y) - 1) - 7 = d_{G/xy}(w) = 12,\]

meaning that \( d_G(x) + d_G(y) = 21 \). Without loss of generality, assume \( d_G(x) \leq d_G(y) \). In the rest of the proof, we consider the following two cases: \( d_G(x) \leq 9 \) and \( d_G(y) \geq 12 \), or \( d_G(x) = 10 \) and \( d_G(y) = 11 \).

**Case 1:** \( d_G(x) \leq 9 \) and \( d_G(y) \geq 12 \). Note that \( d_G(y) \geq 12 \) means that there is at most one vertex in \( N \) that is not adjacent to \( y \). Choose \( y' \in N \) such that \( y \) is adjacent to every vertex in \( N - \{y'\} \). Let \( C_{y'} \) be the island of \( G[N] \) that contains \( y' \). Since \( d_G(x) \geq 8 \), \( x \) has at least 6 neighbors in \( N - \{y'\} \). Since \( G[N] \cong K_{2,2,2,3,3} \), there exist two non-adjacent vertices \( w_1, w_2 \in N - \{y'\} \) that are both adjacent to \( x \). Let \( C_x \) be the island of \( G[N] \) that contains \( w_1 \) and \( w_2 \). Note \( C_x \) and \( C_{y'} \) may or may not be the same island, but it does not matter. Let \( C' \) be an island of \( G[N] - C_x \cup C_{y'} \), and let \( w_3 \in C' \). By contracting \( y'w_3 \) and \( xw_1 \), we can obtain a resulting graph that has a subgraph isomorphic to \( G' = (G[N \cup \{y\}])/y'w_3 + w_1w_2 \). Let \( H = G[N \cup \{y\}]/y'w_3 \), and note it is now enough to prove \( G' = H + w_1w_2 \) has a \( K_{10} \) minor. Since \( y \) is adjacent to every vertex in \( N - \{y'\} \), it is adjacent to \( w_3 \) in \( G \) and thus adjacent to the new vertex of \( H \). It follows that \( H \cong (G[N \cup \{y\}]) + yy'/y'w_3 \). Since \( G[N \cup \{y\}] + yy' \cong K_{1,2,2,3,3} \) and \( y', w_3 \) are in two distinct islands of \( G[N \cup \{y\}] + yy' \) of size at least 2, it follows that \( H \cong (G[N \cup \{y\}]) + yy'/y'w_3 \) is isomorphic to \( K_1 + K_{2,3,3}, K_3 + K_{2,2,3}, \) or \( K_{1,1,2,2,2,2} \). Note that in the graph \( H = (G[N \cup \{y\}])/y'w_3 \), \( w_1 \) and \( w_2 \) are not adjacent. This means that to prove \( G' = H + w_1w_2 \) has a \( K_{10} \) minor, it suffices to show for every \( f \notin E(H) \), \( H + f > K_{10} \). Let \( f \notin E(H) \) be arbitrary. If \( H \cong K_4 + K_{2,3,3}, \) then \( H + f \cong K_6 + K_{3,3} \) or \( K_4 + K_{2,3} + P_3 \); if \( H \cong K_3 + K_{2,2,3}, \) then \( H + f \cong K_5 + K_{2,2,3} \) or \( K_3 + K_{2,2,2} + P_3 \); and if \( H \cong K_{1,1,2,2,2,2} \), then \( H + f \cong K_4 + K_{2,2,2,2} \). In any case,
$H + f$ is isomorphic to a graph that has a $K_{10}$ minor, and it follows that $H + f > K_{10}$.

**Case 2:** $d_G(x) = 10$ and $d_G(y) = 11$. In this case, $x$ has exactly 3 non-neighbors and $y$ and exactly 2 non-neighbors in $N$. Let $x_1, x_2, x_3$ and $y_1, y_2$ be the non-neighbors of $x$ and $y$, respectively. Note that $\{x_1, x_2, x_3\} \cap \{y, y_1\} = \emptyset$, and $\{x_1, x_2, x_3\} \subseteq N_G(y)$ and $\{y_1, y_2\} \subseteq N_G(x)$. Let $N' = \{x_1, x_2, x_3, y_1, y_2\} \subseteq N$.

Assume $y_1$ and $y_2$ are in two distinct islands of $G[N]$. Let these two islands be $C_{y_1}$ and $C_{y_2}$. Note there are exactly three islands of $G[N] - C_{y_1} \cup C_{y_2}$. If some island $C'$ of $G[N] - C_{y_1} \cup C_{y_2}$ is such that every vertex in $C'$ is adjacent to $x$, let the two islands in $G[N] - C_{y_1} \cup C_{y_2} \cup C'$ be $C_{x_1}$ and $C_{x_2}$. Let $w_1 \in C_{x_1}, w_2 \in C_{x_2},$ and $w_3 \in C'$. Contract $y_1w_1,$ $y_2w_2,$ and $xw_3$ in $G$, and we can get a resulting graph isomorphic to $K_7 + K_{2,2} > K_{10}$.

We may then assume that the three islands in $G[N] - C_{y_1} \cup C_{y_2}$ are $C_{x_1}, C_{x_2},$ and $C_{x_3}$, and that $x_i \in C_{x_i}$ for each $i \in \{1, 2, 3\}$. Now, each one of the five islands of $G[N]$ contains exactly one vertex in $N$. Observe that we can choose a vertex $w_0 \in N - N'$ that is in some 3-island of $G[N] - C_{y_2}$. Contract edges $x_1y_1, x_2y_2,$ and $x_3w_0$, and we can then obtain a graph isomorphic to $K_{11} > K_{10}$.

Now we may assume $y_1$ and $y_2$ are in the same island $C_y$ of $G[N]$, meaning that $N'$ can cover up to 4 islands of $G[N]$. This means there exists some island $C_0$ of $G[N]$ such that $N' \cap C_0 = \emptyset$. Choose $C_0$ to be a 3-island if possible, and let $w_0$ be a vertex in $C_0$.

If $N'$ covers exactly four islands of $G[N]$, then $x_1, x_2, x_3$ are in three distinct islands that are distinct from $C_y$. By contracting $x_1y_1, x_2y_2,$ and $x_3w_0$, we can obtain a resulting graph isomorphic to $K_7 + K_{2,2}$ if $|C_y| = 2$ and $K_{11}$ if $|C_y| = 3$. In either case, the resulting graph has a $K_{10}$ minor and hence $G > K_{10}$.

Assume $N'$ covers exactly three islands of $G[N]$, say $C_y, C_1,$ and $C_2$. Without loss of generality, assume $x_i \in C_i$ for $i = 1, 2$. If $x_3 \in C_y$, then $C_y = \{y_1, y_2, x_3\}$. By contracting $x_1y_1, x_2y_2,$ and $xw_0$, we can obtain a resulting graph isomorphic to $K_7 + K_{2,2} > K_{10}$ since we chose $C_0$ to be a 3-island if possible. Without loss of generality, we may then assume $x_3 \in C_1$. If $|C_y| = |C_1| = 2$, then $G \cong G_4$ if $|C_2| = 2$ and $G \cong G_3$ if $|C_2| = 3$. In
particular, let $e$ be the unique non-edge of $G$ such that $G + e \cong K_{2,3,3,3,3}$, then if $|C_2| = 2$ and $G \cong G_4$, $x$ is an end of $e$ in a 3-island of $G + e$ and $y$ is not an end of $e$ in a 3-island $G + e$; if $|C_2| = 3$ and $G \cong G_3$, $x$ is an end of $e$ and $y$ is not an end of $e$ in a 3-island of $G + e$.

We may then assume at least one of $C_y$ and $C_1$ contains 3 vertices. By contracting $x_1y_1$, $x_2y_2$, and $x_3w_0$, we can get a resulting graph isomorphic to $K_{11} > K_{10}$ if $|C_y| = |C_1| = 3$ or $K_7 + K_{2,2} > K_{10}$ if exactly one of $C_y$ and $C_1$ is a 3-island.

Finally, consider that $N'$ covers exactly two islands of $G[N]$. If $|C_y| = 2$, then $x_1, x_2, x_3$ form a 3-island of $G[N]$, meaning that $G \cong K_{2,3,3,3,4}$ with $d(x) = 10$ and $d(y) = 11$. If $|C_y| = 3$, first assume $C_1 = \{x_1, x_2, x_3\}$. By contracting $x_1y_1, x_2y_2,$ and $x_3w_0$, we can obtain a resulting graph isomorphic to $K_7 + K_{2,2} > K_{10}$. We can then assume, without loss of generality, that $C_y = \{y_1, y_2, x_3\}$ and $\{x_1, x_2\} \subseteq C_1$. If $|C_1| = 3$, contract $x_1y_1$, $x_2y_2$, and $xw_0$, and we can obtain a graph isomorphic to $K_7 + K_{2,2} > K_{10}$. If $|C_1| = 2$, then by the choice of $C_0$ we have $|C_0| = 3$. Let $w_1, w_2$ be two vertices from distinct islands in $G[N] - C_y \cup C_1 \cup C_0$. By contracting $y_1w_1, y_2w_2$, and $xw_0$, we will obtain a graph isomorphic to $K_7 + K_{2,2} > K_{10}$.

\[\square\]

**Lemma 2.5.12** (Contraction Lemma 3). Let $G$ be a graph with $\delta(G) \geq 8$. Suppose there is an edge $xy \in E(G)$ such that $x$ and $y$ share exactly 7 common neighbors in $G$. If $G/xy \cong K_{1,1,2,2,2,2,2}$, then either $G > K_{10}$ or $G$ is isomorphic to $K_{1,2,2,2,2,3,3}$, $G_1$, or $G_2$.

In particular, (1) if $G \cong K_{1,2,2,2,2,3,3}$, then $d_G(x) = d_G(y) = 10$, and (2) if $G \cong G_1$ or $G_2$, let $e$ be the unique non-edge of $G$ such that $G + e \cong K_{2,2,2,2,2,3}$; if $G \cong G_1$, one of $x$ and $y$ is in the 3-island in $G + e$ and the other one is an end of $e$; if $G \cong G_2$, one of $x$ and $y$ is the end of $e$ in the 3-island of $G + e$, and that the other one is not an end of $e$ and in a 2-island of $G$.

**Proof.** Let $w$ be the vertex in $G/xy$ obtained by the contraction of $xy$ in $G$. Let $C_w$ be the island of $G/xy$ that contains $w$. By Lemma 2.5.10, we may assume $w$ is adjacent to all other vertices in $G/xy$, meaning that $|C_w| = 1$. Let $N = N_{G/xy}(w) = N_G(x) \cup N_G(y) - \{x, y\}$.
Note $G[N] \cong K_{1,2,2,2,2,2}$. It follows that
\[
(d_G(x) - 1) + (d_G(y) - 1) - 7 = d_{G/xy}(w) = 11,
\]
meaning that $d_G(x) + d_G(y) = 20$. Without loss of generality, assume $d_G(x) \leq d_G(y)$. Note that $\delta(G) \geq 8$. We will proceed the rest of the proof following these three cases:

**Case 1:** $d_G(x) = 8$ and $d_G(y) = 12$. Since $d_G(y) = 12$, $y$ is adjacent to every vertex in $N$ and $x$ has 7 neighbors in $N$. Since $G[N] \cong K_{1,2,2,2,2,2}$ has exactly 6 islands, there exists a 2-island $\{w_1, w_2\}$ of it such that both $w_1$ and $w_2$ are adjacent to $x$. By contracting $xw_1$, we get a resulting graph isomorphic to $K_1 + K_{2,2,2,2} > K_{10}$, and therefore $G > K_{10}$.

**Case 2:** $d_G(x) = 9$ and $d_G(y) = 11$. In this case, $x$ has exactly 8 neighbors in $N$, and that there is a unique $y' \in N$ that is not adjacent to $y$. Let $C_{y'}$ be the island of $G[N]$ that contains $y'$. If $|C_{y'}| = 2$, then $x$ has at least 6 neighbors in $N - C_{y'}$. Since $G[N] - C_{y'} \cong K_{1,2,2,2,2,2}$, there exists a 2-island $C_1 = \{w_1, w_2\}$ of it such that both $w_1$ and $w_2$ are adjacent to $x$. Let $C_2 = \{w_3, w_4\}$ be any 2-island of $G[N] - C_{y'} \cup C_1$. Note that both $w_3$ and $w_4$ are adjacent to $y$ and $y'$. Contracting $xw_1$ and $y'w_3$ in $G$, and we get a resulting graph isomorphic to $K_7 + K_{2,2} > K_{10}$, implying that $G > K_{10}$. We may then assume $|C_{y'}| = 1$. Note $x$ has exactly three non-neighbors in $N$. Call them $x_1, x_2$, and $x_3$. If $x_1, x_2, x_3$ are in three distinct islands in $G[N - y'] \cong K_{2,2,2,2,2,2}$, then there remain two 2-islands $C_1 = \{w_1, w_2\}$ and $C_2 = \{w_3, w_4\}$ that do not contain any non-neighbor of $x$. By contracting $x_1w_1$, $x_2w_3$, and $x_3y'$, we obtain a resulting graph isomorphic to $K_{10}$. Without loss of generality, we may then assume $\{x_1, x_2\}$ is a 2-island of $G[N]$. It follows that $G \cong G_2$. In particular, let $e \notin E(G)$ be the unique non-edge $G + e \cong K_{2,2,2,2,2,3}$. It follows that $x$ is the end of $e$ in the 3-island of $G + e$ and $y$ is not an end of $e$ in a 2-island of $G$.

**Case 3:** $d_G(x) = d_G(y) = 10$. In this case, $x$ and $y$ each have exactly two non-
neighbors in $N$. Let the non-neighbors for $x$ be $x_1$ and $x_2$ and the non-neighbors for $y$ be $y_1$ and $y_2$. Note that $x_1, x_2, y_1, y_2$ are four distinct vertices. Assume $\{x_1, x_2\}$ is a 2-island of $G[N]$ first. If $\{y_1, y_2\}$ is another 2-island of $G[N]$, then $G \cong K_{1,2,2,2,3,3}$ with $d_G(x) = d_G(y) = 10$. If one of $y_1$ and $y_2$ is in the unique 1-island of $G[N]$, then $G \cong G_1$. In particular, let $e \not\in E(G)$ be the unique non-edge such that $G + e \cong K_{2,2,2,2,3,3}$, and then $x$ is in the 3-island of $G + e$ and $y$ is an end of $e$. If $y_1$ and $y_2$ are in two distinct 2-islands in $G[N] - \{x_1, x_2\}$, by contracting $x_1y_1$ and $x_2y_2$ we get a resulting graph isomorphic to $K_7 + K_{2,2} > K_{10}$. We may then assume $x_1, x_2$ are in distinct islands of $G[N]$, and by symmetry $y_1$ and $y_2$ are in distinct islands of $G[N]$ too. At most one vertex among $x_1, x_2, y_1, y_2$ is in the 1-island, so without loss of generality assume that $x_1$ and $x_2$ are in two distinct 2-islands. Note that there exists a 2-island $C_0$ of $G[N]$ such that both vertices in $C_0$ are common neighbors for $x$ and $y$. Let $w_0 \in C_0$. This implies that the new vertex of $G/xyw_0$ is adjacent to both $y_1$ and $y_2$. It follows that $G/xyw_0 \cong K_3 + K_{2,2} + \overline{P}_5$. It is easy to observe that $K_3 + K_{2,2} + \overline{P}_5 > K_{10}$, implying that $G > K_{10}$.

\[ \square \]

**Lemma 2.5.13** (Contraction Lemma 4). Let $G$ be a graph with $\delta(G) \geq 8$. Suppose there is an edge $xy \in E(G)$ such that $x$ and $y$ share exactly 7 common neighbors in $G$. If $G/xy$ is a $(K_{1,1,2,2,2,2,2,7})$-cockade, then either $G > K_{10}$ or $G$ is isomorphic to $K_{1,2,2,2,3,3}, G_1$, or $G_2$.

In particular, (1) if $G \cong K_{1,2,2,2,3,3}$, then $d_G(x) = d_G(y) = 10$, and (2) if $G \cong G_1$ or $G_2$, let $e$ be the unique non-edge of $G$ such that $G + e \cong K_{2,2,2,2,3,3}$: if $G \cong G_1$, one of $x$ and $y$ is in the 3-island in $G + e$ and the other one is an end of $e$; if $G \cong G_2$, one of $x$ and $y$ is the end of $e$ in the 3-island of $G + e$, and that the other one is not an end of $e$ and in a 2-island of $G$.

**Proof.** We will prove the lemma by inducting on $|G/xy|$. The base case is $G/xy \cong K_{1,1,2,2,2,2,2,2}$, which is proved in Lemma 2.5.12. Assume that $G/xy$ is a non-trivial $(K_{1,1,2,2,2,2,2,7})$-cockade. Choose subgraphs $H_1$, $H_2$ of $G/xy$ such that $H_1, H_2$ are both $(K_{1,1,2,2,2,2,2,7})$-
cockades, $H_1 \cup H_2 \cong G/xy$, and $H_1 \cap H_2 \cong K_7$. Let $w$ be the new vertex of $G/xy$. For each $i \in \{1, 2\}$, let $H^*_i = G[(V(H_i) - \{w\}) \cup \{x, y\}]$. Observe that since each $H_i$ is a $(K_{1,1,2,2,2,2,7})$-cockade, $\delta(H_i) \geq 10$. It follows that for every vertex $v \in V(H^*_i) - \{x, y\}$, $d_{H^*_i}(v) \geq d_{H_i}(v) \geq 10$. Suppose $w \in V(H_1) - V(H_2)$. Then $x, y$ have exactly 7 common neighbors in $H^*_1$, implying that $d_{H^*_1}(x) \geq 8$ and $d_{H^*_1}(y) \geq 8$. It follows that $\delta(H^*_1) \geq 8$, and by induction we may assume $H^*_1$ is isomorphic to $K_{1,2,2,2,3,3}$, $G_1$, or $G_2$. This is then a contradiction, since $H^*_1$ contains a subgraph isomorphic to $K_7$ but any one of these three exceptional graphs does not.

By symmetry, we may assume $w \in V(H_1 \cap H_2)$. Let $S = V(H_1 \cap H_2) - \{w\}$, and note that $G[S] \cong K_6$. Let $Z \subseteq V(G)$ be the set of the 7 common neighbors of $x$ and $y$. Assume $Z \subseteq V(H^*_1)$ for a moment. This again implies that both $x$ and $y$ have at least 8 neighbors in $H^*_1$ and therefore $\delta(H^*_1) \geq 8$. By induction, $H^*_1$ is isomorphic to $K_{1,2,2,2,3,3}$, $G_1$, or $G_2$. If $H^*_1 \cong G_1$ or $G_2$, let $f \notin E(H^*_1)$ be the unique non-edge such that $H^*_1 + f \cong K_{2,2,2,2,2,3}$ and let $L = H^*_1 + f$; if $H^*_1 \cong K_{1,2,2,2,3,3}$, let $L = H^*_1$. Note that in either case, $L$ has exactly 6 islands. Since $H^*_1[S] \cong K_6$, every vertex in $S$ is in a distinct island in $L$. Since $x$ and $y$ are adjacent in $L$, they must be in different islands. It follows that there exist unique vertices $x', y' \in S$ such that $x'$ and $x$ are in the same island, and that $y'$ and $y$ are in the same island of $L$. Let $K$ be a component of $H^*_2 - H^*_1$. Note that $N_G(K) \subseteq S \cup \{x, y\}$. By Lemma 2.5.5, since $K_{1,1,2,2,2,2,2}$ is 7-connected and $H_2$ is a $(K_{1,1,2,2,2,2,7})$-cockade, we know that $H_2$ is 7-connected. It follows that $|N_{H_2}(K)| \geq 7$ and therefore $|N_G(K)| = |N_{H_2^*(K)}| \geq 7$. Since $|S \cup \{x, y\}| = 8$, without loss of generality we can assume both $x$ and $x'$ are contained in $N_G(K)$. By contracting all vertices in $K$ to $x$, we can get a resulting graph on $V(H^*_1)$ that contains $H^*_1 + xx'$ as a subgraph. Since $x$ and $x'$ are in the same island of $L$, $xx' \neq f$ in the case $H^*_1 \cong G_1$ or $G_2$. By Lemma 2.5.7, $H^*_1 + xx' \cong K_{10}$ and therefore $G > K_{10}$.

Now, we may assume $Z \nsubseteq V(H^*_1)$, and by symmetry we may also assume $Z \nsubseteq V(H^*_2)$. 46
For each \( i \in \{1, 2\} \), let \( Z_i = Z \cap V(H_i^* - H_{3-i}^*) \), and note that \( |Z_i| \geq 1 \). It follows that

\[
|S - Z| = 6 - |Z \cap S| = 6 - (7 - |Z_1| - |Z_2|) = |Z_1| + |Z_2| - 1 \geq |Z_i|
\]

for every \( i \in \{1, 2\} \). Again by Lemma 2.5.5, \( H_1, H_2 \) are both 7-connected. This implies that \( H_i - \{w\} - S \cap Z \) is \((6 - |Z \cap S|)\)-connected, and equivalently \(|S - Z|\)-connected, for each \( i \in \{1, 2\} \). Since \(|Z_i| \leq |S - Z|\), there exist \(|Z_i|\) disjoint paths from \( Z_i \) to \( S - Z \) in \( H_i - \{w\} - S \cap Z \) for each \( i \in \{1, 2\} \). For each \( i \in \{1, 2\} \), choose disjoint paths \( P_i^1, ..., P_i^{|Z_i|} \) between \( Z_i \) and \( S - Z \) in \( H_i - \{w\} - S \cap Z \), and let \( Z_i' \subseteq S - Z \) be the set of endpoints of these paths in \( S - Z \). Contract \( P_i^1, ..., P_i^{|Z_i|} \), and let \( H_i' \) be the induced subgraph on \( V(H_i^*) \) of the resulting graph. Note now in \( H_i' \), \( x \) and \( y \) have exactly 7 common neighbors, implying that \( \delta(H_i') \geq 8 \). By induction, for each \( i \in \{1, 2\} \), \( H_i' \) is isomorphic to \( K_{1,2,2,2,3,3}, G_1, \) or \( G_2 \), with detailed positions of \( x \) and \( y \) in \( H_i' \) described in the statement of the lemma.

For each \( i \in \{1, 2\} \), if \( H_i' \cong G_1 \) or \( G_2 \), let \( e_i \notin E(H_i') \) be the unique non-edge such that \( H_i' + e_i \cong K_{2,2,2,2,3} \) and let \( L_i = H_i' + e_i \); and if \( H_i' \cong K_{1,2,2,2,3,3} \), let \( L_i = H_i' \). In either case, \( L_i \) contains exactly 6 islands, implying that every vertex in \( S \) is in a distinct island of \( L_i \). By induction, for each \( i \in \{1, 2\} \), if \( H_i' \cong K_{1,2,2,2,3,3} \), then \( d_{H_i'}(x) = d_{H_i'}(y) = 10 \), if \( H_i' \cong G_1 \), one of \( x \) and \( y \) is in the 3-island in \( L_i \) and the other one is an end of \( e_i \); if \( H_i' \cong G_2 \), one of \( x \) and \( y \) is the end of \( e_i \) in the 3-island of \( L_i \), and that the other one is not an end of \( e_i \) and in a 2-island of \( H_i' \). Observe that in any of the three cases, there exist exactly three islands, say \( C_i^1, C_i^2, \) and \( C_i^3 \), that are all 2-islands of \( L_i \) such that both vertices in \( C_i^j \) are common neighbors for \( x \) and \( y \) for \( j = 1, 2, 3 \). For every \( i \in \{1, 2\} \) and every \( j \in \{1, 2, 3\} \), since exactly one vertex in \( C_i^j \) is contained in \( S \), the other one must be in \( V(H_i^* - H_{3-i}^*) \). Note the vertex that is in \( C_i^j \cap V(H_i^* - H_{3-i}^*) \) must be adjacent to both \( x \) and \( y \) in the graph \( H_i^* \), meaning that this vertex is contained in \( Z_i \). It follows that \( |Z_i| \geq 3 \) for both \( i = 1, 2 \). Since \(|Z_1| + |Z_2| \leq |Z| = 7 \), without loss of generality we can assume \(|Z_2| = 3 \), and that the three vertices in \( Z_2 \) are precisely one from each \( C_2^1, C_2^2, \) and \( C_2^3 \). For
each $v \in Z_2$, note that $N_{H^*_2}(v) = N_{H^*_2}(v)$. It follows that in the graph $H^*_2$, each $v \in Z_2$ has exactly one non-neighbor in $S$. Recall $P_1, P_2, P_3$ are three disjoint paths linking $S - Z$ and $Z_2$ in $H_2 - \{w\} - S \cap Z$. Therefore, we may choose each path $P_j$ to have length exactly 1 for $j = 1, 2, 3$.

Now, since $x$ and $y$ are adjacent, they are in distinct islands of $L_1$. Since each vertex in $S$ is in a distinct island of $L_1$, there exist unique vertices $x', y' \in S$ such that $x'$ and $x$ are in the same island of $L_1$, and that $y'$ and $y$ are in the same island of $L_1$. Note if $H'_1 \cong G_1$ or $G_2$, $xx' \not\sim e_1$ and $yy' \not\sim e_1$. By Lemma 2.5.7, $H'_1 + xx' > K_{10}$ and $H'_1 + yy' > K_{10}$. Therefore, it suffices to show that there exists a path $Q$ in $H^*_2$ that is internally contained in $H^*_2 - H^*_1 - Z_2$ and links $x$ and $x'$ or links $y$ and $y'$.

Now, since $H^*_2 \cong K_{1,2,2,2,3,3}, G_1$, or $G_2$, we know that $|H^*_2| = |H^*_2| = 13$ and therefore $|V(H^*_2 - H^*_1) - Z_2| = 2$, meaning that there are exactly two vertices, say $u_1, u_2$, in $H^*_2 - H^*_1$ that are not common neighbors of $x$ and $y$ in $G$. Observe that $N_G(u_i) = N_{H^*_2}(u_i) = N_{H^*_2}(u_i)$ for $i = 1, 2$. Also observe that regardless of which graph $H^*_2$ is isomorphic to, any vertex in it that is not $x$ or $y$ has degree at least 10. It follows that $u_1, u_2$ each has at most two non-neighbors in $H^*_2$. We may then assume, in the graph $H^*_2$, $u_1, u_2$ each has exactly two non-neighbors among $\{x, x', y, y'\}$ and that $u_1, u_2$ are adjacent to each other.

Note that in any case, at least one of $x$ and $y$ is contained in a 3-island of $L_2$. Without loss of generality, assume that $y$ is contained in a 3-island of $L_2$. Call this island $C_y$, and note that exactly one of $u_1, u_2$ is contained in $C_y$. Without loss of generality, say $u_1 \in C_y$. Then, observe that in any case we have $u_1$ adjacent to $y$ in $H^*_2$. This means we may assume $N_{H^*_2}(u_1) \cap \{x, x', y, y'\} = \{x, y'\}$. Since $u_1, u_2$ are adjacent to each other, we may further assume that $N_{H^*_2}(u_2) \cap \{x, x', y, y'\} = \{x, y'\}$ too. It follows that $H^*_2[\{x', y, u_1, u_2\}]$ has a supergraph isomorphic to $\overline{K_{2,2}}$, which is a contradiction, since none of $K_{1,2,2,2,3,3}, G_1$, and $G_2$ contains a subgraph isomorphic to $\overline{K_{2,2}}$.

**Lemma 2.5.14** (Contraction Lemma 5). Let $G$ be a graph with $\delta(G) \geq 8$. Suppose there is an edge $xy \in E(G)$ such that $x$ and $y$ share exactly 7 common neighbors in $G$. Suppose
\[ G/xy \text{ is isomorphic to a } (K_{2,2,2,2,2,3}, 6)\text{-cockade of multiplicity 2. Then, } G > K_{10}. \]

**Proof.** Let \( H = G/xy \). Let \( H_1, H_2 \) be induced subgraphs of \( H \) such that \( H = H_1 \cup H_2 \), \( H_1 \cap H_2 \cong K_6 \), and \( H_i \cong K_{2,2,2,2,2,3} \) for \( i = 1, 2 \). Let \( w \) be the new vertex of \( H = G/xy \). If \( w \in V(H_i - H_{3-i}) \) for some \( i \in \{1, 2\} \), then we can apply Lemma 2.5.10 to \( G[V(H_i - \{w\}) \cup \{x, y\}] \) and show that \( G > K_{10} \). Therefore, we may assume that \( w \in V(H_1 \cap H_2) \). For \( i = 1, 2 \), let \( H_i^* = G[V(H_i - \{w\}) \cup \{x, y\}] \).

Observe that for \( i = 1, 2 \), since \( H_i \cong K_{2,2,2,2,2,3} \), \( d_G(v) \geq d_{H_i}(v) \geq 10 \) for every \( v \in V(G) - \{x, y\} \). Let \( Z \subseteq V(G) - \{x, y\} \) be the subset of 7 vertices that are common neighbors of \( x \) and \( y \) in \( G \). For \( i = 1, 2 \), let \( Z_i = Z \cap V(H_i^* - H_{3-i}) \). If \( |Z_1| = 0 \), then \( x \) and \( y \) have exactly 7 common neighbors in \( H_2^* \), which implies that \( d_{H_2^*}(x), d_{H_2^*}(y) \geq 8 \). It follows that \( \delta(H_2^*) \geq 8 \). Since \( H_2^*/xy = H_2 \cong K_{2,2,2,2,2,3} \), by Lemma 2.5.10 it follows that \( H_2^* > K_{10} \) and therefore \( G > K_{10} \). By symmetry, we may then assume \( |Z_i| \geq 1 \) for \( i = 1, 2 \). Let \( S = V(H_1 \cap H_2) - \{w\} \), and it follows that \( |Z_i| + |S \cap Z| \leq 6 \) for \( i = 1, 2 \). Since \( |S| = 5 \), we have \( |Z_i| + 5 - |S - Z| \leq 6 \) and thus \( |S - Z| \geq |Z_i| - 1 \) for \( i = 1, 2 \).

Assume for a moment that \( |S - Z| = |Z_1| - 1 = |Z_2| - 1 \). Then,

\[ 7 = |Z| = |Z_1| + |Z_2| + |S \cap Z| = 2|S - Z| + 2 + |S \cap Z| = 7 + |S - Z|, \]

and therefore \( |S - Z| = 0 \). It follows that \( S \subseteq Z \) and \( |Z_1| = |Z_2| = 1 \). Let \( z_i \) be the unique vertex in \( Z_i \) for \( i = 1, 2 \). Note now \( x \) and \( y \) have exactly 6 common neighbors in \( H_1^* \), meaning that \( d_{H_1^*}(x), d_{H_1^*}(y) \geq 7 \). Note that if \( N(x) \cap V(H_1^* - H_2^*) - \{z_1\} \neq \emptyset \) and \( N(x) \cap V(H_1^* - H_2^*) - \{z_1\} \neq \emptyset \), then we would have \( d_{H_1^*}(x), d_{H_1^*}(y) \geq 8 \) and therefore \( \delta(H_1^*) \geq 8 \). By Lemma 2.5.10, it follows that \( H_1^* > K_{10} \) and thus \( G > K_{10} \). Without loss of generality, we may assume that in the graph \( G \), every vertex in \( N_{G/xy}(w) \cap V(H_1) - \{z_1\} \) is adjacent to \( x \) only but not adjacent to \( y \). Note this means that \( H_1^* - \{y\} \cong K_{2,2,2,2,2,3} \). Now, note that there exists a unique vertex \( s_1 \in S \) that is not adjacent to \( z_1 \) in \( H_1^* \). Since \( S \subseteq Z \), \( s_1 \) is adjacent to \( y \). It follows that \( (H_1^* - \{y\}) \cup \{z_1 s_1\} \subseteq H_1^*/ys_1 \). Since
$H^*_1 - \{y\} \cong K_{2,2,2,2,3}$, by Lemma 2.5.7 we know that $(H^*_1 - \{y\}) \cup \{z_1s_1\} > K_{10}$. It follows that $H^*_1 > K_{10}$ and thus $G > K_{10}$.

Hence, we may assume there exists some $i \in \{1, 2\}$ such that $|S - Z| \geq |Z_i|$. Without loss of generality, assume that $|S - Z| \geq |Z_1|$. Observe that if there exist $|Z_1|$ disjoint paths linking $Z_1$ and $S - Z$ in $H^*_1 - \{x, y\} \cup (S \cap Z)$, then by contracting each of these paths to its end in $S - Z$, we would obtain a resulting graph $H'_{2}$ on $V(H^*_2)$ such that $\delta(H'_{2}) \geq 8$ and $x, y$ have 7 common neighbors in $H'_{2}$. By Lemma 2.5.10, it follows that $H'_{2} > K_{10}$ and thus $G > K_{10}$. Therefore, it suffices to prove such disjoint paths exist.

Note that $H_1 \cong K_{2,2,2,2,3}$ and $G[S] \cong K_5$. If $|Z_1| \geq 3$, then there exists a complete matching from $Z_1$ to $S - Z$, and therefore the desired disjoint paths exist. If $|Z_1| = 2$, we may assume $|Z_1| = |S - Z| = 2$ and the two vertices $u_1, u_2 \in Z_1$ and one vertex $v_1 \in S - Z$ form a 3-island in $H_1$. Let $v_2$ be the vertex in $S - Z$ that is not equal to $u_3$.

Let $w_1 \in V(H^*_1 - H^*_2) - Z_1$ be the unique vertex that is not adjacent to $v_2$, and let $w_2$ be any vertex in $V(H^*_1 - H^*_2) - Z_1 \cup \{w_1\}$. We can then observe that $w_1$ is adjacent to both $u_i$ and $v_i$ for $i = 1, 2$, and therefore paths going through $u_i, w_i, v_i$ in order for both $i = 1, 2$ are as desired. If $|Z_1| = 1$, then we may assume $|S - Z| = 1$ as well and the vertex $u \in Z_1$ is not adjacent to the vertex $v \in S - Z$. Again since $H_1 \cong K_{2,2,2,2,3}$, there exists some $w \in V(H^*_1 - H^*_2) - \{u\}$ that is a common neighbor for $u$ and $v$. The path going through $u, w, v$ in order is then as desired. \qed
In this chapter, we study the structure of possible minimal counter-examples to Theorem 1.1.5. We will prove a series of lemmas on the number of edges, minimum degree, connectivity, and separations of possible minimal counter-examples to Theorem 1.1.5.

In particular, we say a graph $G$ on $n \geq 8$ vertices is a minimal counter-example to Theorem 1.1.5 if the following statements hold:

1. $e(G) \geq 8n - 35$,
2. $G \not\cong K_{10}$,
3. $G$ is not isomorphic to any exceptional graph,
4. For every graph $G'$ such that $8 \leq |G'| \leq n - 1$ and $e(G') \geq 8|G'| - 35$, either $G' > K_{10}$ or $G'$ is isomorphic to an exceptional graph, and
5. Subject to (1)-(4), $e(G)$ is minimum.

To prove Theorem 1.1.5, for the sake of a contradiction, we assume that a minimal counter-example to Theorem 1.1.5 exists. For convenience, we will use $G$ to denote a fixed minimal counter-example to Theorem 1.1.5 in the rest of Chapter 3, Chapter 4, and Chapter 5.

### 3.1 Basic Properties

**Lemma 3.1.1.** $G$ has the following properties:

1. $|V(G)| = n \geq 11$, $e(G) = 8n - 35$.
2. $\delta(G) \geq 10$, and $\delta(N(x)) \geq 8$ for every $x \in V(G)$.
3. If $G'$ is a proper minor of $G$ such that $|G'| \geq 8$, then $e(G') \leq 8|G'| - 34$ and the equality holds if and only if $G' \cong K_{2,2,2,2,3}$ or $K_{2,3,3,3,3}$. 
Proof. To see (1) is true, first observe that there is no graph on at least $8n - 35$ edges for $n = 8$ or 9, and that the only graph on $n = 10$ vertices with at least $8n - 35$ edges is $K_{10}$. It follows that $n \geq 11$. If $e(G) > 8n - 35$, then by the definition of a minimal counter-example to Theorem 1.1.5, $G \setminus e$ must be an exceptional graph for every $e \in E(G)$, which is a contradiction to Lemma 2.5.7. Hence, $e(G) = 8n - 35$.

To show (2), we first prove $\delta(G) \geq 8$ and $\delta(N(x)) \geq 6$ for every $x \in V(G)$. Suppose there exists an edge $xy \in E(G)$ such that $x$ and $y$ share at most 5 common neighbors. This means that $e(G/xy) \geq 8n - 35 - 6 = 8(n - 1) - 35 + 2$. By Lemma 2.5.4, $G/xy$ is not an exceptional graph, a contradiction to the fact that $G$ is a minimal counter-example to Theorem 1.1.5. It follows that every pair of adjacent vertices in $G$ share at least 6 common neighbors, meaning that $\delta(G) \geq 7$ and $\delta(N(x)) \geq 6$ for every $x \in V(G)$. Suppose there exists some $x \in V(G)$ such that $d(x) = 7$. This implies that $N(x) \cong K_7$, which is a subgraph of $G \setminus x$. Note $e(G \setminus x) = 8(n - 1) - 35 + 1$. Since $G$ is a minimal counter-example to Theorem 1.1.5, $G \setminus x$ must be an exceptional graph. By Lemma 2.5.4, $G \setminus x \cong K_{2,2,2,2,2,2,2}$ or $K_{2,3,3,3,3,3}$, which is a contradiction since neither one of these two exceptional graphs contains a subgraph isomorphic to $K_7$. We conclude that $\delta(G) \geq 8$ and $\delta(N(x)) \geq 6$ for every $x \in V(G)$.

To continue proving (2), for the sake of a contradiction, assume that there exists some $xy \in E(G)$ such that $x$ and $y$ have $k$ common neighbors where $k = 6$ or 7. It follows that $e(G/xy) = 8n - 35 - (k + 1) = 8(n - 1) - 35 + (7 - k) \geq 8|G/xy| - 35$. Again since $G$ is a minimal counter-example to Theorem 1.1.5, $G/xy$ must be isomorphic an exceptional graph. If $k = 6$, then $e(G/xy) = 8|G/xy| - 34$, meaning that $G/xy \cong K_{2,2,2,2,2,2,2,2}$ or $K_{2,3,3,3,3,3,3}$ by Lemma 2.5.4. Since $\delta(G) \geq 8$, Lemma 2.5.10 implies that $G > K_{10}$, a contradiction. We may then assume $k = 7$ and $e(G/xy) = 8|G/xy| - 35$, so again $G/xy$ is isomorphic to an exceptional graph. Since $\delta(G) \geq 8$, by Lemma 2.5.10-Lemma 2.5.14, it follows that either $G > K_{10}$ or $G$ is isomorphic to some exceptional graph, again a contradiction to the fact that $G$ is a minimal counter-example to Theorem 1.1.5. Therefore,
we have so far proved that \( \delta(N(x)) \geq 8 \) for every \( x \in V(G) \), which then implies \( \delta(G) \geq 9 \). Notice that if \( d(x) = 9 \) for some \( x \in V(G) \), then we would immediately have \( |N[x]| > K_{10} \) as \( \delta(N(x)) \geq 8 \), a contradiction to the fact that \( G \not\cong K_{10} \). This completes the proof for (2).

To show (3), assume that \( G' \) is a proper minor of \( G \) with \( |G'| \geq 8 \) and \( e(G') \geq 8|G'| - 34 \). It suffices to prove that \( G' \cong K_{2,2,2,2,2,3} \) or \( K_{2,3,3,3,3} \) and \( e(G') = 8|G'| - 34 \). Since \( G \) is a minimal counter-example to Theorem 1.1.5, we know that \( G' \) is isomorphic to some exceptional graph. By Lemma 2.5.4, it follows that \( G' \cong K_{2,2,2,2,2,3} \) or \( K_{2,3,3,3,3} \), and \( e(G') = 8|G'| - 34 \).

\( \square \)

### 3.2 Separations and Connectivity

The major goal of this section is to prove Lemma 3.2.9 that \( G \) is 7-connected. To prove it, we will first need to prove a series of lemmas on separations of \( G \). Some of these lemmas before Lemma 3.2.9 will be used later in this thesis as well.

**Lemma 3.2.1.** Let \((A_1, A_2)\) be a non-trivial separation of \( G \). For \( i = 1, 2 \), let \( G_i = G[A_i] \), and let \( \mathcal{G}_i \) be a non-empty subset of minors of \( G \) on \( V(G_i) \), i.e. every graph in \( \mathcal{G}_i \) has its set of vertices equal to \( V(G_i) \) and can be obtained from \( G \) by deleting or contracting edges that have at least one end in \( V(G_{3-i} - G_i) \). For \( i = 1, 2 \), define \( d(\mathcal{G}_i) \) and \( r(\mathcal{G}_i) \) as follows:

\[
d(\mathcal{G}_i) = \max_{H_i \in \mathcal{G}_i} \{ e(H_i) - e(G_i) \};
\]

\[
r(\mathcal{G}_i) = 1 \text{ if there exists a graph in } \mathcal{G}_i \text{ isomorphic to } K_{2,2,2,2,2,3} \text{ or } K_{2,3,3,3,3}, \text{ and } r(\mathcal{G}_i) = 0 \text{ otherwise.}
\]

Let \( S = A_1 \cap A_2 \). Then,

\[
8|S| \geq 35 + d(\mathcal{G}_1) + d(\mathcal{G}_2) + e(G[S]) - r(\mathcal{G}_1) - r(\mathcal{G}_2) \geq 33 + d(\mathcal{G}_1) + d(\mathcal{G}_2) + e(G[S])
\]

**Proof.** For convenience, let \( d_i = d(\mathcal{G}_i) \) and \( r_i = r(\mathcal{G}_i) \) for \( i = 1, 2 \). Choose \( H_i \in \mathcal{G}_i \) with \( e(H_i) - e(G_i) = d_i \) such that \( H_i \cong K_{2,2,2,2,2,3} \) or \( K_{2,3,3,3,3} \) if possible, for both \( i = 1, 2 \). It follows that \( e(G_1) + e(G_2) = e(H_1) + e(H_2) - d_1 - d_2 \). By (2) of Lemma 3.1.1, \( \delta(G) \geq 10 \). Since \( (A_1, A_2) \) is a non-trivial separation of \( G \), \( |H_i| = |G_i| \geq 11 \) for both \( i = 1, 2 \). Since \( H_1, H_2 \) are both proper minors of \( G \) with \( |H_i| \geq 11 \) for \( i = 1, 2 \), by (3) of Lemma 3.1.1
it follows that 

\[ e(H_i) \leq 8|H_i| - 35 + r_i \text{ for } i = 1, 2. \]

Since \( e(G) = 8n - 35 \) by (1) of Lemma 3.1.1, it follows that

\[
8n - 35 = e(G_1) + e(G_2) - e(G[S]) \\
= e(H_1) + e(H_2) - d_1 - d_2 - e(G[S]) \\
\leq 8(|H_1| + |H_2|) - 70 + r_1 + r_2 - d_1 - d_2 - e(G[S]) \\
= 8n + 8|S| - 70 + r_1 + r_2 - d_1 - d_2 - e(G[S]).
\]

Therefore,

\[
8|S| \geq 35 + d_1 + d_2 + e(G[S]) - r_1 - r_2.
\]

Since \( r_i = r(G_i) \leq 1 \) for both \( i = 1, 2 \), it follows that

\[
8|S| \geq 35 + d_1 + d_2 + e(G[S]) - r_1 - r_2 \geq 33 + d_1 + d_2 + e(G[S]).
\]

\[ \square \]

**Lemma 3.2.2.** \( G \) is 6-connected.

**Proof.** Let \((A_1, A_2)\) be a non-trivial separation of \( G \) of minimum order. Let \( S = A_1 \cap A_2 \).

For \( i = 1, 2 \), let \( G_i = G[A_i] \) and let \( d_i \) be the maximum number of edges that can be added to \( S \) by contracting edges that have at least one end in \( G_{3-i} - G_i \). By Lemma 3.2.1, \( 8|S| \geq 33 + d_1 + d_2 + e(G[S]) \), which implies \( |S| \geq 5 \). For the sake of a contradiction, assume \( |S| = 5 \). It follows that \( d_1 + d_2 + e(G[S]) \leq 7 \). Let \( \delta = \delta(G[S]) \) for convenience.

Since \( S \) is a minimum separating set of \( G \), every vertex in \( S \) has some neighbor in \( G_{3-i} - G_i \) for both \( i = 1, 2 \). It follows that \( d_i \geq |S| - 1 - \delta = 4 - \delta \) for \( i = 1, 2 \), as we can contract all of \( G_{3-i} - G_i \) to some vertex \( v \in S \) with \( d_{G[S]}(v) = \delta \) to make \( v \) adjacent to all other vertices in \( S \). Since \( e(G[S]) \geq \frac{1}{2} |S| \delta = \frac{5}{2} \delta \), it follows that

\[
7 \geq d_1 + d_2 + e(G[S]) \geq 2(4 - \delta) + \frac{5}{2} \delta = 8 + \frac{1}{2} \delta,
\]
which means that \( \delta \leq -2 \), a contradiction.

Lemma 3.2.3. Let \( U \subseteq V(G) \) such that \( U \neq \emptyset \) and \( |N(U)| \leq \delta(G) - 1 \). If there is no non-trivial separation of \( (G[U \cup N(U)], N(U)) \) of order at most \( |N(U)| - 1 \), then for every \( Z \subseteq N(U) \) such that \( |Z| = 4 \), the following statements are true:

1. If \( |N(U)| \leq \delta(G) - 2 \), then \( G[U \cup Z] \) has a \( K_4 \) minor rooted at \( Z \).

2. If \( |N(U)| = \delta(G) - 1 \), then one of the following two statements is true: (2a) For every two vertices \( y_1, y_2 \in Z \), \( G[U \cup Z] \) has a minor \( L \) such that \( V(L) = Z \) and \( L \cup \{y_1, y_2\} \cong K_4 \). (2b) If \( \delta(G) = 11 \), then for every \( Z' \subseteq N(U) - Z \) such that \( |Z'| = 4 \), \( G[U \cup Z'] \) has a \( K_4 \) minor rooted at \( Z' \).

Proof. Let \( H = G[U \cup Z] \). Note we may assume that \( H \) does not have a \( K_4 \) minor rooted at \( Z \). We will show either a contradiction, or that one of (2a) and (2b) holds true and \( |N(U)| = \delta(G) - 1 \).

Since there is no non-trivial separation of \( (G[U \cup N(U)], N(U)) \) of order at most \( |N(U)| - 1 \), there is no non-trivial \( \leq 3 \)-separation of \( (H, Z) \). Choose \( (X, Y) \) to be a 4-separation of \( (H, Z) \) such that \( Y - X \neq \emptyset \), and subject to that \( |Y| \) is minimum. Note that such a separation exists due to the trivial 4-separation \( (Z, Z \cup U) \) of \( (H, Z) \). Let \( Z^* = X \cap Y \). Then note that the minimality of \( |Y| \) implies that there is no non-trivial \( \leq 4 \)-separation \( (G[Y], Z^*) \). Since there is no non-trivial \( \leq 3 \)-separation of \( (H, Z) \), there exist four disjoint paths linking \( Z \) and \( Z^* \) in \( G[X] \). Since \( H \) does not have a \( K_4 \) minor rooted at \( Z \), it follows that \( G[Y] \) does not have a \( K_4 \) minor rooted at \( Z^* \).

By Theorem 2.2.2, one of (ii)-(iv) is true for \( H \) and \( Z \). Since there is no non-trivial \( \leq 3 \)-separation of \( (H, Z) \), we know (iii) is not true. Therefore, one of (ii) and (iv) is true.

Case 1: (ii) is true, i.e. there is a trisection \( (A_1, A_2, B) \) of \( G[Y] \) of order 2 such that \( |Z^* \cap (A_i - B)| = 1 \) for \( i = 1, 2 \).

Let \( z_i \) be the unique vertex in \( Z^* \) that is in \( A_i - B \) for \( i = 1, 2 \), and let \( a, b \) be the two vertices in \( A_1 \cap A_2 \cap B \). Notice that if \( A_1 - B - \{z_1\} \neq \emptyset \), then \( (A_2 \cup B \cup \{z_1\}, A_1) \) would be a non-trivial 3-separation of \( (G[Y], Z^*) \), a contradiction to the fact that there is
no non-trivial $\leq 4$-separation $(G[Y], Z^*)$. By symmetry, it follows that $A_i - B = \{z_i\}$ for $i = 1, 2$. If $B - A_1 \cup A_2 \cup Z^* = \emptyset$, then similarly we would have a non-trivial $\leq 4$-separation $(A_1 \cup A_2 \cup Z^*, B)$ of $(G[Y], Z^*)$, again a contradiction. It follows that $B - A_1 \cup A_2 \cup Z^* = \emptyset$, and therefore $Y - Z^* \subseteq \{a, b\}$. Notice that vertices in $Y - Z^* = Y - X$ have no neighbor in $X - Y$, meaning that $N_G(Y - Z^*) \subseteq Z^* \cup (N_G(U) - Z)$. It follows that

$$|N_G(Y - Z^*)| \leq |Z^* \cup (N_G(U) - Z)| = |Z^*| + |N_G(U) - Z| = |N_G(U)|.$$

If $|N_G(U)| \leq \delta(G) - 2$, then $|N_G(Y - Z^*)| \leq \delta(G) - 2$. This means that every vertex in $Y - Z^*$ has at least two neighbors in $Y - Z^*$ and thus $|Y - Z^*| \geq 3$, a contradiction to the fact that $Y - Z^* \subseteq \{a, b\}$.

If $|N_G(U)| = \delta(G) - 1$, then $|N_G(Y - Z^*)| \leq \delta(G) - 1$. We will prove that (2a) is true. Note that $|N_G(Y - Z^*)| \leq \delta(G) - 1$ means that every vertex in $Y - Z^*$ has at least one neighbor in $Y - Z^*$. Since $Y - Z^* \subseteq \{a, b\}$, it follows that $Y - Z^* = \{a, b\}, ab \in E(G)$, and that $a, b$ each are adjacent to all vertices in $N_G(Y - Z^*) = Z^* \cup (N_G(U) - Z)$. This means that for every pair of vertices $y_1', y_2' \in Z^*, G[Y] = G[Z^* \cup \{a, b\}]$ has a minor $L'$ on $Z^*$ such that $L' \cup \{y_1'y_2'\} \cong K_4$, as we can simply contract the edges $ay_3'$ and $by_4'$, where $Z^* - \{y_1', y_2'\} = \{y_3', y_4'\}$. Recall that there exist four disjoint paths linking $Z$ and $Z^*$ in $H[X]$. It follows that for every pair of vertices $y_1, y_2 \in Z$, $H$ has a minor $L$ such that $V(L) = Z$ and $L \cup \{y_1y_2\} \cong K_4$. Therefore, (2a) is true.

**Case 2:** (iv) is true, i.e. $G[Y]$ can be drawn in the plane so that every vertex in $Z^*$ is incident with the infinite region.

Since $G[Y]$ can be drawn in the plane so that every vertex in $Z^*$ is incident with the infinite region, there exists a planar graph $J$ that can be obtained from $G[Y]$ by making $J[Z^*]$ isomorphic to $K_4^-$. Note that

$$e(J) = e(J[Z^*]) + e(Z^*, Y - Z^*) + e(G[Y - Z^*]) = 5 + e(Z^*, Y - Z^*) + e(G[Y - Z^*]).$$
Since $J$ is planar, $e(J) \leq 3|J| - 6$. It follows that

$$5 + e(Z^*, Y - Z^*) + e(G[Y - Z^*]) = e(J) \leq 3|J| - 6 = 3(|Y - Z^*| + 4) - 6 = 3|Y - Z^*| + 6.$$ 

Therefore,

$$e(Z^*, Y - Z^*) + e(G[Y - Z^*]) \leq 3|Y - Z^*| + 1.$$ 

If $|N(U)| \leq \delta(G) - 2$, then $\delta(G) - |N(U)| \geq 2$. It follows that

$$e(Z^*, Y - Z^*) + e(G[Y - Z^*]) \leq 3|Y - Z^*| + 1.$$ 

Since $e(Z^*, Y - Z^*) + e(G[Y - Z^*]) \leq 3|Y - Z^*| + 1$, it follows that

$$\frac{1}{2} e(Z^*, Y - Z^*) + 3|Y - Z^*| \leq e(Z^*, Y - Z^*) + e(G[Y - Z^*]) \leq 3|Y - Z^*| + 1,$$

meaning that $e(Z^*, Y - Z^*) \leq 2$. By the minimality of $|Y|$ when choosing $(X, Y)$, every vertex in $Z^*$ has at least one neighbor in $Y - Z^*$, meaning $e(Z^*, Y - Z^*) \geq 4$, a contradiction.

It remains to consider $|N(U)| = \delta(G) - 1$, and we will prove that (2b) is true. To prove (2b), assume that $\delta(G) = 11$. Since $J$ is planar, $G[Y - Z^*]$ which is a subgraph of $J$ is also planar, and therefore $e(G[Y - Z^*]) \leq 3|Y - Z^*| - 6$. Recall that $e(Z^*, Y - Z^*) + e(G[Y - Z^*]) \leq 3|Y - Z^*| + 1$. It follows that

$$e(Z^*, Y - Z^*) + 2e(G[Y - Z^*]) \leq 6|Y - Z^*| - 5.$$ 

57
Since $\delta(G) = 11$, we now have

$$e(N(Y - Z^*), Y - Z^*) + 2e(G[Y - Z^*]) = \sum_{v \in Y - Z^*} d_G(v) \geq 11|Y - Z^*|.$$ 

Notice that

$$e(N(U) - Z, Y - Z^*) = \left(e(N(Y - Z^*), Y - Z^*) + 2e(G[Y - Z^*])\right) - \left(e(Z^*, Y - Z^*) + 2e(G[Y - Z^*])\right) \geq 11|Y - Z^*| - (6|Y - Z^*| - 5) = 5|Y - Z^*| + 5.$$ 

Since $N(U) = \delta(G) - 1 = 10$, we have $|N(U)| = |N(U)| - 4 = 6$. It follows that there exist five distinct vertices $v_1, v_2, v_3, v_4, v_5 \in Y - Z^*$ such that each of them is adjacent all 6 vertices in $N(U) - Z$. Let $Z' \subseteq N(U) - Z$ such that $|Z'| = 4$ be arbitrary, and let $Z' = \{z'_1, z'_2, z'_3, z'_4\}$. By contracting edges $z'_i v_i$ for $i = 1, 2, 3$, we would then obtain a $K_4$ minor of $G[(Y - Z^*) \cup Z']$ rooted at $Z'$. Since $G[(Y - Z^*) \cup Z'] \subseteq G[U \cup Z']$, it follows that $G[U \cup Z']$ has a $K_4$ minor rooted at $Z'$.

**Lemma 3.2.4.** Let $S$ be a separating set of $G$. The following statements are true:

1. There is no $w \in S$ such that $G[S - \{w\}]$ is complete.
2. If $|S| \leq \delta(G) - 2$ and is minimum over all separating sets of $G$, then there is no $Z \subseteq S$ with $|Z| = 4$ such that the graph obtained from $G[S]$ by making $Z$ a clique is complete.
3. $G[S]$ contains an independent set of size 3 or two disjoint non-edges.

**Proof.** Let $(A_1, A_2)$ be a non-trivial separation of $G$. Let $S = A_1 \cap A_2$ and $G_i = G[A_i]$ for $i = 1, 2$.

We first prove that (1) implies (3). Assume (1) is true. Since $G[S - \{w\}]$ is not a complete graph for all $w \in S$, for every non-adjacent vertices $x, y \in S$, neither $G[S - \{x\}]$ nor $G[S - \{y\}]$ is a complete graph. It follows that there exists some $z \in S - \{x, y\}$ such
that $xz, yz \notin E(G)$, or that there exist some $z_1, z_2 \in S - \{x, y\}$ such that $z_1z_2 \notin E(G)$. The former case implies that $G[S]$ contains an independent set of size 3 on $\{x, y, z\}$, and the latter case implies that $G[S]$ has two disjoint non-edges, namely $xy$ and $z_1z_2$.

It now suffices to prove (1) and (2) in the rest of this proof. Let $w \in S$ and $Z \subseteq S$ such that $|Z| = 4$. For $i = 1, 2$, let $H_{1}^w$ be the graph obtained from $G_i$ by making $w$ adjacent to all other vertices in $S$, and let $H^Z_i$ be the graph obtained from $G_i$ by making $Z$ a clique. If $S$ is a minimum separating set of $G$ and $|S| \leq \delta(G) - 2$, let $(H_1, H_2)$ be equal to one of $(H_1^w, H_2^w)$ and $(H_1^Z, H_2^Z)$. Otherwise, let $(H_1, H_2) = (H_1^w, H_2^w)$. Let $H = H_1 \cup H_2$. It suffices to prove that $H[S]$ is not a complete graph. For the sake of a contradiction, assume that it is not.

Observe that we can choose $S$ such that it is a minimal separating set of $G$. This is because, if $S$ is not minimal, we would have $(H_1, H_2) = (H_1^w, H_2^w)$, and we can then replace $S$ with a minimal subset $S' \subseteq S$ such that $S'$ separates $G$, and that there exists some $w' \in S'$ where $G[S' \setminus \{w'\}]$ is complete. It follows that $G > H_i$ for $i = 1, 2$ in both cases: If $(H_1, H_2) = (H_1^w, H_2^w)$, since $S$ is a minimal separating set, we can contract all of $V(G_{3-i} - G_i)$ to $w$ for $i = 1, 2$; and if $(H_1, H_2) = (H_1^Z, H_2^Z)$, we know $|S| \leq \delta(G) - 2$ and $S$ is a minimum separating set of $G$, which then allows us to apply Lemma 3.2.3 to obtain that $G > H_i$ for $i = 1, 2$. Finally, notice that $|S| \leq 8$, since otherwise we could contract all of $V(H_1) - S$ to one single vertex and obtain a $K_{10}$ minor of $H_1$, meaning $G > K_{10}$, a contradiction. By Lemma 3.2.2, it follows that $6 \leq |S| \leq 8$.

**Claim 1.** $H_i \not\cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$ for $i = 1, 2$.

**Proof of Claim 1.** For the sake of a contradiction, assume $H_1 \cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$.

Note that $K_{2,3,3,3,3}$ contains no clique of size greater than 5, and $K_{2,2,2,2,2,3}$ contains no clique of size greater than 6. Since $|S| \geq 6$, it follows that $H_1 \cong K_{2,2,2,2,2,3}$, $|S| = 6$, and $H[S] \cong K_6$. Let $d = e(H[S]) - e(G[S])$. Note $e(G_i) = e(H_i) - d$ for $i = 1, 2$, and that
Lemma 3.2.1 it follows that

\[ e(G[S]) + d = e(H[S]) = e(K_6) = 15. \]

Since \( e(G) = 8n - 35 \), we have

\[ 8n - 35 = e(G_1) + e(G_2) - e(G[S]) = e(H_1) - d + e(G_2) - e(G[S]) = e(H_1) + e(G_2) - 15. \]

Since \( H_1 \cong K_{2,2,2,2,2,2,3} \), \( e(H_1) = 70 \) and \( |G_2| = n - (|K_{2,2,2,2,2,2,3}| - |S|) = n - 7 \). It follows that

\[ e(G_2) = 8n - 35 - 70 + 15 = 8(n - 7) - 34 = 8|G_2| - 34. \]

Note that \(|S| \leq 8\), so \(|G_2| > 8\). Notice that \( G_2, H_2 \) are both proper minors of \( G \) such that \( e(H_2) \geq e(G_2) \). By (3) of Lemma 3.1.1, it follows that \( G_2 = H_2 \cong K_{2,2,2,2,2,2,3} \) or \( K_{2,3,3,3,3,3} \). Since \( H_2[S] \) is a clique on \(|S| \geq 6\) vertices, it follows that \( G_2 = H_2 \cong K_{2,2,2,2,2,2,3} \).

This means that \( G \) is isomorphic to a \((K_{2,2,2,2,2,3,3,3,3})\)-cockade of multiplicity 2, which is an exceptional graph, a contradiction.

Now, let \( d_i \) be the maximum number of edges that can be added to \( S \) by contracting edges that have at least one end in \( G_{3-i} - G_i \) for \( i = 1, 2 \). Since \( H[S] \) is complete, \( d_1 = d_2 = e(H[S]) - e(G[S]) \). Since \( H_i \not\cong K_{2,2,2,2,2,3} \) or \( K_{2,3,3,3,3,3} \) for \( i = 1, 2 \), by Lemma 3.2.1 it follows that

\[ 8|S| \geq 35 + d_1 + d_2 + e(G[S]). \]

**Claim 2:** \( G[S] \cong K_7, K_8, \) or \( K_8^- \)

**Proof of Claim 2.** We consider the case \((H_1, H_2) = (H_1^w, H_2^w)\) and the case \((H_1, H_2) = (H_1^Z, H_2^Z)\) separately.

**Case 1:** \((H_1, H_2) = (H_1^w, H_2^w)\). Let \( \delta = d_{G[S]}(w) \). Then \( d_i = |S| - 1 - \delta \) for \( i = 1, 2 \) and \( e(G[S]) = e(S_2) + \delta \). It follows that \( 8|S| \geq 35 + 2(|S| - 1 - \delta) + e(S_2) + \delta \), meaning that \( \delta \geq 33 + e(S_2) - 6|S| \). Since \( \delta \leq |S| - 1 \) and \( 6 \leq |S| \leq 8 \), it follows that \( \delta \geq 6 \) and \(|S| = 7 \) or 8. Therefore, \( G[S] \cong K_7, K_8, \) or \( K_8^- \)
Case 2: \((H_1, H_2) = (H_1^2, H_2^2)\). Let \(z = e(G[Z])\). Then \(d_i = 6 - z\) for \(i = 1, 2\) and 
\[ e(G[S]) = \binom{|S|}{2} - (6 - z). \]
It follows that \(8|S| \geq 35 + 2(6 - z) + \binom{|S|}{2} - (6 - z)\), meaning that \(z \geq 41 + \binom{|S|}{2} - 8|S|\). Since \(z \leq 6\) and \(6 \leq |S| \leq 8\), we have either \(|S| = 7\) and \(z = 6\), or that \(|S| = 8\) and \(z \geq 5\). Again, it follows that \(G[S] \cong K_7, K_8\), or \(K_8^\sim\).

Consider the case \(G[S] \cong K_7\) first, which implies that \(H_i = G_i\) for \(i = 1, 2\). By Claim 1, \(G_i \not\cong K_{2,2,2,2,2,2}\) or \(K_{3,3,3,3,3}\) for \(i = 1, 2\). By Lemma 3.1.1, \(|G_i| \geq 11\) and therefore \(e(G_i) \leq 8|G_i| - 35\) for both \(i = 1, 2\). Therefore, \(e(G_1) + e(G_2) \leq 8(|G| + 7) - 70 = 8n - 14\). On the other hand, \(e(G_1) + e(G_2) = e(G) + \binom{7}{2} = 8n - 35 + 21 = 8n - 14\).

It follows that for both \(i = 1, 2\), \(e(G_i) = 8|G_i| - 35\) and hence \(G_i\) is isomorphic to some exceptional graph, due Lemma 3.1.1 as \(G_i\) is a proper minor of \(G\) on at least 11 vertices.

Note that \(G_i\) contains a clique of size 7 for both \(i = 1, 2\). It follows that each \(G_i\) for \(i = 1, 2\) is isomorphic to a \((K_{1,1,2,2,2,2,2}, 7)-cockade\) and therefore \(G = G_1 \cup G_2\) is also isomorphic to a \((K_{1,1,2,2,2,2,2}, 7)-cockade\), a contradiction.

We may now assume \(G[S] \cong K_8\) or \(K_8^\sim\). Note that \(H_i\) for \(i = 1, 2\) is a proper minor of \(G\) on at least 11 vertices that contains a clique of size 8. Also note that no exceptional graph contains a clique of size 8. It follows that \(e(H_i) \leq 8|H_i| - 36\) for both \(i = 1, 2\). Let \(t = e(H_i) - e(G_i) = \binom{8}{2} - e(G[S]) = 28 - e(G[S]),\) and note that \(t = 0\) or \(1\). Then,

\[
e(G) = e(G_1) + e(G_2) - e(G[S])
\]
\[
\leq (8|H_1| - 36 - t) + (8|H_2| - 36 - t) - (28 - t)
\]
\[
= 8(|H_1| + |H_2|) - 100 - t
\]
\[
= 8(n + 8) - 100 - t
\]
\[
= 8n - 36 - t.
\]

Since \(t = 0\) or \(1\), \(e(G) \leq 8n - 36\), a contradiction to the fact that \(e(G) = 8n - 35\). \(\square\)
Lemma 3.2.5. Let \((A_1, A_2)\) be a non-trivial separation of \(G\) such that \(|A_1 \cap A_2| \leq \delta(G) - 2\) and \(|A_1 \cap A_2|\) is minimum over all non-trivial separations of \(G\). Let \(w \in A_1 \cap A_2\), and let \(H_i\) for \(i = 1, 2\) be the graph obtained from \(G[A_i]\) by making \(w\) adjacent to all other vertices in \(A_1 \cap A_2\). Then, \(H_i \not\cong K_{2,2,2,2,2,3}\) or \(K_{2,3,3,3,3,3}\) for \(i = 1, 2\).

Proof. Let \(S = A_1 \cap A_2\) and \(G_i = G[A_i]\) for \(i = 1, 2\). By Lemma 3.2.2 and Lemma 3.2.4, \(|S| \geq 6\) and \(H_1[S]\) is not a clique.

For the sake of a contradiction, assume \(H_1 \cong K_{2,2,2,2,2,2,2,3}\) or \(K_{2,3,3,3,3,3,3,3}\). If \(H_1 \cong K_{2,2,2,2,2,2,2,3,3}\), let \(\{x_i, y_i\}\) for \(i = 1, 2, 3, 4, 5\) be the five 2-islands and \(\{r_1, r_2, r_3\}\) be the 3-island of \(H_1\); if \(H_1 \cong K_{2,3,3,3,3,3,3,3}\), let \(\{x_i, y_i, z_i\}\) for \(i = 1, 2, 3, 4\) be the four 3-islands and \(\{r_1, r_2\}\) be the 2-island of \(H_1\). Observe that up to isomorphism, we may assume either \(w = x_1\) or \(w = r_1\) in both cases. Let \(C_w\) be the island of \(H_1\) that contains \(w\). Since \(w\) is adjacent to every vertex in \(S\) in \(H_1\), it follows that \(C_w - \{w\} \subseteq A_1 - S\).

Claim 1. There do not exist four distinct vertices \(a_1, b_1, a_2, b_2 \in S - \{w\}\) such that \(a_ib_i \not\in E(H_1)\) for \(i = 1, 2\).

Proof of Claim 1. For the sake of a contradiction, assume that \(a_1, b_1, a_2, b_2 \in S - \{w\}\) are distinct vertices such that \(a_ib_i \not\in E(H_1)\) for \(i = 1, 2\). Let \(Z_1 = \{a_1, b_1, a_2, b_2\}\), and let \(R_1\) be the graph obtained from \(G_1\) by making \(Z_1\) a clique. By Lemma 3.2.3, we know that \(G > R_1\). We then consider all cases, up to isomorphism, in the following table. Notice that \(R_1 - \{w\} > K_{10}\) in every case in the table. It follows that \(G > R_1 > K_{10}\), a contradiction.

<table>
<thead>
<tr>
<th>(H_1)</th>
<th>(w)</th>
<th>(Z_1)</th>
<th>(R_1 - {w})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K_{2,2,2,2,2,2,3})</td>
<td>(x_1)</td>
<td>({x_2, y_2, x_3, y_3})</td>
<td>(K_5 + K_{2,2,3})</td>
</tr>
<tr>
<td>(K_{2,2,2,2,2,3})</td>
<td>(x_1)</td>
<td>({x_2, y_2, r_1, r_2})</td>
<td>(K_3 + K_{2,2,2} + P_3)</td>
</tr>
<tr>
<td>(K_{2,2,2,2,2,2,3})</td>
<td>(r_1)</td>
<td>({x_1, y_1, x_2, y_2})</td>
<td>(K_4 + K_{2,2,2,2})</td>
</tr>
<tr>
<td>(K_{2,3,3,3,3,3})</td>
<td>(x_1)</td>
<td>({x_2, y_2, x_3, y_3})</td>
<td>(K_{2,2,3} + P_3 + P_3)</td>
</tr>
<tr>
<td>(K_{2,3,3,3,3,3})</td>
<td>(x_1)</td>
<td>({x_2, y_2, r_1, r_2})</td>
<td>(K_{1,1,2,3,3} + P_3)</td>
</tr>
<tr>
<td>(K_{2,3,3,3,3,3,3})</td>
<td>(r_1)</td>
<td>({x_2, y_2, x_3, y_3})</td>
<td>(K_{1,3,3,3} + P_3 + P_3)</td>
</tr>
</tbody>
</table>
Note that $H_1[S]$ is not a clique, and $w$ is adjacent to all other vertices in $S - \{w\}$. It follows that there exists some non-edge in $H_1[S - \{w\}]$. By Claim 1, there is a unique island $C_0$ of $H_1$ such that $|C_0 \cap S| \geq 2$. Choose distinct vertices $p_1, p_2 \in C_0 \cap S$. Notice that $|S - C_0 \cup \{w\}| \geq 6 - 3 - 1 = 2$, meaning that at least two islands if $H_1$ have exactly one vertex in $S - \{w\}$. Let $C_2$ be the island of $H_1$ that contains $x_2$. Without loss of generality, in both cases we can then assume that $C_2 \cap S = \{x_2\}$. Observe that that $Z_2 = \{w, x_2, p_1, p_2\}$ is a set of four distinct vertices by construction. Let $R_2$ be the graph obtained from $G_1$ by making $Z_2$ a clique. Again by Lemma 3.2.3, we have $G > R_2$.

Now, note that $y_2 \in C_2 - \{x_2\}$ and thus $y_2 \in V(H_1) - S$, which then implies that $w, y_2$ are adjacent in $G$ and therefore in $R_2$. Furthermore, observe that the new vertex in $R_2/\{w, y_2\}$ is adjacent to all other vertices in it. In the following table, we consider all cases up to isomorphism and show that $R_2/\{w, y_2\} > K_{10}$ in every case. It follows that $G > R_2 > K_{10}$, a contradiction.

<table>
<thead>
<tr>
<th>$H_1$</th>
<th>$w$</th>
<th>${p_1, p_2}$</th>
<th>$R_2/{w, y_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{2,2,2,2,2,3}$</td>
<td>$x_1$</td>
<td>${x_3, y_3}$</td>
<td>$K_5 + K_{2,2,3}$</td>
</tr>
<tr>
<td>$K_{2,2,2,2,3}$</td>
<td>$x_1$</td>
<td>${r_1, r_2}$</td>
<td>$K_3 + K_{2,2,2} + P_3$</td>
</tr>
<tr>
<td>$K_{2,2,2,2,3}$</td>
<td>$r_1$</td>
<td>${x_3, y_3}$</td>
<td>$K_4 + K_{2,2,2,2}$</td>
</tr>
<tr>
<td>$K_{2,3,3,3,3}$</td>
<td>$x_1$</td>
<td>${x_3, y_3}$</td>
<td>$K_{1,2,2,2,3} + P_3$</td>
</tr>
<tr>
<td>$K_{2,3,3,3,3}$</td>
<td>$x_1$</td>
<td>${r_1, r_2}$</td>
<td>$K_3 + K_{2,2,3}$</td>
</tr>
<tr>
<td>$K_{2,3,3,3,3}$</td>
<td>$r_1$</td>
<td>${x_3, y_3}$</td>
<td>$F_{1,2,3,3,3} + P_3$</td>
</tr>
</tbody>
</table>

$\square$

**Lemma 3.2.6.** Let $(A_1, A_2)$ be a non-trivial separation of $G$ such that $|A_1 \cap A_2| \leq \delta(G) - 2$ and $|A_1 \cap A_2|$ is minimum over all non-trivial separations of $G$. Let $Z \subseteq S$ such that $|Z| = 4$, and let $H_i = G[A_i] \cup \{z_iz_2 : z_1, z_2 \in Z\}$ for $i = 1, 2$. Then, $H_i \not\cong K_{2,2,2,2,2,3}$ or $K_{2,3,3,3,3}$ for $i = 1, 2$.

**Proof.** Let $S = A_1 \cap A_2$ and $G_i = G[A_i]$ for $i = 1, 2$. By Lemma 3.2.2 and Lemma 3.2.4, $|S| \geq 6$ and $H_1[S]$ is not a clique. For the sake of a contradiction, assume $H_1 \cong K_{2,2,2,2,2,3}$.
or \(K_{2,3,3,3,3}\). Label the vertices in \(H\) as in the proof for Lemma 3.2.5. Note that \(H_1[Z] \cong K_4\). Without loss of generality, assume that either \(Z = \{x_1, x_2, x_3, x_4\}\) or \(Z = \{x_1, x_2, x_3, r_1\}\) in both cases.

**Claim 1.** For every \(z \in Z\) and \(z' \in S - Z\), \(zz' \in E(G)\).

**Proof of Claim 1.** For the sake of a contradiction, assume that there are vertices \(z \in Z\) and \(z' \in S - Z\) such that \(zz' \notin E(G)\), and note that this means that \(zz' \notin E(H_1)\) either. Without loss of generality, assume that either \(z = x_1\) and \(z' = y_1\), or that \(z = r_1\) and \(z' = r_2\). Let \(Z_1 = \{z, z', x_2, x_3\}\) and \(R_1\) be the graph obtained from \(G_1\) by making \(Z_1\) a clique. By Lemma 3.2.3, \(G > R_1\). Note that \(|Z \cap Z_1| = 3\) in all cases. Let \(z_0\) be the unique vertex in \(Z - Z_1\).

Let \(z'_0 \in V(H_1) - Z\) such that \(z'_0\) is adjacent to every vertex in \(Z\) in \(H_1\). Then, by the construction of \(H_1\), observe that \(z'_0\) is also adjacent to every vertex in \(Z\) in \(G\) and therefore in \(R_1\) as well. It follows that the new vertex in \(R_1/z_0z'_0\) is adjacent to all vertices in \(Z \cap Z_1\). In the following table, we consider all cases up to isomorphism, and in each case we show that there exists some \(z'_0 \in V(H_1) - Z\) such that \(z'_0\) is adjacent to every vertex in \(Z\) in \(H_1\) and \(R_1/z_0z'_0 > K_{10}\). It follows that \(G > R_1 > K_{10}\), a contradiction.

<table>
<thead>
<tr>
<th>(H_1)</th>
<th>(Z)</th>
<th>((z, z'))</th>
<th>((z_0, z'_0))</th>
<th>(R_1/z_0z'_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K_{2,2,2,2,2,2,2})</td>
<td>(x_1, x_2, x_3, x_4) or (x_1, x_2, x_3, r_1)</td>
<td>((x_1, y_1)) or ((r_1, x_4))</td>
<td>((x_4, r_1))</td>
<td>(K_4 + K_{2,2,2,2,2})</td>
</tr>
<tr>
<td>(K_{2,2,2,2,2,2,2})</td>
<td>(x_1, x_2, x_3, x_4) or (x_1, x_2, x_3, r_1)</td>
<td>((x_1, y_1))</td>
<td>((x_4, r_1))</td>
<td>(K_3 + K_{2,2,2,2,2} + P_3)</td>
</tr>
<tr>
<td>(K_{2,3,3,3,3})</td>
<td>(x_1, x_2, x_3, x_4) or (x_1, x_2, x_3, r_1)</td>
<td>((x_4, r_1))</td>
<td>(K_{1,1,2,3,3} + P_3)</td>
<td></td>
</tr>
<tr>
<td>(K_{2,3,3,3,3})</td>
<td>(x_1, x_2, x_3, x_4) or (x_1, x_2, x_3, r_1)</td>
<td>((x_4, r_1))</td>
<td>(K_3 + K_{2,2,3,3})</td>
<td></td>
</tr>
</tbody>
</table>

Recall that \(H_1[S]\) is not a clique and \(H_1[Z] \cong K_4\). By Claim 1, there exist vertices \(p_1, p_2 \in S - Z\) such that \(p_1p_2 \notin E(H_1)\). Let \(Z_2 = \{x_1, x_2, p_1, p_2\}\), and let \(R_2\) be the graph obtained from \(G_1\) by making \(Z_2\) a clique. Again we have \(G > R_2\) by Lemma 3.2.3, and therefore it suffices to prove \(R_2 > K_{10}\). We will consider the case \(H_1 \cong K_{2,2,2,2,2,2}\) and the case \(H_1 \cong K_{2,3,3,3,3,3}\) separately in the remaining proof.
First assume that $H_1 \cong K_{2,2,2,2,3}$. In this case, $Z \cap Z_2 = \{x_1, x_2\}$. Let $u_1, u_2$ be the two distinct vertices in $Z - Z_2$, and note that $N_{G_1}(v) = N_{H_1}(v)$ for every $v \in V(G_1) - Z$. Therefore, if there exist vertices $u'_1, u'_2 \in V(H_1) - Z \cup Z_2$ both contained in some island of $H_1$ that is disjoint from $Z \cup Z_2$, then by contracting edges $u_1u'_1$ and $u_2u'_2$ in $R_2$, we can then have the two new vertices $w_1, w_2$ in the resulting graph satisfying that $w_1, w_2$ are adjacent to each other and $w_i$ for $i = 1, 2$ is adjacent to both $x_1$ and $x_2$. It follows that $(R_2/u_1u'_1)/u_2u'_2 = ((H_1/u_1u'_1)/u_2u'_2) \cup \{vv' : v, v' \in Z_2\}$. In the table below, we show that in each case we can always find some $u'_1, u'_2 \in V(H_1) - Z \cup Z_2$ contained in some island of $H_1$ disjoint from $Z \cup Z_2$ such that the corresponding graph $(R_2/u_1u'_1)/u_2u'_2 = ((H_1/u_1u'_1)/u_2u'_2) \cup \{vv' : v, v' \in Z_2\}$ has a $K_{10}$ minor. It follows that $G > R_2 > K_{10}$, a contradiction.

<table>
<thead>
<tr>
<th>$H_1$</th>
<th>$Z$</th>
<th>$(p_1, p_2)$</th>
<th>$(u_1, u_2)$</th>
<th>$(u'_1, u'_2)$</th>
<th>$(R_2/u_1u'_1)/u_2u'_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{2,2,2,2,3}$</td>
<td>${x_1, x_2, x_3, x_4}$</td>
<td>${x_5, y_5}$</td>
<td>${x_3, x_4}$</td>
<td>${r_1, r_2}$</td>
<td>$K_7 + K_{2,2}$</td>
</tr>
<tr>
<td>$K_{2,2,2,2,3}$</td>
<td>${x_1, x_2, x_3, x_4}$</td>
<td>${r_1, r_2}$</td>
<td>${x_3, y_4}$</td>
<td>${x_5, y_5}$</td>
<td>$K_7 + K_{2,2}$</td>
</tr>
<tr>
<td>$K_{2,2,2,2,3}$</td>
<td>${x_1, x_2, x_3, r_1}$</td>
<td>${x_4, y_4}$</td>
<td>${x_3, r_1}$</td>
<td>${x_5, y_5}$</td>
<td>$K_7 + K_{2,2}$</td>
</tr>
</tbody>
</table>

Now, assume that $H_2 \cong K_{2,3,3,3,3}$. If $Z = \{x_1, x_2, x_3, x_4\}$, then $(p_1, p_2) = \{r_1, r_2\}$.

Observe that $x_3, x_4$ each are adjacent to all of $y_1, y_2, z_1, z_2$ in $G_1$. It follows that $(R_2/x_3y_1)/x_4y_2$ contains $H_1[V(G_1) - \{x_3, x_4\}] \cup \{y_1z_1, y_2z_2, r_1r_2\} \cong K_{1,1,2,2} + \overline{P}_3 + \overline{P}_3$ as a subgraph.

Since $K_{1,1,2,2} + \overline{P}_3 + \overline{P}_3 > K_{10}$, it follows that $G > R_2 > K_{10}$, a contradiction. If $Z = \{x_1, x_2, x_3, r_1\}$, then without loss of generality assume that $(p_1, p_2) = \{x_4, y_4\}$.

Now, $x_3, r_1$ each are adjacent to all of $y_1, y_2, z_1, z_2$ in $G_1$. It follows that $(R_1/x_3y_1)/x_4y_2$ contains $H_1[V(G_1) - \{x_3, r_1\}] \cup \{y_1z_1, y_2z_2, x_4y_4\} \cong K_{1,2} + \overline{P}_3 + \overline{P}_3 + \overline{P}_3$ as a subgraph.

Since $K_{1,2} + \overline{P}_3 + \overline{P}_3 + \overline{P}_3K_{10}$, it follows that $G > R_2 > K_{10}$, again a contradiction.

We can now combine Lemma 3.2.5 and Lemma 3.2.6 to form the next lemma.

**Lemma 3.2.7.** Let $(A_1, A_2)$ be a non-trivial separation of $G$. Let $S = A_1 \cap A_2$. If $|S| \leq \delta(G) - 2$ and $|S|$ is minimum over all non-trivial separations of $G$, then the following
Therefore, by Lemma 3.2.1, we have that

(1) For every \( w \in S \), \( G[A_i] \cup \{ wr : r \in A_1 \cap A_2 - \{ w \} \} \not\cong K_{2,2,2,2,2,3} \) or \( K_{2,3,3,3,3} \).

(2) For every \( Z \subseteq S \) of size 4, \( G[A_i] \cup \{ z_1z_2 : z_1, z_2 \in Z \} \not\cong K_{2,2,2,2,2,3} \) or \( K_{2,3,3,3,3} \).

The next lemma is an immediate consequence of Lemma 3.2.1 and Lemma 3.2.7.

**Lemma 3.2.8.** Let \( (A_1, A_2) \) be a non-trivial separation of \( G \) and \( S = A_1 \cap A_2 \) such that \( |S| \leq \delta(G) - 2 \) and \( |S| \) is minimum over all non-trivial separations of \( G \). Let \( S = A_1 \cap A_2 \).

Let \( d' = \max_{Z:Z \subseteq S, |Z| = 4} \{ 6 - e(G[Z]) \} \) and \( d = \max \{ d', |S| - 1 - \delta(G[S]) \} \). Then, \( 8|S| \geq 35 + 2d + e(G[S]) \).

**Proof.** For \( i = 1, 2 \), let \( G_i' \) be the set of graphs obtained from \( G_i \) by making some \( w \in S \) adjacent to all other vertices in \( S \), and let \( G_i'' \) be the set of graphs obtained from \( G_i \) by making some subset \( Z \subseteq S \) of size 4 a clique. Let \( G_i = G_i' \cup G_i'' \) for \( i = 1, 2 \). By Lemma 3.2.3, every graph in \( G_i \) for \( i = 1, 2 \) is a minor of \( G \) with vertex set \( V(G_i) \).

Note that \( |S| - 1 - \delta(G[S]) = \max_{H \in G'_i} \{ e(H) - e(G_i) \} \) and \( d' = \max_{H \in G''_i} \{ e(H) - e(G_i) \} \) for \( i = 1, 2 \). It follows that for both \( i = 1, 2 \),

\[
  d = \max \{ d', |S| - 1 - \delta(G[S]) \} = \max_{H \in G_i} \{ e(H) - e(G_i) \}.
\]

By Lemma 3.2.7, no graph in \( G_i \) is isomorphic to \( K_{2,2,2,2,2,3} \) or \( K_{2,3,3,3,3} \) for \( i = 1, 2 \). Therefore, by Lemma 3.2.1, we have that \( 8|S| \geq 35 + 2d + e(G[S]) \). \( \square \)

**Lemma 3.2.9.** \( G \) is 7-connected.

**Proof.** Let \( (A_1, A_2) \) be a non-trivial separation of \( G \) of minimum order. Let \( S = A_1 \cap A_2 \) and \( G_i = G[A_i] \) for \( i = 1, 2 \). By Lemma 3.2.2, \( |S| \geq 6 \), so we may assume \( |S| = 6 \) for the sake of a contradiction. Note that \( \delta(G) \geq 10 \), so \( |S| < \delta(G) - 2 \), which allows us to apply Lemma 3.2.8. Let \( \delta = \delta(G[S]) \). Let \( d = \max \{ 5 - \delta, \max_{Z:Z \subseteq S, |Z| = 4} \{ 6 - e(G[Z]) \} \} \). By Lemma 3.2.8, \( 8|S| \geq 35 + 2d + e(G[S]) \). With \( |S| = 6 \), it follows that
\[
  2d \leq 13 - e(G[S]).
\]
Since \( d \geq 5 - \delta \), we have \( e(G[S]) \leq 13 - 2d \leq 3 + 2\delta \). Since \( e(G[S]) \geq \frac{1}{2} \delta |S| = 3\delta \), it follows that \( 3\delta \leq e(G[S]) \leq 3 + 2\delta \) and hence \( \delta \leq 3 \).

**Claim 1.** Either \( e(G[S]) \leq 1 \) or \( E(G[S]) \) is precisely a perfect matching of size 3.

**Proof of Claim 1.** By the definition of \( d \), \( e(G[Z]) \geq 6 - d \) for every \( Z \subseteq S \) with \( |Z| = 4 \). Note that each pair of vertices is contained in exactly \( \binom{4}{2} \) subsets of size 4 of \( S \). It follows that

\[
e(G[S]) = \sum_{Z: Z \subseteq S, |Z|=4} e(G[Z]) \geq \binom{6}{4} (6-d) \binom{4}{2} = 15 - \frac{5}{2} d.
\]

Since \( 2d \leq 13 - e(G[S]) \), we know \( 15 - \frac{5}{2} d \leq e(G[S]) \leq 13 - 2d \) and therefore \( d \geq 4 \).

Note that \( e(G[S]) \geq \frac{1}{2} |S| \delta = 3\delta \). It follows that \( 3\delta \leq e(G[S]) \leq 13 - 2d \leq 13 - 8 = 5 \), which shows \( \delta \leq \frac{5}{3} \) and therefore \( \delta \leq 1 \).

If \( d = 4 \), observe that \( 5 = 15 - \frac{5}{2} \cdot 4 \leq e(G[S]) \leq 13 - 2 \cdot 4 = 5 \). It follows that (i) \( e(G[S]) = 5 \) and (ii) \( e(G[Z]) = 2 \) for every \( Z \subseteq S \) with \( |Z| = 4 \). Note that (i) implies \( \Delta(G) \geq 2 \) and (ii) implies \( \Delta(G) \leq 2 \), so we can choose \( v \in S \) such that \( d_{G[S]}(v) = 2 \).

Let \( N_{G[S]}(v) = \{u_1, u_2\} \) and \( S - N_{G[S]}[v] = \{w_1, w_2, w_3\} \). Let \( Z_1 = \{v, u_1, u_2, w_1\} \).

Since \( e(G[Z_1]) = 2 \), \( w_i \) is adjacent to neither \( u_1 \) nor \( u_2 \). By symmetry, it follows that \( w_i \) is adjacent to neither \( u_1 \) nor \( u_2 \) for \( i = 1, 2, 3 \). Let \( Z_2 = \{u_1, u_2, w_1, w_2\} \). We then see that \( e(G[Z_2]) \leq 1 \), a contradiction.

Hence, \( d \geq 5 \) and \( e(G[S]) \leq 13 - 2d \leq 3 \), meaning that \( \delta \leq 1 \). For the sake of a contradiction, assume that \( e(G[S]) = 2 \) or 3 and \( E(G[S]) \) is not a perfect matching of size 3. It follows that \( G[S] \) is isomorphic to one of the following graphs: the union of two disjoint edges, a path of length 2 or 3, a 3-star, and the disjoint union of a path of length 2 and an isolated edge. In every one of these 5 graphs, observe that there always exists an independent set \( Z \) of size 4 in \( S \). It follows that \( d = \max_{Z: Z \subseteq S, |Z|=4} \{6 - e(G[Z])\} = 6 \) and therefore \( e(G[S]) \leq 13 - 2d = 1 \), a contradiction.

**Claim 2.** For \( i = 1, 2 \), \( G \) has a minor \( L_i \) on \( V(G_i) \) obtained by contracting edges that have at least one end in \( G_{3-i} - G_i \) such that
(1) \( e(L_i) - e(G_i) \geq 7 \) if \( e(G[S]) \leq 1 \), and \( e(L_i) - e(G_i) \geq 6 \) if \( e(G[S]) = 3 \), and

(2) there exists a vertex in \( L_i[S] \) adjacent to all other vertices in \( S \) in \( L_i \).

Proof of Claim 2. We will prove (1) and (2) hold for \( i = 1 \), and the case of \( i = 2 \) will follow by symmetry. Let \( S = \{s_1, s_2, s_3, s_4, s_5, s_6\} \). Without loss of generality, assume that \( E(G[S]) = \{s_1s_2\} \) if \( e(G[S]) = 1 \) in all cases, and \( E(G[S]) = \{s_1s_2, s_3s_4, s_5s_6\} \) if \( e(G[S]) = 3 \). Since \( \delta(G) \geq 10 \), \( |G_2 - G_1| \geq 5 \). Choose two distinct vertices \( x, y \in V(G_2 - G_1) \). Since \( G \) is 6-connected, there exist 6 paths \( P_1, \ldots, P_6 \) in \( G_2 \) between \( \{x\} \) and \( S \) that are disjoint except for \( x \). Without loss of generality, assume that \( P_j \) links \( x \) and \( s_j \) for \( j = 1, \ldots, 6 \).

Assume that every \( P_j \) has length exactly 1 for a moment. Since \( G - x \) is 5-connected, there exist five paths \( Q_1, \ldots, Q_5 \) in \( G_2 - x \) between \( \{y\} \) to \( S \) that are disjoint except for \( y \). If \( e(G[S]) \leq 1 \), without loss of generality, assume \( s_4 \) is an end of \( Q_1 \). By contracting the edge \( xs_3 \), contracting all vertices on \( Q_1 \) to \( s_4 \), and contracting other \( Q_j \) paths properly, we could obtain a minor \( L_1 \) of \( G \) on \( V(G_1) \), which is isomorphic to some graph obtained from \( G_1 \) by making \( s_3 \) adjacent to all other vertices in \( S \) and making \( s_4 \) adjacent to at least four other vertices in \( S \). It follows that \( e(L_1) - e(G_1) \geq 8 \), and \( s_3 \) is adjacent to all other vertices in \( S \) in the graph \( L_1 \). If \( e(G[S]) = 3 \), without loss of generality, assume \( s_3 \) is an end of \( Q_1 \). By contracting the edge \( xs_1 \), contracting all vertices on \( Q_1 \) to \( s_3 \), and contracting other \( Q_j \) paths properly, we could obtain a minor \( L_1 \) of \( G \) on \( V(G_1) \), which is isomorphic to some graph obtained from \( G_1 \) by making \( s_1 \) adjacent to all other vertices in \( S \) and \( s_3 \) adjacent to at least two vertices other than \( s_1 \) and \( s_4 \) in \( S \). It follows that \( e(L_1) - e(G_1) \geq 6 \).

We can now assume \( P_j \) has length at least 2 for some \( j \in \{1, \ldots, 6\} \). Since now, for every path \( P \), we will use \( I(P) \) to denote the set of internal vertices of \( P \).

Consider \( e(G[S]) \leq 1 \) first. Without loss of generality, we can assume one of \( P_1, P_2, P_3 \) has length at least 2. Let \( X_1 = I(P_1) \cup I(P_2) \cup I(P_3) \) and \( Y_1 = V(P_4 \cup P_5 \cup P_6 - x) \), and note that \( X_1 \neq \emptyset \) and \( Y_1 \neq \emptyset \). Since \( G \) is 6-connected, \( C_1 = \{x, s_1, s_2, s_3\} \) is not a cut of \( G \). It follows that there exists a path \( Q_1 \) in \( G_2 - C_1 \) that links some \( x_1 \in X_1 \) and
\[ y_1 \in Y_1 \] such that no internal vertex of \( Q_1 \) is in \( X_1 \) or \( Y_1 \). Without loss of generality, assume \( y_1 \in V(P_1 - x) \). Let \( X_2 = X_1 \cup V(P_4 \cup Q_1) - \{x, s_4\} \) and \( Y_2 = V(P_5 \cup P_6) - \{x\} \). Note that \( X_2 \neq \emptyset \) and \( Y_2 \neq \emptyset \), and that \( C_2 = \{x, s_1, s_2, s_3, s_4\} \) is not a cut of \( G \). It follows that there exists a path \( Q_2 \) in \( G_2 - C_2 \) linking some \( x_2 \in X_2 \) and \( y_2 \in Y_2 \) such that no internal vertex of \( Q_2 \) is in \( X_2 \) or \( Y_2 \). Without loss of generality, assume \( y_2 \in V(P_5 - x) \). Observe that \( Q_1 \) and \( Q_2 \) are internally disjoint, and that \( Q_j \) is internally disjoint from \( P_i \) for every \( j \in \{1, 2\}, i \in \{1, 2, 3, 4, 5, 6\} \). For \( j = 1, 2 \), contract edges on \( Q_j \) such that \( Q_j \) eventually becomes a path of length 1 linking \( x_j \) and \( y_j \). Then, contract edges on \( P_1, \ldots, P_6 \) such that vertices in \( V(P_i) - \{x\} \) are identified as one vertex at \( s_i \) for \( i = 1, \ldots, 5 \), and all vertices of \( P_6 \) are identified as one vertex at \( s_6 \). It follows that in the resulting graph, there is an edge between \( s_6 \) and every other vertex in \( S \), an edge between \( s_4 \) and \( \{s_1, s_2, s_3\} \), and an edge between \( s_5 \) and \( \{s_1, s_2, s_3, s_4\} \). Let \( L_1 \) be the resulting graph induced on \( V(G_1) \). We see that \( e(L_1) - e(G_1) \geq 5 + 2 = 7 \), and \( s_6 \) is adjacent to every other vertex in \( S \) in \( L_1 \).

Now, consider \( e(G[S]) = 3 \) and \( E(G[S]) = \{s_1s_2, s_3s_4, s_5s_6\} \). Without loss of generality, assume \( I(P_1) \neq \emptyset \). Let \( X_1 = I(P_1) \cup I(P_2) \) and \( Y_1 = V(\bigcup_{i=3}^6 P_i - x) \). Similarly to the previous case, since \( C_1 = \{x, s_1, s_2\} \) is not a cut of \( G \), there exists a path \( Q_1 \) in \( G_2 - C_1 \) linking some \( x_1 \in X_1 \) and \( y_1 \in Y_1 \) such that no internal vertex of \( Q_1 \) is in \( X_1 \) or \( Y_1 \). Without loss of generality, assume \( y_1 \in V(P_3 - x) \). Let \( X_2 = X_1 \cup V(P_3 \cup P_4 \cup Q_1) - \{x, s_3, s_4\} \), \( Y_2 = V(P_5 \cup P_6) - \{x\} \), and \( C_2 = \{x, s_1, s_2, s_3, s_4\} \). Since \( C_2 \) is not a cut of \( G \), there exists a path \( Q_2 \) linking some \( x_2 \in X_2 \) and \( y_2 \in Y_2 \) such that no internal vertex of \( Q_2 \) is in \( X_2 \) or \( Y_2 \). Without loss of generality, assume \( y_2 \in V(P_5 - x) \). By contracting edges on \( Q_j \) such that \( Q_j \) becomes a path of length 1 for \( j = 1, 2 \) and contracting edges on \( P_1, \ldots, P_6 \) such that every internal vertex of \( P_i \) gets identified to \( s_i \) for \( i = 1, \ldots, 6 \), and \( x \) gets identified to \( s_6 \), we can obtain a resulting graph in which there is an edge between \( s_6 \) and every other vertex in \( S \), an edge between \( s_3 \) and \( \{s_1, s_2\} \), and an edge between \( s_5 \) and \( \{s_1, s_2, s_3, s_4\} \). Let \( L_1 \) be the induced subgraph of the resulting graph on \( V(G_1) \). It follows that \( e(L_1) - e(G_1) \geq 4 + 2 = 6 \), and that \( s_6 \) is adjacent to every other vertex in \( S \) in \( L_1 \).
For \( i = 1, 2 \), define \( d_i \) to be the maximum number of edges that can be added to \( G[S] \) by contracting edges that have at least one end in \( G_{3-i} - G_i \), and let \( J_i \) be a graph on \( V(G_i) \) obtained from contracting edges that have at least one end in \( G_{3-i} - G_i \) such that \( e(J_i) = e(G_i) + d_i \). For \( i = 1, 2 \), define \( r_i \) and \( d_i' \) as follows: \( r_i = 1 \) if \( J_i \cong K_{2,2,2,2,2,3} \) or \( K_{2,3,3,3,3} \), and \( r_i = 0 \) otherwise; \( d_i' = 7 \) if \( e(G[S]) \leq 1 \), and \( d_i' = 6 \) if \( e(G[S]) = 3 \). By Claim 2, \( d_i \geq d_i' \) for both \( i = 1, 2 \). By Lemma 3.2.1, it follows that

\[
d_1 + d_2 + e(G[S]) \leq 8|S| - 35 + r_1 + r_2 = 13 + r_1 + r_2 \leq 15.
\]

**Claim 3.** If \( r_i = 1 \) for some \( i \in \{1, 2\} \), then \( e(G[S]) = 3 \), \( d_1 = d_2 = 6 \), \( J_i \cong K_{2,3,3,3,3} \), and \( J_i[S] \cong K_{3,3} \).

**Proof of Claim 3.** Without loss of generality assume \( r_1 = 1 \) and \( J_1 \cong K_{2,2,2,2,2,3} \) or \( K_{2,3,3,3,3} \). Observe that every induced subgraph of \( K_{2,2,2,2,2,3} \) on 6 vertices has at least 11 edges, and every induced subgraph of \( K_{2,3,3,3,3} \) on 6 vertices has at least 9 edges. It follows that \( e(G[S]) + d_1 = e(J_1[S]) \geq 9 \) and therefore \( d_2 \leq 6 \). Recall that \( e(G[S]) \leq 3 \) and \( d_i \geq d_i' \geq 6 \) for \( i = 1, 2 \) by Claim 1 and Claim 2. It follows that \( e(G[S]) = 3 \) and \( d_1 = d_2 = 6 \), which then implies that \( J_1 \cong K_{2,3,3,3,3} \) and \( J_1[S] \cong K_{3,3} \). ⊞

Now, if \( e(G[S]) \leq 1 \), by Claim 3 we know that \( r_1 = r_2 = 0 \) and therefore \( d_1 + d_2 + e(G[S]) \leq 13 \), a contradiction to the fact that \( d_i \geq d_i' = 7 \) for \( i = 1, 2 \). We may then assume \( e(G[S]) = 3 \). By Claim 2, \( d_i \geq d_i' = 6 \) for \( i = 1, 2 \). Since \( d_1 + d_2 + e(G[S]) \leq 13 + r_1 + r_2 \), it follows that \( d_i = d_i' = 6 \) and \( r_i = 1 \) for both \( i = 1, 2 \). By Claim 2, for both \( i = 1, 2 \) there exists a minor \( L_i \) of \( G \) on \( V(G_i) \) obtained by contracting edges of \( G \) that have at least one end in \( G_{3-i} - G_i \) such that \( e(L_i) \geq e(G_i) + 6 \), and that there exists a vertex in \( S \) adjacent to all other vertices in \( S \) in \( L_i \). Since \( d_i = d_i' = 6 \), we may choose \( J_i \) to be equal to \( L_i \) for both \( i = 1, 2 \). This is then a contradiction to Claim 3 that \( J_1[S] \cong J_2[S] \cong K_{3,3} \), since no vertex in \( K_{3,3} \) is adjacent to all other vertices in \( S \). ⊡
3.3 Bounding Minimum Degree

We proved earlier in Lemma 3.1.1 that \( \delta(G) \geq 10 \). In this section, we will show \( \delta(G) \geq 11 \).

**Lemma 3.3.1.** \( \delta(G) \geq 11 \).

**Proof.** For the sake of a contradiction, assume that there exists a vertex \( x \in V(G) \) such that \( d(x) \leq 10 \). By Lemma 3.1.1, \( d(x) \geq 10 \) and \( \delta(N(x)) \geq 8 \). It follows that \( \delta(G) = 10 \), and that \( N(x) \) contains a subgraph isomorphic to \( K_{2,2,2,2,2} \). Let \( N(x) = \bigcup_{i=1}^{t} \{ s_i, t_i \} \), and assume that \( N[s_i] \cap N(x) \supseteq N(x) - \{ t_i \} \) and \( N[t_i] \cap N(x) \supseteq N(x) - \{ s_i \} \) for all \( i = 1, 2, 3, 4, 5 \).

Note that if there exist at most two non-edges in \( N(x) \), then \( N[x] \) would have a subgraph isomorphic to \( K_7 + K_{2,2} \), which has a \( K_{10} \) minor, a contradiction. It follows there exist at least three non-edges in \( N(x) \), meaning that \( e(N[x]) \leq \binom{11}{2} - 3 = 52 < 53 = 8 \cdot 11 - 35 \).

Since \( e(G) = 8n - 35 \), it follows that \( |G - N[x]| > 0 \). Let \( (A_1, A_2) \) be a non-trivial separation of \( (G, N[x]) \) of minimum order. Let \( S = A_1 \cap A_2 \). Since \( (N[x], V(G) - \{ x \}) \) is a non-trivial 10-separation of \( (G, N[x]) \), we know that \( |S| \leq 10 \). By Lemma 3.2.9, it follows that \( 7 \leq |S| \leq 10 \).

For \( i = 1, 2 \), let \( G_i = G[A_i] \), and let \( J_i \) be a minor of \( G \) on \( V(G_i) \) that can be obtained from \( G \) by contracting edges that have at least one end in \( G_{3-i} - G_i \) such that \( d_i = e(J_i) - e(G_i) \) is maximum. Let \( r \) be the number of graphs among \( J_1 \) and \( J_2 \) that are isomorphic to \( K_{2,2,2,2,2,3} \) or \( K_{2,3,3,3,3,3} \). By Lemma 3.2.1, we know that

\[
d_1 + d_2 + e(G[S]) \leq 8|S| - 35 + r.
\]

Since \( \delta(G) = 10 \), by the minimality of \( |S| \) and Lemma 3.2.3, if \( |S| \leq 8 \) then for every \( Z \subseteq S \) with \( |Z| = 4 \) we know \( G[(A_2 - A_1) \cup Z] \) has a \( K_4 \) minor rooted at \( Z \). By Lemma 3.2.4, \( G[S] \) contains an independent set of size 3 or two disjoint non-edges. It follows that \( d_1 \geq 2 \) if \( |S| \leq 8 \).
Observe that by the minimality of $|S|$, $x \notin S$ and that there exist $|S|$ disjoint paths $P_1, ..., P_8$ in $G_1 - \{x\}$ linking $N(x)$ and $S$. Let $U \subseteq N(x)$ be the subset of vertices that are ends of $P_1, ..., P_8$ in $N(x)$. We now prove the following claim that $|S| \geq 8$, and that if $|S| = 8$, then $G[U] \cong K_{2,2,2,2}$.

**Claim 1.** $|S| \geq 8$; and if $|S| = 8$, then $G[U] \cong K_{2,2,2,2}$.

**Proof of Claim 1.** First assume that $|S| = 7$, for the sake of a contradiction. Recall that this implies $d_1 \geq 2$. With $|S| = 7$, we have $d_1 + d_2 + e(G[S]) \leq 21 + r$. Since $|U| = |S| = 7$, it follows that $G[U]$ has a subgraph isomorphic to $K_{1,2,2,2}$. Furthermore, note that if there exist $s_i, t_i \in U$ that are not adjacent for some $i \in \{1, 2, 3, 4, 5\}$, then $s_i, t_i$ each are adjacent to all four vertices in $N[x] - U$. Therefore, by contracting edges between $U$ and $N[x] - U$ properly and then contracting each path among $P_1, ..., P_7$ to a single vertex, we would obtain a clique on $S$. It follows that $J_2[S] \cong K_7$ and $d_2 + e(G[S]) = \binom{7}{2} = 21$. Since neither $K_{2,2,2,2,2,2}$ nor $K_{2,3,3,3,3}$ contains a clique of size 7, $J_2 \not\cong K_{2,2,2,2,2,2}$ or $K_{2,3,3,3,3}$, and therefore $r \leq 1$. It follows that $d_1 \leq 21 + r - (d_2 + e(G[S])) = 21 + r - 21 = r \leq 1$, a contradiction to the fact that $d_1 \geq 2$.

We may now assume that $|S| = 8$, and it suffices to prove $G[U] \cong K_{2,2,2,2}$. With $|S| = 8$, we have $d_1 + d_2 + e(G[S]) \leq 29 + r$. Since $|U| = |S| = 8$, $G[U]$ contains a subgraph isomorphic to $K_{2,2,2,2}$. For the sake of a contradiction, assume $G[U] \not\cong K_{2,2,2,2}$.

We next prove that $J_1 \cong K_{2,2,2,2,2,2}$. Note that $G[U] \not\cong K_{2,2,2,2}$ implies that $G[U]$ contains a subgraph isomorphic to $K_{1,1,2,2,2}$. Also note that every two non-adjacent vertices in $U$ are adjacent to all vertices in $N[x] - U$, where $|N[x] - U| = 3$ as $|U| = |S| = 8$. Therefore, by contracting edges between $U$ and $N[x] - U$ properly and then contracting each path among $P_1, ..., P_8$ to a single vertex, we would obtain a clique on $S$. It follows that $J_2[S] \cong K_8$, $d_2 + e(G[S]) = \binom{8}{2} = 28$, and $r \leq 1$. Hence, $d_1 \leq 29 + r - (d_2 + e(G[S])) = r + 1 \leq 2$. Since $d_1 \geq 2$, it follows that $d_1 = 2, r = 1$, and therefore $J_1 \cong K_{2,2,2,2,2,2}$ or $K_{2,3,3,3,3}$. Note that $J_1 \supseteq G_1$ which has $N[x]$ as a subgraph. Since $N_{G}[x]$ contains a subgraph isomorphic to $K_{1,2,2,2,2,2}$ and $K_{2,3,3,3,3}$ does not contain a subgraph isomorphic to
\(K_{1,2,2,2,2,2}\), it follows that \(J_1 \cong K_{2,2,2,2,2,3}\).

Now, since \(J_1 \cong K_{2,2,2,2,2,3}\) does not have a clique of size 7, we know that \(N_{J_1}[x] = N_G[x] \cong K_{1,2,2,2,2,2}\). Let the two vertices in \(V(G_1) - N[x]\) be \(u\) and \(w\). Since \(u, w\) are not contained in \(N(x)\) and \(x \notin S\), \(x\) is adjacent to neither \(u\) nor \(w\) in \(J_1\). It follows that \(\{x, u, w\}\) is exactly the 3-island of \(J_1 \cong K_{2,2,2,2,2,3}\), meaning that these three vertices are pairwise non-adjacent in both \(G_1\) and \(J_1\). Since \(N_{J_1}[x] = N_{G_1}[x]\) and \(e(J_1) = e(G_1) + d_1 = e(G_1) + 2\), we know that the two edges in \(E(J_1 - G_1)\) each has exactly one end in \(\{u, w\}\), and therefore \(\{u, w\} \cap S \neq \emptyset\).

Let \(S' = S - \{u, w\}\). Assume \(\{u, w\} \subseteq S\) for a moment. Note that at most two vertices in \(S\) are not adjacent to both \(u, w\), meaning that \(|(S' - N_G(u)) \cup (S' - N_G(w))| \leq 2\). Choose distinct vertices \(y_1, y_2 \in S'\) such that \((S' - N_G(u)) \cup (S' - N_G(w)) \subseteq \{y_1, y_2\}\). Note that \(e(G[\{u, w, y_1, y_2\}]) \leq 3\). On the other hand, since \(|S| = 8\) and \(d_1 = 2\), by Lemma 3.2.3, \(e(G[Z]) \geq 6 - 2 = 4\) for every \(Z \subseteq S\) with \(|Z| = 4\), a contradiction. We may then assume that \(S \cap \{u, w\} = \{u\}\), without loss of generality. Note this then implies that \(w\) is adjacent to every vertex in \(N(x)\). Since \(|S'| = 7\), without loss of generality we assume that \(\{s_1, t_1, s_2, t_2\} \subseteq S'\). By Lemma 3.2.3, the graph \(G'_1\) obtained from \(G_1\) by adding the edges \(s_1t_1\) and \(s_2t_2\) is a minor of \(G\). Then, in the graph \(G'_1 - \{u\}\), by contracting the edges \(ws_3\) and \(s_4s_5\) we could obtain a \(K_{10}\) minor. This means that \(G > G'_1 - \{u\} > K_{10}\), a contradiction.

By Claim 1, without loss of generality, we can assume that \(\{s_1, t_1, \ldots, s_4, t_4\} \subseteq U\).

For each \(i = 1, 2, 3, 4, 5\), if \(s_i\) (or \(t_i\)) is in \(U\), we let the vertex in \(S\) corresponding to it via the paths \(P_1, \ldots, P_{|S|}\) be \(s_i'\) (or \(t_i'\), respectively). Let \(I = \{1, 2, 3, 4\}\) if \(|S| \leq 9\) and \(I = \{1, 2, 3, 4, 5\}\) if \(|S| = 10\).

Observe that if there exist three disjoint paths \(Q_1, Q_2, Q_3\) internally contained in \(A_2 - A_1\) such that, for some distinct indices \(i, j, k \in I\), \(Q_1\) links \(s_i', t_i'\), \(Q_2\) links \(s_j', t_j'\), and \(Q_3\) links \(s_k', t_k'\), then by contracting the paths \(Q_1, Q_2, Q_3, P_1, \ldots, P_{|S|}\) properly, we can obtain a resulting graph on \(N[x]\) that contains a subgraph isomorphic to \(K_7 + K_{2,2} > K_{10}\), which is
Let $X = \{s_i', t_i, s_j', t_j, s_k', t_k'\}$ where $i, j, k$ are distinct indices in $I$. For the sake of a contradiction, we may assume that for every $X' \subseteq S : X = \{s_i', t_i, s_j', t_j, s_k', t_k'\}$ where $i, j, k \in I$ are distinct. The following statements are true:

1. \((|I| - 2) e(S', A_2 - A_1) + (|I| - 3) e(G[A_2 - A_1]) \leq (|I| - 3) \cdot (5|A_2 - A_1| + 3),\)
2. \((10 - t)|A_2 - A_1| \leq e(S', A_2 - A_1) + 2e(G[A_2 - A_1]),\)
3. \((8 - t)|A_2 - A_1| + 1 \leq e(S', A_2 - A_1) + e(G[A_2 - A_1]).\)

**Proof of Claim 2.** Let $X \in \mathcal{X}$. By the minimality of $|S|$, $(G[(A_2 - A_1) \cup X], X)$ does not have separation of order at most 5. Since $(G[(A_2 - A_1) \cup X], X)$ is not 3-linked, by Theorem 2.3.2 we know that $\rho_{G[(A_2 - A_1) \cup X]}(A_2 - A_1) \leq 5|A_2 - A_1| + 3$. Since $X \in \mathcal{X}$ was arbitrary and $|\mathcal{X}| = \binom{|I|}{3}$, we have

$$\sum_{X \in \mathcal{X}} \rho_{G[(A_2 - A_1) \cup X]}(A_2 - A_1) \leq \binom{|I|}{3} \cdot (5|A_2 - A_1| + 3).$$

Since every pair $\{s_i, t_i\} \subseteq S'$ is contained in exactly $\binom{|I| - 2}{2}$ sets in $\mathcal{X}$, it follows that

$$\left(\binom{|I| - 1}{2}\right) e(S', A_2 - A_1) + \left(\binom{|I|}{3}\right) e(G[A_2 - A_1]) = \sum_{X \in \mathcal{X}} \rho_{G[(A_2 - A_1) \cup X]}(A_2 - A_1) \leq \binom{|I|}{3} \cdot (5|A_2 - A_1| + 3),$$

and this proves (1).

Note $t = |S - S'|$. Since $\delta(G) = 10$, every vertex in $A_2 - A_1$ has at least $10 - t$
neighbors in \(G[(A_2 - A_1) \cup S']\). It follows that

\[
(10 - t)|A_2 - A_1| \leq \sum_{v \in A_2 - A_1} d_{G[(A_2 - A_1) \cup S']}(v) = e(S', A_2 - A_1) + 2e(G[A_2 - A_1]),
\]

and this proves (2).

To see (3), first observe that \(e(S, A_2 - A_1) + e(G[A_2 - A_1]) = e(G) - e(G_1)\). Since \(G > G_1\), we know \(e(G_1) \leq 8|G_1| - 34\). Assume \(e(G_1) \geq 8|G_1| - 35\), and note this means that \(G_1\) is isomorphic to some exceptional graph. By Lemma 3.2.4, there exist two distinct pairs of non-adjacent vertices \(\{a_1, b_1\}\) and \(\{a_2, b_2\}\) in \(G[S]\), i.e. \(\{a_1, b_1\} \neq \{a_2, b_2\}\) and \(a_i, b_i\) are not adjacent in \(G\) for \(i = 1, 2\). By the minimality of \(|S|\), we know that \(G > G_1 + \{a_i, b_i\}\) for both \(i = 1, 2\). On the other hand, by Lemma 2.5.7, there exists at most one pair of non-adjacent vertices in \(G_1\) such that after adding an edge between them, the resulting graph does not have a \(K_{10}\) minor. By the minimality of \(S\), it follows that \(G > G_1 + \{a_i, b_i\} > K_{10}\) for some \(i \in \{1, 2\}\), a contradiction. Hence, we conclude that \(e(G_1) \leq 8|G_1| - 36\), and therefore

\[
e(S, A_2 - A_1) + e(G[A_2 - A_1]) = e(G) - e(G_1) \geq (8n - 35) - (8|G_1| - 36) = 8|A_2 - A_1| + 1.
\]

Since \(e(S', A_2 - A_1) + e(G[A_2 - A_1]) = e(S, A_2 - A_1) + e(G[A_2 - A_1]) - e(S - S', A_2 - A_1)\), and \(e(S - S', A_2 - A_1) \leq |S - S'| \cdot |A_2 - A_1| = t|A_2 - A_1|\), it follows that

\[
e(S', A_2 - A_1) + e(G[A_2 - A_1]) \geq (8 - t)|A_2 - A_1| + 1,
\]

which completes the proof of (3).

\[
\]

**Claim 3.** \(|S| = 10\) and \(|A_2 - A_1| \leq 4\).

**Proof of Claim 3.** We first assume \(|S| \leq 9\), which means that \(|I| = 4\). By (1) of Claim 2,

\[
3e(S', A_2 - A_1) + 4e(G[A_2 - A_1]) \leq 4(5|A_2 - A_1| + 3) = 20|A_2 - A_1| + 12.
\]
Observe that
\[
3e(S', A_2 - A_1) + 4e(G[A_2 - A_1]) = (e(S', A_2 - A_1) + 2e(G[A_2 - A_1])) \\
+ 2((e(S', A_2 - A_1) + e(G[A_2 - A_1]))).
\]

By (2) and (3) of Claim 2, we have

\[
(10 - t)|A_2 - A_1| + 2((8 - t)|A_2 - A_1| + 1) \leq 20|A_2 - A_1| + 12,
\]

which can be simplified to \((6 - 3t)|A_2 - A_1| \leq 10\). If \(|S| = 8\), then \(t = 0\) and thus \(|A_2 - A_1| \leq 1\), a contradiction to the fact that \(\delta(G) = 10\).

If \(|S| = 9\), then \(t = 1\) and thus \(|A_2 - A_1| \leq 3\). Since \(\delta(G) = 10\), \(|A_2 - A_1| = 2\) or \(3\). If \(|A_2 - A_1| = 2\), then every vertex in \(A_2 - A_1\) is adjacent to all vertices in \(S\) and the other vertex in \(A_2 - A_1\). Without loss of generality, assume that \(s_5 \in U\) and \(t_5 \notin U\). By contracting each path \(P_i\) to a single vertex for \(i = 1, \ldots, 9\) and contracting the edges \(xs_1, s_2s_3, \) and \(s_4t_5\), we would obtain a \(K_{10}\) minor, a contradiction. If \(|A_2 - A_1| = 3\), let \(v_1, v_2, v_3\) be the three vertices in \(A_2 - A_1\). Since \(\delta(G) = 10\), \(v_i\) is adjacent to at least 7 vertices in \(S'\) for \(i = 1, 2, 3\). This means that for each \(i = 1, 2, 3\), there exists three distinct indices \(i^1, i^2, i^3 \in I\) such that \(v_i\) is adjacent to \(s_{ij}'\) and \(t_{ij}'\) for all \(j = 1, 2, 3\). By relabeling vertices in \(S'\), we may assume that \(v_i\) is adjacent to both \(s_{ij}'\) and \(t_{ij}'\) for \(i = 1, 2, 3\).

By contracting each path \(P_i\) to a single vertex for \(i = 1, \ldots, 9\) and contracting each edge \(v_j s_{ij}'\) for \(j = 1, 2, 3\), we would obtain a resulting graph on \(N[x]\) that contains a subgraph isomorphic to \(K_7 + K_{2,2} > K_{10}\), a contradiction.

We now assume \(|S| = 10\), meaning \(|I| = 5\), \(t = 0\), and \(S = S'\). By (1) of Claim 2,

\[
6e(S, A_2 - A_1) + 10e(G[A_2 - A_1]) \leq 10(5|A_2 - A_1| + 3) = 50|A_2 - A_1| + 30.
\]
Note that

\[ 6e(S, A_2 - A_1) + 10e(G[A_2 - A_1]) = 4(e(S, A_2 - A_1) + 2e(G[A_2 - A_1])) + 2(e(S, A_2 - A_1) + e(G[A_2 - A_1])). \]

By (2) and (3) of Claim 2, it follows that

\[ 4(10|A_2 - A_1|) + 2(8|A_2 - A_1| + 1) \leq 50|A_2 - A_1| + 30. \]

This means that \( 6|A_2 - A_1| \leq 28 \), and therefore \( |A_2 - A_1| \leq 4. \)

Claim 3 shows that \( |S| = 10 \) and \( |A_2 - A_1| \leq 4 \). Note that we may just choose \( A_1 = N[x], A_2 = V(G) - \{x\}, \) and \( S = N(x) \). It follows that \( |G - N[x]| \leq 4 \). Let \( l = |G - N[x]| \), and let \( v_1, \ldots, v_l \) be the vertices in \( G - N[x] \).

If \( l = 1 \), then the only vertex \( v_1 \) in \( G - N[x] \) is adjacent to every vertex in \( N(x) \) since \( \delta(G) = 10 \). Observe that \( e(N(x)) = e(G) - d(x) - d(v_1) = 8 \cdot 12 - 35 - 10 - 10 = 41 \), meaning \( N(x) \cong K_{1,1,2,2,2,2} \). It follows that \( G \cong K_{1,1,2,2,2,2,2} \), which is an exceptional graph, a contradiction.

If \( l = 2 \), since \( \delta(G) = 10 \), we know \( v_i \) is adjacent to at least 9 vertices in \( N(x) \) for \( i = 1, 2 \). Without loss of generality, we can assume that \( v_1, v_2 \) each are adjacent to \( s_i \) and \( t_i \) for \( i = 1, 2, 3 \). Note that if \( s_it_i \in E(G) \) for some \( i \in \{1, 2, 3, 4, 5\} \), then by contracting \( v_1s_j \) and \( v_2s_k \) for some distinct \( j, k \in \{1, 2, 3\} \) we would obtain a new graph on \( N[x] \) that has a subgraph isomorphic to \( K_7 + K_{2,2} > K_{10} \), a contradiction. Hence, \( s_it_i \notin E(G) \) for all \( i = 1, \ldots, 5 \). It follows that \( N[x] \cong K_{1,1,2,2,2,2,2} \), and therefore \( e(N(x), G - N[x]) + e(G[\{v_1, v_2\}]) = e(G) - e(K_{1,2,2,2,2,2}) = 8 \cdot 13 - 35 - 50 = 19 \). Since \( \delta(G) = 10 \), it follows that \( v_1, v_2 \) are adjacent and they each are adjacent to exactly 9 vertices in \( N(x) \). Without loss of generality, assume \( s_5 \) is the unique vertex in \( N(x) \) not adjacent to \( v_1 \), and that either \( s_4 \) or \( t_5 \) is the one that is not adjacent to \( v_2 \). If \( s_4, v_2 \) are not adjacent, then by contracting
xs_1, s_2s_3, and s_4s_5 we could obtain a K_{10} minor, a contradiction. If t_5, v_2 are not adjacent, the G \cong K_{2,2,2,2,2} + C_5, which is an exceptional graph, again a contradiction.

We may then assume l = 3 or 4. Say s_j and t_j form a pair for j = 1, 2, 3, 4, 5. Since d(v_i) \geq 10 for each i, each v_i has at least 10 - (l - 1) = 11 - l neighbors in N(x). Note that 11 - l \geq 7 since l \leq 4, and this means that each v_i is a common neighbor for at least two pairs in N(x). Also note that if some v_i is a common neighbor for 3 pairs in N(x), then, by relabeling the vertices in N(x) and G - N[x], we may assume that v_i is a common neighbor for s_i and t_i for i = 1, 2, 3. By contracting v_i s_i for i = 1, 2, 3, we would then obtain a new graph having a subgraph isomorphic to K_7 + K_{2,2} > K_{10}, a contradiction. Therefore, we may assume that every v_i is a common neighbor for exactly two pairs in N(x), meaning that l = 4 and every v_i has exactly 7 neighbors in N(x). Since d(v_i) \geq 10 for every i = 1, 2, 3, 4, it follows that G - N[x] \cong K_4. Therefore, e(N[x]) = e(G) - (e(N(x), G - N[x]) + e(G - N[x])) = (8 \cdot 15 - 35) - (4 \cdot 7 + \binom{4}{2}) = 85 - 32 = 53, meaning that N[x] \cong K_7 + K_{2,2} > K_{10}, a contradiction. \qed
CHAPTER 4
MAIN TECHNICAL LEMMA

The goal of the entire Chapter 4 is to prove Lemma 4.1.1, the main technical lemma in our proof for Theorem 1.1.5.

4.1 Statements and Proof Outline

We state the main technical lemma and give an outline of its proof in this section.

**Lemma 4.1.1.** Let \( x \in V(G) \) such that \( 11 \leq d(x) \leq 15 \). Let \( M \) be the subset of vertices of \( N(x) \) that are not adjacent to all other vertices of \( N(x) \), i.e. \( M = \{ v \in N(x) : vu \not\in E(G) \text{ for some } u \in N(x) - \{v\} \} \). Then, for every component \( K \) of \( G - N[x] \), there exists some component \( K' \) of \( G - N[x] \) such that \( N(K') \cap M \not\subseteq N(K) \).

To prove Lemma 4.1.1, for the sake of a contradiction, assume that there exists some \( x \in V(G) \) with \( 11 \leq d(x) \leq 15 \) and a component \( K \) of \( G - N[x] \) such that for every component \( K' \) of \( G - N[x] \), \( N(K') \cap M \subseteq N(K) \) where \( M = \{ v \in N(x) : vu \not\in E(G) \text{ for some } u \in N(x) - \{v\} \} \). We choose such a pair \( (x, K) \) such that \( d(x) \) is minimum over all choices.

The rest of our proof for Lemma 4.1.1 can be outlined as follows.

In Section 4.2, we first prove \( M \subseteq N(K) \) and \( N(x) \not\supseteq K_8 \cup K_1 \) in Lemma 4.2.1 and Lemma 4.2.2. Notice that \( 11 \leq d(x) \leq 15 \) and \( \delta(N(x)) \geq 8 \) by Lemma 3.1.1. It follows that \( N(x) \) is isomorphic to some graph \( H \) such that (i) \( 11 \leq |H| \leq 15 \), (ii) \( \delta(H) \geq 8 \), and (iii) \( H \not\supseteq K_8 \cup K_1 \). Note that there are only finitely many graphs satisfying (i)-(iii). In Lemma 4.2.3, we present all edge-minimal graphs satisfying (i)-(iii), which are generated by a computer program. There are precisely 101 such graphs, up to isomorphism, and we call them problem graphs. It remains to show that if \( N(x) \) has a subgraph isomorphic to
some problem graph, then we can contract edges that have at least one end in \(G - N[x]\) such that the resulting graph on \(N[x]\) has a \(K_{10}\) minor. It turns out that if \(N(x)\) has a subgraph isomorphic to the three problem graphs \(K_{2,3,3,3}, K_{3,3} + C_5\), or \(K_{4,4,4}\), we would need to spend some more effort to find the desired \(K_{10}\) minor; and if \(N(x)\) does not have a subgraph isomorphic to those three graphs, a \(K_{10}\) minor is relatively easier to be found.

In Section 4.3, we consider the case that \(N(x)\) has a subgraph \(N' \cong K_{2,3,3,3}, K_{3,3} + C_5\), or \(K_{4,4,4}\). We first prove that \(N(x) = N'\), so \(N(x)\) itself is isomorphic to one of the three problem graphs in Lemma 4.3.1. We then prove \(|G - N[x]| \geq 3\) in Lemma 4.3.2 and a quite technical result on 2-separations of each component of \(G - N[x]\) in Lemma 4.3.4. Then, we show that a \(K_{10}\) minor can be found if \(G - N[x]\) is 2-connected in Lemma 4.3.5 and Lemma 4.3.6, and that a \(K_{10}\) minor can be found if \(G - N[x]\) is NOT 2-connected in Lemma 4.3.7.

In Section 4.4, we consider the case that \(N(x)\) does NOT have a subgraph isomorphic to \(K_{2,3,3,3}, K_{3,3} + C_5\), or \(K_{4,4,4}\). We use computer programs to verify that every problem graph that is NOT isomorphic to \(K_{2,3,3,3}, K_{3,3} + C_5\), or \(K_{4,4,4}\) satisfies one of the properties (A1) and (A2) in Lemma 4.4.1 and one of the properties (B1)-(B6) in Lemma 4.4.3. Finally, we use properties (A1) and (A2) to show that a \(K_{10}\) minor can be found if \(G - N[x]\) is 2-connected and has at least two vertices in Lemma 4.4.2, and we use properties (B1)-(B6) to show that a \(K_{10}\) minor can be found otherwise in Lemma 4.4.4.

### 4.2 Problem Graphs

We will prove \(M \subseteq N(K)\) in Lemma 4.2.1 and \(N(x) \not\subseteq K_8 \cup K_1\) in Lemma 4.2.2.

**Lemma 4.2.1.** \(M \subseteq N(K)\).

**Proof.** For the sake of a contradiction, assume that \(M - N(K) \neq \emptyset\).

We first observe that for every \(v \in M - N(K)\), \(v\) does not have any neighbor in \(G - N[x]\), since otherwise there would exist some component \(K'\) of \(G - N[x]\) such that \(y \in N(K') \cap M\) but \(y \not\in N(K)\), which is a contradiction to the choice of \(x\) and \(K\).
Choose \( y \in M - N(K) \) such that \( d(y) \) is minimum. Let \( M_y = \{ v \in N(y) : vu \not\in E(G) \text{ for some } u \in N(y) - \{v\} \} \). By the previous observation, it follows that \( y \) has no neighbor in \( G - N[x] \) and therefore \( N[y] \subseteq N[x] \). Since \( y \in M \), \( y \) is not adjacent to every vertex in \( N(x) \), and it follows that \( d(y) < d(x) \). Let \( J \) be the component of \( G - N[x] \) that contains \( K \). We will complete the proof by considering the following two cases: \( N(x) - N[y] \subseteq V(J) \), or \( N(x) - N[y] \not\subseteq V(J) \).

**Case 1:** \( N(x) - N[y] \subseteq V(J) \).

Since \( d(y) < d(x) \), by the choice of \( x \) and \( K \) we know that \( J \) is not the only component of \( G - N[y] \). Let \( J' \) be any component of \( G - N[y] \) such that \( J' \neq J \). Since \( J' \) is chosen arbitrarily, it suffices to show a contradiction by proving that \( N(J') \cap M \subseteq N(J) \).

Observe that since \( N[y] \subseteq N[x] \), \( G - N[y] \) is an induced subgraph of \( G - N[y] \). Since \( N(x) - N[y] \subseteq V(J) \), every vertex in \( G - N[y] \) but not in \( G - N[x] \) is contained in the component \( J \) of \( G - N[y] \). It follows that \( J' \) itself is also a component of \( G - N[x] \). By the choice of \( x \) and \( K \), it follows that \( N(J') \cap M \subseteq N(K) \). Since \( N[y] \subseteq N[x] \), we know that \( x \not\in M_y \) and therefore \( M_y \subseteq M \). It follows that

\[
N(J') \cap M_y \subseteq N(J') \cap M \subseteq N(K).
\]

Now, observe that since \( V(K) \subseteq V(J) \), every neighbor of \( K \) is either in \( J \) or in \( N(J) \), and therefore

\[
N(J') \cap M_y \subseteq N(K) = (N(K) \cap V(J)) \cup (N(K) \cap N(J)).
\]

Notice that \( N(J') \cap M_y \subseteq N(y) \) and \( N(K) \cap V(J) \subseteq V(J) \) which is disjoint from \( N(y) \). It follows that

\[
N(J') \cap M_y \subseteq N(K) \cap N(J) \subseteq N(J).
\]

**Case 2:** \( N(x) - N[y] \not\subseteq V(J) \).
Let $H$ be a component of $N(x) - N[y] \cup V(J)$. Note that $H \subseteq G - N[y] \cup V(J)$, meaning that $H$ is contained in some component of $G - N[y]$ disjoint from $J$.

We will first show that $H$ itself is a component of $G - N[y]$. For every $z \in V(H)$, note that $z \in N(x) - N[y]$, meaning that $z \in M$ and therefore $z \in M - N(K)$. By the observation at the beginning of this proof, it follows that $z$ has no neighbor in $G - N[x]$. It follows that $N(H) \subseteq N[x]$. Let $J'$ be the component of $G - N[y]$ that contains $H$. Note that if $V(J' - H) \neq \emptyset$, since $N[y] \subseteq N[x]$ then every vertex in $J' - H$ must be contained in some component $K'$ of $G - N[x]$ such that $K' \neq K$, a contradiction to the fact that $N(H) \subseteq N[x]$. Hence, we conclude that $H$ itself is a component of $G - N[y]$.

Now, since $V(K) \subseteq V(J)$ and $J, H$ are disjoint components of $G - N[y]$, it follows that for every $z \in V(H)$ we have $z \not\in N(K)$ and therefore $z \in M - N(K)$. By the choice of $y$, $d(z) \geq d(y)$ for every $z \in V(H)$. Let $t = |H|$. Assume $t = 1$ for a moment, and let $z^*$ be the unique vertex in $H$. It follows that $N(H) = N(z^*) = N(y)$. Since $d(y) < d(x)$, this means that $(y, H)$ contradicts the choice of $(x, K)$. Therefore, $t \geq 2$.

On the other hand, since $H \subseteq N(x) - N[y] = N(x) - (N[y] - \{x\})$, we know that $t \leq d(x) - d(y) \leq 15 - 11 = 4$. Hence, we have $2 \leq t \leq 4$.

Now, let $L = G[N[y] \cup V(H)]$. Note $|L| = d(y) + t + 1$, and that

$$e(L) = d(y) + e(N(y)) + e(N(y), V(H)) + e(H).$$

Note that $x \in N(y)$ is adjacent to all other vertices in $N(y)$. Since $\delta(N(y)) \geq 8$, we know $\delta(N(y) - \{x\}) \geq 7$. It follows that

$$e(N(y)) = e(\{x\}, N(y) - \{x\}) + e(N(y) - \{x\}) \geq d(y) - 1 + \frac{1}{2} \cdot 7(d(y) - 1) = \frac{9}{2}d(y) - \frac{9}{2}.$$
For every $z \in V(H)$, since $N(z) \subseteq V(H) \cup N(y)$ and $d(z) \geq d(y)$, we have

\[ e(N(y), V(H)) + 2e(H) = \sum_{z \in V(H)} d(z) \geq \sum_{z \in V(H)} d(y) = td(y), \]

and therefore $e(N(y), V(H)) + e(H) \geq td(y) - e(H) \geq td(y) - \left(\frac{t}{2}\right)$. Hence, we have

\[ e(L) \geq d(y) + \frac{9}{2}d(y) - \frac{9}{2} + td(y) - \left(\frac{t}{2}\right) \]

\[ = \left(\frac{11}{2} + t\right)d(y) - \frac{1}{2}t^2 + \frac{1}{2}t - \frac{9}{2} \]

\[ = 8d(y) + t + 1 + \left(t - \frac{5}{2}\right)d(y) - \frac{1}{2}t^2 - \frac{15}{2}t - \frac{25}{2} \]

\[ = 8|L| + \left(t - \frac{5}{2}\right)d(y) - \frac{1}{2}t^2 - \frac{15}{2}t - \frac{25}{2}. \]

Let $G_1 = G[V(J) \cup N(J)]$, $S = N(J)$, and $G_2 = G - V(J)$. Observe that $G_1 \cup G_2 = G$, $G_1 \cap G_2 = G[S]$, and $L \subseteq G_2$. Let $d_2$ be the maximum number of edges that can be added to $G[S]$ by contracting edges that have at least one edge in $V(G_1 - G_2)$. Let $L'$ be a graph with $V(L') = V(L)$ that can be obtained from $G$ by deleting vertices in $V(G_2 - L)$ and contracting edges that have at least one end in $J$ such that $e(L') = e(L) + d_2$. We then have

\[ e(L') = e(L) + d_2 \geq 8|L'| + \left(t - \frac{5}{2}\right)d(y) - \frac{1}{2}t^2 - \frac{15}{2}t - \frac{25}{2} + d_2. \]

If $t = 3$ or $4$, then $t - \frac{5}{2} > 0$. Since $d(y) \geq 11$,

\[ e(L') \geq 8|L'| + 11\left(t - \frac{5}{2}\right) - \frac{1}{2}t^2 - \frac{15}{2}t - \frac{25}{2} + d_2 = 8|L'| - \frac{1}{2}(t - \frac{7}{2})^2 - \frac{271}{8} + d_2 = 8|L'| - 34 + d_2. \]

If $t = 2$, note $N(y) \subseteq N[x] - \{y\} - H$, meaning that $d(y) \leq d(x) - t \leq 15 - 2 = 13$. Then,

\[ e(L') \geq 8|L'| - \frac{1}{2}13 - \frac{1}{2}t^2 - \frac{15}{2}t - \frac{25}{2} + d_2 = 8|L'| - 36 + d_2. \]
Note that \(|L'| = d(y) + t + 1 \geq 11 + 2 + 1 = 14 > 8\). If \(e(L') \geq 8|L'| - 35\), then by induction we know that \(L'\) is isomorphic to some exceptional graph. If \(e(L') \geq 8|L'| - 35\), then by induction we know that \(L'\) is isomorphic to some exceptional graph. This is not possible, because \(x\) is adjacent to all other vertices in \(L'\), and there is no exceptional graph on at least 14 vertices in which there is a vertex adjacent to all other vertices. It follows that \(e(L') \leq 8|L'| - 36\), which could happen when \(t = 2\) and \(d_2 = 0\). However, by (3) of Lemma 3.2.4, the connectivity of \(G_1 - G_2 = J\) implies that by contracting all of \(J\) to a vertex in \(S\) that has the minimum degree inside \(G[S]\) we would obtain at least one extra edge on \(G[S]\). It follows that \(d_2 \geq 1\), a contradiction.

Lemma 4.2.2. \(N(x) \not\cong K_8 \cup K_1\).

Proof. For the sake of a contradiction, assume \(N(x) > K_8 \cup K_1\). Choose \(y \in N(x)\) such that \(N(x) - \{y\} > K_8\). We may assume that \(y\) is not adjacent to some vertex in \(N(x)\), since otherwise \(N(x) > K_9\) which then implies \(G > K_{10}\), a contradiction. It follows that \(y \in M \subseteq N(K)\). By contracting all vertices in \(K\) to \(y\), in the resulting graph on \(N(x)\), \(y\) would be adjacent to all other vertices in \(M\), meaning that \(y\) would be adjacent to all other vertices in \(N(x)\). It follows that the new graph on \(N(x)\) has a \(K_9\) minor and therefore \(G > K_{10}\), a contradiction.

Lemma 4.2.3 (computer-assisted). \(N(x)\) has a subgraph that is isomorphic to one of the 101 graphs listed in Appendix.

We call each one of these 101 graphs a problem graph.

4.3 \(K_{2,3,3,3}, K_{3,3} + C_5\), and \(K_{4,4,4}\)

Now, assume that \(N(x)\) contains a subgraph \(N'\) isomorphic to \(K_{2,3,3,3}, K_{3,3} + C_5\), or \(K_{4,4,4}\) such that \(|N(x)| = |N'|\).

We first define three types of minors of \(G\). For a minor \(H\) of \(G - x\) rooted at \(N(x)\), say \(H\) is a minor of \(G\) of type I if there exist distinct vertices \(s_1, s_2, s_3, t_1, t_2, t_3 \in N(x)\) such
that for some distinct islands $C_1, C_2, C_3$ of $N'$, $s_i, t_i \in C_i$ and $s_it_i \in E(H) - E(N')$ for $i = 1, 2, 3$; say $H$ is a minor of $G$ of type II if there are distinct vertices $x_1, x_2, x_3, y_1, y_2$ such that $x_1, x_2, x_3 \in C_1$ and $y_1, y_2 \in C_2$ for some distinct islands $C_1, C_2$ of $N'$ and $x_1x_2, x_1x_3, x_2x_3, y_1y_2 \in E(H) - E(N')$; say $H$ is a minor of $G$ of type III if $N' \cong K_{3,3} + C_5$ and there are distinct vertices $x_1, x_2, y_1, y_2, y_3, y_4 \in N(x)$ such that $x_1, x_2$ are contained in a 3-island of $N'$, $y_i$ for $i = 1, 2, 3, 4$ are in the 5-island of $N'$, and that $x_1x_2, y_1y_2, y_2y_3, y_3y_4 \in E(H) - E(N')$.

Observe that since $N' \cong K_{2,3,3,3}, K_{3,3} + C_5,$ or $K_{4,4,4},$ if $G$ has a minor $H$ of one of the three types defined above, then $G > K_{10},$ a contradiction. Hence, $G$ does not have a minor of type I, type II, or type III.

**Lemma 4.3.1.** $N(x) \cong N'.$

**Proof.** For the sake of a contradiction, assume that $N(x) \not\cong N'.$

We first prove that $G - N[x] \not\cong \emptyset$. Let $E' = E(N(x)) - E(N')$. Note that the end vertices of edges in $E'$ are in at most two islands of $N'$, since otherwise $G$ would have a minor of type I, a contradiction. If $G - N[x] = \emptyset$, then $|E'| = e(G) - (d(x) + e(N')) = 8 \cdot |G| - 35 - (d(x) + e(N'))$. This means that if $N' \cong K_{2,3,3,3}$, then $|E'| = (8 \cdot 12 - 35) - (11 + 45) = 5$; if $N' \cong K_{3,3} + C_5$, then $|E'| = (8 \cdot 12 - 35) - (11 + 44) = 6$; and if $N' \cong K_{4,4,4}$, then $|E'| = (8 \cdot 13 - 35) - (12 + 48) = 9$. One can then check that in all cases, $G$ would have a minor of type II or type III, a contradiction. Hence, $G - N[x] \not\cong \emptyset$.

Now, let $s_1, t_1 \in N(x)$ be such that $s_1t_1 \in E(N') - E(G)$, and let $C_1$ be the island of $N'$ containing $s_1, t_1$. Recall that $K$ is a component of $G - N[x]$ such that $M \subseteq N(K)$ where $M = \{v \in N(x) : vu \not\in E(G) \text{ for some } u \in N(x) - \{v\}\}$.

We next show that $N' - C_1 = N(x) - C_1$. For the sake of a contradiction, assume that there exist $s_2, t_2 \in N(x)$ and an island $C_2$ of $N' - C_1$ such that $s_2, t_2 \in C_2$ and $s_2t_2 \in E(N(x)) - E(N')$. Then, we see that $N' - C_1 \cup C_2 \cong N(x) - C_1 \cup C_2$, since otherwise $G$ would have a minor of type I, a contradiction. Since $N'$ has at least three
islands in all cases, there exist two vertices \( s_3, t_3 \in N(x) - C_1 \cup C_2 \) that are not adjacent to each other in both \( N' \) and \( N(x) \). It follows that \( s_3, t_3 \in M \subseteq N(K) \). By contracting all of \( K \) to one of \( s_3 \) and \( t_3 \), we would then obtain a resulting graph on \( N(x) \) that has edges \( s_1t_1, s_2t_2, s_3t_3 \), meaning that \( G \) has a minor of type I, a contradiction. Therefore, \( N' - C_1 = N(x) - C_1 \). Observe that in all cases, every vertex in \( N' \) has some non-neighbor in it. It follows that \( N(x) - C_1 \subseteq M \subseteq N(K) \).

In the rest of the proof, we consider the case \(|K| \geq 2\) and the case that \(|K| = 1\).

**Case 1:** \(|K| \geq 2\).

Observe that since \( N' \cong K_{2,3,3,3}, K_{3,3} + C_5 \), or \( K_{4,4,4} \), in all cases there exists a subset of vertices \( X = \{x_1, x_2, x_3\} \subseteq N(x) - C_1 \) such that \( N'[X] = G[X] \cong K_3 \). Let \( C_2 \) be the island of \( N' \) containing \( X \), and note that \( X \subseteq C_2 \subseteq N(K) \). Note that \( G[V(K) \cup X] \) is connected, and it does not have a \( K_3 \) minor rooted at \( X \), since otherwise \( G \) would have a minor of type II due to the edge \( s_1t_1 \), a contradiction. By Lemma 2.2.1, \( G[V(K) \cup X] \) has a cut vertex \( w \), and that there are components \( J_1, J_2, J_3 \) of \( G[V(K) \cup X] - \{w\} \) such that \( x_i \in V(J_i) \) for \( i = 1, 2, 3 \). Notice that \( w \in V(K) \), as \( K \) itself is a connected subgraph of \( G \).

Since \(|K| \geq 2\), \( K - \{w\} \neq \emptyset \). Without loss of generality, assume that either \( J_1 - \{x_1\} \neq \emptyset \) or \( K' = G[V(K) \cup X] - V(J_1 \cup J_2 \cup J_3) \cup \{w\} \neq \emptyset \). Let \( L \) be a non-trivial component of \( J_1 - \{x_1\} \) if \( J_1 - \{x_1\} \neq \emptyset \), and let \( L \) be a component of \( K' \) if \( K' \neq \emptyset \). In both cases, notice that \( L \) is a non-trivial subgraph of \( K - V(J_2 \cup J_3) \cup \{w, x_1\} \) such that \( N(L) \cap (V(G) - N[x]) \subseteq \{w\} \), meaning that \( L \) has at most one neighbor in \( G - N[x] \). Since \( G \) is 7-connected, \( L \) has at least six neighbors in \( N(x) \). Observe that \( L \) does not have non-adjacent neighbors in some island \( C_3 \) of \( N(x) - C_1 \cup C_2 \), since otherwise we could just contract all of \( V(J_2 \cup J_3 - \{x_3\}) \cup \{w\} \) to \( x_2 \) and contract all of \( L \) to one of its non-adjacent neighbors in \( C_3 \) to obtain a minor of \( G \) of type I, a contradiction. It follows that

\[
|N(L) \cap C_1| \geq 6 - |N(L) \cap C_2| - \omega(N' - C_1 \cup C_2).
\]
If $N' \cong K_{2,3,3,3}$ or $K_{3,3} + C_5$, then $C_2 = X$ and $N(L) \cap C_2 = N(L) \cap X \subseteq \{x_1\}$, meaning that $|N(L) \cap C_2| \leq 1$. If $C_1$ is an independent set, then one can observe that in all cases we have $\omega(N' - C_1 \cup C_2) = 2$ and thus $|N(L) \cap C_1| \geq 6 - 1 - 2 = 3$. It follows that $C_1$ is an independent set of size 3 and $C_1 \subseteq N(L)$. Then, by contracting all of $L$ to the unique vertex in $C_1 - \{s_1, t_1\}$ and contracting all of $V(J_2 \cup J_3 - \{x_3\}) \cup \{w\}$ to $x_2$, we then obtain a minor of $G$ of type II, a contradiction. It follows that $C_1$ is not an independent set, and this means that $N' = K_{3,3} + C_5$ and $N'[C_1]$ is the 5-cycle. Note now $N' - C_1 \cup C_2$ is simply an independent set of size 3, so $\omega(N' - C_1 \cup C_2) = 1$. Therefore, $|N(L) \cap C_1| \geq 4$. Without loss of generality, we can assume that $s_1, s_3, t_3 \in C_1$ are all neighbors of $L$ where $s_3, t_3$ are in the positions such that $s_1 s_3, s_3 t_3 \not\in E(N')$. Then, by contracting all of $L$ to $s_3$ we can then obtain a resulting graph that includes edges $s_1 t_1, s_1 s_3, s_3 t_3$. By further contracting all of $V(J_2 \cup J_3 - \{x_3\}) \cup \{w\}$ to $x_2$, we would then obtain a minor of $G$ of type III, a contradiction.

We may then assume that $N' \cong K_{4,4,4}$. In this case, every island of $N'$ is an independent set of size 4, and thus $|N(L) \cap C_2| \leq |\{x_1\} \cup (C_2 - X)| = 2$ and $\omega(N' - C_1 \cup C_2) = 1$. It follows that $|N(L) \cap C_1| \geq 6 - 2 - 1 = 3$. Note that if $L$ has all four vertices in $C_1$ as its neighbors, then by contracting all of $L$ to one vertex in $C_1 - \{s_1, t_1\}$, we would then obtain a clique of size 3 in $C_1$, and this implies that $G$ has a minor of type II due to $J_2, J_3$, and $w$. It follows that $L$ has exactly three neighbors in $C_1$, two neighbors in $C_2$, one neighbor in $N(x) - C_1 \cup C_2$, and one neighbor in $G - N[x]$, and this means that $|N(L)| = 7$. By Lemma 3.2.3, it follows that $G[V(L) \cup (N(L) \cap C_1)]$ has a $K_3$ minor rooted at $N(L) \cap C_1$. Therefore, $G$ has a minor of type II due to $J_2, J_3$, and $w$, a contradiction.

**Case 2:** $|K| = 1$.

**Case 2.1:** $G - N[x]$ is disconnected.

Let $K'$ be a component of $G - N[x]$ such that $K \neq K'$. Assume for a moment that there exists an island $C_2$ of $N(x) - C_1$ such that $K'$ has some non-adjacent neighbors $s_2, t_2 \in C_2$. Since $N'$ has at least three islands in all cases, there exists an island $C_3$
of \( N' - C_1 \cup C_2 \). Let \( s_3, t_3 \in C_3 \) be two non-adjacent vertices, and note this means that \( \{s_3, t_3\} \subseteq M \subseteq N(K) \). Then, by contracting all of \( K' \) to \( s_2 \) and contracting all of \( K \) to \( s_3 \), we would then obtain a minor of \( G \) of type I, a contradiction. Therefore, 
\[ N(K') \cap (N(x) - C_1) \] is a clique. One can then check that unless \( N' \cong K_{3,3} + C_5 \) and \( C_1 \) is the 5-cycle in it, \( |N(K')| \leq |C_1| + \omega(N(x) - C_1) \leq 6 \), which is a contradiction to the 7-connectivity of \( G \). It follows that \( N' \cong K_{3,3} + C_5 \) and \( N'[C_1] \) is a 5-cycle. Due to the 7-connectivity of \( G \), \( K' \) has exactly one neighbor in each 3-island of \( N' \) and has all five vertices in \( C_1 \) as its neighbors. Let \( s_2, t_2 \in C_1 - \{s_1, t_1\} \) be such that \( s_1s_2, s_2t_2 \notin E(N') \). Then, by contracting all of \( K' \) to \( s_2 \) and contracting all of \( K \) to any vertex in \( N(x) - C_1 \), we would then obtain a minor of type III, a contradiction.

**Case 2.2:** \( G - N[x] \) is connected.

Since \( |K| = 1 \), let \( y \) be the unique vertex in \( K \), and let \( t = e(G[C_1]) - e(N'[C_1]) \). We then have \( 8n - 35 = e(G) = d(x) + d(y) + e(N') + t \), and therefore

\[ t = (8n - 35) - (d(x) + d(y) + e(N')). \]

If \( N' \cong K_{2,3,3,3} \) or \( K_{3,3} + C_5 \), then \( n = 13 \) and \( d(x) = d(y) = 11 \) since \( \delta(G) \geq 11 \). It follows that \( t = 8 \cdot 13 - 35 - (11 + e(N')) - 11 = 47 - e(N') \). If \( N' \cong K_{2,3,3,3} \), then \( e(N') = 45 \) and therefore \( t = 2 \). This implies that \( C_1 \) is a 3-island of \( N' \), and that \( G[C_1] \) is a path of length 2. It follows that \( G \cong K_{1,2,2,2,3,3} \), which is an exceptional graph, a contradiction. If \( N' \cong K_{3,3} + C_5 \), then \( e(N') = 44 \) and therefore \( t = 3 \). If \( C_1 \) is a 3-island, then \( G \cong K_3 + K_{2,3} + C_5 \) and \( N'[C_1] \cong C_5 \), note \( G[C_1] \) is either a path of length 3 or a disjoint union of an edge and a path of length 2. It follows that \( G \cong K_{1,1,2,3,3} + P_3 \) or \( K_{1,2,2,2,3,3} \). This is a contradiction, since \( K_{1,1,2,3,3} + P_3 > K_{10} \) and \( K_{1,2,2,2,3,3} \) is an exceptional graph.

If \( N' \cong K_{4,4,4} \), then \( n = 14 \), \( d(x) = 12 \), \( d(y) \leq 12 \), and \( e(N') = 48 \). It follows that \( t \geq (8 \cdot 14 - 35) - 12 - 48 - 12 = 5 \). Since every island of \( N' \) is a 4-island, \( G[C_1] \)
has at least five edges on 4 vertices, meaning that there is a clique of size 3 on $C_1$ in $G$. Note that $V(K) = \{y\}$ has all vertices in $N(x) - C_1$ as its neighbors. By contracting an edge between $y$ and any vertex in $N(x) - C_1$, we would then obtain a minor of type II, a contradiction. \hfill \Box

Lemma 4.3.2. $|G - N[x]| \geq 3$.

Proof. For the sake of a contradiction, assume $|G - N[x]| \leq 2$. By Lemma 4.3.1, we know that $N(x) = N' \cong K_{2,3,3,3}, K_{3,3} + C_5$, or $K_{4,4,4}$. Notice that if $N(x) \cong K_{4,4,4}$, then every two adjacent vertices in $N(x)$ have exactly five common neighbors in $G - N[x]$, so they must have at least three common neighbors in $G - N[x]$, a contradiction to the assumption that $|G - N[x]| \leq 2$. Hence, either $N(x) \cong K_{2,3,3,3}$ or $N(x) \cong K_{3,3} + C_5$.

If $N(x) \cong K_{2,3,3,3}$, then observe that every two adjacent vertices each contained in some 3-island of $N(x)$ have exactly 6 common neighbors in $G - N[x]$ and therefore have at least two common neighbors in $G - N[x]$. It follows that $|G - N[x]| = 2$, and that every vertex in a 3-island of $N(x)$ is adjacent to both vertices in $G - N[x]$. Let $V(G) - N[x] = \{a, b\}$. Note now $|G| = 14$, and

$$d(a) + d(b) - e(G(\{a, b\})) = (8 \cdot 14 - 35) - e(K_{1,2,3,3,3}) = 21.$$ 

Since $\delta(G) \geq 11$, $ab \in E(G)$ and $a, b$ each have exactly 10 neighbors in $N(x)$. Recall that every vertex in a 3-island of $N(x)$ is adjacent to both $a$ and $b$. Since $\delta(G) \geq 11$ and every vertex in a 3-island of $N(x)$ has exactly 10 neighbors in $N[x]$, it follows that the set of edges between $\{a, b\}$ and the 2-island of $N(x)$ is precisely a perfect matching. It follows that $G \cong K_{3,3} + C_5$, which is an exceptional graph, a contradiction.

If $N(x) \cong K_{3,3} + C_5$, then observe that every two vertices from distinct islands of $N(x)$ have exactly six common neighbors in $N[x]$ and thus have at least two common neighbors in $G - N[x]$. It follows that $|G - N[x]| = 2$, $|G| = 14$, and every vertex in $G - N[x]$ is adjacent to all vertices in $N(x)$. Since $e(G) = 8 \cdot 14 - 35$, one can then check that
there is no edge between the two vertices in $G - N[x]$ and thus $G \cong K_{3,3,3} + C_5$, again a contradiction since $K_{3,3,3} + C_5$ is an exceptional graph.

If $N(x) \cong K_{2,3,3,3}$ or $K_{3,3} + C_5$, let $T_1, T_2$ be two distinct 3-islands of $N(x)$; if $N(x) \cong K_{4,4,4}$, let $T_1, T_2$ be two disjoint independent sets of size 3 each in a 4-island of $N(x)$. In all cases, let $T_1 = \{x_1, x_2, x_3\}$ and $T_2 = \{y_1, y_2, y_3\}$.

**Lemma 4.3.3.** In all cases, the following statements are true:

1. There is no clique of size 5 in $N(x)$.
2. For all $i, j \in \{1, 2, 3\}$, $x_i, y_j$ have at least two common neighbors in $G - N[x]$.

**Proof.** (1) is simply true, as there is no clique of size 5 in $K_{2,3,3,3}$, $K_{3,3} + C_5$, or $K_{4,4,4}$. To see (2) is true, note that for any $i, j \in \{1, 2, 3\}$, $x_i y_j \in E(G)$ and they have at least eight common neighbors in $G$ by Lemma 3.1.1. One can observe that $x_i, y_j$ have at most five common neighbors in $N(x)$ in all cases, due to the construction of $T_1, T_2$. It follows that $x_i, y_j$ have at most six common neighbors in $N(x)$ and therefore at least two common neighbors in $G - N[x]$. 

Let $H \subseteq G$ be a subgraph, and let $S \subseteq V(H)$ be a subset of vertices. Say a vertex $v \in V(G)$ is associated with $S$ with respect to $H$ if there is a path $P$ linking $v$ and some vertex $u \in S$ such that $P$ is otherwise disjoint from $H$.

### 4.3.1 Proof of Lemma 4.3.4

**Lemma 4.3.4.** Let $(A, B)$ be a 2-separation of a component of $G - N[x]$ such that $A \cap B = \{a, b\}$, $B - A \neq \emptyset$, and there is a path linking $a, b$ in $G[B]$. If no vertex in $G - N[x]$ is a common neighbor for two non-adjacent vertices in $N(x)$, then there exist two disjoint paths in $G[B]$ such that one links $a, b$ and the other one links a neighbor of $y_1$ and a neighbor of $y_2$ for some non-adjacent vertices $y_1, y_2 \in N(x) \cap N(B - A)$. 

90
Proof. Say a 2-separation $(A', B')$ of $G[A \cup B]$ satisfies property $\mathcal{P}$ if there exist two disjoint paths in $G[B']$ such that one links $a'$ and $b'$ and the other one links a neighbor of $y'_1$ and a neighbor of $y'_2$ for some $y'_1, y'_2 \in N(x) \cap N(B' - A')$ such that $y'_1y'_2 \not\in E(G)$. For the sake of a contradiction, assume that $(A, B)$ fails property $\mathcal{P}$ such that $|B|$ is minimum among all 2-separations of $G[A, B]$ that fail property $\mathcal{P}$. With these assumptions, we make the following claim.

Claim 1. The following statements are true:

(1) $N(x)$ does not contain a clique of size more than 4.

(2) For every $v \in B - A$, $|N(v) \cap N(x)| \leq 4$ and $d_{G[B]}(v) \geq 7$.

(3) For any $a - b$ path $P$ in $G[B]$, there is no 2-separation $(B_1, B_2)$ of $G[B]$ such that $V(P) \subseteq B_1$ and $B_2 - B_1 \neq \emptyset$.

(4) There is no cut vertex of $G[B]$ that separates $a$ and $b$.

Proof of Claim 1. (1) is simply true because $N(x) \cong K_{2,3,3,3}, K_{3,3} + C_5$, or $K_{4,4,4}$ by Lemma 4.3.1, and none of these three graphs has a clique of size more than 4. Then, (2) and (3) follow immediately from (1), due to the facts that $\delta(G) \geq 11$, $G$ is 7-connected, and that no vertex in $G - N[x]$ is a common neighbor for two non-adjacent vertices in $N(x)$.

To see (4) is true, for the sake of a contradiction, assume that there is a cut vertex $w \in B - A$ of $G[B]$ that separates $a, b$. Let $B_1, B_2$ be the components of $G[B] - \{w\}$ such that $a \in V(B_1)$ and $b \in V(B_2)$. Since there is a path linking $a, b$ in $G[B]$, there exists a path $P_a$ lining $a, w$ in $G[B_1 \cup \{w\}]$ and a path $P_b$ linking $b, w$ in $G[B_2 \cup \{w\}]$. By the minimality of $|B|$ when choosing $(A, B)$, we know that $B = V(B_1 \cup B_2) \cup \{w\}$. By (2), $w$ has at least 7 neighbors in $B$ and thus one of $B_1 - \{a\}$ and $B_2 - \{b\}$ is non-empty. Without loss of generality, say $B_1 - \{a\} \neq \emptyset$. Let $A' = A \cup B - (B_1 - \{a\})$ and $B' = B_1 \cup \{w\}$. Notice that $(A', B')$ is a 2-separation of $G[A \cup B]$ such that $B' - A' \neq \emptyset$ and there is a path linking the two vertices $a, w$ in $A' \cup B'$. Then by the minimality of $|B|$ when choosing $(A, B)$, it follows that $(A', B')$ satisfies property $\mathcal{P}$, meaning that there exist two disjoint paths in $G[B']$ such that one links $a, w$ and the other one links a neighbor of $y'_1$ and a neighbor of $y'_2$. 

91
for some non-adjacent vertices $y'_1, y'_2 \in N(x) \cap N(B' - A')$. Then, by extending the $a - w$ path along $P_b$ to make it an $a - b$ path, we would then have a path linking $a, b$ and a path linking $y'_1, y'_2$ that are disjoint from each other. This means that $(A, B)$ satisfies property $\mathcal{P}$, a contradiction. It follows that $B_1 = \{a\}$ and therefore $B_2 = \{b\}$ by symmetry, meaning that $B - A = \{w\}$. Since $\delta(G) \geq 11$, $w$ has at least 9 neighbors in $N(x)$, a contradiction to (2).

Since $G$ is 7-connected, $B - A$ has at least 5 neighbors in $N'$. By (1) of Claim 1, there exist two non-adjacent vertices $v_1, v_2 \in N(x) \cap N(B - A)$. Let $u_1, u_2$ be neighbors of $v_1, v_2$ in $B - A$, respectively. Note that $u_1 \neq u_2$, as no vertex in $G - N[x]$ is a common neighbor for two non-adjacent vertices in $N(x)$.

**Claim 2.** There exist two internally disjoint $a - b$ paths $L_1, L_2$ in $G[B]$ such that $u_1 \in V(L_1)$ and $u_2 \in V(L_2)$.

**Proof of Claim 2.** By (4) of Claim 1, there exist two disjoint internally disjoint $a - b$ paths in $G[B]$. Let them be $Q_1$ and $Q_2$. Observe that if $u_1, u_2$ are both included in $Q_1$, then $u_1Q_1u_2$ would be a $u_1 - u_2$ path that is disjoint from the $a - b$ path $Q_2$, which implies property $\mathcal{P}$, a contradiction. By symmetry, it follows that $|V(Q_i \cap \{u_1, u_2\})| \leq 1$ for $i = 1, 2$, so we may assume that one of $u_1, u_2$ is not included in $V(Q_1 \cup Q_2)$.

Next, we show that we may assume that one of $Q_1, Q_2$ goes through $u_1$. To see it, assume that $u_1 \notin V(Q_1 \cup Q_2)$. By (3) of Claim 1, there exist three paths $S_1, S_2, S_3$ linking $u_1$ and $V(Q_1 \cup Q_2)$ in $G[B]$ that are pairwise disjoint except for $u_1$. Let $s_i$ be the end of $S_i$ in $V(Q_1 \cup Q_2)$ for $i = 1, 2, 3$. Notice that, without loss of generality, we can assume that $s_1, s_2$ are both on $Q_1$, and that $Q_1$ goes through $a, s_1, s_2, b$ in order. (It is possible that $s_i = a$ or $b$ for $i = 1, 2$, but it does not matter.) Replace $Q_1$ with the path $aQ_1s_1 \cup S_1 \cup S_2 \cup s_2Q_1b$, and it follows that $Q_1$ now goes through $u_1$.

Now, since $u_1 \in V(Q_1)$ and one of $u_1, u_2$ is not included in $V(Q_1 \cup Q_2)$, it follows that $u_2 \notin V(Q_1 \cup Q_2)$. Again by (4) of Claim 1, there exist three paths $R_1, R_2, R_3$ linking $u_2$
and $V(Q_1 \cup Q_2)$ in $G[B]$ that are pairwise disjoint except for $u_2$. Let $r_i$ be the end of $R_i$ in $V(Q_1 \cup Q_2)$ for $i = 1, 2, 3$. If $r_i \in V(Q_1) - \{a, b\}$ for some $i \in \{1, 2, 3\}$, then we would have the $u_1 - u_2$ path $R_i \cup r_i Q_1 u_1$ being disjoint from the $a - b$ path $Q_2$, implying property $\mathcal{P}$, a contradiction. It follows that $r_i \in V(Q_2)$ for $i = 1, 2, 3$. Without loss of generality, say $Q_2$ goes through $a, r_1, r_2, b$ in order (possible that $r_1 = a$ or $r_2 = b$). Then, the $a - b$ paths $L_1 = Q_1$ and $L_2 = aQ_2r_1 \cup R_1 \cup R_2 \cup r_2Q_2b$ are as desired.

Claim 3. There exists a non-trivial 3-separation $(D, E)$ of $G[B]$ such that

1. $a, b \in D$ and $N(v_i) \cap B \subseteq D$ for $i = 1, 2$,
2. $G[E]$ and $G[E - D]$ are both connected, and
3. there is no non-trivial $\leq 3$-separation of $(G[E], D \cap E)$.

Proof of Claim 3. We first prove that $G[B]$ is not a planar graph. By (2) of Claim 1, $d_{G[B]}(v) \geq 11 - 4 = 7$ for all $v \in B - A$, and it follows that

$$e(G[B]) \geq \frac{7}{2}|B - A| + \frac{1}{2}(d_{G[B]}(a) + d_{G[B]}(b)).$$

If $G[B]$ is planar, then

$$\frac{7}{2}|B - A| + \frac{1}{2}(d_{G[B]}(a) + d_{G[B]}(b)) \leq e(G[B]) \leq 3|B| - 6 = 3|B - A|,$$

which means that $\frac{1}{2}|B - A| + \frac{1}{2}(d_{G[B]}(a) + d_{G[B]}(b)) \leq 0$, a contradiction.

Let $H$ be the multigraph obtained from $G[B \cup \{v_1, v_2\}]$ by adding the four edges $an_1$, $n_1b$, $bn_2$, $n_2a$ and eliminating the edge $v_1v_2$ if $v_1v_2 \in E(G)$. Since $G[B]$ is a subgraph of $H$, $H$ is not planar either. Observe that any $v_1 - v_2$ path $P$ in $H$ such that $a, b \notin V(P)$ has a subgraph $P'$ that links a neighbor of $v_1$ and a neighbor of $v_2$. Since $(A, B)$ does not satisfy property $\mathcal{P}$, it follows that there do not exist two disjoint paths in $H$ such that one links $v_1$ and $v_2$ and the other one links $a$ and $b$. Let $C$ be the cycle in $H$ that goes through $a, v_1, b, v_2$ in order. By Theorem 2.3.1, there exists a non-trivial $C$-reduction of $H$ that
can be drawn on the plane such that $C$ bounds the infinite region. This means that there exists a non-trivial $\leq 3$-separation $(D', E')$ of $(H, V(C))$. Choose $(D', E')$ such that $|E'|$ is minimum over all such $\leq 3$-separations.

Let $D = D' \cap B$ and $E = E'$. It follows that $(D, E)$ is a $\leq 3$-separation of $G[B]$ such that $a, b \in D$ and $E - D \neq \emptyset$. (2) and (3) can then be simply implied by the minimality of $|E'|$ when choosing $(D', E')$. Observe that if $N(v_i) \cap B \subseteq D$ for $i = 1, 2$, then we know $D - E \neq \emptyset$ since $a, b, u_1, u_2$ are four distinct vertices, and therefore $(D, E)$ is a non-trivial separation of $G[B]$. Hence, in the remaining of the proof, it suffices to show that $|D \cap E| = 3$ and $N(v_i) \cap B \subseteq D$ for $i = 1, 2$.

To see $|D \cap E| = 3$, assume $|D \cap E| \leq 2$ for the sake of a contradiction. Since $a, b \in D$, it follows that at least one of the $a - b$ paths $L_1$ and $L_2$ is included in $D$ completely, a contradiction to (3) of Claim 1. Since $D \cap E = (D' \cap E') \setminus \{v_1, v_2\}$ and $|D' \cap E'| \leq 3$, it follows that $|D' \cap E'| = 3$ and $v_1, v_2 \in D' - E'$. Therefore, no neighbor of $v_1$ or $v_2$ is included in $E' - D'$. Since $E - D = E' - D'$, it follows that $N(v_i) \cap B \subseteq D$ for $i = 1, 2$.

The next goal is to prove that $V(L_i) \cap (E - D) \neq \emptyset$ for $i = 1, 2$ in Claim 5. To prove it, we need to introduce a few definitions first and make some observations first in Claim 4.

Let $Y = \{y \in N(x) : v_1y, v_2y \in E(G)\} \subseteq N(x)$. We will need to consider explicit positions of $v_1, v_2$ in $N'$, and observe that it suffices for us to consider the following five cases, up to isomorphism:

**Case 1:** $N(x) \cong K_{2,3,3,3}$ and $v_1, v_2$ are both contained in a 3-island.

**Case 2:** $N(x) \cong K_{2,3,3,3}$ and $v_1, v_2$ are both contained in the 2-island.

**Case 3:** $N(x) \cong K_{3,3} + C_5$ and $v_1, v_2$ are both contained in a 3-island.

**Case 4:** $N(x) \cong K_{3,3} + C_5$ and $v_1, v_2$ are both contained in the 5-island.

**Case 5:** $N(x) \cong K_{4,4,4}$ and $v_1, v_2$ are both contained in a 4-island.

**Claim 4.** The following statements are true.

(2) Let \( v_3, v_4 \in Y \) be non-adjacent and \( Y' = \{ y \in Y : v_3y, v_4y \in E(G) \} \subseteq Y \). Then, \( G[Y'] \) does not have a clique of size 3.

(3) In Case 2 and Case 5, \( N(x) - \{ v_1, v_2 \} \) does not have a clique of size 4.

(4) In Cases 1, 3, and 4, if \( Z \subseteq N(x) - \{ v_1, v_2 \} \) such that \( G[Z] \cong K_4 \), then there exist vertices \( v'_1, v'_2 \in Z \) such that \( v_iv'_i \not\in E(G) \) for \( i = 1, 2 \), where \( v'_1 \) and \( v'_2 \) are not necessarily distinct.

Proof of Claim 4. In Cases 1-5, \( G[Y] \) is isomorphic to \( K_{2,3,3}, K_{3,3,3}, \overline{K_3} + C_5, K_{1,3,3}, \) and \( K_{4,4} \) respectively. Since none of these five graphs have a clique of size 4, it follows that (1) is true. Observe that \( N(x) - \{ v_1, v_2 \} \cong K_{3,3,3} \) in Case 2, and that \( N(x) - \{ v_1, v_2 \} \cong K_{2,4,4} \) in Case 5. Since neither \( K_{3,3,3} \) nor \( K_{2,4,4} \) has a clique of size 4, it follows that (3) is true.

To see (2), observe that \( v_3, v_4 \) are in the same island of \( G[Y] \), as they are non-adjacent. In Case 3, \( G[Y] \cong \overline{K_3} + C_5 \) and therefore \( G[Y'] \cong C_5 \) or \( K_{1,3} \). Neither of these two graphs has a clique of size 3, meaning that (2) is true in Case 3. In other cases, observe that \( G[Y] \) has at most three islands and each island is an independent set. This means that no vertex in \( Z \) is in the same island with \( v_3 \) and \( v_4 \) in \( G[Y] \). By (1), \( G[Y'] \) does not contain a clique of size 3.

To see (4), let \( C_0 \) be the island of \( N(x) \) that contains \( v_1 \) and \( v_2 \). Observe that \( C_0 \) is an independent set of size 3 in Case 1 and Case 3. Also observe that \( N(x) - C_0 \cong K_{2,3,3} \) in Case 1 and \( N(x) - C_0 \cong \overline{K_3} + C_5 \) in Case 3, and both of these two graphs have their maximum clique of size 3. It follows that if \( Z \subseteq N(x) - \{ v_1, v_2 \} \) such that \( G[Z] \cong K_4 \), then \( Z \) must contain the unique vertex in \( C_0 - \{ v_1, v_2 \} \) which is adjacent to neither of \( v_1, v_2 \).

In Case 4, if \( Z \subseteq N(x) - \{ v_1, v_2 \} \) such that \( G[Z] \cong K_4 \), then \( Z \) must contain exactly one vertex in each one of the two 3-islands and two vertices in the 5-island \( C_0 \). Since the two vertices in \( Z \cap C_0 \) are adjacent to each other, it follows that one of them is not adjacent to \( v_1 \) and the other one is not adjacent to \( v_2 \).

Claim 5. \( V(L_i) \cap (E - D) \neq \emptyset \) for \( i = 1, 2 \).
Proof of Claim 5. For the sake of a contradiction, assume that \( V(L_1) \subseteq D \). By Claim 3, we know that \( v_1, v_2 \not\in N(E - D) \cap N(x) \). Since \( G[E - D] \) is connected by Claim 3, \( N(E - D) \cap N(\mathcal{L}(x)) \) must be a clique, because otherwise there would be a path in \( G[E - D] \) linking neighbors of two non-adjacent vertices in \( N(x) \) which is disjoint from the \( a - b \) path \( L_1 \), meaning that property \( \mathcal{P} \) holds, a contradiction. By (1) of Claim 1 and the 7-connectivity of \( G \), it follows that \( N(E - D) \cap N(x) \cong K_4 \). By (3) of Claim 4, Case 2 and Case 5 are not possible, so it remains to consider Cases 1, 3, and 4. By (4) of Claim 4, we can choose \( v'_i \in N(E - D) \cap N(x) \) for \( i = 1, 2 \) such that \( v_i v'_i \not\in E(G) \), where \( v'_1, v'_2 \) are not necessarily distinct.

Assume \( V(L_2) \cap (E - D) \neq \emptyset \) for a moment. Since \( G[E - D] \) is connected, there exists a path \( R \subseteq G[E - D] \) linking some \( u'_2 \in N(v'_2) \cap (E - D) \) and a vertex on \( L_2 \) such that \( R \) is otherwise disjoint from \( L_2 \). We can then extend \( R \) to obtain a \( v_2 - v'_2 \) path which is disjoint from the \( a - b \) path \( L_1 \). This implies property \( \mathcal{P} \) is true, a contradiction.

We may then assume that \( V(L_1 \cup L_2) \subseteq D \). Let \( r \in E - D \). By (3) of Claim 1, there is no 2-cut of \( G[B] \) separating \( r \) from \( V(L_1 \cup L_2) \). It follows that there exist three paths linking \( r \) and \( V(L_1 \cup L_2) \) that are pairwise disjoint except for \( r \) and are disjoint from \( V(L_1 \cup L_2) \) otherwise. Note that at least one of these three paths has an end in \( V(L_1 \cup L_2) - \{a, b\} \). Without loss of generality, assume that one of these three paths has an end in \( V(L_1) - \{a, b\} \). It follows that there is a path \( R_1 \) linking \( r \) and \( u_1 \) that is disjoint from \( L_2 \). Since \( G[E - D] \) is connected and contains some vertex \( u'_1 \in N(v'_1) \), it follows that there exists a path \( R_2 \) linking \( u_1 \) and \( u'_1 \) that is disjoint from the \( a - b \) path, meaning that property \( \mathcal{P} \) holds true, a contradiction.

By Claim 5, since the end vertices \( a, b \) of \( L_i \) are both contained in \( D \), we know \( |V(L_i) \cap (D \cap E)| \geq 2 \). Since \( |D \cap E| = 3 \), without loss of generality, we can assume that \( a \in D \cap E \), \( b \in D - E \), and \( |V(L_i) \cap (D \cap E) - \{a\}| = 1 \) for \( i = 1, 2 \). Let \( D \cap E = \{w_1, w_2, w_3\} \) where \( a = w_3 \) and \( w_i \) is the unique vertex in \( V(L_i) \cap (D \cap E) - \{a\} \) for \( i = 1, 2 \).

Claim 6. There exists a \((D \cap E)\)-tripod in \( G[E] \) such that every leg of the tripod is trivial.
Proof of Claim 6. We first prove that $G[E]$ is a non-planar graph. By (2) of Claim 1, every vertex in $E - D$ has at least 7 neighbors in $G[E]$. It follows that

$$e(G[E]) = \frac{1}{2} \sum_{v \in E} d_G[v] \geq \frac{1}{2} \sum_{v \in D \cap E} d_G[v] + \frac{7}{2} |E - D| = \frac{1}{2} \sum_{v \in D \cap E} d_G[v] + \frac{7}{2} |E| - \frac{21}{2}.$$ If $G[E]$ is planar, then

$$\frac{1}{2} \sum_{v \in D \cap E} d_G[v] + \frac{7}{2} |E| - \frac{21}{2} \leq e(G[E]) \leq 3|E| - 6,$$

meaning that $\sum_{v \in D \cap E} d_G[v] + |E| \leq 9$. Since $E - D \neq \emptyset$ and every vertex in $E - D$ has at least 7 neighbors in $G[E]$, we know that $|E| \geq 8$. It follows that $\sum_{v \in D \cap E} d_G[v] \leq 1$, meaning that some vertex in $D \cap E$ has no neighbor in $E - D$, a contradiction to Claim 5.

Now, recall that there is no non-trivial $\leq 3$-separation of $(G[E], D \cap E)$ by (3) of Claim 3. By Lemma 2.4.3 and (4) of Lemma 2.4.4, there exists some $(D \cap E)$-tripod $T \subseteq G[E]$ that is split by some 3-separation $(E_1, E_2)$ of $G[E]$. This means that $D \cap E \subseteq \mathcal{L}(T) \subseteq E_1$ and $E_2 - E_1 \neq \emptyset$. Since there is no non-trivial $\leq 3$-separation of $(G[E], D \cap E)$, it follows that $D \cap E = \mathcal{L}(T) = E_1$ and therefore every leg of $T$ is trivial. Hence, the $(D \cap E)$-tripod $T$ is as desired.

Claim 7. $N(E - D) \cap N(x) \subseteq Y$ where $Y = \{y \in N(x) : v_1 y, v_2 y \in E(G)\}$ as defined before Claim 4.

Proof of Claim 7. For the sake of a contradiction, assume that $v'_1 \in N(E - D) \cap N(x)$ is not adjacent to $v_1$.

By Claim 6, let $T \subseteq G[E]$ be a $(D \cap E)$-tripod such that every leg of $T$ is trivial. Let $p, q \in V(T) - D \cap E$ and paths $P_i, Q_i$ for $i = 1, 2, 3$ be such that $T$ is the union of the internally disjoint paths $P_1, P_2, P_3, Q_1, Q_2, Q_3$ such that $P_i$ links $p, w_i$ and $Q_i$ links $q, w_i$ for $i = 1, 2, 3$.

Note that $v'_1 \in N(E - D) \cap N(x)$ has some neighbor $r_1 \in E - D$. By (2) of Claim 1,
$r_1$ has at least 7 neighbors in the connected subgraph $G[E]$. Therefore, without loss of generality, we can assume that there is a path $R$ linking $r_1$ and some $r'_1 \in V(P_1 \cup P_2 \cup P_3) - D \cap E$ that is disjoint from $T$ otherwise. Recall that $u_1$ is a neighbor of $v_1$ in $B - A$ included in the path $L_1$. By Claim 3, $u_1 \in V(L_1) \cap D - \{b\}$. It follows that $R \cup P_1 \cup P_2 \cup P_3 - \{a, w_2\}$ has a subpath $R'$ linking $r_1$ and $w_1$ that is disjoint from $Q_2 \cup Q_3$. Then, $R' \cup u_1L_1w_1$ would be a path linking $r_1, u_1$ which is disjoint from the $a - b$ path $Q_3 \cup Q_2 \cup w_2L_2b$, meaning that property $P$ holds, a contradiction.

Claim 8. Let $T \subseteq G[E]$ be a $(D \cap E)$-tripod such that every leg of $T$ is trivial. Let $p, q \in V(T) - D \cap E$ and paths $P_i, Q_i$ for $i = 1, 2, 3$ be such that $T$ is the union of the internally disjoint paths $P_1, P_2, P_3, Q_1, Q_2, Q_3$ such that $P_i$ links $p, w_i$ and $Q_i$ links $q, w_i$ for $i = 1, 2, 3$. If there exist non-adjacent vertices $y_1, y_2 \in N(E - D) \cap N(x)$, then every neighbor of $y_1$ or $y_2$ in $E - D$ is not associated with $V(P_1 \cup P_2 \cup Q_1 \cup Q_2) - \{p, q\}$ with respect to $T$.

Proof of Claim 8. Let $t_i \in E - D$ be a neighbor of $y_i$ for $i = 1, 2$. For the sake of a contradiction, assume that $t_1$ is associated with $V(P_1 \cup P_2 \cup Q_1 \cup Q_2) - \{p, q\}$. Without loss of generality, say $t_1$ is associated with $P_1 - \{p\}$.

For convenience, let $P' = P_1 \cup P_2 \cup P_3 - D \cap E$ and $Q' = Q_1 \cup Q_2 \cup Q_3 - D \cap E$. By (2) of Claim 1, $t_1$ has at least 7 neighbors in the connected subgraph $G[E]$, so $t_2$ is associated with $V(P')$ or $V(Q')$ with respect to $T$. If $t_2$ is associated with $V(P')$ with respect to $T$, then there exists a path linking $t_1$ and $t_2$ that is disjoint from $Q_1 \cup Q_2 \cup Q_3$. If $t_2$ is associated with $V(Q')$ with respect to $T$, then there is a path linking $t_1, t_2$ that goes through $w_1$ (possibly $q$ as well) and is disjoint from $P_2 \cup P_3$. In both cases, we can find a $t_1 - t_2$ path that is disjoint from some $a - b$ path, which implies property $P$, a contradiction. 

Now, let $T \subseteq G[E]$ be a fixed $(D \cap E)$-tripod. Let vertices $p, q$ and paths $P_i, Q_i$ be labeled as in Claim 8. Since $G$ is 7-connected and $G[E - D]$ is connected, we know that $|N(E - D) \cap N(x)| \geq 4$. By Claim 7 and (1) of Claim 4, $N(E - D) \cap N(x)$ is not a
clique. Hence, we choose non-adjacent vertices \( v_3, v_4 \in N(E - D) \cap N(x) \) and vertices \( u_3, u_4 \in E - D \) such that \( u_i v_i \in E(G) \) for \( i = 3, 4 \).

**Claim 9.** There exists a \( \leq 2 \)-separation \((E_1, E_2)\) of \( G[E]\) such that \( V(P_1 \cup P_2 \cup Q_1 \cup Q_2) \subseteq E_1 \) and \( a \in E_2 - E_1 \).

**Proof of Claim 9.** For the sake of a contradiction, assume that there exist three paths \( R_1, R_2, R_3 \) in \( G[E] \) linking \( a \) and \( V(P_1 \cup P_2 \cup Q_1 \cup Q_2) \) that are pairwise disjoint except for \( a \). Let \( r_i \) be the end of \( R_i \) on \( P_1 \cup P_2 \cup Q_1 \cup Q_2 \) for \( i = 1, 2, 3 \). Due to Corollary 2.3.4 and the existence of the paths \( P_3, Q_3 \), we may choose \( R_1, R_2, R_3 \) such that \( p = r_2 \) and \( q = r_3 \). Then, without loss of generality, assume that \( r_1 \in V(P_1) - \{p\} \).

Observe that \( T' = P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup R_2 \cup R_3 \subseteq G[E] \) is also a \((D \cap E)\)-tripod such that every leg of \( T' \) is trivial. By Claim 8 and the construction of \( u_3, u_4 \), it follows that both \( u_3, u_4 \) are associated with \( V(R_2 \cup R_3) - \{a\} \) with respect to \( T' \). It follows that, for \( i = 3, 4 \), there exists some path \( S_i \subseteq G[E] \) linking \( u_i \) and some vertex \( V(R_2 \cup R_3) - \{a\} \) that is otherwise disjoint from \( T' \). Observe that \( V(S_i) \cap V(R_1) = \emptyset \) for \( i = 3, 4 \), since otherwise \( u_i \) would be associated with \( V(P_1) - \{p\} \) with respect to \( T' \), a contradiction to Claim 8. Therefore, we can find a subpath of \( S_3 \cup S_4 \cup P_2 \cup R_2 \cup Q_2 \cup R_3 - \{a\} \) linking \( u_3 \) and \( u_4 \) that is disjoint from the \( a - b \) path \( R_1 \cup r_1 P_1 w_1 \cup w_1 L_1 b \). This implies property \( \mathcal{P} \), a contradiction.

By Claim 9, we choose \((E_1, E_2)\) to be a \( \leq 2 \)-separation of \( G[E] \) such that \( V(P_1 \cup P_2 \cup Q_1 \cup Q_2) \subseteq E_1 \) and \( a \in E_2 - E_1 \) such that \( |E_2| \) is maximum over all choices. Observe that \(|E_1 \cap E_2| = 2\) due to the paths \( P_3 \) and \( Q_3 \). Furthermore, we can write \( E_1 \cap E_2 = \{p_0, q_0\} \) such that \( p_0 \in V(P_3) - \{a\}, q_0 \in V(Q_3) - \{a\}, V(p P_3 p_0 \cup q Q_3 q_0) \subseteq E_1, \) and \( V(p_0 P_3 a \cup q_0 Q_3 a) \subseteq E_2 \).

**Claim 10.** Every vertex in \( N(E_1 - \{p_0, q_0, w_1, w_2\}) \cap N(x) \) is adjacent to all other vertices in \( N(E - D) \cap N(x) \).

**Proof of Claim 10.** Assume that there exist two vertices \( y_1, y_2 \in N(E - D) \cap N(x) \) such
that \( y_1 y_2 \not\in E(G) \). It suffices to prove that neither \( y_1 \) nor \( y_2 \) has a neighbor in \( E_1 - \{p_0, q_0, w_1, w_2\} \). Observe that if \( p = p_0 \) and \( q = q_0 \), then \( V(P_3 \cup Q_3) \subseteq E_2 \), and it follows that \( y_1, y_2 \) have no neighbor in \( E_1 - E_2 \) by Claim 8. Therefore, without loss of generality, we may assume that \( p \neq p_0 \).

Let \( W = V(P_1 \cup P_2 \cup Q_1 \cup Q_2) \), just for convenience. By the maximality of \( |E_2| \) when choosing \((E_1, E_2)\), there is no non-trivial \( \leq 2 \)-separation \((F_1, F_2)\) of \( G[E_1] \) such that \( W \subseteq F_1 \) and \( \{p_0, q_0\} \subseteq F_2 \). By Corollary 2.3.5, it follows that there exist internally disjoint paths \( S_1, S_2, S_3 \) in \( G[E_1] \) satisfying the following properties: (1) \( S_i \) for \( i = 1, 2 \) each link \( p_0 \) and some vertex in \( W \), (2) \( S_3 \) links \( q_0 \) and some vertex in \( W \), and (3) the end vertices of \( S_1, S_2, S_3 \) in \( W \) are distinct and include both \( p, q \). Observe that in all cases, there exist distinct vertices \( p', q' \in W - \{w_1, w_2, p_0\} \) and seven internally disjoint paths \( P'_1, P'_2, Q'_1, Q'_2, P''_3, Q''_3, R \) in \( G[E_1] \) such that (i) \( P'_i \) for \( i = 1, 2 \) links \( p', w_i \), (ii) \( Q'_i \) for \( i = 1, 2 \) links \( q', w_i \), (iii) \( P''_3 \) links \( p', p_0 \), (iv) \( Q''_3 \) links \( q', q_0 \), and (v) \( R \) links \( p_0 \) and some \( r \in V(P'_1 \cup P'_2 \cup Q'_1 \cup Q'_2) - \{p', q'\} \). Let \( P'_3 = P''_3 \cup p_0 P_3 p \) and \( Q'_3 = Q''_3 \cup q_0 Q_3 q \). It follows that \( T' = \bigcup_{i=1,2,3}(P'_i \cup Q'_i) \) is a \((D \cap E)\)-tripod in \( G[E] \) such that every leg of \( T' \) is trivial, and that \( R \) is a path linking \( p_0, r \) that is disjoint from \( T' \) otherwise.

Let \( t_1, t_2 \in E - D \) be neighbors of \( y_1, y_2 \), respectively. Note that it suffices to prove \( t_1, t_2 \not\in E_1 - E_2 \). For the sake of a contradiction, assume that \( t_1 \in E_1 - E_2 \). Note that for both \( i = 1, 2 \), \( t_i \in E - D \), meaning that \( t_i \) has at least \( 11 - 4 - 3 = 4 \) neighbors in the connected subgraph \( G[E - D] \) by Claim 1, and therefore there exists some non-empty \( S_i \subseteq V(T') - E \cap D \) such that \( t_i \) is associated with \( S_i \) with respect to \( T' \). By Claim 8, \( t_i \) for \( i = 1, 2 \) is not associated with \( V(P'_i \cup P'_2 \cup Q'_1 \cup Q'_2) - \{p', q'\} \) with respect to \( T' \). Since \( t_1 \in E_1 - E_2 \), without loss of generality, assume that \( t_1 \) is associated with \( V(p' P_3 p_0) - \{p_0\} \) with respect to \( T' \). Note that \( t_2 \) is not associated with \( V(P'_3) - \{a\} \) with respect to \( T' \), since otherwise there would exist a path linking \( t_1, t_2 \) that is disjoint from an \( a - b \) path obtained by extending \( Q'_2 \cup Q'_3 \), which implies property \( P \), a contradiction. It follows that \( t_2 \) is associated with \( V(Q'_3) - \{a\} \) with respect to \( T' \).
Recall that $R$ is a path linking $p_0$ and $r \in V(P'_1 \cup P'_2 \cup Q'_1 \cup Q'_2) - \{p', q'\}$ that is otherwise disjoint from $T'$. Without loss of generality, assume that either $r \in V(P_1) - \{p'\}$ or $r \in V(Q'_1) - \{q'\}$. Note that $R$ is disjoint from $T_1$ and $T_2$, since otherwise $t_1$ or $t_2$ would be associated with $V(P'_1 \cup P'_2 \cup Q'_1 \cup Q'_2) - \{p', q'\}$ with respect to $T'$, a contradiction to Claim 8.

Let $b_1, b_2$ be the end vertices of paths $T_1, T_2$, respectively, on $p'P'_3p_0-\{p_0\}$ and $Q'_3-\{a\}$, respectively. Let $O_1 = T_1 \cup b_1P'_3p_0 \cup P'_2 \cup Q'_2 \cup q'Q'_3b_2 \cup T_2$. Then, observe that in both the case $r \in V(P_1) - \{p'\}$ and the case $r \in V(Q'_1) - \{q'\}$, $O_1$ is a $t_1 - t_2$ path disjoint from the subgraph $O'_2 = (P'_1 \cup Q'_1 - \{p', q'\}) \cup R \cup aP'_3p_0$. If $r \in V(P'_1) - \{p'\}$, let $O_2 = aP'_3p_0 \cup R \cup rP'_1w_1 \cup w_1L_1b$; and if $r \in V(Q'_1) - \{q'\}$, let $O_2 = aP'_3p_0 \cup R \cup rQ'_1w_1 \cup w_1L_1b$. In both cases, we see that $O_2$ is a subpath of $O'_2$ linking $a$ and $b$, and therefore $O_1, O_2$ are disjoint paths. This implies property $\mathcal{P}$ again, a contradiction.

To finish the proof, we first show that $E_1 = \{w_1, w_2, p, q\}$. Note that it suffices to prove that $E_1 = \{w_1, w_2, p_0, q_0\}$. For the sake of a contradiction, assume that $E_1 - \{w_1, w_2, p_0, q_0\} \neq \emptyset$. Let $K_1$ be a component of $G[E_1] - \{w_1, w_2, p_0, q_0\}$. Since $G$ is 7-connected, $|N(K_1) \cap N(x)| \geq 3$. Recall that $Y = \{y \in N(x) : v_1y, v_2y \in E(G)\}$. Since $V(K_1) \subseteq E - D$, we have $N(K_1) \cap N(x) \subseteq N(E - D) \cap N(x) \subseteq Y$ by Claim 7. By Claim 10, $N(K_1) \cap N(x)$ is a clique such that every vertex in it is adjacent all other vertices in $N(E - D) \cap N(x)$. Recall that $v_3, v_4 \in N(E - D) \cap N(x)$ are non-adjacent by construction. It follows that $N(K_1) \cap N(x)$ is a clique contained in the subset $Y' = \{y \in Y : v_3y, v_4y \in E(G)\}$. By (2) of Claim 4, it follows that $|N(K_1) \cap N(x)| \leq 2$, a contradiction.

Now, $p \in E - D$ has at least 3 neighbors in $E_1$, and that $p$ has at least 4 neighbors in $N(x)$. Since $\delta(G) \geq 11$, $p$ has at least 4 neighbors in $E_2 - E_1$, meaning that $E_2 - E_1 \cup \{a\} \neq \emptyset$. Let $F_1 = E_1 \cup \{a\}$ and $F_2 = E_2$. It follows that $(F_1, F_2)$ is a non-trivial 3-separation of $(G[E], D \cap E)$, a contradiction to (3) of Claim 3.
4.3.2 Proof of Lemma 4.3.5

Lemma 4.3.5. If $G - N[x]$ is 2-connected and no vertex in $G - N[x]$ is a common neighbor for two non-adjacent vertices in $N(x)$, then $G - x$ has a minor $J$ rooted at $N(x)$ such that $J > K_9$.

Proof. For the sake of a contradiction, assume that such a minor $J$ does not exist. Recall the definitions of $T_1 = \{x_1, x_2, x_3\}$ and $T_2 = \{y_1, y_2, y_3\}$ right before Section 4.3.1. By Lemma 4.3.3, for all $i, j \in \{1, 2, 3\}$, $x_i, y_j$ have at least two common neighbors in $G - N[x]$. Since no vertex in $G - N[x]$ is a common neighbor for two non-adjacent vertices in $N(x)$, there exist a subset of three vertices $X = \{v_1, v_2, v_3\} \subseteq V(G) - N[x]$ and a subset of nine vertices $A \subseteq V(G) - N[x] \cup X$ such that $v_i$ for $i = 1, 2, 3$ is a common neighbor for $x_i, y_i$, and that every vertex in $A$ is a common neighbor for $x_i, y_j$ for some unique ordered pair $(i, j)$ where $i, j \in \{1, 2, 3\}$.

Our proof for Lemma 4.3.5 is a bit lengthy. An outline of the proof is as follows. In Claim 1 and Claim 2, we will make a series of observations on the structure of $G - N[x]$, given that it is 2-connected and no vertex in it is adjacent to two non-adjacent vertices in $N(x)$. In Claim 3, we find an $X$-tripod $T$ satisfying a few desired extremal properties. Then in Claim 4 and Claim 5, we prove that $T$ has at least one non-trivial leg. Without loss of generality, assume that the leg $L_1$ of $T$ on $v_1$ is non-trivial. In Claim 6-Claim 10, we prove that there is a non-trivial $T$-bridge $B_1$ that attaches to $v_1$ and exactly one of $v_2, v_3$ as its only attachment on $T$ outside $L_1$. Without loss of generality, say $B_1$ attaches to $v_2$. Let $A_3 \subseteq A$ be the subset of vertices in $A$ that are adjacent to $x_3$ or $y_3$. We will then use the $T$-bridge $B_1$ to show that every vertex in $A_3$ is not associated with $T - \{v_2\} \cup V(L_3)$ in Claim 11-Claim 14. Finally, we use these vertices in $A_3$ and the $T$-bridge $B_1$ to show a minor of type II exists, a contradiction.

We start the proof with observing a few properties of $G - N[x]$.

Claim 1. The following statements are true:
(1) For every two vertices $a_1, a_2 \in A$, there exist distinct vertices $v_1, v_2 \in T_j$ for some $j \in \{1, 2\}$ such that $a_iv_i \in E(G)$ for $i = 1, 2$.

(2) There do not exist two disjoint connected subgraphs $G_1, G_2$ of $G - N[x]$ and a subset $X' \subseteq V(G_1)$ of three vertices such that for some $j \in \{1, 2\}$, each vertex in $X'$ has a unique neighbor in $T_j$, $G_1$ has a $K_3$-minor rooted at $X'$, and $G_2$ has two non-adjacent neighbors in $N(x) - T_j$.

(3) There do not exist two disjoint connected subgraphs $G_1, G_2$ of $G - N[x]$ such that $X \subseteq V(G_1)$, $G_1$ has a $K_3$-minor rooted at $X$, and $G_2$ has two non-adjacent neighbors in $N(x)$.

(4) If $G_1 \subseteq G - N[x]$ such that $X \subseteq V(G_1)$ and $G_1$ has a $K_3$-minor rooted at $X$, then $|V(K) \cap A| \leq 1$ for every component $K$ of $G - N[x] \cup V(G_1)$.

Proof of Claim 1. One can observe that (1) is simply true by the definition of $A$, and that (3) and (4) immediately follow (1) and (2). So it suffices to prove (2). For the sake of a contradiction, assume that such subgraphs $G_1, G_2$ and $X' \subseteq V(G_1)$ exist. Without loss of generality, say each vertex in $X'$ has a unique neighbor in $T_1$, and $w_1, w_2 \in N(x) - T_1$ are non-adjacent and both neighbors of $G_2$. It follows that $G[V(G_1) \cup T_1]$ has a $K_3$ minor rooted at $T_1$. By contracting edges in $G[V(G_1) \cup T_1]$ properly and contracting all vertices in $G_2$ to one of $w_1, w_2$, we could then obtain a minor of $G$ of type II, a contradiction. Therefore, (2) is true. \[\dashrightarrow\]

In the next claim, we make observations on some properties of $X$-tripods in $G - N[x]$.

Claim 2. If $G - N[x]$ has an $X$-tripod $T$ as a subgraph, then the following statements are true.

(1) $T$ has a $K_3$-minor rooted at $X$.

(2) For every component $K$ of $G - (N[x] \cup V(T))$, $N(K) \cap N(x)$ is a clique.

(3) Every non-trivial $T$-bridge in $G - N[x]$ has at least three attachments on $T$.

(4) Let vertices $p, q$ and $z_i$ for $i = 1, 2, 3$ and paths $L_i, P_i, Q_i$ for $i = 1, 2, 3$ be labeled for $T$ as in Definition 2.4.2. Then for each $i \in \{1, 2, 3\}$, there exists at most one vertex in
\( \mathcal{A} \) associated with \( V(P_i) - \{z_i, p\} \) with respect to \( T \) and at most one vertex in \( \mathcal{A} \) associated with \( V(Q_i) - \{z_i, q\} \) with respect to \( T \).

**Proof of Claim 2.** One can easily check that (1) is true, and that (2) can be implied by (1) and Claim 1.

To prove (3), recall that \( N(x) \) has no clique of size greater than 4 by Lemma 4.3.3. By (2), it follows that every non-trivial \( T \)-bridge \( B \) has at most four neighbors in \( N(x) \). Since \( G \) is 7-connected, it follows that \( B \) has at least three attachments on \( T \).

To see (4) is true, let \( G_1 = T - (V(P_i) \setminus \{z_i, p\}) \). Observe that \( G_1 \) has a \( K_3 \) minor rooted at \( X \). Let \( K \) be the component of \( G - N[x] \cup V(G_1) \) such that \( P_i - \{z_i, p\} \subseteq K \). It follows that \( |V(K) \cap A| \leq 1 \), meaning that at most one vertex in \( A \) is associated with \( V(P_i) - \{z_i, p\} \) with respect to \( T \). By symmetry, it follows that at most one vertex in \( A \) is associated with \( V(P_i) - \{z_i, p\} \) with respect to \( T \). ⊢

**Claim 3.** There exists an \( X \)-tripod \( T \) in \( G - N[x] \) split by some 3-separation of \( G - N[x] \) such that every \( T \)-bridge in \( G - N[x] \) is stable, and that there is no \( X \)-tripod in \( G - N[x] \) that can be obtained from \( T \) by a tripod-transformation.

**Proof of Claim 3.** Let \((A_1, A_2)\) be a 2-separation of \((G - N[x], X)\) such that \(|A_1|\) is minimum. Note that we know there exists some 2-separation, as we know \(|G - N[x]| \geq 3 \) by Lemma 4.3.2 and we do not require the 2-separation to be non-trivial. By the choice of \((A_1, A_2)\), there is no non-trivial \( \leq 2 \)-separation of \((G[A_1], X)\).

We next prove that \( G[A_1] \) is non-planar. By Lemma 4.3.3, there is no clique of size 5 in \( N(x) \). Since no vertex in \( G - N[x] \) is a common neighbor for two non-adjacent vertices in \( N(x) \), it follows that every vertex in \( V(G) - N[x] \) has at most 4 neighbors in \( N(x) \). Since \( \delta(G) \geq 11 \) by Lemma 3.1.1, we have \( d_{G - N[x]}(v) \geq 7 \) for every \( v \in V(G) - N[x] \), meaning that all but at most 2 vertices in \( G[A_1] \) has degree at least 7 inside \( G[A_1] \). Therefore, \( e(G[A_1]) \geq \frac{7}{2}(|A_1| - 2) = \frac{7}{2}|A_1| - 7 \). If \( G[A_1] \) is planar, then

\[
\frac{7}{2}|A_1| - 7 \leq e(G[A_1]) \leq 3|A_1| - 6,
\]

104
meaning that $|A_1| \leq 2$. This is a contradiction, since $X \subseteq A_1$ and therefore has cardinality at least 3.

Now, since $G[A_1]$ is non-planar and $(G[A_1], X)$ has no non-trivial $\leq 2$-separation, by Lemma 2.4.3 and Lemma 2.4.5, there exists an $X$-tripod $T$ in $G[A_1]$ such that some 3-separation of $G[A_1]$ splits $T$, every $T$-bridge in $G[A_1]$ is stable, and that there is no $X$-tripod in $G[A_1]$ that can be obtained from $T$ by a tripod-transformation. It now suffices to prove that $A_2 - A_1 = \emptyset$.

For the sake of a contradiction, assume that $A_2 - A_1 \neq \emptyset$. Let $a, b$ be the two vertices in $A_1 \cap A_2$. Then, since $G - N[x]$ is 2-connected, there exists some $a$-$b$ path in $G[A_2]$. By Lemma 4.3.4, there exists two disjoint paths $P, Q$ in $G[A_2]$ such that $P$ links $a, b$ and $Q$ links a neighbor of $w_1$ and a neighbor of $w_2$ for some non-adjacent vertices $w_1, w_2 \in N(x)$. Then, the path $Q$ excludes $a$ and $b$ and is therefore disjoint from the $X$-tripod $T$ in $G - N[x]$, a contradiction to (2) of Claim 2.

Now, fix $T \subseteq G - N[x]$ to be an $X$-tripod and fix $(A, B)$ to be a 3-separation of $G - N[x]$ such that $(A, B)$ splits $T$, every $T$-bridge in $G - N[x]$ is stable, and that there is no $X$-tripod in $G - N[x]$ that can be obtained from $T$ by a tripod-transformation. Let vertices $z_1, z_2, z_3, p, q \in V(T)$ and paths $L_1, L_2, L_3, P_1, P_2, P_3, Q_1, Q_2, Q_3$ are labeled as in Definition 2.4.2 for $T$.

**Claim 4.** If every leg of $T$ is trivial, then there is a trisection $(A_1, A_2, A_3)$ of order 2 of $G - N[x]$ such that $\{p, q\} = A_1 \cap A_2 \cap A_3$ and $V(P_i \cup Q_i) \subseteq A_i$ for $i = 1, 2, 3$.

**Proof of Claim 4.** For the sake of a contradiction, assume that there exists a path $R$ linking some $r_1 \in V(P_1 \cup Q_1) - \{p, q\}$ and $r_2 \in V(P_2 \cup Q_2) - \{p, q\}$ that is otherwise disjoint from $T$. Note that up to symmetry, we can assume that $r_1 \in V(P_1) - \{p\}$ and either $r_2 \in V(P_2) - \{p\}$ or $V(Q_2) - \{q\}$. One can observe that both $T \cup R - \{p\}$ and $T \cup R - \{q\}$ have a $K_3$ minor rooted at $X$ in all cases. It follows that at most one vertex in $A$ is associated with $s$ for $s \in \{p, q\}$ with respect to $T$.

Note that $|A| = 9$, and it follows that at least 7 vertices in $A$ are associated with
V(T) − \{p, q\}, with respect to T. By (4) of Claim 2, there exists some vertex \(a_0 \in A\) that is not associated with \(V(T) − X\) with respect to T. Since \(X \cap A = \emptyset\), it follows that \(a_0\) is contained in some non-trivial \(T\)-bridge \(B_0 \subseteq G − N[x]\) whose set of attachments is a subset of \(X\). By Claim 2, the set of attachments of \(B_0\) on \(T\) is precisely \(X\). Note that \(P_1 \cup P_2 \cup P_3 \cup B_1\) now has a \(K_3\)-minor rooted at \(X\), and therefore there is at most one vertex in \(A\) associated with \(V(Q_1 \cup Q_2 \cup Q_3) − X\) with respect to \(T\). By symmetry, there is at most one vertex in \(A\) associated with \(V(P_1 \cup P_2 \cup P_3) − X\) with respect to \(T\), too.

By Claim 2, it follows that there exist 7 vertices \(a_1, \ldots, a_7 \in A\) such that for \(i = 1, \ldots, 7\), each \(a_i\) is contained in a unique non-trivial \(T\)-bridge \(B_i \subseteq G − N[x]\) whose attachments on \(T\) are precisely \(v_1, v_2, v_3\). Without loss of generality, assume that \(a_i\) is a common neighbor for \(x_3\) and \(y_i\) for \(i = 1, 2, 3\). It follows that \(P_1 \cup P_2 \cup B_1 − \{v_3\}\) has a \(K_3\) minor rooted on \(v_1, v_2, a_1\) and \(B_2 \cup B_3 − \{v_1, v_2\}\) contains a path linking \(a_2, a_3\). By contracting edges that have at least one end in \(P_1 \cup P_2 \cup B_1 − \{v_3\}\) properly and contracting all of \(B_2 \cup B_3 − \{v_1, v_2\}\) to \(y_2\), we can then obtain a clique on \(T_1\) and the edge \(y_2y_3\) in \(T_2\), meaning \(G\) has a minor of type II, a contradiction.

\[\text{Claim 5. Some leg of } T \text{ is non-trivial.}\]

**Proof of Claim 5.** For the sake of a contradiction, assume every leg of \(T\) is trivial for some \(X\)-tripod \(T \subseteq G − N[x]\). Let vertices \(p, q \in V(T)\) and paths \(P_1, P_2, P_3, Q_1, Q_2, Q_3\) be labeled as in Definition 2.4.2 for \(T\). By Claim 4, there is a trisection \((A_1, A_2, A_3)\) of order 2 of \(G − N[x]\) such that \(\{p, q\} = A_1 \cap A_2 \cap A_3\) and \(V(P_i \cup Q_i) \subseteq A_i\) for \(i = 1, 2, 3\). Since \(|A| = 9, |A \cap (A_i − \{p, q\})| \geq 3\) for some \(i \in \{1, 2, 3\}\). Without loss of generality, say there are distinct vertices \(a_1, a_2, a_3 \in A\) that are all contained \(A_1 − \{p, q\}\).

Assume for a moment that there exist pairwise disjoint subsets \(S_1, S_2, S_3 \subseteq A_1 − \{v_1, p, q\}\) such that for \(i = 1, 2, 3\), \(a_i \in S_i\), \(G[S_i]\) is connected, and \(\{v_1, p, q\} \subseteq N(S_i)\). If \(a_1\) is adjacent to \(x_1\), then note that the subgraph \(G_1 = G[S_1 \cup V(P_2 \cup Q_2 \cup Q_3)]\) has a \(K_3\) minor rooted at \(\{a_1, v_2, v_3\}\). By contracting edges inside this subgraph properly and contracting edges between \(T_1\) and \(\{a_1, v_2, v_3\}\), we can then obtain a clique on \(T_1\). Note that
the subgraph $G[S_2 \cup S_3 \cup \{v_1\}]$ is disjoint from $G_2$. It follows that $N(a_i) \cap T_2 = \{y_1\}$ for both $i = 2, 3$, since otherwise we would be able to obtain a minor of $G$ of type II, a contradiction.

Without loss of generality, we then assume that $a_2$ is adjacent to $x_2$ and $y_1$. Now since $a_2$ is adjacent to $y_1$, by the same argument as above, $a_1, a_3$ are both adjacent to $x_1$. It follows that $a_3$ is adjacent to both $x_1$ and $y_1$. Using the same argument again, we then have $a_i$ is adjacent to both $x_1, y_1$ for all $i = 1, 2, 3$, a contradiction to the construction of $A$. Hence, $a_1$ is not adjacent to $x_1$, and it follows that $a_i$ is not adjacent to $x_1$ or $y_1$ for all $i = 1, 2, 3$. Without loss of generality, we can then assume that $a_1x_2, a_1y_2, a_2x_2, a_2y_2, a_3 \in E(G)$. It follows that the subgraph of $G$ induced on $S_1 \cup S_2 \cup \{v_1, p\}$ has a $K_3$ minor rooted at $\{v_1, a_1, a_2\}$. By contracting edges in $G[T_2 \cup S_1 \cup S_2 \cup \{v_1, p\}]$ properly we can then obtain a clique on $T_2$, and by contracting edges in $G[\{x_2, x_3\} \cup V(Q_2 \cup Q_3)]$ properly we can obtain the edge $x_2x_3$. This again shows that $G$ has a minor of type II, a contradiction.

Therefore, it now suffices to prove that there exist pairwise disjoint subsets $S_1, S_2, S_3 \subseteq A_1 - \{v_1, p, q\}$ such that for $i = 1, 2, 3$, $a_i \in S_i$, $G[S_i]$ is connected, and $\{v_1, p, q\} \subseteq N(S_i)$. Let $A' \subseteq \{a_1, a_2, a_3\}$ be the subset of vertices that are not associated with $V(T) - \{v_1, p, q\}$. Note that $|A'| \leq 3$. Also note that $a_i \in A_1 - \{p, q, v_1\}$ for all $i = 1, 2, 3$, which means that every vertex in $A'$ is contained in a unique non-trivial $T$-bridge inside $G[A_1]$ whose set of attachments on $T$ is a subset of $\{v_1, p, q\}$. By (3) of Claim 2, the attachments of this bridge are precisely $v_1, p, q$. It follows that if $|A'| = 3$, then we have found the desired three pairwise disjoint subsets already. So we may assume that $|A'| \leq 2$. Notice that every vertex in $A'$ is associated with $V(P_1) - \{v_1, p\}$ or $V(Q_1) - \{v_1, q\}$ with respect to $T$. By (4) of Claim 2, $|A'| \geq 1$ and thus $|A'| = 1$ or $2$. We will discuss the case $|A'| = 1$ and the case $|A'| = 2$ separately in the rest of this proof.

**Case 1:** $|A'| = 2$.

Without loss of generality, assume that $A' = \{a_1, a_2\}$, and that $a_3$ is associated with $V(P_1 \cup Q_1) - \{v_1, p, q\}$. Let $B_i \subseteq G[A_1]$ for $i = 1, 2$ be the non-trivial $T$-bridge such that $a_i \in V(B_i)$. Note that the attachments of $B_i$ for $i = 1, 2$ on $T$ are precisely $v_1, p, q$. 

107
Also note that \( a_3 \notin V(B_1 \cup B_2) \), since otherwise one of \( a_1, a_2 \) would be associated with \( V(P_1 \cup Q_1) - \{v_1, p, q\} \) with respect to \( T \) as well, a contradiction to the fact that \( \mathcal{A}' = \{a_1, a_2\} \).

Let \( H = (T - V(P_1 \cup Q_1)) \cup B_1 \cup B_2 \), and let \( R_1 \) be a path linking \( v_1, p \) and \( R_2 \) be a path linking \( v_1, q \) such that \( R_1, R_2 \) are internally disjoint and \( V(R_i) \subseteq V(B_i) \) for \( i = 1, 2 \). Let \( T' \) be the graph obtained from \( T \) by substituting \( P_1, Q_1 \) with \( R_1, R_2 \), respectively. Observe that \( T' \subseteq H \) is an \( X \)-tripod such that \( a_1, a_2 \) are associated with \( R_1 - \{v_1, p\} \) and \( R_2 - \{v_1, q\} \), respectively, with respect to \( T' \). Note that \( a_3 \notin V(H) \) and therefore \( a_3 \notin V(T') \). By Claim 2, \( a_3 \) is contained in a non-trivial \( H \)-bridge \( B_3 \subseteq G[A_1] \) whose attachments on \( H \) are all included in \( V(B_1 \cup B_2) \). Notice that \( B_3 \) has no attachment in \( V(B_i) - \{v_1, p, q\} \), since otherwise \( a_3 \) would be associated with \( V(R_1) - \{v_1, p\} \) or \( V(R_2) - \{v_1, q\} \) with respect to \( T' \), a contradiction to (4) of Claim 2. By (3) of Claim 2, it follows that the attachments of \( B_3 \) on \( H \) are precisely \( v_1, p, q \). Let \( S_i = V(B_i) - \{v_1, p, q\} \) for \( i = 1, 2, 3 \), and it follows that \( S_1, S_2, S_3 \) are as desired.

**Case 2:** \( |\mathcal{A}'| = 1 \).

Without loss of generality, assume that \( \mathcal{A}' = \{a_1\} \), \( a_2 \) is associated with \( V(P_1) - \{v_1, p\} \), and \( a_3 \) is associated with \( V(Q_1) - \{v_1, q\} \). Let \( B_1 \subseteq G[A_1] \) be the non-trivial \( T \)-bridge whose attachments are precisely \( v_1, p, q \). Let \( T' \) be the \( X \)-tripod obtained from \( T \) by replacing \( P_1 \) with a \( v_1 - p \) path \( P_1' \) that is in internally contained in \( B_1 \). Let \( H = (T - V(P_1)) \cup B_1 \). Note that \( T' \subseteq H \) and \( a_2 \notin V(H) \) is not associated with \( V(P_1') - \{v_1, p\} \) or \( V(Q_1) - \{v_1, q\} \) with respect to \( T' \). It follows that \( a_2 \) is contained in non-trivial \( H \)-bridge \( B_2 \) such that the attachments of \( B_2 \) on \( H \) are precisely \( v_1, p, q \). Let \( H' = (T' - V(Q_1)) \cup B_2 \), and let \( T'' \) be the \( X \)-tripod obtained from \( T' \) by replacing \( Q_1 \) with a \( v_1 - q \) path \( Q_1' \) that is internally contained in \( B_2 \). It follows that \( T'' \subseteq H' \) and \( a_3 \notin V(H') \) is contained in a non-trivial \( H' \)-bridge \( B_3 \) such that the attachments of \( B_3 \) on \( H \) are precisely \( v_1, p, q \). It follows that \( S_i = V(B_i) - \{v_1, p, q\} \) for \( i = 1, 2, 3 \) are as desired.

Now, let \( R_i = V(L_i \cup P_i \cup Q_i) - \{p, q, v_i\} \) for \( i = 1, 2, 3 \).
Claim 6. At least two vertices in \( A \) are associated with \( R_i \) with respect to \( T \) for some \( i \in \{1, 2, 3\} \) such that \( z_i \neq v_i \).

Proof of Claim 6. We first prove that there exist at most two vertices in \( A \) that are not associated with \( V(T) - X \) with respect to \( T \). For the sake of a contradiction, assume that there are distinct \( a_1, a_2, a_3 \in A \) that are not associated with \( V(T) - X \) with respect to \( T \). Note that \( X \cap A = \emptyset \). By Claim 2, \( a_i \) for each \( i = 1, 2, 3 \) is contained in a unique non-trivial \( T \)-bridge \( B_i \) such that set of attachments of \( B_i \) on \( T \) is equal to \( X \). Without loss of generality, assume that \( a_1 \) is adjacent to \( x_3 \). Note that \( \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{B}_1 - \{v_3\} \) has a \( K_3 \) minor rooted at \( \{v_1, v_2, x_3\} \), and therefore by contracting this subgraph properly and contracting edges between \( T_1 \) and \( \{v_1, v_2, x_3\} \) we can obtain a clique on \( T_1 \). Since \( v_3 \) is adjacent to \( y_3 \), it follows that \( a_2, a_3 \) are both adjacent to \( y_3 \). Since \( a_2, a_3 \) are both adjacent to \( y_3 \), by the same argument that we applied to \( a_1 \), it follows that \( a_1, a_2, a_3 \) are all adjacent to \( x_3 \). This means that \( a_2, a_3 \) are both common neighbors for \( x_3, y_3 \), a contradiction to the construction of \( A \).

Since \( |A| = 9 \), it follows that at least 7 vertices in \( A \) are associated with \( V(T) - X \). Note \( V(T) - X = R_1 \cup R_2 \cup R_3 \cup \{p, q\} \), and therefore at least 5 vertices in \( A \) are associated with \( R_1 \cup R_2 \cup R_3 \). Note that if \( z_i \neq v_i \) for some \( i \in \{1, 2, 3\} \), we may assume that at most one vertex in \( A \) is associated with \( R_i \). If \( z_i = v_i \) for some \( i \in \{1, 2, 3\} \), then by (4) of Claim 2, at most one vertex in \( A \) is associated with \( V(P_i) - \{v_i, p\} \) and at most one vertex in \( A \) is associated with \( V(Q_i) - \{v_i, q\} \), and therefore at most two vertices in \( A \) are associated with \( R_i \). By Claim 5, without loss of generality, we can assume that \( v_1 \neq z_1 \), \( v_i = z_i \) for \( i = 2, 3 \), and for any subset of vertices \( V' \in \{R_1, V(P_2) - \{v_2, p\}, V(Q_2) - \{v_2, q\}, V(P_3) - \{v_3, p\}, V(Q_3) - \{v_3, q\}\} \), there exists exactly one vertex in \( A \) associated with \( V' \). In particular, note that there exist two vertices \( a_1, a_2 \in A \) that are both associated with \( V(Q_2 \cup Q_3) - \{v_2, v_3\} \).

Note that \( \{z_1, v_2, v_3\} \) now separates \( V(L_1) \) from \( V(P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup Q_3) \) in \( G - N[x] \). Since \( G - N[x] \) is 2-connected, there exist two paths \( S_1, S_2 \) linking \( v_1 \) and
\{z_1, v_2, v_3\} such that \(S_1, S_2\) are disjoint except for \(v_1\) and they are both internally disjoint from \(T - V(L_1)\). It follows that \(G_1 = P_1 \cup P_2 \cup P_3 \cup S_1 \cup S_2\) has a \(K_3\) minor rooted at \(X\). Note that \(a, a_1 \in \mathcal{A}\) are both associated with \(V(Q_2 \cup Q_3) - \{v_2, v_3\}\) with respect to \(T\), and that \(G_1\) is disjoint from \(V(Q_2 \cup Q_3) - \{v_2, v_3\}\). This is then a contradiction to (4) of Claim 1.

By Claim 5 and Claim 6, without loss of generality, assume that \(v_1 \neq z_1\) and there exist two vertices \(b_1, b_2 \in \mathcal{A}\) such that \(b_1, b_2\) are both associated with \(R_1\). For \(i = 1, 2\), let \(W_i\) be a path in \(G - N[x]\) linking \(b_i\) and some vertex in \(R_1\) that is otherwise disjoint from \(T\). It follows that \(G[R_1] \cup W_1 \cup W_2\) contains a path linking \(b_1, b_2\) in \(G - N[x]\).

From Claim 7 to Claim 10, we will show that there exists some non-trivial \(T\)-bridge attaching to \(v_1\), and that every such \(T\)-bridge attaches to exactly one of \(v_2\) and \(v_3\) as its only attachment on \(T\) outside \(L_1\).

Claim 7. The following statements are true.

1. There does not exist a non-trivial \(T\)-bridge with attachments \(u_i \in V(L_i)\) for \(i = 1, 2, 3\) such that \(u_i \neq z_i\) for some \(i \in \{1, 2, 3\}\).

2. There do not exist two disjoint paths \(S_1, S_2\) in \(G - N[x]\) such that for some distinct indices \(j, k \in \{1, 2, 3\}\), \(S_1\) links \(v_j\) and some vertex on \(L_k - \{v_k\}\), \(S_2\) links \(v_k\) and some vertex on \(L_j - \{v_j\}\), and that \(S_1, S_2\) are both internally disjoint from \(T\).

3. If a non-trivial \(T\)-bridge \(D\) attaches to \(v_1\) and at least two vertices on \(L_2 \cup L_3\), then \(b_i \in V(D) - V(T)\) for some \(i \in \{1, 2\}\).

4. If a non-trivial \(T\)-bridge \(D\) attaches to \(v_1\) and a vertex on \(L_j - \{v_j\}\) for some \(j \in \{2, 3\}\) and there is a path \(S\) linking \(v_j\) and some vertex on \(L_{5-j}\) such that \(S\) is otherwise disjoint from \(T\), then \(b_i \in V(D) - V(T)\) for some \(i \in \{1, 2\}\).

Proof of Claim 7. (1) and (2) are simply true since there is no \(X\)-tripod that can be obtained from \(T\) by any tripod-transformation.

To see (3), assume some non-trivial \(T\)-bridge \(D\) attaches to \(v_1\) and at least two vertices on \(L_2 \cup L_3\). Then notice that \(G_1 = (D \setminus (L_1 - \{v_1\})) \cup L_2 \cup L_3 \cup P_2 \cup P_3\) has a \(K_3\) minor rooted
at \( X \) and is disjoint from \( R_1 \). If \( b_1, b_2 \) neither are in \( D - V(T) \), then \( G_2 = G[R_1] \cup W_1 \cup W_2 \) would be disjoint from \( G_1 \) and contain a path linking \( b_1 \) and \( b_2 \), a contradiction to (4) of Claim 1. It follows that \( b_i \in V(D) - V(T) \) for some \( i \in \{1, 2\} \).

To see (4), without loss of generality, assume that a non-trivial \( T \)-bridge \( D \) attaches to \( v_1 \) and a vertex on \( L_2 - \{v_2\} \), and that a path \( S \) links \( v_2 \) and some vertex on \( L_3 \) such that \( S \) is otherwise disjoint from \( T \). By (1), \( D \) has no attachment on \( L_3 \) and therefore \( S \) is internally disjoint from \( T \cup D \). Then, observe that \( G'_1 = (D \setminus (L_1 - \{v_1\})) \cup L_2 \cup L_3 \cup S \cup P_2 \cup P_3 \) has a \( K_3 \) minor rooted at \( X \) and is disjoint from \( R_1 \). By a similar argument as above, it follows that \( b_i \in V(D) - V(T) \) for some \( i \in \{1, 2\} \).

Claim 8. The following statements are true.

1. If \( v_i \neq z_i \) for \( i \in \{1, 2, 3\} \), then \( v_i \) has at least 6 neighbors in \( A - V(L_i) \).

2. There exists a non-trivial \( T \)-bridge attaching to \( v_1 \).

Proof of Claim 8. We first prove (1). Since \( (A, B) \) splits \( T \) and \( v_i \neq z_i \), we know that \( v_i \in A - B \). Since no vertex in \( G - N[x] \) is a common neighbor for two non-adjacent vertices in \( N(x) \), by Lemma 4.3.3, \( v_i \) has at most four neighbors in \( N(x) \). Since every \( T \)-bridge is stable, \( v_i \) has exactly one neighbor on \( L_i \). As \( \delta(G) \geq 11 \) by Lemma 3.1.1 and \( v_i \in A - B \), it follows that \( v_i \) has at least 6 neighbors in \( A - V(L_i) \).

We now prove (2). Since \( v_1 \neq z_1 \), by (1) it follows that \( v_1 \) has at least 6 neighbors in \( A - V(L_1) \). If there is no non-trivial \( T \)-bridge attaching to \( v_1 \), then \( v_1 \) must have at least 6 neighbors on \( L_2 \cup L_3 \). It follows that the subgraph \( G[\{v_1\} \cup V(L_2 \cup L_3 \cup P_2 \cup P_3)] \subseteq G - N[x] \) has \( K_3 \) minor rooted at \( X \) and is disjoint from \( G[R_1] \cup W_1 \cup W_2 \) which contains a path linking \( b_1, b_2 \), a contradiction to (3) of Claim 1. Hence, there exists some non-trivial \( T \)-bridge attaching to \( v_1 \).

Claim 9. If a non-trivial \( T \)-bridge \( B_1 \) attaches to \( v_1 \) and some \( u_2 \in V(L_j) - \{v_j\} \) for some \( j \in \{2, 3\} \), then the following statements are true.

1. There exists a path \( S \) linking \( v_j \) and \( v_1 \) or some vertex on \( L_{5-j} \). If \( v_1 \) is an end of \( S \),
then $B_1$ attaches to $v_j$ and $S$ is internally contained in $B_1 - V(T)$.

(2) $b_i \in V(B_1) - V(T)$ for some $i \in \{1, 2\}$.

(3) Neither one of $b_1$, $b_2$ is a common neighbor for $x_1$ and $y_1$.

(4) $b_1, b_2$ are the only vertices in $A$ that are associated with $(L_1 - \{v_1\}) \cup (R(T) - \{z_2, z_3\})$.

**Proof of Claim 9.** Without loss of generality, assume that $u_2 \in V(L_2) - \{v_2\}$.

We first prove (1) and (2). Note that there is a $v_1$-$u_2$ path internally contained in $B_1 - V(T)$. By (2) of Claim 7, there is no path linking $v_2$ and some vertex on $L_1 - \{v_1\}$ that is otherwise disjoint from $T \cup B_1$. Note that the fact that $B_1$ attaches to $u_2 \in V(L_2) - \{v_2\}$ implies that $v_2 \neq z_2$. By (1) of Claim 8, $v_2$ has at least 6 neighbors in $A - (V(L_1 \cup L_2) \setminus \{v_1\})$. It follows that there exists a path $S$ linking $v_2$ and either $v_1$ or some vertex on $L_3$ such that $S$ is otherwise disjoint from $T \cup B_1$. Moreover, if $v_1$ is an end of $S$, then $B_1$ attaches to $v_2$ and $S$ is internally contained in $B_1 - V(T)$. Then, due to the existence of the path $S$, by (3) - (4) in Claim 7, it follows that $b_i \in V(B_1) - V(T)$ for some $i \in \{1, 2\}$. We are now done with proving (1) and (2).

Now assume that $b_1 \in V(B_1) - V(T)$ without loss of generality. Since $b_1$ is associated with $R_1 = V(L_1 \cup P_1 \cup Q_1) - \{v_1, p, q\}$ and $(A, B)$ splits $T$, we know that $B_1$ attaches to some $u_1 \in V(L_1) - \{v_1\}$. Recall that $W_i$ for $i = 1, 2$ is a path linking $b_i$ and some vertex in $R_i$ in $G - N[x]$ that is otherwise disjoint from $T$.

To prove (3), note that $(T - \{v_1\}) \cup W_2$ has a $K_3$ minor rooted at $\{b_2, v_2, v_3\}$ and is disjoint from some path linking $v_1$ and $b_1$ inside $B_1$. If $b_2$ is a common neighbor for $x_1$ and $y_1$, then $b_1$ is not adjacent to both $x_1$ and $y_1$ by the definition of $A$ and thus $v_1$ and $b_1$ have distinct neighbors in at least one of $T_1$ and $T_2$. This is then a contradiction to (2) of Claim 1. Therefore, $b_2$ is not a common neighbor for $x_1, y_1$. Note that either $S$ is contained in the $T$-bridge $B_1$, or that $S$ links $v_2$ and some vertex on $L_3$ and is otherwise disjoint from $T \cup B_1$. In both cases, $G_1 = (B_1 - V(L_1)) \cup S \cup L_2 \cup L_3 \cup P_2 \cup P_3$ has a $K_3$ minor rooted at $\{b_1, v_2, v_3\}$ and is disjoint from $G_2 = G[R_1 \cup \{v_1\}] \cup W_2$ which contains a path linking
$v_1$ and $b_2$, again a contradiction to (2) of Claim 1. To conclude, $b_1$ is neither a common neighbor for $x_1$ and $y_1$.

To prove (4), let $G_1' = (B_1 - R_1) \cup S \cup L_2 \cup L_3 \cup P_2 \cup P_3$. Observe that $G_1'$ has a $K_3$ minor rooted at $X$ and is disjoint from $b_2$ and $G[R_1] \cup Q_2 \cup Q_3 - \{z_2, z_3\}$, which includes all vertices in $R_1$. Since $b_2 \not\in V(G_1')$ and $b_2$ is associated with $R_1$, by (4) of Claim 1, $b_2$ is the only vertex in $\mathcal{A} - \{b_1\}$ that is associated with $R_1 \cup V(Q_2 \cup Q_3) - \{z_2, z_3\}$. Let $G''_1 = (B_1 - R_1) \cup S \cup L_2 \cup L_3 \cup Q_2 \cup Q_3$. Then notice that $G_1'$ has a $K_3$ minor rooted at $X$ too and is disjoint from $b_2$ and $G[R_1] \cup P_2 \cup P_3 - \{z_2, z_3\}$. Therefore, $b_2$ is also the only vertex in $\mathcal{A} - \{b_1\}$ that is associated with $R_1 \cup V(P_2 \cup P_3) - \{z_2, z_3\}$. Combining the two observations, we conclude that $b_1, b_2$ are the only vertices in $\mathcal{A}$ associated with $R_1 \cup \mathcal{R}(T) - \{z_2, z_3\}$, the union of $(R_1 \cup V(P_2 \cup P_3) - \{z_2, z_3\})$ and $(R_1 \cup V(Q_2 \cup Q_3) - \{z_2, z_3\})$.

Claim 10. Every non-trivial $T$-bridge attaching to $v_1$ attaches to exactly one of $v_2, v_3$ as its only attachment on $T$ outside $L_1$.

Proof of Claim 10. Let $B_1$ be any $T$-bridge attaching to $v_1$. Since every $T$-bridge is stable, $B_1$ has some attachment on $L_2 \cup L_3$. By (1) of Claim 7, attachments of $B_1$ outside $L_1$ are all included in one of $L_2$ and $L_3$. Without loss of generality, say attachments of $B_1$ outside $L_1$ are all on $L_2$. It then suffices to prove that $B_1$ does not attach to any vertex on $L_2 - \{v_2\}$.

For the sake of a contradiction, assume that $B_1$ attaches to some $u_2 \in V(L_2) - \{v_2\}$, and it follows that (1)-(4) in Claim 9 are true. Let $S$ be a path linking $v_2$ and $v_1$ or a vertex on $L_3$ as described in (1) of Claim 9.

Let $a_1 \in \mathcal{A}$ be the vertex adjacent to both $x_1$ and $y_1$. By Claim 9, $a_1 \neq b_1$ or $b_2$, and $a_1$ is not associated with $(L_1 - \{v_1\}) \cup (\mathcal{R}(T) - \{z_2, z_3\}))$. Assume for a moment that there exist distinct $m_1, m_2 \in V(L_2 \cup L_3)$ and two paths $M_1, M_2$ in $G - N[x]$ such that $M_i$ for $i = 1, 2$ links $a_1$ and $m_i$ and is otherwise disjoint from $T$. It follows that $L_2 \cup L_3 \cup M_1 \cup M_2 \cup P_2 \cup P_3$ has a $K_3$ rooted minor at $\{a_1, v_2, v_3\}$ and is disjoint from the subgraph $B_1 - V(L_2)$ which contains a path linking $v_1$ and $b_1$. Since $a_1$ is adjacent to both $x_1, y_1$, we know $v_1$ and $b_1$ have distinct neighbors in $T_1$ or $T_2$. This is then a contradiction to (2) of Claim 1.
Hence, \( a_1 \) is associated with at most one vertex on \( L_2 \cup L_3 \). Note that \( a_1 \not\in X \) and every non-trivial \( T \)-bridge has at least three attachments on \( T \). Since \( a_1 \) is not associated with \( (L_1 - \{v_1\}) \cup (R(T) - \{z_2, z_3\}) \), it follows that \( a_1 \) is simply a vertex on \( L_i - \{v_i\} \) for some \( i \in \{2, 3\} \). In the remaining proof, we will discuss the case \( a_1 \in V(L_2) - \{v_2\} \) and the case \( a_1 \in V(L_3) - \{v_3\} \) separately.

**Case 1:** \( a_1 \in V(L_2) - \{v_2\} \).

In this case, first observe that if the path \( S \) links \( v_2 \) and some vertex on \( L_3 \), then \( L_2 \cup L_3 \cup S \cup P_2 \cup P_3 \) would have a \( K_3 \) minor rooted at \( \{a_1, v_2, v_3\} \) and be disjoint from some \( v_1 \)-\( b_1 \) path contained in \( B_1 - V(L_2) \), a contradiction to (2) of Claim 1. By (1) of Claim 9, \( S \) links \( v_1, v_2 \) and \( B_1 \) attaches to \( v_2 \). Now, note that \( (B_1 - V(L_1)) \cup L_2 \) has a \( K_3 \) minor rooted at \( \{a_1, v_2, b_1\} \) and is disjoint from the \( v_1 \)-\( v_3 \) path \( L_1 \cup L_3 \cup P_1 \cup P_3 \). It follows that \( b_1 \) is not adjacent to \( x_3 \) or \( y_3 \). Note that \( b_1 \) is not a common neighbor for \( x_1 \) and \( y_1 \) by (3) of Claim 9. It follows that \( b_1 \) is adjacent to one of \( x_2 \) and \( y_2 \). Then, one can observe that subgraph \( (B_1 - V(L_2)) \cup L_3 \cup P_1 \cup P_3 \) has a \( K_3 \) minor rooted at \( \{v_1, b_1, v_3\} \) and is disjoint from the path \( v_2 L_2 a_1 \) linking \( a_1 \) and \( v_2 \), a contradiction to (2) of Claim 1.

**Case 2:** \( a_1 \in V(L_3) - \{v_3\} \).

In this case, first observe that \( B_1 \cup L_1 \cup L_2 \cup P_1 \cup Q_1 \cup P_2 \) has a \( K_3 \) minor rooted at \( \{v_1, v_2, b_1\} \) and is disjoint from the path \( v_3 L_3 a_1 \). By (2) of Claim 1, \( b_1 \) is not adjacent to \( x_3 \) or \( y_3 \). By (3) of Claim 9, \( b_1 \) is not a common neighbor for \( x_1, y_1 \). So without loss of generality, we can assume that \( b_1 \) is adjacent to \( x_2 \).

We next prove that there is no path linking \( v_3 \) and some vertex in \( L_1 \cup L_2 - \{v_1\} \) that is otherwise disjoint from \( T \). If a path \( S' \) links \( v_3 \) and a vertex on \( L_1 - \{v_1\} \) and is otherwise disjoint from \( T \), then the subgraph \( S' \cup (L_1 - \{v_1\}) \cup L_2 \cup L_3 \cup R(T) \) would have a \( K_3 \) minor rooted at \( \{a_1, v_2, v_3\} \) and is disjoint from some \( v_1 \)-\( b_1 \) path contained in \( B_1 \), a contradiction to (2) of Claim 1. If a path \( S'' \) links \( v_3 \) and some vertex on \( L_2 \) and is otherwise disjoint from \( T \), then \( L_2 \cup L_3 \cup S' \cup P_2 \cup P_3 \) would have a \( K_3 \) minor rooted at \( \{a_1, v_2, v_3\} \) and is disjoint from some \( v_1 \)-\( b_1 \) path in \( B_1 \), again a contradiction to (2) of Claim 1.
Now, since \( a_1 \in V(L_3) - \{v_3\} \), we know \( v_3 \neq z_3 \). By (1) of Claim 8, \( v_3 \) has at least 6 neighbors in \( A - V(L_3) \). As there is no path linking \( v_3 \) and some vertex in \( L_1 \cup L_2 - \{v_1\} \) that is otherwise disjoint from \( T \), there is a non-trivial \( T \)-bridge \( B_2 \neq B_1 \) attaching to \( v_1 \) and \( v_3 \). Since \((A,B)\) splits \( T \) and every \( T \)-bridge has at least three attachments on \( T \), \( B_2 \) also attaches to some vertex on \( L_3 - \{v_3\} \). It follows that \( B_2 \cup L_2 \cup L_3 \cup P_2 \cup P_3 \) has a \( K_3 \) minor rooted at \( X \) and is disjoint from \( G[R_1] \cup W_1 \cup W_2 \) which contains a path linking \( b_1 \) and \( b_2 \), a contradiction to (4) of Claim 1.

Recall that every non-trivial \( T \)-bridge has at least three attachments on \( T \). By Claim 10, since the \( X \)- tripod \( T \) is split, every \( T \)-bridge attaches to some vertex on \( L_1 - \{v_1\} \). By (2) Claim 8, we now fix a vertex \( v'_1 \in V(L_1) - \{v_1\} \) and a non-trivial \( T \)-bridge \( B_1 \) attaching to \( v_1 \) such that there is no non-trivial \( T \)-bridge \( B'_1 \) attaching to a vertex on \( v'_1 \cup L_1 - \{v'_1\} \). By Claim 10, assume that \( v_2 \) is the only attachment of \( B_1 \) on \( T \) outside \( L_1 \). Let \( L'_1 = v_1 L_1 v'_1 \) for notation. Let \( A_3 = A \) be the subset of five vertices in \( A \) that are adjacent to \( x_3 \) or \( y_3 \). In Claim 11-Claim 14, we will show that every vertex in \( A_3 \) is not associated with anywhere on \( T \) outside \( V(L_3) \cup \{v_2\} \) due to the existence of the \( T \)-bridge \( B_1 \).

**Claim 11.** \( (V(B_1) - V(T)) \cap A_3 = \emptyset \).

**Proof of Claim 11.** For the sake of a contradiction, assume there exists some \( c \in (V(B_1) - V(T)) \cap A_3 \). Without loss of generality, say \( c \) is adjacent to \( x_3 \). For \( i = 1, 2 \), let \( a_i \in A \) be the vertex adjacent to both \( x_2 \) and \( y_i \). By Claim 2, \( a_i \not\in V(B_1) - V(T) \) for \( i = 1, 2 \).

If \( a_i \) for some \( i \in \{1, 2\} \) is associated with \( (V(L_1) - \{v_1\}) \cup (V(P_1 \cup P_2 \cup P_3) - \{z_2, z_3\}) \), then \( G_1 = L_1 \cup (B_1 - \{v_2\}) \cup P_1 \cup P_2 \cup P_3 - \{z_2, z_3\} \) would have a \( K_3 \) minor rooted at \( \{v_1, a_i, c\} \). Note that \( G_2 = L_2 \cup L_3 \cup Q_2 \cup Q_3 \) is a path linking \( v_2, v_3 \) that is disjoint from \( G_1 \). By contracting edges in \( G_1, G_2 \) and edges between \( T_1 \) and \( G_i \) for \( i = 1, 2 \) properly, we can then obtain a clique on \( T_1 \) and the edge \( y_2 y_3 \) in \( T_2 \), a contradiction. Therefore, \( a_1, a_2 \) are not associated with \( (V(L_1) - \{v_1\}) \cup (V(P_1 \cup P_2 \cup P_3) - \{z_2, z_3\}) \). By symmetry, they are not associated with \( (V(L_1) - \{v_1\}) \cup (V(Q_1 \cup Q_2 \cup Q_3) - \{z_2, z_3\}) \) either. It follows
that $a_1, a_2$ are not associated with $(V(L_1) - \{v_1\}) \cup (R(T) - \{z_2, z_3\})$.

Now note that $L_1 \cup L_2 \cup B_1 \cup P_1 \cup P_2$ has a $K_3$ minor rooted at $\{v_1, v_2, c\}$. Since $v_3$ is adjacent to $y_3$, it follow that neither $a_1$ nor $a_2$ is associated with $V(L_3)$. Therefore, $a_1, a_2$ are not associated with $V(T) - V(L_2) \cup \{v_1\}$. Notice that $\{a_1, a_2\} \cap \{v_1, v_2\} = \emptyset$.

By Claim 2, $a_1, a_2$ are both associated with $V(L_2) - \{v_2\}$. It follows that for $i = 1, 2$, there exists some path $S_i$ linking $a_i$ and some vertex on $L_2 - \{v_2\}$ such that $S_i$ is otherwise disjoint from $T \cup B_1$. Let $G'_1 = L_1B_1 \cup P_1 \cup P_3 \cup L_3$ and $G'_2 = (L_2 - \{v_2\}) \cup S_1 \cup S_2$.

It follows that $G'_1, G'_2$ are disjoint connected subgraphs of $G - N[x]$ such that $G'_1$ has a $K_3$ minor rooted at $X$, and that $G'_2$ contains a path linking $a_1, a_2$, a contradiction to (4) of Claim 1.

Claim 12. There is no vertex in $A_3$ associated with $V(L'_1) - \{v_1\}$.

Proof of Claim 12. For the sake of a contradiction, assume that there exists some $c \in A_3$ associated with $V(L'_1) - \{v_1\}$. Without loss of generality, say $c$ is adjacent to $x_3$. By Claim 11, $c \notin V(B_1) - V(T)$. Let $S_c$ be a path linking $c$ and some vertex in $V(L'_1) - \{v_1\}$ such that $S_c$ is otherwise disjoint from $T \cup B_1$. Let $G_1 = B_1 \cup L'_1 \cup S_c$, and note that $G_1$ has a $K_3$ minor rooted at $\{v_1, v_2, c\}$ and is disjoint from the connected subgraph $T - (V(L'_1) \cup \{v_2\})$ of $G - N[x]$. By contracting edges in $G_1$ and edges between $T_1$ and $G_1$, we can obtain a clique on $T_1$. Since $v_3 \in V(T) - (V(L'_1) \cup \{v_2\})$, by (2) of Claim 1, no vertex adjacent to $y_1$ or $y_2$ is associated with $V(T) - (V(L'_1) \cup \{v_2\})$.

Let $A' \subseteq A$ be the subset of vertices that are adjacent to $y_1$ or $y_2$ and are not contained in $B_1 - V(T)$. Notice that $|A'| \geq 5$, and that every vertex in $A'$ is either a vertex on $L'_1$ or contained in a non-trivial $T$-bridge whose attachments are contained in $V(L'_1) \cup \{v_2\}$.

Notice that $B_1 \cup T - (V(L'_1) \setminus \{v_1, v'_1\})$ has a $K_3$ minor rooted at $X$ and is disjoint from the interior of $L'_1$. By (4) of Claim 1, at most one vertex in $A$ is associated with the interior of $L'_1$. Also, at most one vertex in $A'$ is equal to $v'_1$. Since every non-trivial $T$-bridge is stable and has at least three attachments on $T$, it follows that there exist distinct vertices $a_i \in A'$ for $i = 1, 2, 3$ each contained in a unique non-trivial $T$-bridge $D_i$ whose
attachments on $T$ are precisely $v_1, v_2,$ and $v'_1$.

Assume for a moment that $a_1$ is adjacent to $x_3$. Recall that $A' \cap (V(B_1) - V(T)) = \emptyset$ by definition. Let $G_1 = B_1 \cup D_1 - (V(L'_1) \setminus \{v_1\})$. Then observe that $G_1$ has a $K_3$ minor rooted at $\{v_1, v_2, a_1\}$ and therefore a clique on $T_1$ can be obtained by contracting edges in $G_1$ and edges between $G_1$ and $T_1$ properly. Let $G_2 = (D_2 - \{v_1, v_2\}) \cup v'_1 L_1 z_1 \cup L_3 \cup P_1 \cup P_3$, and note that $G_2$ contains a path linking $a_2, v_3$ and is disjoint from $G_1$. Since $a_2$ is adjacent to $y_1$ or $y_2$, by the definition of $A'$, it follows that by contracting edges in $G_2$ and edges between $T_2$ and $G_2$ properly, we can then obtain the edge $y_2y_3$, which then implies a minor of type II, a contradiction. By symmetry, it follows that none of $a_1, a_2, a_3$ is adjacent to $x_3$.

Now, each $a_i$ for $i = 1, 2, 3$ is a common neighbor for one of $x_1, x_2$ and one of $y_1, y_2$. Without loss of generality, assume that $a_1$ is a common neighbor for $x_j$ and $y_j$ for some $j \in \{1, 2\}$. If $j = 1$, let $G_1 = (D_1 - \{v_1\}) \cup v'_1 L_1 z_1 \cup L_2 \cup L_3 \cup P_1 \cup P_2 \cup P_3$, $G_2 = D_2 - \{v'_1, v_2\}$, and $X' = \{a_1, v_2, v_3\}$. Then, $G_1, G_2$ are disjoint subgraphs of $G - N[x]$ such that $G_1$ has a $K_3$ minor rooted at $X'$ and $G_2$ contains a path linking $v_1$ and $a_2$. Since $a_1$ is adjacent to both $x_1$ and $y_1$, $a_2$ is adjacent to $x_2$ or $y_2$. This is then a contradiction to (2) of Claim 1. If $j = 2$, let $G_1 = (D_1 - \{v_2\}) \cup L_1 \cup L_3 \cup P_1 \cup P_3$, $G_2 = D_2 - \{v_1, v'_1\}$, and $X' = \{v_1, a_1, v_3\}$. Then, $G_1$ has a $K_3$ minor rooted at $X'$ and $G_2$ contains a path linking $v_2$ and $a_2$. Since $a_1$ is adjacent to $x_2$ and $y_2$, $a_2$ is adjacent to $x_1$ or $y_1$. Again, a contradiction to (2) of Claim 1.

Claim 13. There is no vertex in $A_3$ associated with $V(v'_1 L_1 z_1) \cup R(T) - \{z_2, z_3\}$.

Proof of Claim 13. For the sake of a contradiction, assume that $c \in A_3$ is associated with $V(v'_1 L_1 z_1) \cup R(T) - \{z_2, z_3\}$. Without loss of generality, assume that $c$ is adjacent to $x_3$ and is associated with $V(v'_1 L_1 z_1 \cup P_1 \cup P_2 \cup P_3) - \{z_2, z_3\}$. Let $s \in V(v'_1 L_1 z_1 \cup P_1 \cup P_2 \cup P_3) - \{z_2, z_3\}$ such that there exists a path $S_1$ linking $c$ and $s$ that is otherwise disjoint from $T$, and let $S_2$ be the subpath of $v'_1 L_1 z_1 \cup P_1 \cup P_2 \cup P_3 - \{z_2, z_3\}$ linking $v'_1$ and $s$. Note that we may choose $c, s$ and the paths $S_1, S_2$ such that no vertex in $A_3$ is associated with $V(S_2) - \{s\}$, and subject to this, $|S_1|$ is minimum.
Now, let \( G_1 = B_1 \cup L'_1 \cup S_1 \cup S_2 \). Observe that \( G_1 \) has a \( K_3 \) minor rooted at \( \{v_1, v_2, c\} \) and \( v_3 \not\in V(G_1) \). By (2) of Claim 1, no vertex adjacent to \( y_1 \) or \( y_2 \) in \( G - N[x] \) is associated with \( V(T) - V(G_1) \). By the definition of \( A_3 \), there exists some \( c' \in A_3 - \{c\} \) that is adjacent to \( y_1 \) or \( y_2 \). Observe that \( V(G_1) \cap V(T) = \{v_2\} \cup V(L'_1) \cup V(S_2) \), so either \( c' \in \{v_2\} \cup V(L'_1) \cup V(S_2) \) or \( c' \) is contained in a non-trivial \( T \)-bridge whose attachments on \( T \) are all included in \( \{v_2\} \cup V(L'_1) \cup V(S_2) \). By Claim 13 and the choice of \( c, s \) and \( S_1, S_2 \), no vertex in \( A_3 \) is associated with \( V(L'_1) \cup V(S_2) - \{v_1, s\} \) and \( c' \neq s \). Since \( \{v_1, v_2\} \cap A = \emptyset \) and every non-trivial \( T \)-bridge has at least three attachments on \( T \), it follows that \( c' \) is contained in a non-trivial \( T \)-bridge, say \( B' \), whose attachments on \( T \) are precisely \( v_1, v_2 \), and \( s \).

Since \( (A, B) \) splits \( T \) and \( v_1 \in A - B \), it follows that \( B' \) is contained in \( A \) and therefore \( s \in V(v'_1 L_1 z_1) \). By Claim 12, \( c \) is not associated with \( v'_1 \), so \( s \not\in V(v'_1 L_1 z_1) - \{v'_1\} \). It follows that \( B' \) is a non-trivial \( T \)-bridge whose attachments on \( T \) are precisely \( v_1, v_2 \), and \( s \) where \( s \not\in V(v'_1 L_1 z_1) - \{v'_1\} \), a contradiction to the choice of \( B_1 \) and \( v'_1 \). 

\textit{Claim 14.} There is no vertex in \( A_3 \) associated with \( v_1 \) or \( V(L_2) - \{v_2\} \).

\textit{Proof of Claim 14.} Note that if some vertex in \( A_3 \) is associated with \( v_1 \), then it is contained in \( V(D) - V(T) \) for some non-trivial \( T \)-bridge \( D \), since \( v_1 \not\in A \). By Claim 10, since \( (A, B) \) splits \( T \) and every non-trivial \( T \)-bridge has at least three attachments on \( T \), \( D \) attaches to some vertex on \( L_1 - \{v_1\} \) and therefore some vertex in \( A_3 \) is associated with \( V(L_1) - \{v_1\} \), a contradiction to Claim 12. Therefore, no vertex in \( A_3 \) associated with \( v_1 \).

It remains to prove that no vertex is associated with \( V(L_2) - \{v_2\} \). Notice that \( B_1 \cup L_1 \cup L_3 \cup P_1 \cup P_3 \) has a \( K_3 \) minor rooted at \( X \) and is disjoint from \( L_2 - \{v_2\} \). If two vertices \( c_1, c_2 \in A_3 \) are both associated with \( V(L_2) - \{v_2\} \), note that neither of them is contained in \( B_1 \), as \( B_1 \) has no attachment on \( L_2 - \{v_2\} \). It follows that we can extend some subpath of \( L_2 - \{v_2\} \) to a \( c_1 - c_2 \) path that is disjoint from \( B_1 \cup L_1 \cup L_3 \cup P_1 \cup P_3 \), a contradiction to (4) of Claim 1. Therefore, at most one vertex in \( A_3 \) is associated with \( V(L_2) - \{v_2\} \).

For the sake of a contradiction, assume \( c \in A_3 \) is associated with \( V(L_2) - \{v_2\} \), and
let $S$ be a path linking $c$ and some vertex on $L_2 - \{v_2\}$ such that $S$ is otherwise disjoint from $T$. Without loss of generality, say $c$ is adjacent to $x_3$. By the definition of $A_3$, there exists some $c' \in A_3$ such that $c' \neq c$ and $c'$ is adjacent to $x_3$ and one of $y_1, y_2$. Notice that $c'$ is not associated with $V(T) - (V(L_3) \cup \{v_2\})$ by Claim 11-Claim 13 and the fact that $c$ is the only vertex in $A_3$ associated with $V(L_2) - \{v_2\}$. Since $v_2 \notin A_3$ and every $T$-bridge is stable, it follows that $c'$ is associated with $V(L_3)$, meaning that there exists a path $S'$ linking $c'$ and some vertex on $L_3$ such that $S'$ is otherwise disjoint from $T$. Let $G_1 = B_1 \cup L_1 \cup L_2 \cup S \cup P_1 \cup P_2$ and $G_2 = L_3 \cup S'$. Then, $G_1$ has a $K_3$ minor rooted at $X$ and $G_2$ contains a path linking $c'$ and $v_3$, a contradiction to (3) of Claim 1.

By Claim 11-Claim 14, every vertex in $A_3$ is either a vertex on $L_3 - \{v_3\}$ or contained in $D - V(T)$ for some non-trivial $T$-bridge $D$ attaching to $v_2$ and at least two vertices on $L_3$, as every non-trivial $T$-bridge is stable and has at least three attachments on $T$. It follows that every vertex in $A_3$ is associated with $V(L_3) - \{v_3\}$. Choose $c \in A_3$ and $u_3 \in V(L_3) - \{v_3\}$ such that some $c$-$u_3$ path $S$ is disjoint from $T$ except for $u_3$ and no vertex in $A_3$ is associated with $V(u_3 L_3 z_3) - \{u_3\}$, and subject to these, $|S|$ is minimum. Without loss of generality, say $c$ is adjacent to $x_3$. Notice that $G_1 = B_1 \cup L_1 \cup S \cup u_3 L_3 z_3 \cup P_1 \cup P_3$ has a $K_3$ minor rooted at $\{v_1, v_2, c\}$ and is disjoint from $v_1 L_3 u_3 - \{u_3\}$, so a clique on $T_1$ can be obtained by contracting edges in $G_1$ and edges between $T_1$ and $G_1$ properly. By the definition of $A_3$, choose $c' \in A_3$ such that $c' \neq c$ and $c'$ is adjacent to $x_3$ and one of $y_1, y_2$. It follows that $c'$ is not associated with $V(v_1 L_3 u_3) - \{u_3\}$, since otherwise there would exist a path linking $v_3$ and $c'$ disjoint from $G_1$, a contradiction to (2) of Claim 1. Since $c'$ is associated with some vertex on $L_3 - \{v_3\}$ and no vertex in $A_3$ is associated with $V(u_3 L_3 z_3) - \{u_3\}$ by the choice of $u_3$, it follows that $c'$ associated with exactly $u_3$ on $L_3$. Moreover, $c' = u_3$, as every non-trivial $T$-bridge is stable and thus has at least two attachments on $L_3$. By the minimality of $|S|$ when choosing $c'$ and $u_3$, this means that we should have chosen $c'$ and $u_3$ instead of $c$ and $u_3$, a contradiction. $\square$
4.3.3 Proof of Lemma 4.3.6

Lemma 4.3.6. Suppose $G - N[x]$ is 2-connected. If there exists a vertex $u \in V(G) - N[x]$ that is common neighbor for two non-adjacent vertices in $N(x)$, then $G - x$ has a minor $J$ rooted at $N(x)$ such that $J > K_9$.

Proof. For $i = 1, 2$, let $C_i$ be the island of $N(x)$ such that $T_i \subseteq C_i$.

Claim 1. There do not exist vertices distinct $a, b \in V(G) - N[x]$ such that $a$ is adjacent to at least two vertices in each one of $T_1, T_2$, and that $b$ has at least five neighbors in $T_1 \cup T_2$.

Proof of Claim 1. For the sake of a contradiction, assume that such $a, b \in V(G) - N[x]$ exist, where $b$ is adjacent to all vertices in $T_1$ and at least two vertices in $T_2$ without loss of generality. Since $|G - N[x]| \geq 3$ by Lemma 4.3.2, there exists some component $L$ of $G - N[x] \cup \{a, b\}$. Since $G - N[x]$ is 2-connected and $G$ is 7-connected, $a, b$ are both neighbors of $L$ and $|N(L) \cap N(x)| \geq 5$. By Lemma 4.3.3, there is no clique of size 5 in $N(x)$. Therefore, there exist $z_1, z_2 \in N(L) \cap N(x)$ that are not adjacent to each other.

Note that if $\{z_1, z_2\} \cap (C_1 \cup C_2) = \emptyset$, then by contracting an edge between $T_1$ and $a$, an edge between $T_2$ and $b$, and contracting all of $L$ to $z_1$, we would then obtain a minor of $G$ of type I, a contradiction. It follows that $\{z_1, z_2\} \subseteq C_i$ for some $i = 1, 2$.

Assume for a moment that $z_1, z_2$ are both contained in $C_1$. Note that $|C_1 - T_1| \leq 1$, so at least one of $z_1, z_2$ is contained in $T_1$. Since $|T_1| = 3$, without loss of generality, assume that $z_1 = x_1$ and $z_2 \neq x_2$. This means that $z_1, z_2, x_2$ are three distinct vertices in $C_1$. Note that $b$ is adjacent to $x_1, x_2, x_3$ and $b \in N(L)$, so by contracting all of $L$ to $z_2$ and contracting the edge $bx_2$, we would then obtain a clique on $\{z_1, z_2, x_2\}$. Then, by contracting an edge between $a$ and a neighbor of it in $T_2$, we then obtain a minor of $G$ of type II, a contradiction.

We may then assume $\{z_1, z_2\} \subseteq C_2$. Note that $a$ has at least two neighbors in $T_1$, so without loss of generality, we say $a$ is adjacent to $x_1$ and $x_2$. Then, by contracting $ax_1$ and $bx_3$ and contracting all of $L$ to $z_1$, we would then obtain a clique on $T_1$ and the edge $z_1z_2$ in $C_2$. This means that we obtained a minor of $G$ of type II again, a contradiction. \[\square\]
Claim 2. There exist distinct vertices $v_1, v_2, v_3 \in V(G) - N[x] \cup \{u\}$ such that $v_i$ is a common neighbor of $x_i$ and $y_{\sigma(i)}$ for $i = 1, 2, 3$, for some permutation $\sigma \in S_3$.

Proof of Claim 2. Note that by Lemma 4.3.3, every vertex in $T_1$ and every vertex in $T_2$ have at least two common neighbors in $G - N[x]$.

By Claim 1, there do not exist two vertices $G - N[x]$ that are both adjacent to all six vertices in $T_1 \cup T_2$. This means that $|\left((V(G) - N[x]) \cap \left(\bigcup_{i=1,2,3} N(x_i) \cap N(y_{\sigma(i)})\right)\right)| \geq 3$ for every permutation $\sigma \in S_3$. Therefore, there are three distinct vertices $v_1, v_2, v_3 \in V(G) - N[x]$ such that $v_i$ is a common neighbor for $x_i$ and $y_i$ for $i = 1, 2, 3$. Similarly, there are distinct vertices $u_1, u_2, u_3 \in V(G) - N[x]$ such that $u_1 \in N(x_1) \cap N(y_2)$, $u_2 \in N(x_2) \cap N(y_3)$, and $u_3 \in N(x_3) \cap N(y_1)$. Hence, we may assume that $u \in \{v_1, v_2, v_3\}$ and $u \in \{u_1, u_2, u_3\}$.

Without loss of generality, assume that $u = v_1$. Note that if there exists some $v_4 \in V(G) - \{v_1, v_2, v_3\}$ such that $v_4$ is adjacent to both $x_1$ and $y_1$, then $v_2, v_3, v_4$ are as desired. Therefore, we may assume that all common neighbors of $x_1, y_1$ in $G - N[x]$ are included in $\{v_1, v_2, v_3\}$. Since $x_1, y_1$ have at least two common neighbors in $G - N[x]$, without loss of generality, we assume that $v_2$ is adjacent to both $x_1$ and $y_1$. By the same argument as above, it follows that all common neighbors of $x_2, y_2$ are contained in $\{v_1, v_2, v_3\}$ and that one of $v_1$ and $v_3$ is a common neighbor for $x_2, y_2$.

Assume for a moment that $v_3$ is a common neighbor for $x_2$ and $y_2$. Then, by the previous argument, one of $v_1$ and $v_2$ is a common neighbor for $x_3$ and $y_3$. Observe that $v_2$ is not adjacent to $x_3$ or $y_3$, since otherwise this would be a contradiction to Claim 1 due to the fact that $v_3$ has at least two neighbors in $T_i$ for both $i = 1, 2$. It follows that $v_1$ is adjacent to both $x_3$ and $y_3$. Since $G - N[x]$ is 2-connected, there exists a path $P$ linking $v_2$ and $v_3$ in $G - N[x]$ that does not include $v_1$. Note that $v_1$ is adjacent to both $x_1, x_3$ in $T_1$, $v_2$ is adjacent to $y_1, y_2 \in T_2$, and that $v_3$ is adjacent to $y_2, y_3 \in T_2$. By contracting the edges $v_1x_1, v_2y_1, v_3y_3$ and contracting the path $P$ to a single edge, we would then obtain a minor of type II of $G$, a contradiction.
Hence, $v_1$ is a common neighbor for $x_2$ and $y_2$. Notice now both $v_1$ and $v_2$ are adjacent to all four vertices $x_1, y_1, x_2, y_2$, meaning that they each have at least two neighbors in $T_i$ for $i = 1, 2$. By Claim 1, we can then make the following observation.

**Observation.** The following statements are true.

1. $v_3$ has at most four neighbors in $T_1 \cup T_2$.
2. $v_i$ is not adjacent to $x_3$ or $y_3$ for $i = 1, 2$.

Recall that $u_1, u_2, u_3 \in V(G) - N[x]$ are distinct vertices such that $u_1 \in N(x_1) \cap N(y_2)$, $u_2 \in N(x_2) \cap N(y_3)$, and $u_3 \in N(x_3) \cap N(y_1)$. Also recall that $u \in \{u_1, u_2, u_3\}$. By (2) of Observation, $u = v_1$ is not adjacent to $x_3$ or $y_3$. This means that $u \neq u_2$ or $u_3$, and therefore $u_1 = u = v_1$. If there exists some $u_4 \in V(G) - \{u_1, u_2, u_3\}$ such that $u_4$ is a common neighbor for $x_1$ and $y_2$, then we would have $u_2, u_3, u_4$ as desired. Since $x_1, y_2$ have at least two common neighbors in $G - N[x]$, we may assume that one of their common neighbors in $G - N[x]$ is $u_2$ or $u_3$.

If $u_2$ is a common neighbor for $x_1, y_2$, then $u_2$ is adjacent to both $x_2$ and $y_2$. Since all common neighbors of $x_2, y_2$ in $G - N[x]$ are contained $\{v_1, v_2, v_3\}$, we know that $u_2 \in \{v_1, v_2, v_3\}$. Note that $u_2 \neq v_1$ since $v_1 = u = u_1$ and $u_1 \neq u_2$. Since $v_2$ is not adjacent to $y_3$ by (2) of Observation but $u_2, y_3$ are adjacent, it follows that $u_2 \neq v_2$ and therefore $u_2 = v_3$. This means that $u_2 = v_3$ is adjacent to all of $x_1, x_2, y_2, x_3, y_3$, a contradiction to (1) of Observation due to the fact that $u_1$ is adjacent to all four of $x_1, y_1, y_2$. If $u_3$ is a common neighbor for $x_1, y_2$, then $u_3$ is adjacent to both $x_1$ and $y_1$. Since all common neighbors of $x_1, y_1$ in $G - N[x]$ are contained $\{v_1, v_2, v_3\}$ and $v_1 = u = u_1 \neq u_3$, it follows that $u_3 = v_2$ or $v_3$. Then, we can again find contradictions to (2) and (1) of Observation in cases $u_3 = v_2$ and $u_3 = v_1$, respectively.

By Claim 2, without loss of generality (by relabeling the vertices in $T_1, T_2$), we can assume that $v_i$ is a common neighbor for $x_i$ and $y_i$ for $i = 1, 2, 3$. Let $w_1, w_2 \in N(x)$ be the two non-adjacent vertices that have $u$ as a common neighbor. Without loss of generality, assume that $w_1, w_2 \notin C_1$. 

122
Note that this means \( G[(V(G) - N[x] \cup \{u\}) \cup T_1] \) does not have a \( K_3 \) minor rooted at \( T_1 \), since otherwise we would obtain a minor of \( G \) of type II due to the common neighbor \( u \) for \( w_1, w_2 \not\in T_1 \). Since \( G - N[x] \) is 2-connected, we know \( G - N[x] \cup \{u\} \) is connected. By Lemma 2.2.1, since \( T_1 \) is an independent set, it follows that \( G[(V(G) - N[x] \cup \{u\}) \cup T_1] \) has a cut vertex \( w \in V(G) - N[x] \cup \{u\} \), and that there are distinct components \( J_1, J_2, J_3 \) of \( G[(V(G) - N[x] \cup \{u\}) \cup T_1] \) such that \( x_i \in V(J_i) \) for \( i = 1, 2, 3 \). Without loss of generality, assume that \( w \neq v_1 \) or \( v_2 \), and it is possible that \( w = v_3 \). Since \( v_i \) is adjacent to \( x_i \) for \( i = 1, 2, 3 \), it follows that \( v_1 \in V(J_1), v_2 \in V(J_2), \) and \( v_3 \in V(J_3) \cup \{w\} \).

For \( i = 1, 2 \), let \( L_i \) be the component of \( J_i - \{x_i\} \) such that \( v_i \in L_i \). If \( w = v_3 \), then let \( L_3 = \{v_3\} = \{w\} \); otherwise, let \( L_3 \) be the component of \( J_3 - \{x_3\} \) such that \( v_3 \in L_3 \). Note that since \( G - N[x] \) is 2-connected, \( \{u, w\} \subseteq N(L_i) \) for \( i = 1, 2 \), and that \( \{u, w\} \subseteq N(L_3) \) if \( w \neq v_3 \).

**Claim 3.** The following statements are true.

1. \( x_2, x_3 \notin N(L_1), x_1, x_3 \notin N(L_2), \) and if \( w \neq v_3 \) then \( x_1, x_2 \notin V(L_3) \).
2. \( V(G) - N[x] \cup \{u\} = V(L_1 \cup L_2 \cup L_3) \cup \{w\} \).

**Proof of Claim 3.** Since \( w \in V(G) - N[x] \cup \{u\} \) is a cut vertex of \( G[(V(G) - N[x] \cup \{u\}) \cup T_1] \) and \( J_1, J_2, J_3 \) are distinct components of \( G[(V(G) - N[x] \cup \{u\}) \cup T_1] \) such that \( x_i \in V(J_i) \) for \( i = 1, 2, 3 \), we know that \( N(J_i - \{x_i\}) \cap T_1 = \{x_i\} \) for \( i = 1, 2 \), and that if \( w \neq v_3 \) then \( N(J_3 - \{x_3\}) \cap T_1 = \{x_3\} \). By the definition of \( L_i \) for \( i = 1, 2, 3 \), it follows that (1) is true.

To prove (2), for the sake of a contradiction, assume that there is some component \( L' \) of \( G - N[x] \cup \{u, w\} \) such that \( V(L') \cap \{v_1, v_2, v_3\} = \emptyset \). Note that \( u, w \) are the only neighbors of \( L' \) in \( G - N[x] \). Since \( G \) is 7-connected and \( N(x) \) does not have a clique of size 5, it follows that \( L' \) has at least 5 neighbors in \( N(x) \) and therefore it has two non-adjacent neighbors \( r_1, r_2 \in N(x) \). Note that we can fix \( j \in \{1, 2\} \) such that \( r_1, r_2 \not\in C_j \). Then, observe that we can contract edges in \( G[V(L_1 \cup L_2 \cup L_3) \cup \{u, w\}] \) property to become a cycle that goes through \( v_1, v_2, v_3, \) and this means that \( G[[V(L_1 \cup L_2 \cup L_3) \cup \{u, w\}] \cup T_j] \)

123
has a $K_3$ minor rooted at $T_j$. Since $L'$ has non-adjacent neighbors $r_1, r_2 \in N(x) - T_j$, it follows that $G$ has a minor of type II, a contradiction. 

In the rest of the proof, we consider the case $w = v_3$ and the case $w \neq v_3$ separately.

**Case 1:** $w = v_3$.

By Claim 3, there is no neighbor of $x_3$ in $L_1$ or $L_2$ and therefore $N(x_3) \cap (V(G) - N[x]) \subseteq \{u, w\}$. Note that $x_3$ and $y_i$ for $i = 1, 2, 3$ have at least two common neighbors in $G - N - [x]$. It follows that $u$ and $w$ are precisely the common neighbors for $x_3$ and $y_i$ in $G - N[x]$ for $i = 1, 2, 3$. Hence, both $u$ and $w$ are adjacent to $x_3$ and all three vertices in $T_2$.

Assume for a moment that $y_3 \in N(L_1)$. Then, by contracting all of $L_1$ to $y_1$ and contracting the edge $y_2u$, we would then obtain a clique on $T_2$. By contracting all of $L_2 \cup \{w\}$ to $x_2$, we can obtain the edge $x_2x_3$. It follows that $G$ has a minor of type II, a contradiction. This means that $y_3 \notin N(L_1)$, and furthermore by symmetry we know $y_3 \notin N(L_2)$ either.

By (2) of Claim 3, we have $N(y_3) \cap (V(G) - N[x]) \subseteq \{u, w\}$. For $i = 1, 2, 3$, since there are at least two common neighbors for $x_i$ and $y_3$ in $G - N[x]$, it follows that they are precisely $u$ and $w$. This means that $u$ and $w$ each are adjacent to all six vertices in $T_1 \cup T_2$, a contradiction to Claim 1.

**Case 2:** $w \neq v_3$.

By (1) of Claim 3, we know that $x_2, x_3 \notin N(L_1), x_2, x_3 \notin N(L_2)$, and $x_1, x_2 \notin V(L_3)$. Assume for a moment that $y_2, y_3 \notin N(L_1), y_1, y_3 \notin N(L_2)$, and $y_1, y_2 \notin N(L_3)$. This would then imply that $u$ and $w$ are precisely the common neighbors for $x_i$ and $y_j$ in $G - N[x]$ for $i, j \in \{1, 2, 3\}$ such that $i \neq j$. This means that $u$ and $w$ each are adjacent to all six vertices in $T_1 \cup T_2$, a contradiction to Claim 1. It follows that $y_i \in N(L_j)$ for some $i, j \in \{1, 2, 3\}$ such that $i \neq j$. Without loss of generality, we assume that $y_2 \in N(L_1)$.

Recall that $w_1, w_2 \in N(x)$ are two non-adjacent neighbors of $u$, and we previously assumed that they are not in $C_1$ without loss of generality. Now observe that if $w_1, w_2$
are contained in an island of $N(x)$ outside $C_1 \cup C_2$, then by contracting all of $L_1$ to $y_1$, contracting all of $L_2 \cup L_3 \cup \{w\}$ to $x_2$, and contracting the edge $uw_2$, we would then obtain a minor of $G$ of type I, a contradiction. It follows that $w_1, w_2 \in C_2$. In the rest of the proof, we will consider the cases $|\{w_1, w_2\} \cap \{y_1, y_2\}| = 0, 1, or 2, separately.

Assume $|\{w_1, w_2\} \cap \{y_1, y_2\}| = 0$, and note that this is only possible when $N(x) \cong K_{4,4,4}$ and $\{w_1, w_2\} = C_2 - \{y_1, y_2\}$. Without loss of generality, say $w_1 = y_3$ and $y_4$ is the unique vertex in $C_2 - T_2$. By contracting all of $L_i$ to $y_i$ for $i = 1, 2$ and contracting all of $L_3 \cup \{w\}$ to $y_3$, we first obtain a clique on $T_1 = \{y_1, y_2, y_3\}$. Then, by contracting the edge $uy_4$, we see that $y_4$ would then be adjacent to all three vertices $y_1, y_2, y_3$ due to the fact that $u \in N(L_i)$ for $i = 1, 2, 3$. This means that the resulting graph now on $N[x]$ is isomorphic to $K_5 + K_{4,4}$, which has a $K_{10}$ minor. It follows that $G > K_{10}$, a contradiction.

Assume $|\{w_1, w_2\} \cap \{y_1, y_2\}| = 1$. Then, without loss of generality, assume that $y_1 = w_1$ and $y_2 \neq w_2$. Notice that $y_1 = w_1, y_2$, and $w_2$ are now three distinct vertices in $C_2$. By contracting contracting the edge $uw_2$, contracting all of $L_1$ to $y_2$, and contracting all of $L_2 \cup L_3 \cup \{w\}$, we would then obtain a clique on $\{y_1, y_2, w_2\}$ in $C_2$ and the edge $x_2x_3$ in $C_1$. This means that $G$ has a minor of type II, a contradiction.

Finally, assume $|\{w_1, w_2\} \cap \{y_1, y_2\}| = 2$. Then, without loss of generality, assume that $w_1 = y_1$ and $w_2 = y_2$. Note that if $y_3 \in N(L_1)$, then by contracting all of $L_1$ to $y_3$, contracting the edge $uy_3$, and contracting all of $L_2 \cup L_3 \cup \{w\}$ to $x_2$, we would then obtain a clique on $T_2$ and the edge $x_2x_3$ in $T_1$. This means that $G$ has a minor of type II, a contradiction. Therefore, $y_3 \notin N(L_1)$. Since $y_3 \notin N(L_1 \cup L_2)$ by Claim 3, $N(y_3) \cap (V(G) - N[x]) \subseteq \{u, w\}$. Since $x_1$ and $y_3$ have at least two common neighbors in $G - N[x]$, it follows that their common neighbors in $G - N[x]$ are exactly $u$ and $w$, and this means that $u$ is in fact adjacent to all three vertices in $T_2$. Then, by contracting all of $L_1$ to $y_1$, contracting the edge $uy_3$, and contracting all of $L_2 \cup L_3 \cup \{w\}$ to $x_2$, we can obtain a clique on $T_2$ and the edge $x_2x_3$ in $T_1$. This means that $G$ has a minor of type II, a contradiction. 

\[\square\]
4.3.4 Proof of Lemma 4.3.7

Lemma 4.3.7. If $G - N[x]$ is not 2-connected, then $G - x$ has a minor $J$ rooted at $N(x)$ such that $J > K_9$.

Proof. Recall that by Lemma 4.2.1, $M \subseteq N(K)$ where $K$ is a component of $G - N[x]$ and $M = \{v \in N(x) : vu \not\in E(G) \text{ for some } u \in N(x) - \{v\}\}$. Since $N(x) \cong K_{2,3,3,3}$, $K_{3,3} + C_5$, or $K_{4,4,4}$ by Lemma 4.3.1, we see that $N(x) = M$ and therefore $N(K) = N(x)$.

If $G - N[x]$ is disconnected, let $L_1, L_2$ be two distinct components of $G - N[x]$ such that $N(L_1) = N(x)$ and, subject to that, $|L_2|$ is maximum. If $G - N[x]$ has a cut vertex $w$, let $L_1, L_2$ be two distinct components of $G - N[x] \cup \{w\}$ such that $|L_1| + |L_2|$ are maximum among all choices. In both cases, let $A_i = V(L_i) \cap N(x)$ and $H_i = G[A_i \cup V(L_i)]$ for $i = 1, 2$. We first make some simple observations in Claim 1 and Claim 2.

Claim 1. The following statements are true:

1. $L_1, L_2$ are disjoint connected induced subgraphs of $G$, and $V(H_1 \cap H_2) = A_1 \cap A_2 \subseteq N(x)$.
2. $|A_i| \geq 6$ for $i = 1, 2$.
3. $A_i \subseteq N_G(L_i)$ and $|N_G(L_i) - A_i| \leq 1$ for $i = 1, 2$.
4. $|L_i| \geq 2$ for some $i \in \{1, 2\}$.

Proof of Claim 1. (1) is simply true by the construction of $L_i, A_i$, and $H_i$ for $i = 1, 2$. Observe that in all cases, $L_i$ has at most one neighbor in $G - N[x]$ for $i = 1, 2$, and it follows that (2) and (3) are true since $G$ is 7-connected.

To prove (4), for the sake of a contradiction, assume that $|L_i| = 1$ for $i = 1, 2$. By the choice of $L_1$ and $L_2$, it follows that either $G - N[x]$ is a star or $V(G) - N[x]$ is just an independent set in $G$. Recall that $|G - N[x]| \geq 3$ by Lemma 4.3.2. Therefore, in both cases, there exist three distinct vertices $v_1, v_2, v_3 \in V(G) - N[x]$ such that $v_1, v_2$ each have at most one neighbor in $G - N[x]$, and $v_3$ has at most two neighbors in $G - N[x]$. Since $\delta(G) \geq 11$, $v_1, v_2$ each have at least 10 neighbors in $N(x)$ and $v_3$ has at least 9
neighbors in \( N(x) \). Since \( N(x) \cong K_{2,3,3,3}, K_{3,3} + C_5 \), or \( K_{4,4,4} \), we observe that \( v_1, v_2 \) each have non-adjacent neighbors in three distinct islands of \( N(x) \), and that \( v_3 \) has non-adjacent neighbors in at least two distinct islands of \( N(x) \). Hence, there exist distinct vertices \( s_1, t_1, s_2, t_2, s_3, t_3 \in N(x) \) such that \( s_i, t_i \) for \( i = 1, 2, 3 \) are non-adjacent, contained in a distinct island of \( N(x) \), and both adjacent \( v_i \). By contracting edges \( v_is_i \) for \( i = 1, 2, 3 \), we would then obtain a minor of \( G \) of type I, a contradiction. \( \dashv \)

Now, let \( G_1 = L_1 \). Let \( G_2 = L_2 \) if \( G - N[x] \) is disconnected, and let \( G_2 = G - N[x] \cup V(L_1) \) if \( G - N[x] \) has a cut vertex.

Claim 2. The following statements are true:

(1) \( G_1, G_2 \) are disjoint connected induced subgraphs of \( G - N[x] \).

(2) \( A_1 = N(G_1) \cap N(x), A_2 \subseteq N(G_2) \cap N(x) \).

(3) \( N(x) = (N(G_1) \cap N(x)) \cup (N(G_2) \cap N(x)) = A_1 \cup (N(G_2) \cap N(x)) \).

Proof of Claim 2. (1) and (2) are simply due to the construction of \( L_1, L_2 \) and \( G_1, G_2 \). If \( G - N[x] \) is disconnected, recall that we chose \( L_1 \) such that \( N(L_1) = N(x) \), and therefore (3) is true. If \( G - N[x] \) has a cut vertex, then \( G_1 \cup G_2 = G - N[x] \). Since \( V(G - N[x]) = N(x) \), it follows that (3) is true. \( \dashv \)

Claim 3. If \( G[A_i] \) for some \( i \in \{1, 2\} \) does not have an independent set of size 3, then the following statements are true.

(1) For \( S \subseteq A_i \) such that \( |S| \geq 6 \), there exists a subset \( Z = \{z_1, z_2, z_3, z_4\} \subseteq S \) of size 4 such that \( z_1z_2, z_3z_4 \notin E(G) \).

(2) \( |A_i| \leq 9 \), where the equality holds only if \( N(x) \cong K_{3,3} + C_5 \) and \( G[A_i] \cong K_{2,2} + C_5 \).

Proof of Claim 3. Let \( S \subseteq A_i \) be such that \( |S| \geq 6 \) be arbitrary. Since there is no dependent set of size 3 in \( G[A_i] \), \( A_i \) includes at most two vertices in each island of \( N(x) \) that is an independent set, and so does \( S \).

Since \( |S| \geq 6 \), we see that if \( N(x) \cong K_{2,3,3,3} \), then \( G[A_i] \) and \( G[S] \) each are isomorphic to one of \( K_{2,2,2}, K_{1,2,2,2}, \) or \( K_{2,2,2,2} \); and if \( N(x) \cong K_{4,4,4} \), then \( G[A_i] = G[S] \cong K_{2,2,2} \).
This shows that $N(x) \cong K_{2,3,3,3}$ or $K_{3,3} + C_5$, then the desired subset $Z$ exists and $|A_i| \leq 8$.

If $N(x) \cong K_{3,3} + C_5$, let $C_1, C_2$ be the two 3-islands and let $C_3$ be the 5-island. Since $A_i$ does not contain an independent set of size 3, $|A_i \cap C_j| \leq 2$ for $j = 1, 2$. Furthermore, we may assume $|A_i \cap C_j| \leq 1$ for some $j$ in $\{1, 2\}$, since otherwise we can find a subset $Z \subseteq A_i \cap (C_1 \cup C_2)$ of size 4 that is as desired. Without loss of generality, say $|A_i \cap C_1| \leq 2$ and $|A_i \cap C_2| \leq 1$. Since $|A_i| \geq 6$, it follows that $|A_i \cap C_3| \geq 3$ and thus there are non-adjacent vertices $s_3, t_3 \in A_i \cap C_3$. Now, if $|A_i \cap C_1| = 2$, then $Z = (A_i \cap C_1) \cup \{s_3, t_3\}$ is as desired. If $|A_i \cap C_1| \leq 1$, then $|A_i \cap C_3| \geq 4$. It follows that any subset $Z \subseteq A_i \cap C_3$ such that $|Z| = 4$ satisfies that $G[Z]$ is a path of length 3 and therefore is as desired. Finally, if $|A_i| \geq 9$, one can simply observe that this is only possible if $|A_i| = 9$ and $G[A_i] \cong K_{2,2} + C_5$.

Claim 4. The following statements are true about $A_i$ for both $i = 1, 2$.

1. If $|A_i| \leq \delta(G) - 3$, there exists a subset $S \subseteq A_i$ such that $|S| \geq 6$ and for every $Z \subseteq S$ such that $|Z| = 4$, $G[V(L_i) \cup Z]$ has a $K_4$ minor rooted at $Z$.

2. If $G[A_i]$ does not contain an independent set of size 3, then there exists $Z = \{z_1, z_2, z_3, z_4\} \subseteq A_i$ such that $z_1z_2, z_3z_4 \notin E(G)$ and $G[V(L_i) \cup Z]$ has a $K_4$ minor rooted at $Z$.

Proof of Claim 4. Recall that we defined $H_i = G[A_i \cup V(L_i)]$ for $i = 1, 2$. Let $(A, B)$ be a separation of $(H_i, A_i)$ such that (i) $B - A \neq \emptyset$, (ii) $|A \cap B|$ is minimum subject to (i), and (iii) $|B|$ is minimum subject to (i) and (ii). Since $(A', B') = (A_i, V(L_i))$ is a separation of $(H_i, A_i)$ where $B' - A' \neq \emptyset$, it follows that $|A \cap B| \leq |A_i|$. Note that $B - A \subseteq V(L_i)$. By Claim 1, we know $|N_G(B - A)| \leq |A \cap B| + 1$. Since $G$ is 7-connected, $|N_G(B - A)| \geq 7$ and therefore $|A \cap B| \geq 6$.

By the minimality of $|A \cap B|$, there exist disjoint paths $P_1, \ldots, P_{|A \cap B|}$ linking $A_i$ and $A \cap B$ in $G[A]$. Let $S \subseteq A_i$ be the collection of end vertices of the disjoint paths $P_1, \ldots, P_{|A \cap B|}$ in $A_i$. It follows that $|S| = |A \cap B| \geq 6$. By the minimality of $|A \cap B|$ and $|B|$ when choosing $(A, B)$, there is no non-trivial separation of $(G[B], A \cap B)$ of order at most $|A \cap B|$.
Since $A \cap B \subseteq N_G(B - A)$, there is no non-trivial separation of $(G[(B - A) \cup N_G(B - A)]]$, $N_G(B - A))$ of order at most $|A \cap B|$. Since $|N_G(B - A)| \leq |A \cap B| + 1$, it follows that there is no non-trivial separation of $(G[(B - A) \cup N_G(B - A)], N_G(B - A))$ of order at most $|N_G(B - A)| - 1$.

To prove (1), assume that $\delta(G) - 3$, and we will show that $S$ is as desired. Let $Z \subseteq S$ be any subset such that $|Z| = 4$. Without loss of generality, say $P_1, P_2, P_3, P_4$ are the disjoint paths whose end vertices in $S$ are $Z$. Let $Z_1' \subseteq A \cap B$ be the collection of end vertices of $P_1, P_2, P_3, P_4$ in $A \cap B$. Note that by Claim 1, $|A_i| \leq \delta(G) - 3$ implies that $|N_G(B - A)| \leq \delta(G) - 2$. Since there is no non-trivial separation of $(G[(B - A) \cup N_G(B - A)], N_G(B - A))$ of order at most $|N_G(B - A)| - 1$, by Lemma 3.2.3 it follows that $G[(B - A) \cup Z_1]$ has a $K_4$ minor rooted at $Z_1$. Due to the disjoint paths $P_1, P_2, P_3, P_4$ linking $Z$ and $Z_1$ in $G[A]$, $G[V(L_i) \cup Z]$ has a $K_4$ minor rooted at $Z$. This completes the proof of (1).

To prove (2), assume that $G[A_i]$ does not contain an independent set of size 3. By Claim 3, $|A_i| \leq 9$ and there exists $Z = \{z_1, z_2, z_3, z_4\} \subseteq S$ such that $z_1z_2, z_3z_4 \notin E(G)$. Again without loss of generality, assume that for $j = 1, 2, 3, 4$, $P_i$ links $z_j \in Z \subseteq A_i$ and $z_j' \in A \cap B$. Let $Z_1 = \{z_1', z_2', z_3', z_4'\} \subseteq A \cap B$. Observe that if $|N_G(B - A)| \leq \delta(G) - 2$, then by the same argument above we can show that $G[(B - A) \cup Z_1]$ has a $K_4$ minor rooted at $Z_1$, and therefore $G[V(L_i) \cup Z]$ has a $K_4$ minor rooted at $Z$ due to the disjoint paths $P_1, P_2, P_3, P_4$. Thus, we may assume that $|N_G(B - A)| \geq \delta(G) - 1 \geq 10$ as $\delta(G) \geq 11$. Note that by Claim 1, $|N_G(B - A)| \leq |A_i| + 1 \leq 10$. If follows that $|N_G(B - A)| = 10$ and $|A_i| = 9$. By Claim 3, this is only possible if $N(x) \cong K_{3,3} + C_5$ and $G[A_i] \cong K_{2,2} + C_5$.

Now, let $Y_1' \subseteq A_i$ be the union of the two 2-islands of $G[A_i]$, and let $Y_2'$ be four vertices in the 5-islands of $G[A_i]$. Without loss of generality, assume that $Z = Y_1'$ and the disjoint paths $P_1, P_2, P_3, P_4$ each have an end vertex in $Y_1'$. Let $Y_1 \subseteq A \cap B$ be the collection of end vertices of these paths in $A \cap B$. Also assume that $P_5, P_6, P_7, P_8$ each have an end vertex in $Y_2'$, and let $Y_2 \subseteq A \cap B$ be the collection of end vertices of these paths in $A \cap B$. Note that $G[Y_1']$ and $G[Y_2']$ each contain two disjoint pairs of non-adjacent vertices, so it suffices
to prove that one of $Y'_1$ and $Y'_2$ is as desired.

By Lemma 3.2.3, one of (2a) and (2b) is true with respect to $G[(B - A) \cup N_G(B - A)]$ and $Y_1$. First assume that (2a) is true with respect to $G[(B - A) \cup N_G(B - A)]$ and $Y_1$. Note that $G[Y'_1] \cong K_{2,2}$, so there exist $y_1, y_2 \in Y'_1$ such that $y_1y_2 \in E(G)$. Without loss of generality, say $P_j$ for $j = 1, 2$ links $y_j \in Y'_1$ and $y'_j \in Y_j$. By (2a), $G[(B - A) \cup Y_1]$ has an $H$-minor rooted at $Y'_1$ such that $H \cup \{y'_1y'_2\} \cong K_4$. Due to the edge $y_1y_2 \in E(G)$ and the disjoint paths $P_1, P_2, P_3, P_4$ linking $Y'_1$ and $Y_1$, it follows that $G[V(L_i) \cup Y'_1]$ has a $K_4$ minor rooted at $Y'_1$. This completes the proof of (2) as $G[Y'_1] \cong K_{2,2}$ contains two disjoint pairs of non-adjacent vertices. We may then assume (2b) is true with respect to $G[(B - A) \cup N_G(B - A)]$ and $Y_1$. Since $Y_1 \cap Y_2 = \emptyset$, by (2b) it follows that $G[(B - A) \cup Y_2]$ has a $K_4$ minor rooted at $Y_2$, implying that $G[V(L_i) \cup Y'_2]$ has a $K_4$ minor rooted at $Y'_2$ due to the disjoint paths $P_5, P_6, P_7, P_8$. This again completes the proof of (2) as $G[Y'_2] \cong K_{2,2}$ contains two disjoint pairs of non-adjacent vertices.

\[ \therefore \]

Claim 5. Let $C_1, C_2$ be two islands of $N(x)$ that are not necessarily distinct. Suppose that for some $i \in \{1, 2\}$, there is a subset $Z = \{s_1, t_1, s_2, t_2\} \subseteq A_i$ of size 4 such that $s_j, t_j \in C_j$ for $j = 1, 2$ and $G[V(L_i) \cup Z]$ has a $K_4$ minor rooted at $Z$. Then, the following statements are true.

1. $A_{3-i} - C_1 \cup C_2$ is a clique.

2. If $C_1 \neq C_2$ and $C_j$ is an independent set for some $j \in \{1, 2\}$, then $|A_{3-i} \cap C_j| \leq \max\{2, |C_j| - 1\}$.

3. If $C_1 \neq C_2$ and $C_1, C_2$ are both independent sets, then there exists a subset $Z' = \{s'_1, t'_1, s'_2, t'_2\} \subseteq A_{3-i}$ of size 4 such that $s'_j, t'_j \in C_j$ for $j = 1, 2$ and $G[(V(L_{3-i}) \cup Z')]$ has a $K_4$ minor rooted at $Z'$.

Proof of Claim 5. To see (1) is true, assume for the sake of a contradiction that there exist $s_3, t_3 \in A_{3-i} - C_1 \cup C_2$ such that $s_3t_3 \notin E(G)$. Note that if $C_1 \neq C_2$, then by contracting edges in $G[V(L_i) \cup Z]$ properly to obtain a clique on $Z = \{s_1, t_1, s_2, t_2\}$ and contracting all of $L_{3-i}$ to $s_3$, we would then obtain a resulting graph on $N[x]$ that contains edges $s_it_i$. 

130
for $i = 1, 2, 3$. This means that $G$ has a minor of type I, a contradiction. It follows that $C_1 = C_2$, and this means that either $N(x) \cong K_{2,3,3} + C_5$ and $C_1 = C_2$ is the 5-island in it, or that $N(x) \cong K_{4,4,4}$ and $C_1 = C_2$ is a 4-island in it. In the former case, by contracting edges in $G[V(L_i) \cup Z]$ properly to obtain a clique on $Z = \{s_1, t_1, s_2, t_2\}$ and contracting all of $L_{3-i}$ to $s_3$, we would then obtain a minor of $G$ of type III, a contradiction. In the latter case, we see that the graph obtained from $N(x) \cong K_{4,4,4}$ by making one of its islands a clique is isomorphic to $K_4 + K_{4,4}$, which has a $K_9$ minor. It follows that by contracting edges in $G[V(L_i) \cup Z]$ properly to obtain a clique on $Z$, the resulting graph on $N[x]$ would have a $K_{10}$ minor, a contradiction.

To see (2) is true, without loss of generality, assume that $C_1$ is an independent set. Note that if $|A_{3-i} \cap C_1| > \max\{2, |C_1| - 1\}$, then $|C_1| \geq |A_{3-i} \cap C_1| \geq 3$. Let $r_1 \in A_{3-i} \cap C_1$ such that $r_1 \neq s_1$ or $t_1$. Then, by contracting edges in $G[V(L_i) \cup Z]$ properly to obtain a clique on $Z = \{s_1, t_1, s_2, t_2\}$ and contracting all of $L_{3-i}$ to $r_1$, we would then obtain a clique on $\{s_1, t_1, r_1\}$ in $C_1$ and the edge $s_2t_2$, meaning that $G$ has a minor of type II, a contradiction.

It remains to prove (3). If $N(x) \cong K_{2,3,3,3}$, let $C_3, C_4$ be the two islands of $N(x) - C_1 \cup C_2$. We would then have $|A_{3-i} \cap C_j| \leq 1$ for $j = 3, 4$ by (1) and $|A_{3-i} \cap C_j| \leq 2$ for $j = 1, 2$ by (1). Since $|A_{3-i}| \geq 6$, it follows that $|A_{3-i} \cap C_j| = 2$ for $j = 1, 2, |A_{3-i} \cap C_j| = 1$ for $j = 3, 4$, and $|A_{3-i}| = 6$. By (1) of Claim 4, $Z' = A_{3-i} \cap (C_1 \cup C_2)$ is as desired. If $N(x) \cong K_{2,3,3,3}$, then $C_1, C_2$ are precisely the two 3-islands of $N(x)$. Let $C_3$ be the 5-island of $N(x)$. Note that the maximum independent set in a 5-cycle has size 2. By (1) and (2), it follows that $|A_{3-i} \cap C_j| \leq 2$ for $j = 1, 2, 3$. Since $|A_{3-i}| \geq 6$, it follows that $|A_{3-i}| = 6$ and $|A_{3-i} \cap C_j| = 2$ for $j = 1, 2, 3$. By (1) of Claim 4, $Z' = A_{3-i} \cap (C_1 \cup C_2)$ is as desired. Finally, if $N(x) \cong K_{4,4,4}$, let $C_3$ be the 4-island of $N(x) - C_1 \cup C_2$. By (1) and (2), it follows that $|A_{3-i} \cap C_j| \leq 3$ for $j = 1, 2$ and $|A_{3-i} \cap C_3| \leq 1$. Note that this shows that $|A_{3-i}| \leq 7$ and $|S \cap C_j| \geq 2$ for every $S \subseteq A_{3-i}$. By Claim 1, $|A_{3-i}| \leq 7$ implies that $|N_G(L_{3-i})| \leq |A_{3-i}| + 1 \leq 8 \leq \delta(G) - 3$ as $\delta(G) \geq 11$. Therefore, by (1) of Claim 4,
there exists some $Z' = \{s'_1, t'_1, s'_2, t'_2\} \subseteq A_{3-i}$ such that $s'_j, t'_j \in C_j$ for $j = 1, 2$.

**Claim 6.** For $i = 1, 2$, for every subset $Z \subseteq A_i$ of size 4 that is the union of two disjoint pairs of non-adjacent vertices, $G[V(L_i) \cup Z]$ does not have a $K_4$ minor rooted at $Z$.

**Proof of Claim 6.** For the sake of a contradiction, assume that for some $i \in \{1, 2\}$, $G[V(L_i) \cup Z]$ has a $K_4$ minor rooted at $Z$ where $Z \subseteq A_i$ such that $Z$ has size 4 and is the union of two disjoint pairs of non-adjacent vertices. Let $Z = \{s_1, t_1, s_2, t_2\}$ where $s_jt_j \notin E(G)$ for $j = 1, 2$. Let $C_j$ be the island of $N(x)$ containing $s_j$ and $t_j$ for $j = 1, 2$. Note it is possible that $C_1 = C_2$, and we will consider the case $C_1 = C_1$ and the $C_1 \neq C_2$ separately in the rest of the proof.

**Case 1: $C_1 = C_2$.**

Note that $C_1 = C_2$ means that either $N(x) \cong K_{3,3} + C_5$ and $C_1 = C_2$ is the the 5-island of $N(x)$, or that $N(x) \cong K_{4,4,4}$ and $C_1 = C_2 = Z$ is the 4-island of $N(x)$. Observe that in the latter case, we can just contract edges in $G[V(L_i) \cup Z]$ to obtain a $K_4$ minor rooted at $Z$, and the resulting graph on $N[x]$ would be isomorphic to $K_5 + K_{4,4}$ which has a $K_{10}$ minor, a contradiction.

We may then assume that $N(x) \cong K_{3,3} + C_5$ and $C_1 = C_2$ is the the 5-island of $N(x)$. By (1) Claim 5, $A_{3-i} - C_1 \cup C_2$ is a clique, meaning that $A_{3-i}$ has at most one vertex in each of the 3-island of $N(x)$. Since $|A_{3-i}| \geq 6$, we have $|A_{3-i} \cap C_1| \geq 4$. Furthermore, this means that for every $S \subseteq A_{3-i}$ such that $|S| \geq 6$, $S$ contains two disjoint pairs of non-adjacent vertices. By Claim 4, there exists a subset $Z' \subseteq A_{3-i} \cap C_1$ of size 4 such that $G[(V(L_{3-i}) \cup Z')]$ has a $K_4$ minor rooted at $Z'$. We can then apply the same argument back to $A_i$ to show that $A_i$ has at most one vertex in each 3-island of $N(x)$. It follows that $A_1$ and $A_2$ each have at most one vertex in each 3-island of $N(x)$. By Claim 2, $N(x) \subseteq A_1 \cup N(G_2)$, and it follows that $N(G_2)$ contains at least two vertices in each 3-island of $N(x)$. Therefore, by contracting edges that have at least one end in $L_1$ properly to obtain a clique of size 4 in the 5-island of $N(x)$ and contracting all of $G_2$ to a neighbor of it.
in a 3-island of \(N(x)\), we would eventually obtain a minor of \(G\) of type III, a contradiction.

**Case 2:** \(C_1 \neq C_2\).

We will consider the case that one of \(C_1, C_2\) is not an independent set and the case that both \(C_1, C_2\) are independent sets separately.

**Case 2.1:** One of \(C_1, C_2\) is not an independent set

Observe that one of \(C_1, C_2\) is not an independent set only if \(N(x) \cong K_{3,3} + C_5\) and one of \(C_1, C_2\) is the 5-island. Without loss of generality, assume that \(C_1\) is a 3-island and \(C_2\) is a 5-island. Let \(C_3\) be the 3-island in \(N(x) - C_1 \cup C_2\). By (1) and (2) of Claim 5, we know that \(|A_{3-i} \cap C_3| \leq 1\) and \(|A_{3-i} \cap C_1| \leq 2\). Since \(|A_{3-i}| \geq 6\), it follows that \(3 \leq |A_{3-i} \cap C_2| \leq 5\).

Furthermore, observe that for every \(S \subseteq A_{3-i}\) such that \(|S| \geq 6\), \(S\) has some subset of size 4 that is the union of two disjoint pairs of non-adjacent vertices. By (1) of Claim 4, there exists a subset \(Z' \subseteq A_{3-i}\) such that \(Z'\) is the union of two disjoint pairs of non-adjacent vertices and that \(G[(V(L_{3-i}) \cup Z')]\) has a \(K_4\) minor rooted at \(Z'\). Since \(|A_{3-i} \cap C_3| \leq 1\), \(|Z' \cap C_3| \leq 1\). By applying (1) in Claim 5 back to \(A_1\), we see that \(|A_{i} \cap C_3| \leq 3\). Now by Claim 2, \(N(x) \subseteq A_1 \cup N(G_2)\), and therefore \(N(G_2)\) contains at least two vertices in \(C_3\). Then, by contracting edges that have at least one end in \(L_1\) to obtain a \(K_4\) minor rooted at \(Z\) or \(Z'\) and contracting all of \(G_2\) to one of its vertex in \(C_3\), we would then obtain a minor of \(G\) of type I or type III, a contradiction.

**Case 2.2:** \(C_1, C_2\) are both independent sets.

By (3) of Claim 5, there exists a subset \(Z' = \{s'_1, t'_1, s'_2, t'_2\} \subseteq A_{3-i}\) of size 4 such that \(s'_j, t'_j \in C_j\) for \(j = 1, 2\) and \(G[(V(L_{3-i}) \cup Z')]\) has a \(K_4\) minor rooted at \(Z'\). Observe that regardless of which graph \(N(x)\) is isomorphic to, there exists an island \(C_3\) of \(N(x) - C_1 \cup C_2\) such that \(|C_3| \geq 3\). By (1) of Claim 5, it follows that for both \(j = 1, 2\), \(|A_j \cap C_3| \leq 1\) if \(C_3\) is an independent set, and that \(|A_j \cap C_3| \leq 2\) if \(G[C_3] \cong C_5\). By Claim 2, we know that \(C_3 \subseteq A_1 \cup N(G_2)\), and therefore \(|N(G_2) \cap C_3| \geq 2\) if \(C_3\) is an independent set, and \(|N(G_2) \cap C_3| \geq 3\) if \(G[C_3] \cong C_5\). In both cases, observe that \(G_2\) has two non-adjacent neighbors \(s_3, t_3 \in C_3\). By contracting edges that have at least one end in \(L_1\) properly to
obtain a $K_4$ minor rooted at one of $Z$ and $Z'$ and contracting all of $G_2$ to $s_3$, we would then obtain a type I, a contradiction.

By Claim 6 and (2) of Claim 4, we can now conclude that $A_i$ contains an independent set of size 3 for both $i = 1, 2$.

**Claim 7.** If $|L_i| \geq 2$ for some $i \in \{1, 2\}$ and $X \subseteq A_i$ such that $G[X] \cong K_3$, then $G[V(L_i) \cup X]$ has a $K_3$ minor rooted at $X$.

**Proof of Claim 7.** Let $x_1, x_2, x_3$ be the three vertices in $X$. For the sake of a contradiction, assume that $G[V(L_i) \cup X]$ does not have a $K_3$ minor rooted at $X$. By Lemma 2.2.1, there exists a cut vertex $u \in V(L_i)$ of $G[V(L_i) \cup X]$, and there are distinct components $J_1, J_2, J_3$ of $G[V(L_i) \cup X] - \{u\}$ such that $x_j \in V(J_j)$ for $j = 1, 2, 3$.

Since $|L_i| \geq 2$, there exists some component $R$ of $L_i - \{u\}$. Observe that due to the components $J_1, J_2, J_3$ of $G[V(L_i) \cup X] - \{u\}$ where $x_j \in V(J_j)$ for $j = 1, 2, 3$, we can assume that $R \cap J_2 = R \cap J_3 = \emptyset$ without loss of generality. Therefore, $|N(R) \cap X| \leq 1$. We also have $|N(R) \cap (V(G) - N[x])| \leq 2$ since $|N(L_i) \cap (V(G) - N[x])| \leq 1$. Since $G$ is 7-connected, it follows that $|N(R) \cap (N(x) - X)| \geq 4$. Let $C_X$ be the island of $N(x)$ that contains $X$, and note that $X = C_X$ if $N(x) \cong K_{2,3,3,3}$ or $K_{3,3} + C_5$, and that $C_X$ is a 4-island if $N(x) \cong K_{4,4,4}$. It follows that $|N(R) \cap (N(x) - C_X)| \geq 4$ if $N(x) \cong K_{2,3,3,3}$ or $K_{3,3} + C_5$, and $|N(R) \cap (N(x) - C_X)| \geq 3$ if $N(x) \cong K_{4,4,4}$. Observe that $N(x) - C_X \cong K_{2,3,3}$ or $K_3 + C_5$ if $N(x) \cong K_{2,3,3,3}$ or $K_{3,3} + C_5$, and $N(x) \cong K_{4,4}$ if $N(x) \cong K_{4,4,4}$. Hence, in all cases, $R$ has two non-adjacent neighbors $r_1, r_2 \in N(x) - C_X$. By contracting all of $G[V(J_2 \cup J_3) \cup \{w\} - \{x_3\}]$ to $x_2$ and contracting all of $R$ to $r_1$, we would then obtain a clique of size four on $\{x_2, x_3, r_1, r_2\}$ where $x_2x_3, r_1r_2 \not\in E(G)$, a contradiction to Claim 6.

**Claim 8.** Suppose $|L_i| \geq 2$ for some $i \in \{1, 2\}$. Then, $|A_{3-i}| = 6$. Furthermore, for any $X \subseteq A_i$ such that $G[X] \cong K_3$, let $C_X$ is the island of $N(x)$ that contains $X$, then $C_X \subseteq A_{3-i}$ and $A_{3-i} - C_X$ is a maximum clique in $N(x) - C_X$. 


**Proof of Claim 8.** Note that we showed that there exists some independent set of size 3 in $A_i$. Let $X \subseteq A_i$ that $G[X] \cong \overline{K_3}$ be arbitrary. We will prove that $|A_{3-i}| = 6$, $C_X \subseteq A_{3-i}$, and $A_{3-i} - C_X$ is a maximum clique in $N(x) - C_X$. By Claim 7, $G[V(L_i) \cup X]$ has a $K_3$ minor rooted at $X$. This implies that $A_{3-i} - C_X$ is a clique, since otherwise we would have a minor of $G$ of type II, which is a contradiction. It follows that $|A_{3-i} - C_X| \leq 3$ if $N(x) \cong K_{2,3,3,3}$ or $K_{3,3} + C_5$, and $|A_{3-i} - C_X| \leq 2$ if $N(x) \cong K_{4,4,4}$. Since $|A_{3-i}| \geq 6$ by Claim 1, it follows that $A_{3-i}$ is precisely the union of $C_X$ and a maximum clique of $A_{3-i} - C_X$. Therefore, $|A_{3-i}| = 6$, $C_X \subseteq A_{3-i}$, and $A_{3-i} - C_X$ is a maximum clique in $N(x) - C_X$.

To finish the proof, we first show that $|L_i| \geq 2$ for $i = 1, 2$. For the sake of a contradiction, assume that $|L_i| = 1$ for some $i \in \{1, 2\}$. Since $\delta(G) \geq 11$ and $L_i$ has at most one neighbor in $G - N[x]$, we have $|A_i| = |N(L_i) \cap N(x)| \geq 10$. By Claim 1, $|L_i| = 1$ implies that $|L_{3-i}| \geq 2$. It follows that $|A_i| = 6$ by Claim 8, a contradiction.

Now, let $X \subseteq A_1$ such that $G[X] \cong \overline{K_3}$, and let $C_1$ be the island of $N(x)$ that contains $X$. By Claim 8, since $|L_1| \geq 2$, it follows that $X \subseteq C_1 \subseteq A_2$. Since $|L_2| \geq 2$ and $X \subseteq A_2$, by Claim 8 again, $A_1 - C_1$ is a maximum clique in $N(x) - C_1$. Note that in all cases, there exists some island $C_2$ of $N(x) - C_1$ that is an independent set of size at least 3. Since $A_1 - C_1$ is a maximum clique in $N(x) - C_1$, we have $|A_1 \cap C_2| = 1$. By Claim 1, $N(x) = A_1 \cup (N(G_2) \cap N(x))$. It follows that there exist non-adjacent vertices $y_1, y_2 \in N(G_2) \cap N(x)$ that are both contained in $C_2$. By Claim 7, we can contract edges that have at least one end in $L_1$ to obtain a clique on $X$ in $C_1$. Then, by contracting all of $G_2$ to one of $y_1, y_2$, we would then obtain the edge $y_1y_2$ in $C_2$. This shows that $G$ has a minor of type II, a contradiction.

\[\square\]

### 4.4 Other Problem Graphs

**Lemma 4.4.1** (computer-assisted). *Let $H$ be a problem graph such that $H \not\cong K_{2,3,3,3}, K_{3,3} + C_5$, or $K_{4,4,4}$. Then, there exists a subset $Z = \{a_1, a_2, b_1, b_2\} \subseteq V(H)$ of size 4 such...*
that \(a_1b_1, a_2b_2 \in E(H), a_i, b_i \text{ for } i = 1, 2 \) share at most \(4 + i \) common neighbors in \(H\), and one of the following statements is true:

(A1) \(a_2, b_2 \) share at most 5 common neighbors in \(H\), and that there exists some \(z \in Z\) and \(v \in V(H) - Z\) such that \(v\) has at most 9 neighbors in \(H\) and \(H \cup \{a_1a_2, a_1b_2, b_1a_2, b_1b_2, zv\} > K_9\).

(A2) \(b_2\) has at most 8 neighbors in \(H\), there exists some \(v \in V(H) - Z\) such that \(a_2\) and \(v\) are adjacent and share at most 6 common neighbors in \(H\), and that \(H \cup \{a_1a_2, a_1b_2, a_1v, b_1b_2\} > K_9\).

**Lemma 4.4.2.** If \(G - N[x]\) is 2-connected or has at most two vertices, then \(N(x) \not\cong H\) where \(H\) is a problem graph and \(H \not\cong K_{2,3,3,3}, K_{3,3} + C_5, \text{ or } K_{4,4,4}\).

**Proof.** For the sake of a contradiction, assume that \(N(x) \cong H\) for some problem graph \(H\) such that \(H \not\cong K_{2,3,3,3}, K_{3,3} + C_5, \text{ or } K_{4,4,4}\). By Lemma 4.4.1, there exists a subset \(Z = \{a_1, a_2, b_1, b_2\} \subseteq V(H)\) of size 4 such that \(a_1b_1, a_2b_2 \in E(G), a_i, b_i \text{ for } i = 1, 2 \) share at most \(4 + i \) common neighbors in \(N(x)\), and one of the properties (A1) and (A2) in Lemma 4.4.1 is true.

Since \(a_i, b_i \text{ share at most } 4 + i \text{ common neighbors in } N(x)\) for \(i = 1, 2\), they share at most \(5 + i \text{ common neighbors in } N[x]\) and thus share at least \(3 - i \text{ common neighbors in } G - N[x]\). This means that there exist distinct vertices \(u_1, u_2 \in V(G) - N[x]\) that are both common neighbors for \(a_1\) and \(b_1\). In the rest of the proof, we will consider the case (A1) is true and the case (A2) is true separately.

**Case 1:** (A1) in Lemma 4.4.1 is true.

In this case, \(a_2, b_2 \) share at most 5 common neighbors in \(N(x)\), and that there exists some \(z \in Z\) and \(v \in N(x) - Z\) such that \(v\) has at most 9 neighbors in \(N(x)\) and \(N(x) \cup \{a_1a_2, a_2b_2, b_1a_2, b_1b_2, zv\} > K_9\). To show a contradiction, it suffices to prove that we can contract some edges that have at least one end in \(G - N[x]\) in a way to obtain the edges \(a_1a_2, a_2b_2, b_1a_2, b_1b_2, zv\) in \(N(x)\), since this would then imply \(G - \{x\} > K_9\) and thus \(G > K_{10}\).
Observe that the fact that $a_2, b_2$ share at most 5 common neighbors in $N(x)$ means that they share at least two common neighbors in $G - N[x]$, so there exist distinct vertices $w_1, w_2 \in N(G)$ that are both common neighbors for $a_2$ and $b_2$. Note that $a_i, b_i$ now share at least two common neighbors in $G - N[x]$ for both $i = 1, 2$, so without loss of generality, we can assume $z$ to be equal to any vertex in $Z$, say $z = a_2$. Since $|G - N[x]| \leq 2$ or $G - N[x]$ is 2-connected, there exist two disjoint paths $Q_1, Q_2$ between $\{u_1, u_2\}$ and $\{w_1, w_2\}$ in $G - N[x]$. Without loss of generality, assume that $Q_i$ joins $u_i, w_i$ for $i = 1, 2$. Since $v$ has at most 9 neighbors in $N(x)$, it has some neighbor in $G - N[x]$. So without loss of generality, we can assume that there exists a path $R$ linking $v$ and some vertex on $Q_1$ such that $R$ is contained in $G - N[x]$ except for $v$ and $R$ is disjoint from $Q_2$. Then, by contracting all of $V(Q_1 \cup R) - \{v\}$ to $a_2 = z$, we would obtain the edges $a_1a_2, b_1a_2$, and $a_2v = zv$, and by contracting all of $V(Q_2)$ to $b_2$, we would then obtain $a_1b_2$ and $b_1b_2$.

**Case 2: (A2) in Lemma 4.4.1 is true.**

In this case, $b_2$ has at most 8 neighbors in $N(x)$, there exists some $v \in N(x) - Z$ such that $a_2$ and $v$ are adjacent and share at most 6 common neighbors in $N(x)$, and that $N(x) \cup \{a_1a_2, a_1b_2, a_1v, b_1b_2\} > K_9$. To show a contradiction, it suffices to prove that we can contract some edges that have at least one end in $G - N[x]$ in a way to obtain the edges $a_1a_2, a_1b_2, a_1v, b_1b_2$.

Since $a_2$ and $v$ are adjacent and share at most 6 common neighbors in $N(x)$, they share at most 7 common neighbors in $N[x]$ and thus at least one common neighbor in $G - N[x]$. Let $w_1 \in V(G) - N[x]$ be a common neighbor for $a_2$ and $v$. Since $b_2$ has at most 8 neighbors in $N(x)$, it has at most 9 neighbors in $N[x]$ and therefore at least two neighbors in $G - N[x]$ due to the fact that $\delta(G) \geq 11$. Therefore, there exists some $w_2 \in V(G) - N[x]$ such that $w_1 \neq w_2$ and $b_2w_2 \in E(G)$. Recall that $a_i, b_i$ share at least $3 - i$ common neighbors in $G - N[x]$ for $i = 1, 2$, meaning that there exists some common neighbor of $a_2, b_2$ in $G - N[x]$. Therefore, if $w_1$ is not a common neighbor for $a_2, b_2$, we choose $w_2$ to be a common neighbor for them. This means that one of $w_1, w_2$ is a common neighbor for
Again since \(|G - N[x]| \leq 2\) or \(G - N[x]\) is 2-connected, there exist two disjoint paths \(Q_1, Q_2\) between \(\{u_1, u_2\}\) and \(\{w_1, w_2\}\) in \(G - N[x]\), and without loss of generality we assume that \(Q_1\) joins \(u_i, w_i\) for \(i = 1, 2\). If \(w_1\) is a common neighbor for \(a_2, b_2\), then \(w_1 \in V(Q_1)\) is adjacent to all three of \(a_2, b_2, v\). This means that by contracting all of \(Q_1\) to \(a_1\), we can obtain edges \(a_1a_2, a_1b_2\), and \(a_1v\). Then, by contracting all of \(Q_2\) to \(b_1\), we can obtain the edge \(b_1b_2\). We may now assume that \(w_2\) is a common neighbor for \(a_2, b_2\). Then, by contracting all of \(Q_1\) to \(a_1\) we can obtain edges \(a_1a_2\) and \(a_1v\), and by contracting all of \(Q_2\) to \(b_2\) we can obtain edges \(a_1b_2\) and \(a_2b_2\). \(\square\)

**Lemma 4.4.3** (computer-assisted). Let \(H\) be a problem graph such that \(H \not\cong K_{2,3,3,3}, K_{3,3} + C_5,\) or \(K_{4,4,4}\). Let \(M\) be the subset of vertices in \(H\) that are not adjacent to all other vertices in \(H\), i.e. \(M = \{v \in V(H) : \exists u \in V(H) - \{v\} \text{ such that } vu \not\in E(H)\}\). Then, for every \(B_1, B_2 \subseteq V(H)\) such that \(|B_i| \geq 6\) for \(i = 1, 2\), \(M \subseteq B_1 \cup B_2\), and neither \(H[B_1]\) nor \(H[B_2]\) is a clique, one of the following statements is true:

(B1) There exist \(b_1 \in B_1 - B_2\) and \(b_2 \in B_2 - B_1\) such that \(b_1, b_2\) are adjacent and share at most 6 common neighbors in \(H\).

(B2) There exist \(b_1 \in B_1\) and \(b_2 \in B_2\) such that \(H' > K_9\), where \(H'\) is the graph obtained from \(H\) by making \(b_1\) adjacent to all other vertices in \(B_1\) and making \(b_2\) adjacent to all other vertices in \(B_2\).

(B3) For some \(i \in \{1, 2\}\), \(|B_i| \leq 8\) and for every \(B'_i \subseteq B_i\) such that \(|B'_i| \geq 6\), there exist some \(Z \subseteq B'_i\) with \(|Z| = 4\) and \(b \in B_{3-i}\) such that \(H' > K_9\), where \(H'\) is the graph obtained from \(H\) by making \(Z\) a clique and making \(b\) adjacent to all other vertices in \(B_{3-i}\).

(B4) One of \(B_1, B_2\) is contained in the other such that \(|B_1 \cap B_2| = 6\), and there exists some \(Z \subseteq B_1 \cap B_2\) such that \(|Z| = 4\) and \(e(H[B_1 \cap B_2]) - e(H[Z]) - \delta(H[B_1 \cap B_2]) \geq 6\).

(B5) One of \(B_1, B_2\) is contained in the other such that \(|B_1 \cap B_2| = 6\) and \(G[B_1 \cap B_2] \cong K_{6-}^\circ\).

(B6) \((B_1, B_2)\) is a non-trivial separation of \(H\) of order \(k \leq 7\) such that \(e(H[B_1 \cap B_2]) = 4k - 20 + \binom{k-5}{2}\), and that edges with at least one end in \(B_i - B_{3-i}\) for \(i = 1, 2\) can be
contracted in a way such that the new graph on \( B_1 \cap B_2 \) has at most 3 non-edges.

**Lemma 4.4.4.** If \( G - N[x] \) is not 2-connected and \(|G - N[x]| \geq 3\), then \( N(x) \not\cong H \) where \( H \) is a problem graph and \( H \not\cong K_{2,3,3,3}, K_{3,3} + C_5\), or \( K_{4,4,4} \).

**Proof of Lemma 4.4.4.** For the sake of a contradiction, assume that \( N(x) \cong H \) for some problem graph \( H \) that is not isomorphic to \( K_{2,3,3,3}, K_{3,3} + C_5\), or \( K_{4,4,4} \). Let \( M = \{v \in N(x) : \exists u \in N(x) - \{v\} \text{ such that } vu \not\in E(G)\} \subseteq N(x) \). Recall that by the choice of \( x \) and Lemma 4.2.1, there exists a component \( K \) of \( G - N[x] \) such that \( M \subseteq N(K) \) and \( N(K') \cap M \subseteq N(K) \) for every component \( K' \) of \( G - N[x] \).

If \( G - N[x] \) is disconnected, choose \( G_1 \) to be one component of \( G - N[x] \) and let \( G_2 = G - N[x] \cup V(G_1) \). If \( G - N[x] \) is connected and has a cut vertex \( w \), choose \( G_1 \) to be one component of \( G - N[x] \cup \{w\} \) and let \( G_2 = G - N[x] \cup V(G_1) \). Let \( B_i = N(G_i) \cap N(x) \) for \( i = 1, 2 \) in both cases. We now make the following observation.

**Observation.** The following statements are true.

1. \( V(G_1) \cap V(G_2) = \emptyset \) and \( V(G_1) \cup V(G_2) = V(G) - N[x] \).
2. \(|N(G_i) \cap N(x)| = |B_i| \geq 6 \) for \( i = 1, 2 \).
3. If \(|B_i| = 6 \) for some \( i \in \{1, 2\} \), then \( G - N[x] \) is connected and has a cut vertex.
4. \( M \subseteq B_1 \cup B_2 \).
5. \( M \subseteq B_i \) for some \( i \in \{1, 2\} \) if \( G - N[x] \) is disconnected.
6. For \( i = 1, 2 \), there exists a connected induced subgraph \( L_i \) of \( G_i \) such that \(|N_G(L_i) \cap (V(G) - N[x])| \leq 1 \) and \(|N_G(L_i) \cap N(x)| \geq 6 \).
7. Neither \( G[B_1] \) nor \( G[B_2] \) is a clique.

**Proof of Observation.** (1)-(3) are simply true due to the construction of \( G_1, G_2 \) and the 7-connectivity of \( G \). (4) and (5) true because there exists some component \( K \) of \( G - N[x] \) such that \( M \subseteq N(K) \) by Lemma 4.2.1.

To see (6) is true, let \( L_1 = G_1 \) in both cases. If \( G - N[x] \) is disconnected, let \( L_2 \) be one single component of \( G_2 \); and if \( G - N[x] \) has a cut vertex \( w \), let \( L_2 \) be a component of
Since $G$ is 7-connected, $L_1$, $L_2$ are as desired and therefore (6) is true in both cases.

To see (7) is true, let $L_i \subseteq G_i$ for $i = 1, 2$ be as in (6), and let $A_i = N_G(L_i) \cap N(x) \subseteq B_i$ for $i = 1, 2$. Then, notice that $A_i$ for both $i = 1, 2$ is a separator of $G$ if $G$ is disconnected, and that $A_i \cup \{w\}$ is a separator of $G$ if $w$ is a cut vertex of $G - N[x]$ that defines $G_1$ and $G_2$. By Lemma 3.2.4, $G[A_i]$ is a clique for neither $i = 1, 2$. Since $A_i \subseteq B_i$ for $i = 1, 2$, it follows that neither $G[B_1]$ nor $G[B_2]$ is a clique.

Hence, by (2), (4), and (7) in Observation and Lemma 4.4.3, one of the properties in Lemma 4.4.3 is true about $B_1$ and $B_2$. We will consider each one of them separately in the rest of this proof.

**Case 1: (B1) in Lemma 4.4.3 is true**

In this case, there exist $b_1 \in B_1 - B_2$ and $b_2 \in B_2 - B_1$ such that $b_1$, $b_2$ are adjacent and share at most 6 common neighbors in $N(x)$, meaning that they share at most 7 common neighbors in $N[x]$ and thus at least one common neighbor $u \in V(G) - N[x]$. Since $b_1 \in B_1 - B_2$ and $b_2 \in B_2 - B_1$, we know $b_1 \notin N(G_2)$ and $b_2 \notin N(G_1)$. It follows that $u \notin V(G_1) \cup V(G_2)$, a contradiction to (1) in Observation.

**Case 2: (B2) in Lemma 4.4.3 is true**

In this case, there exist $b_1 \in B$ and $b_2 \in B_2$ such that $J > K_9$, where $J$ is the graph obtained from $N(x)$ by making $b_1$ adjacent to all other vertices in $B_1$ and making $b_2$ adjacent to all other vertices in $B_2$. This means that by contracting all of $G_1$ to $b_1 \in B_1$ and contracting all of $G_2$ to $b_2 \in B_2$, we can then obtain a resulting graph on $N(x)$ that has a $K_9$ minor, and therefore $G > K_{10}$, a contradiction.

**Case 3: (B3) in Lemma 4.4.3 is true**

In this case, $|B_i| \leq 8$ for some fixed $i \in \{1, 2\}$, and for every $B_i' \subseteq B_i$ such that $|B_i'| \geq 6$, there exist some $Z \subseteq B_i'$ with $|Z| = 4$ and $b \in B_{3-i}$ such that $J > K_9$, where $J$ is the graph obtained from $N(x)$ by making $Z$ a clique and making $b$ adjacent to all other
vertices in $B_{3 - i}$.

By (6) in Observation, choose a connected induced subgraph $L_i$ of $G$ such that $|N_G(L_i) \cap (V(G) - N[x])| \leq 1$ and $|N_G(L_i) \cap N(x)| \geq 6$. Then, in the graph $G[V(L_i) \cup N_G(L_i)]$, choose a separation $(X, Y)$ of $(G[V(L_i) \cup N_G(L_i)], N_G(L_i))$ such that $Y - X \neq \emptyset$ and $|X \cap Y|$ is minimum over all choices of $(X, Y)$. Notice that the minimality of $|X \cap Y|$ implies that there exist disjoint paths $P_1, \ldots, P_{|X \cap Y| - 1}$ linking $N_G(L_i)$ and $X \cap Y$ in $G[X]$. Note that $Y - X \neq \emptyset$ and $N_G(Y - X) = X \cap Y$. Since $G$ is 7-connected, it follows that $|X \cap Y| = |N_G(Y - X)| \geq 7$.

Since $|N_G(L_i) \cap (V(G) - N[x])| \leq 1$, without loss of generality, we can assume that the end vertices of $P_1, \ldots, P_{|X \cap Y| - 1}$ in $N_G(L_i)$ are all contained in $N_G(L_i) \cap N(x)$. Let $U \subseteq N_G(L_i) \cap N(x)$ and $U' \subseteq X \cap Y$ be the sets of end vertices of $P_1, \ldots, P_{|X \cap Y| - 1}$ in $N_G(L_i) \cap N(x)$ and $X \cap Y$, respectively. Note that $|U| = |U'| = |X \cap Y| - 1 \geq 6$. Since $U \subseteq N_G(L_i) \cap N(x) \subseteq N_G(G_i) \cap N(x) = B_i$, it follows that $U$ is a subset of $B_i$ of size at least 6. By (B3) in Lemma 4.4.3, there exist $Z \subseteq U$ with $|Z| = 4$ and $b \in B_{3 - i}$ such that $J > K_9$, where $J$ is the graph obtained from $N(x)$ by making $Z$ a clique and making $b$ adjacent to all other vertices in $B_{3 - i}$. Without loss of generality, say vertices in $Z$ are precisely the end vertices of $P_1, P_2, P_3, P_4$ in $U$, and say $Z'$ is the set of end vertices of $P_1, P_2, P_3, P_4$ in $U' \subseteq X \cap Y$.

Note now we have $|N_G(L_i) \cap (V(G) - N[x])| \leq 1$, $N_G(L_i) \cap N(x) \subseteq B_i$, and $|B_i| \geq 8$. It then follows that

$$|N_G(L_i)| \leq |N_G(L_i) \cap N(x)| + 1 \leq |B_i| + 1 \leq 9 \leq \delta(G) - 2.$$

By the choice of $(X, Y)$, since the trivial separation $(X', Y') = (N_G(L_i), V(L_i) \cup N_G(L_i))$ satisfies that $Y' - X' \neq \emptyset$, we know that $|X \cap Y| \leq |X' \cap Y'| = |N_G(L_i)|$ due to the minimality of $|X \cap Y|$. It follows that $|X \cap Y| \leq \delta(G) - 2$. Observe that the choice of $(X, Y)$ also implies that there is no non-trivial separation of $(G[Y], X \cap Y)$ of order at
most $|X \cap Y| - 1$. Hence, by Lemma 3.2.3, $G[(Y - X) \cup Z']$ has a $K_4$ minor rooted at $Z'$.

Now, by contracting all of $G_{3-i}$ to $b \in B_{3-i}$ and by contracting edges in $G[Y]$ properly and contracting each path $P_i$ for $i = 1, 2, 3, 4$ to a single vertex, we can then eventually obtain $u$ to $w$ a minimum separator of $G$.

By (3) and (6) of Observation and by the 7-connectivity of $G$, $G - N[x]$ is connected and has a cut vertex $w$, and that $N(G_i) = B_i \cup \{w\}$ is a minimum separator of $G$.

Let $H_1 = G[V(G_i) \cup B_i \cup \{w\}]$ and $H_2 = G - V(G_i)$. Notice that $H_1, H_2$ defines a non-trivial 7-separation of $G$, where $V(H_1 \cap H_2) = B_i \cup \{w\}$ is a separator of order 7 of $G$.

By Lemma 3.2.3, $G[V(G_i) \cup Z]$ has a rooted-$K_4$ minor at $Z$ and therefore we can contract edges that have at least one end in $G_i$ to $Z$ properly to obtain $6 - e(G[Z])$ extra edges. Let this new graph on $V(H_2)$ be $H'_2$. By Lemma 3.2.4, $H'_2 \not\cong K_{2.2.2,2.3}$ or $K_{2.3.3,3,3}$. On the other hand, note that $x \in V(H_2 - H_1)$ and $G_{3-i} - \{w\} \subseteq H_2 - H_1$. Since $B_i \cup \{w\}$ is a minimum separator of $G$, by contracting all vertices in one component of $G_{3-i} - \{w\}$ to $w$, we can have $w$ adjacent to all six vertices in $B_i$ in the new graph. Furthermore, let $u \in B_i$ such that $d_{G[B_i]}(u) = \delta(G[B_i])$. Since $B_i \subseteq N(x)$, by contracting the edge $xu$ we can then have $u$ adjacent to all other vertices in $B_i$. Let $\delta = \delta(G[B_i])$. Then, by contracting all vertices in one component of $G_{3-i} - \{w\}$ to $w$ and contracting the edge $xu$, we are able to obtain $6 + e(G[B_i]) + (5 - \delta) = 11 + e(G[B_i]) - \delta$ edges on $B_i \cup \{w\}$ in the new graph.

By Lemma 3.2.1, it follows that

$$8 \cdot 7 \geq 35 + (6 - e(G[Z])) + (11 + e(G[B_i]) - \delta) - 1 = 51 + e(G[B_i]) - e(G[Z]) - \delta,$$

meaning that $e(G[B_i]) - e(G[Z]) - \delta \leq 5$, a contradiction to the inequality $e(G[B_i]) -
$e(G[Z]) - \delta \geq 6$ in property (B4).

**Case 5: (B5) in Lemma 4.4.3 is true**

In this case, $B_i \subseteq B_{3-i}$ for some $i \in \{1, 2\}$ such that $|B_i| = 6$ and $e(G[B_i]) \cong K_6^-$. By (3) and (6) in Observation and by the 7-connectivity of $G$, it follows that $G - N[x]$ is connected and has a cut vertex $w$ such that $N(G_i) = B_i \cup \{w\}$. Notice that $N(G_i) = B_i \cup \{w\}$ is a minimum separator of $G$. Since $G[B_i] \cong K_6^-$, there is a unique missing edge in $G[B_i]$. Let $t = d_{N(G_i)}(w)$. It follows that $e(N(G_i)) = 14 + t$. Since $N(G_i) = B_i \cup \{w\}$ is a minimum separator of $G$, we know that by contracting all vertices in any component of $G - N(G_i)$ to $w$, we can obtain $6 - t$ extra edges on $N(G_i)$. By Lemma 3.2.8, it follows that

$$8 \cdot 7 \geq 35 + 2(6 - t) + 14 + t = 61 - t.$$  

This means that $t \geq 5$ and therefore $w$ has at most one non-neighbor in $B_i$. By (1) in Lemma 3.2.4, $w$ has exactly one non-neighbor in $B_i$. Since $G[B_i] \cong K_6^-$, there exists some $Z \subseteq B_i \cup \{w\}$ such that $|Z| = 4$ and that the graph obtained from $N(G_i)$ by making $Z$ a clique is isomorphic to $K_7$, a contradiction to (2) in Lemma 3.2.4.

**Case 6: (B6) in Lemma 4.4.3 is true**

In this case, $(B_1, B_2)$ is a non-trivial separation of $N(x)$ of order $k \leq 7$ such that $e(G[B_1 \cap B_2]) = 4k - 20 + \binom{k-5}{2}$, and that edges with at least one end in $B_i - B_{3-i}$ for $i = 1, 2$ can be contracted in a way such that the new graph on $B_1 \cap B_2$ has at most 3 non-edges.

Notice that if $G - N[x]$ is disconnected, then $(B_1 \cap B_2) \cup \{x\}$ separates $V(G_1) \cup (B_1 - B_2)$ from $V(G_2) \cup (B_2 - B_1)$; and if $G - N[x]$ has a cut vertex $w$, then $(B_1 \cap B_2) \cup \{x, w\}$ separates $V(G_1) \cup (B_1 - B_2)$ from $(V(G_2) - \{w\}) \cup (B_2 - B_1)$. Let $H_1 = G[V(G_1) \cup B_1 \cup \{x\}]$ in the former case, and let $H_1 = G[V(G_1) \cup B_1 \cup \{x, w\}]$ in the latter case. In both cases, let $H_2 = G - V(G_1) \cup (B_1 - B_2)$. It follows that $(V(H_1), V(H_2))$ is a separation of $G$ in both cases. Let $S = V(H_1 \cap H_2)$. Notice that if $G - N[x]$ is disconnected,
then $S = (B_1 \cap B_2) \cup \{x\}$ and $|S| = k + 1$; and if $G - N[x]$ has a cut vertex $w$, then $S = (B_1 \cap B_2) \cup \{x, w\}$ and $|S| = k + 2$. We will again apply the inequality in Lemma 3.2.1 to this separation to show contractions.

We first prove an upper bound to $e(G[S])$ in both cases. Since $B_1 \cap B_2 \subseteq N(x)$, we know that $x$ is adjacent to all vertices in $S$. Therefore, if $G - N[x]$ is disconnected, then $e(G[S]) = e(G[B_1 \cap B_2]) + k$; and if $G - N[x]$ has a cut vertex $w$, then $e(G[S]) = e(G[B_1 \cap B_2]) + k + d_G|S|(w) \leq e(G[B_1 \cap B_2]) + 2k$, where the last inequality is due to the facts that $w \notin N(x)$ and $d_G|S|(w) \leq |B_1 \cap B_2| = k$. Since $e(G[B_1 \cap B_2]) = 4k - 20 + \binom{k - 5}{2}$, it follows that

$$e(G[S]) \leq (4 + |S| - k)k - 20 + \binom{k - 5}{2}$$

in both cases.

We now prove a lower bound to the number of edges in $H'_i$ for $i = 1, 2$, a supgraph on $V(H_i)$ that can be obtained from $G$ by contracting edges that have at least one end in $H_{3-i} - H_i$. In the case $G - N[x]$ has a cut vertex $w$, note that $w \in S = V(H_1 \cap H_2)$ and for both $i = 1, 2$, $G_i - \{w\} \subseteq H_i - H_{3-i}$ and $w$ is a neighbor for every component of $G_i - \{w\}$. This means that for $i = 1, 2$, by contracting all of $G_i - \{w\}$ to $w$, we would have $w$ to be adjacent to all vertices in $B_1 \cap B_2 \subseteq B_i$ in the new graph on $S$. Recall that for $i = 1, 2$, edges with at least one end in $B_i - B_{3-i}$ can be contracted in a way such that the new graph on $B_1 \cap B_2$ has at most 3 non-edges. Therefore, for $i = 1, 2$, there is a supgraph $H'_i$ on $V(H_i)$ obtained by contracting edges that have at least one end in $H_{3-i} - H_i$ such that if $G - N[x]$ is disconnected, then

$$e(H'_i[S]) = e(H'_i[B_1 \cap B_2]) + d_{H'_i[S]}(x) \geq \binom{k}{2} - 3 + k;$$

and that if $G - N[x]$ has a cut vertex, then

$$e(H'_i[S]) = e(H'_i[B_1 \cap B_2]) + d_{H'_i[S]}(x) + d_{H'_i[S]}(w) \geq \binom{k}{2} - 3 + 2k.$$
To summarize, in both cases we have
\[ e(H'_i[S]) \geq \left( \frac{k}{2} \right) - 3 + (|S| - k)k. \]

By Lemma 3.2.1, we know that \( 8|S| \geq 33 + e(H'_1[S]) + e(H'_2[S]) - e(G[S]). \) Due to the upper bound for \( e(G[S]) \) and the lower bounds for \( e(H'_1[S]) \) and \( e(H'_2[S]) \) above, it follows that
\[ 8|S| \geq 33 + 2\left( \frac{k}{2} - 3 + (|S| - k)k - ((4 + |S| - k)k - 20 + \left( \frac{k - 5}{2} \right) ) = 47 + (|S| - 5)k - \left( \frac{k - 5}{2} \right) . \]

If \( G - N[x] \) is disconnected, then \( |S| = k + 1 \) and thus
\[ 8(k + 1) \geq 47 + (k - 4)k - \left( \frac{k - 5}{2} \right) = 47 + k^2 - 4k - \left( \frac{k - 5}{2} \right), \]
meaning that \( k^2 - 12k - \left( \frac{k - 5}{2} \right) + 39 \leq 0 \), a contradiction to the fact that \( k \leq 7 \).

If \( G - N[x] \) has a cut vertex, then \( |S| = k + 2 \) and thus
\[ 8(k + 2) \geq 47 + (k - 3)k - \left( \frac{k - 5}{2} \right) = 47 + k^2 - 3k - \left( \frac{k - 5}{2} \right), \]
again a contradiction to the fact that \( k \leq 7 \).
CHAPTER 5
CONCLUSION

Finally, in this chapter we apply Lemma 4.1.1, the main technical lemma, to complete the proof for Theorem 1.1.5.

Proof of Theorem 1.1.5. By Lemma 3.1.1 and Lemma 3.3.1, there exists a vertex \( x \in V(G) \) with \( 11 \leq d(x) \leq 15 \). We choose such a vertex \( x \) with a component \( K \) of \( G - N[x] \) such that \( |K| \) is minimum over all choices of \( x \) and \( K \). In the next Claim, we prove that every vertex in \( K \) has degree at least 16 in \( G \).

Claim 1. For every \( y \in V(K) \), \( d_G(y) \geq 16 \).

Proof of Claim 1. Choose \( y \in V(K) \) such that \( d_G(y) \) is minimum among all vertices in \( K \). For the sake of a contradiction, assume that \( d_G(y) \leq 15 \). Note that \( x \in V(G) - N[y] \), as \( x,y \) are not adjacent to each other. Let \( L \) be the component of \( G - N[y] \) that contains \( x \), and let \( M_y \subseteq N(y) \) be the subset of vertices that are not adjacent to all other vertices in \( N(y) \).

Since \( d_G(y) \leq 15 \), by Lemma 4.1.1 there exists some vertex \( z \in M_y - N(L) \). Note this implies that \( z \notin N(x) \) and therefore \( z \in V(K) - \{y\} \). By the choice of \( y \), we know \( d_G(z) \geq d_G(y) \). Since \( z \in M_y \) is not adjacent to some vertex in \( N(y) \), \( z \) must be adjacent to some \( z' \in V(G) - N[y] \cup V(L) \). Let \( L' \) be the component of \( G - N[y] \) that contains \( z' \), and note that \( L \neq L' \) since \( z \notin N(L) \). It follows that \( z' \notin N(x) \). Since \( z' \) is a neighbor of \( z \) and \( z \in V(K) \), it follows that \( z' \in V(K) \) as well. Furthermore, since \( x \in V(L) \) and \( L, L' \) are two distinct components of \( G - N[y] \), it follows that \( V(L') \cap N(x) = \emptyset \) and thus some component of \( G - N[x] \) includes all vertices in \( L' \). Since \( z' \in V(L') \) is contained in \( K \), it follows that \( V(L') \subseteq V(K) \). Notice that \( y \in V(K) \) and \( y \notin V(L') \), and it follows that \( |L'| < |K| \). Since \( d_G(y) \leq 15 \), the fact that \( |L'| < |K| \) is a contradiction to the choice
of $x$ and $K$. 

Now, let $G_1 = G - V(K)$ and $G_2 = G[N(K) \cup V(K)]$. Let $d_2$ be the maximum number of edges that can be added to $G_2$ by contracting edges that have at least one end in $G_1 - G_2$, and let $J_2$ be a minor of $G$ on $V(G_2)$ such that $e(J_2) = e(G_2) + d_2$. By Claim 1, $d_G(y) \geq 16$ for every $y \in V(K)$. Since $|K| > 0$, we know $\Delta(J_2) \geq \Delta(G_2) \geq 16$, and therefore $J_2 \not\cong K_{2,2,2,2,2}$ or $K_{2,3,3,3,3}$. Since $J_2$ is a proper minor of $G$, it follows that $e(J_2) \leq 8|G_2| - 35$. Let $\delta = \delta(N(K))$ and choose $z \in N(K)$ such that $d_{N(K)}(z) = \delta$. 

Note that $N(K) \subseteq N(x)$, and therefore by contracting the edge $xz$ we could have the new vertex adjacent to all other vertices in $N(K)$. This shows that $d_2 \geq |N(K)| - 1 - \delta$, and therefore 

$$e(G_2) = e(J_2) - d_2 \leq (8|G_2| - 35) - (|N(K)| - 1 - \delta) = 8|K| + 7|N(K)| + \delta - 34.$$ 

By Claim 1, $d_G(y) \geq 16$ for every $y \in V(K)$, and this implies that $16|K| \leq \sum_{y \in V(K)} d_G(y) = 2e(K) + e(K, N(K))$. Note that $e(N(K)) \geq \frac{1}{2} \delta |N(K)|$. For simplicity, let $k = |K|$ and $N = |N(K)|$. It follows that 

$$e(K) = (2e(K) + e(K, N(K))) + e(N(K)) - e(G_2)$$

$$\geq 16k + \frac{1}{2} \delta N - (8k + 7N + \delta - 34)$$

$$= 8k + \frac{1}{2} \delta (N - 2) - 7N + 34.$$ 

Since $\delta(N(x)) \geq 8$, $\delta = \delta(N(K)) \geq \delta(N(x)) - (d(x) - |N(K)|) \geq 8 - d(x) + N$. Let $d = d(x)$. It follows that 

$$e(K) \geq 8k + \frac{1}{2}(8 - d + N)(N - 2) - 7N + 34.$$
Observe that

\[2 \cdot \left( \frac{1}{2}(8-d+N)(N-2)-7N+34 \right) = N^2-(8+d)N+2d(x)+52 = (N-\frac{1}{2}(8+d))^2 - \frac{1}{4}(d+4)^2 + 40.\]

Therefore,

\[2e(K) \geq 16k + (N-\frac{1}{2}(8+d))^2 - \frac{1}{4}(d+4)^2 + 40.\]

Assume \(|K| \geq 8\) for a moment. Since \(G > K\), we know that \(e(K) \leq 8k - 34\), and therefore

\[16k + (N-\frac{1}{2}(8+d))^2 - \frac{1}{4}(d+4)^2 + 40 \leq 2e(K) \leq 16k - 68,\]

meaning that \((N-\frac{1}{2}(8+d))^2 - \frac{1}{4}(d+4)^2 + 108 \leq 0\), and therefore \(-\frac{1}{4}(d+4)^2 + 108 \leq 0\).

Note that \(d = d(x) \leq 15\). It follows that \(-\frac{1}{4}(d+4)^2 + 108 \geq -\frac{192}{4} + 108 > 0\), a contradiction.

We may then assume \(|K| \leq 7\). Note that

\[16k + (N-\frac{1}{2}(8+d))^2 - \frac{1}{4}(d+4)^2 + 40 \leq 2e(K) \leq 2\binom{k}{2} = k^2 - k.\]

It follows that

\[k^2 - 17k \geq (N-\frac{1}{2}(8+d))^2 - \frac{1}{4}(d+4)^2 + 40 \geq 0 - \frac{(15+4)^2}{4} + 40 = -\frac{201}{4},\]

where the second inequality is due to \(d = d(x) \leq 15\). Since \(k^2 - 17k \geq -\frac{201}{4}\) and \(k \leq 7\), it follows that \(k \leq 3\). By Lemma 4.1.1, we know \(|N(K)| \leq d(x) - 1 \leq 14\). Since \(d_G(y) \geq 16\) for every \(y \in K\), it follows that \(|K| = k = 3\), \(|N(K)| = 14\), \(d(x) = 15\), and that every vertex in \(K\) is adjacent to all other vertices in \(G_2 = G[V(K) \cup N(K)]\).

Recall that \(e(G_2) + d_2 = e(J_2) \leq 8|G_2| - 35\). Since \(|G_2| = 3 + 14 = 17\), it follows
that $e(G_2) + d_2 \leq 8 \cdot 17 - 35 = 101$. Therefore,

$$e(N(K)) + d_2 = (e(G_2) + d_2) - (e(K) + e(K, N(K))) \leq 101 - (3 + 3 \cdot 14) = 56.$$  

Note that $\delta \geq \delta(N(x)) - 1 \geq 7$. This means that $e(N(K)) \geq \frac{1}{2}\delta |N(K)| \geq \frac{1}{2} \cdot 7 \cdot 14 = 49$, and thus $d_2 \leq 56 - 49 = 7$. If $\delta \geq 8$, then $e(N(K)) \geq \frac{1}{2} \cdot 8 \cdot 14 = 56$. It follows that $d_2 = 0$, $e(N(K)) = 56$, and $d_{N(K)}(v) = 8$ for every $v \in N(K)$. This then implies that, by contracting an edge between $x$ and any vertex in $N(K)$, we would obtain exactly 5 extra edges on $N(K)$, meaning $d_2 \geq 5$, a contradiction. We then conclude that $\delta = 7$. Note now $d_2 \geq |N(K)| - 1 - \delta = 14 - 1 - 7 = 6$. Since $d_2 \leq 7$, it follows that $d_2 = 6$ or 7 and $e(N(K)) = 49$ or 50.

Now, note that $N(K)$ is either a 7-regular graph or obtained from a 7-regular graph by adding one more edge. Let $U = \{u \in N(K) : d_{N(K)}(u) = 7\}$, and notices that $|U| = 12$ or 14. Choose distinct vertices $u_1, u_2 \in U$ such that $u_1u_2 \notin E(G)$. Let $w$ be the unique vertex in $N(x) - N(K)$. Since $\delta(N(x)) \geq 8$, every vertex $v \in N(K)$ such that $d_{N(K)}(v) = 7$ must be adjacent to $w$ and therefore $U \subseteq N(w)$. Observe that $|N(K) - N[u_1]| = 14 - 1 - 7 = 6$, and $|N(w) \cap N(K) - N[u_2]| \geq |U - N[u_2]| \geq |U| - |N[u_2]| \geq 12 - 8 = 4$. Therefore, by contracting $xu_1$ and $wu_2$, we can obtain $|N(K) - N[u_1]| + |N(w) \cap N(K) - N[u_2]| - 1 \geq 6 + 4 - 1 = 9$ extra edges on $N(K)$. It follows that $d_2 \geq 9$, a contradiction to the fact that $d_2 \leq 7$. 

$\square$
Appendices
APPENDIX A
PROBLEM GRAPHS

We present the 101 problem graphs explicitly here, as mentioned in Lemma 4.2.3. There are 13 problems graphs on 11 vertices, 35 problem graphs on 12 vertices, 33 problem graphs on 13 vertices, 11 problem graphs on 14 vertices, and 9 problem graphs on 15 vertices.

Here is how to read the problem graphs in this appendix: For $k = 11, 12, 13, 14, 15$, for each problem graph on $k$ vertices, the vertices are 0-indexed, and we give the full list of neighbors of each vertex from vertex-0 to vertex-$(k-1)$. For example, the third line of the matrix for graph 1 on 11 vertices, as shown below, says “2 : 3 4 5 6 7 8 9 10”, and this means that the neighborhood of vertex-2 in this graph is precisely the set of vertices indexed 3, 4, 5, 6, 7, 8, 9, 10 in this graph.

A.1 Problem Graphs on 11 vertices

There are 13 problem graphs on 11 vertices, up to isomorphism.

Graph 1, on 11 vertices

0 : 3 4 5 6 7 8 9 10
1 : 3 4 5 6 7 8 9 10
2 : 3 4 5 6 7 8 9 10
3 : 0 1 2 6 7 8 9 10
4 : 0 1 2 6 7 8 9 10
5 : 0 1 2 6 7 8 9 10
6 : 0 1 2 3 4 5 9 10
7 : 0 1 2 3 4 5 9 10
8 : 0 1 2 3 4 5 9 10
9 : 0 1 2 3 4 5 6 7 8
Graph 2, on 11 vertices

0 : 3 4 5 6 7 8 9 10
1 : 3 4 5 6 7 8 9 10
2 : 3 4 5 6 7 8 9 10
3 : 0 1 2 6 7 8 9 10
4 : 0 1 2 6 7 8 9 10
5 : 0 1 2 6 7 8 9 10
6 : 0 1 2 3 4 5 8 10
7 : 0 1 2 3 4 5 9 10
8 : 0 1 2 3 4 5 6 10
9 : 0 1 2 3 4 5 7 10
10 : 0 1 2 3 4 5 6 7 8 9

Graph 3, on 11 vertices

0 : 3 4 5 6 7 8 9 10
1 : 3 4 5 6 7 8 9 10
2 : 3 4 5 6 7 8 9 10
3 : 0 1 2 6 7 8 9 10
4 : 0 1 2 6 7 8 9 10
5 : 0 1 2 6 7 8 9 10
6 : 0 1 2 3 4 5 8 9
7 : 0 1 2 3 4 5 9 10
8 : 0 1 2 3 4 5 6 10
9 : 0 1 2 3 4 5 6 7
10 : 0 1 2 3 4 5 7 8

Graph 4, on 11 vertices

0 : 3 4 5 6 7 8 9 10
Graph 5, on 11 vertices
0 : 2 4 5 6 7 8 9 10
1 : 3 4 5 6 7 8 9 10
2 : 0 4 5 6 7 8 9 10
3 : 1 4 5 6 7 8 9 10
4 : 0 1 2 3 6 8 9 10
5 : 0 1 2 3 7 8 9 10
6 : 0 1 2 3 4 8 9 10
7 : 0 1 2 3 5 8 9 10
8 : 0 1 2 3 4 5 6 7
9 : 0 1 2 3 4 5 6 7
10 : 0 1 2 3 4 5 6 7

Graph 6, on 11 vertices
0 : 2 3 5 6 7 8 9 10
1 : 3 4 5 6 7 8 9 10
2 : 0 4 5 6 7 8 9 10
3 : 0 1 5 6 7 8 9 10
Graph 7, on 11 vertices

0 : 2 3 5 6 7 8 9 10
1 : 3 4 5 6 7 8 9 10
2 : 0 4 5 6 7 8 9 10
3 : 0 1 5 6 7 8 9 10
4 : 1 2 5 6 7 8 9 10
5 : 0 1 2 3 4 7 8 10
6 : 0 1 2 3 4 8 9 10
7 : 0 1 2 3 4 5 9 10
8 : 0 1 2 3 4 5 6 10
9 : 0 1 2 3 4 6 7 10
10 : 0 1 2 3 4 5 6 7 8 9

Graph 8, on 11 vertices

0 : 2 3 4 6 7 8 9 10
1 : 3 4 5 6 7 8 9 10
2 : 0 4 5 6 7 8 9 10
3 : 0 1 5 6 7 8 9 10
4 : 0 1 2 6 7 8 9 10
5 : 1 2 3 6 7 8 9 10
6 : 0 1 2 3 4 5 8 9
Graph 9, on 11 vertices
0 : 2 3 4 5 7 8 9 10
1 : 3 4 5 6 7 8 9 10
2 : 0 4 5 6 7 8 9 10
3 : 0 1 5 6 7 8 9 10
4 : 0 1 2 6 7 8 9 10
5 : 0 1 2 3 7 8 9 10
6 : 1 2 3 4 7 8 9 10
7 : 0 1 2 3 4 5 6 10
8 : 0 1 2 3 4 5 6 10
9 : 0 1 2 3 4 5 6 10
10 : 0 1 2 3 4 5 6 7 8 9

Graph 10, on 11 vertices
0 : 2 3 4 5 7 8 9 10
1 : 3 4 5 6 7 8 9 10
2 : 0 4 5 6 7 8 9 10
3 : 0 1 5 6 7 8 9 10
4 : 0 1 2 6 7 8 9 10
5 : 0 1 2 3 7 8 9 10
6 : 1 2 3 4 7 8 9 10
7 : 0 1 2 3 4 5 6 9
8 : 0 1 2 3 4 5 6 10
9 : 0 1 2 3 4 5 6 7

155
10: 0 1 2 3 4 5 6 8

**Graph 11, on 11 vertices**

0: 2 3 4 5 6 8 9 10
1: 3 4 5 6 7 8 9 10
2: 0 4 5 6 7 8 9 10
3: 0 1 5 6 7 8 9 10
4: 0 1 2 6 7 8 9 10
5: 0 1 2 3 7 8 9 10
6: 0 1 2 3 4 8 9 10
7: 1 2 3 4 5 8 9 10
8: 0 1 2 3 4 5 6 7
9: 0 1 2 3 4 5 6 7
10: 0 1 2 3 4 5 6 7

**Graph 12, on 11 vertices**

0: 2 3 4 5 6 7 9 10
1: 3 4 5 6 7 8 9 10
2: 0 4 5 6 7 8 9 10
3: 0 1 5 6 7 8 9 10
4: 0 1 2 6 7 8 9 10
5: 0 1 2 3 7 8 9 10
6: 0 1 2 3 4 8 9 10
7: 0 1 2 3 4 5 9 10
8: 0 1 2 3 4 5 6 7
9: 0 1 2 3 4 5 6 7 8
10: 0 1 2 3 4 5 6 7 8

**Graph 13, on 11 vertices**

0: 2 3 4 5 6 7 8 9
A.2 Problem Graphs on 12 vertices

There are 35 problem graphs on 12 vertices, up to isomorphism.

Graph 1, on 12 vertices
0 : 4 5 6 7 8 9 10 11
1 : 4 5 6 7 8 9 10 11
2 : 4 5 6 7 8 9 10 11
3 : 4 5 6 7 8 9 10 11
4 : 0 1 2 3 8 9 10 11
5 : 0 1 2 3 8 9 10 11
6 : 0 1 2 3 8 9 10 11
7 : 0 1 2 3 8 9 10 11
8 : 0 1 2 3 4 5 6 7
9 : 0 1 2 3 4 5 6 7
10 : 0 1 2 3 4 5 6 7
11 : 0 1 2 3 4 5 6 7

Graph 2, on 12 vertices
Graph 3, on 12 vertices
0: 4 5 6 7 8 9 10 11
1: 4 5 6 7 8 9 10 11
2: 4 5 6 7 8 9 10 11
3: 4 5 6 7 8 9 10 11
4: 0 1 2 3 7 8 10 11
5: 0 1 2 3 8 9 10 11
6: 0 1 2 3 8 9 10 11
7: 0 1 2 3 4 9 10 11
8: 0 1 2 3 5 6 10 11
9: 0 1 2 3 4 5 6 7
10: 0 1 2 3 4 5 6 7
11: 0 1 2 3 4 5 6 7 8

Graph 4, on 12 vertices
0: 4 5 6 7 8 9 10 11
1: 4 5 6 7 8 9 10 11
2: 4 5 6 7 8 9 10 11
3: 4 5 6 7 8 9 10 11
4: 0 1 2 3 7 8 10 11
5: 0 1 2 3 8 9 10 11
6: 0 1 2 3 8 9 10 11
7: 0 1 2 3 4 9 10 11
8: 0 1 2 3 4 5 6 11
9: 0 1 2 3 5 6 7 11
10: 0 1 2 3 4 5 6 7
11: 0 1 2 3 4 5 6 7 8 9

Graph 4, on 12 vertices
0: 4 5 6 7 8 9 10 11
Graph 5, on 12 vertices

0 : 4 5 6 7 8 9 10 11
1 : 4 5 6 7 8 9 10 11
2 : 4 5 6 7 8 9 10 11
3 : 4 5 6 7 8 9 10 11
4 : 0 1 2 3 7 8 9 10 11
5 : 0 1 2 3 8 9 10 11
6 : 0 1 2 3 8 9 10 11
7 : 0 1 2 3 4 9 10 11
8 : 0 1 2 3 4 5 6 10
9 : 0 1 2 3 5 6 7 11
10 : 0 1 2 3 4 5 6 7 8
11 : 0 1 2 3 4 5 6 7 9

Graph 6, on 12 vertices

0 : 4 5 6 7 8 9 10 11
1 : 4 5 6 7 8 9 10 11
1 : 4 5 6 7 8 9 10 11
Graph 7, on 12 vertices

0: 4 5 6 7 8 9 10 11
1: 4 5 6 7 8 9 10 11
2: 4 5 6 7 8 9 10 11
3: 4 5 6 7 8 9 10 11
4: 0 1 2 3 7 8 9 10 11
5: 0 1 2 3 7 9 10 11
6: 0 1 2 3 8 9 10 11
7: 0 1 2 3 4 5 10 11
8: 0 1 2 3 4 6 10 11
9: 0 1 2 3 5 6 10 11
10: 0 1 2 3 4 5 6 7 8 9
11: 0 1 2 3 4 5 6 7 8 9

Graph 8, on 12 vertices

0: 4 5 6 7 8 9 10 11
1: 4 5 6 7 8 9 10 11
2: 4 5 6 7 8 9 10 11
3: 4 5 6 7 8 9 10 11
4: 0 1 2 3 7 8 9 11
5: 0 1 2 3 7 9 10 11
6: 0 1 2 3 8 9 10 11
7: 0 1 2 3 4 5 10 11
8: 0 1 2 3 4 6 10 11
9: 0 1 2 3 4 5 6 11
10: 0 1 2 3 5 6 7 8
11: 0 1 2 3 4 5 6 7 8 9

160
Graph 9, on 12 vertices
0 : 4 5 6 7 8 9 10 11
1 : 4 5 6 7 8 9 10 11
2 : 4 5 6 7 8 9 10 11
3 : 4 5 6 7 8 9 10 11
4 : 0 1 2 3 7 8 10 11
5 : 0 1 2 3 7 9 10 11
6 : 0 1 2 3 8 9 10 11
7 : 0 1 2 3 4 5 10 11
8 : 0 1 2 3 4 6 10 11
9 : 0 1 2 3 4 5 6 11
10 : 0 1 2 3 4 5 6 7 8
11 : 0 1 2 3 5 6 7 8 9

Graph 10, on 12 vertices
0 : 4 5 6 7 8 9 10 11
1 : 4 5 6 7 8 9 10 11
2 : 4 5 6 7 8 9 10 11
3 : 4 5 6 7 8 9 10 11
4 : 0 1 2 3 7 8 10 11
5 : 0 1 2 3 7 9 10 11
6 : 0 1 2 3 8 9 10 11
7 : 0 1 2 3 4 5 9 10
8 : 0 1 2 3 4 6 10 11
9 : 0 1 2 3 5 6 7 11
10 : 0 1 2 3 4 5 6 7 8
11 : 0 1 2 3 4 5 6 8 9
Graph 11, on 12 vertices

0: 4 5 6 7 8 9 10 11
1: 4 5 6 7 8 9 10 11
2: 4 5 6 7 8 9 10 11
3: 4 5 6 7 8 9 10 11
4: 0 1 2 3 6 8 9 10
5: 0 1 2 3 7 9 10 11
6: 0 1 2 3 8 9 10 11
7: 0 1 2 3 4 5 9 10
8: 0 1 2 3 4 6 9 11
9: 0 1 2 3 5 6 7 8
10: 0 1 2 3 4 5 6 7
11: 0 1 2 3 4 5 6 8

Graph 12, on 12 vertices

0: 4 5 6 7 8 9 10 11
1: 4 5 6 7 8 9 10 11
2: 4 5 6 7 8 9 10 11
3: 4 5 6 7 8 9 10 11
4: 0 1 2 3 6 8 9 10
5: 0 1 2 3 7 9 10 11
6: 0 1 2 3 4 8 9 11
7: 0 1 2 3 5 9 10 11
8: 0 1 2 3 4 6 10 11
9: 0 1 2 3 4 5 6 7
10: 0 1 2 3 4 5 7 8
11: 0 1 2 3 5 6 7 8
Graph 13, on 12 vertices

0 : 3 4 6 7 8 9 10 11
1 : 4 5 6 7 8 9 10 11
2 : 4 5 6 7 8 9 10 11
3 : 0 5 6 7 8 9 10 11
4 : 0 1 2 6 7 8 10 11
5 : 1 2 3 7 8 9 10 11
6 : 0 1 2 3 4 9 10 11
7 : 0 1 2 3 5 9 10 11
8 : 0 1 2 3 4 5 10 11
9 : 0 1 2 3 4 5 6 7
10 : 0 1 2 3 4 5 6 7 8
11 : 0 1 2 3 4 5 6 7 8

Graph 14, on 12 vertices

0 : 3 4 6 7 8 9 10 11
1 : 4 5 6 7 8 9 10 11
2 : 4 5 6 7 8 9 10 11
3 : 0 5 6 7 8 9 10 11
4 : 0 1 2 6 7 8 10 11
5 : 1 2 3 7 8 9 10 11
6 : 0 1 2 3 4 9 10 11
7 : 0 1 2 3 5 9 10 11
8 : 0 1 2 3 4 5 10 11
9 : 0 1 2 3 4 5 6 7
10 : 0 1 2 3 4 5 6 7 8
11 : 0 1 2 3 4 5 6 7 8
### Graph 15, on 12 vertices

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Neighbors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3 4 5 6 7 9 10 11</td>
</tr>
<tr>
<td>1</td>
<td>4 5 6 7 8 9 10 11</td>
</tr>
<tr>
<td>2</td>
<td>4 5 6 7 8 9 10 11</td>
</tr>
<tr>
<td>3</td>
<td>0 5 6 7 8 9 10 11</td>
</tr>
<tr>
<td>4</td>
<td>0 1 2 7 8 9 10 11</td>
</tr>
<tr>
<td>5</td>
<td>0 1 2 3 8 9 10 11</td>
</tr>
<tr>
<td>6</td>
<td>0 1 2 3 8 9 10 11</td>
</tr>
<tr>
<td>7</td>
<td>0 1 2 3 4 9 10 11</td>
</tr>
<tr>
<td>8</td>
<td>1 2 3 4 5 6 10 11</td>
</tr>
<tr>
<td>9</td>
<td>0 1 2 3 4 5 6 7</td>
</tr>
<tr>
<td>10</td>
<td>0 1 2 3 4 5 6 7 8</td>
</tr>
<tr>
<td>11</td>
<td>0 1 2 3 4 5 6 7 8</td>
</tr>
</tbody>
</table>

### Graph 16, on 12 vertices

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Neighbors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3 4 5 6 7 8 9 10</td>
</tr>
<tr>
<td>1</td>
<td>4 5 6 7 8 9 10 11</td>
</tr>
<tr>
<td>2</td>
<td>4 5 6 7 8 9 10 11</td>
</tr>
<tr>
<td>3</td>
<td>0 5 6 7 8 9 10 11</td>
</tr>
<tr>
<td>4</td>
<td>0 1 2 7 8 9 10 11</td>
</tr>
<tr>
<td>5</td>
<td>0 1 2 3 8 9 10 11</td>
</tr>
<tr>
<td>6</td>
<td>0 1 2 3 8 9 10 11</td>
</tr>
<tr>
<td>7</td>
<td>0 1 2 3 4 9 10 11</td>
</tr>
<tr>
<td>8</td>
<td>1 2 3 4 5 6 10 11</td>
</tr>
<tr>
<td>9</td>
<td>0 1 2 3 4 5 6 7</td>
</tr>
<tr>
<td>10</td>
<td>0 1 2 3 4 5 6 7 8</td>
</tr>
<tr>
<td>11</td>
<td>0 1 2 3 4 5 6 7 8</td>
</tr>
</tbody>
</table>
Graph 17, on 12 vertices
0 : 3 4 5 6 7 8 9 10 11
1 : 4 5 6 7 8 9 10 11
2 : 4 5 6 7 8 9 10 11
3 : 0 5 6 7 8 9 10 11
4 : 0 1 2 6 7 9 10 11
5 : 0 1 2 3 8 9 10 11
6 : 0 1 2 3 4 8 9 10
7 : 0 1 2 3 4 8 10 11
8 : 0 1 2 3 5 6 7 11
9 : 0 1 2 3 4 5 6 11
10 : 0 1 2 3 4 5 6 7
11 : 1 2 3 4 5 7 8 9

Graph 18, on 12 vertices
0 : 3 4 5 6 7 8 9 11
1 : 4 5 6 7 8 9 10 11
2 : 4 5 6 7 8 9 10 11
3 : 0 4 5 6 7 8 10 11
4 : 0 1 2 3 7 8 9 10
5 : 0 1 2 3 8 9 10 11
6 : 0 1 2 3 8 9 10 11
7 : 0 1 2 3 4 9 10 11
Graph 19, on 12 vertices
0 : 3 4 6 7 8 9 10 11
1 : 3 5 6 7 8 9 10 11
2 : 4 5 6 7 8 9 10 11
3 : 0 1 6 7 8 9 10 11
4 : 0 2 6 7 8 9 10 11
5 : 1 2 6 7 8 9 10 11
6 : 0 1 2 3 4 5 9 10
7 : 0 1 2 3 4 5 10 11
8 : 0 1 2 3 4 5 10 11
9 : 0 1 2 3 4 5 6 11
10 : 0 1 2 3 4 5 6 7 8
11 : 0 1 2 3 4 5 7 8 9

Graph 20, on 12 vertices
0 : 3 4 6 7 8 9 10 11
1 : 3 5 6 7 8 9 10 11
2 : 4 5 6 7 8 9 10 11
3 : 0 1 6 7 8 9 10 11
4 : 0 2 6 7 8 9 10 11
5 : 1 2 6 7 8 9 10 11
6 : 0 1 2 3 4 5 9 10
7 : 0 1 2 3 4 5 9 11
8 : 0 1 2 3 4 5 9 11
9 : 0 1 2 3 4 5 6 11
10 : 0 1 2 3 4 5 6 7 8
11 : 0 1 2 3 4 5 7 8 9
Graph 21, on 12 vertices

<table>
<thead>
<tr>
<th>vertex</th>
<th>neighbors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3 4 5 6 8 9 10 11</td>
</tr>
<tr>
<td>1</td>
<td>3 5 6 7 8 9 10 11</td>
</tr>
<tr>
<td>2</td>
<td>4 5 6 7 8 9 10 11</td>
</tr>
<tr>
<td>3</td>
<td>0 1 6 7 8 9 10 11</td>
</tr>
<tr>
<td>4</td>
<td>0 2 6 7 8 9 10 11</td>
</tr>
<tr>
<td>5</td>
<td>0 1 2 7 8 9 10 11</td>
</tr>
<tr>
<td>6</td>
<td>0 1 2 3 4 9 10 11</td>
</tr>
<tr>
<td>7</td>
<td>1 2 3 4 5 9 10 11</td>
</tr>
<tr>
<td>8</td>
<td>0 1 2 3 4 5 10 11</td>
</tr>
<tr>
<td>9</td>
<td>0 1 2 3 4 5 6 7</td>
</tr>
<tr>
<td>10</td>
<td>0 1 2 3 4 5 6 7 8</td>
</tr>
<tr>
<td>11</td>
<td>0 1 2 3 4 5 6 7 8</td>
</tr>
</tbody>
</table>

Graph 22, on 12 vertices

<table>
<thead>
<tr>
<th>vertex</th>
<th>neighbors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3 4 5 6 7 8 10 11</td>
</tr>
<tr>
<td>1</td>
<td>3 5 6 7 8 9 10 11</td>
</tr>
<tr>
<td>2</td>
<td>4 5 6 7 8 9 10 11</td>
</tr>
<tr>
<td>3</td>
<td>0 1 6 7 8 9 10 11</td>
</tr>
<tr>
<td>4</td>
<td>0 2 6 7 8 9 10 11</td>
</tr>
<tr>
<td>5</td>
<td>0 1 2 7 8 9 10 11</td>
</tr>
<tr>
<td>6</td>
<td>0 1 2 3 4 9 10 11</td>
</tr>
<tr>
<td>7</td>
<td>0 1 2 3 4 5 9 10</td>
</tr>
<tr>
<td>8</td>
<td>0 1 2 3 4 5 9 11</td>
</tr>
<tr>
<td>9</td>
<td>1 2 3 4 5 6 7 8</td>
</tr>
</tbody>
</table>
Graph 23, on 12 vertices

0: 3 4 5 6 7 8 9 10
1: 3 4 5 6 8 9 10 11
2: 4 5 6 7 8 9 10 11
3: 0 1 6 7 8 9 10 11
4: 0 1 2 6 7 8 9 11
5: 0 1 2 7 8 9 10 11
6: 0 1 2 3 4 9 10 11
7: 0 2 3 4 5 9 10 11
8: 0 1 2 3 4 5 10 11
9: 0 1 2 3 4 5 6 7
10: 0 1 2 3 5 6 7 8
11: 1 2 3 4 5 6 7 8

Graph 24, on 12 vertices

0: 3 4 5 6 8 9 10 11
1: 3 5 6 7 8 9 10 11
2: 4 5 6 7 8 9 10 11
3: 0 1 4 6 7 9 10 11
4: 0 2 3 7 8 9 10 11
5: 0 1 2 7 8 9 10 11
6: 0 1 2 3 8 9 10 11
7: 1 2 3 4 5 9 10 11
8: 0 1 2 4 5 6 10 11
9: 0 1 2 3 4 5 6 7
10: 0 1 2 3 4 5 6 7 8
Graph 25, on 12 vertices
0 : 3 4 5 7 8 9 10 11
1 : 3 5 6 7 8 9 10 11
2 : 4 5 6 7 8 9 10 11
3 : 0 1 4 6 7 8 9 10
4 : 0 2 3 6 8 9 10 11
5 : 0 1 2 6 7 8 10 11
6 : 1 2 3 4 5 8 9 10
7 : 0 1 2 3 5 9 10 11
8 : 0 1 2 3 4 5 6 11
9 : 0 1 2 3 4 6 7 11
10 : 0 1 2 3 4 5 6 7
11 : 0 1 2 3 4 5 6 7 8

Graph 26, on 12 vertices
0 : 2 4 5 6 7 9 10 11
1 : 3 4 6 7 8 9 10 11
2 : 0 4 5 6 8 9 10 11
3 : 1 5 6 7 8 9 10 11
4 : 0 1 2 7 8 9 10 11
5 : 0 2 3 7 8 9 10 11
6 : 0 1 2 3 8 9 10 11
7 : 0 1 3 4 5 9 10 11
8 : 1 2 3 4 5 6 10 11
9 : 0 1 2 3 4 5 6 7
10 : 0 1 2 3 4 5 6 7 8
11 : 0 1 2 3 4 5 6 7 8
Graph 27, on 12 vertices

0 : 2 4 5 6 7 9 10 11
1 : 3 4 6 7 8 9 10 11
2 : 0 4 5 7 8 9 10 11
3 : 1 5 6 7 8 9 10 11
4 : 0 1 2 6 8 9 10 11
5 : 0 2 3 7 8 9 10 11
6 : 0 1 3 4 8 9 10 11
7 : 0 1 2 3 5 9 10 11
8 : 1 2 3 4 5 6 10 11
9 : 0 1 2 3 4 5 6 7
10 : 0 1 2 3 4 5 6 7 8
11 : 0 1 2 3 4 5 6 7 8

Graph 28, on 12 vertices

0 : 2 4 5 6 8 9 10 11
1 : 3 4 6 7 8 9 10 11
2 : 0 4 5 7 8 9 10 11
3 : 1 5 6 7 8 9 10 11
4 : 0 1 2 6 7 9 10 11
5 : 0 2 3 7 8 9 10 11
6 : 0 1 3 4 8 9 10 11
7 : 1 2 3 4 5 9 10 11
8 : 0 1 2 3 5 6 10 11
9 : 0 1 2 3 4 5 6 7
10 : 0 1 2 3 4 5 6 7 8
11 : 0 1 2 3 4 5 6 7 8

Graph 29, on 12 vertices
Graph 30, on 12 vertices

0 : 2 3 5 6 8 9 10 11
1 : 3 4 5 6 7 8 10 11
2 : 0 4 5 6 7 8 10 11
3 : 1 5 6 7 8 9 10 11
4 : 0 1 2 5 8 9 10 11
5 : 0 2 3 4 8 9 10 11
6 : 0 1 2 3 7 8 9 11
7 : 0 1 2 3 6 9 10 11
8 : 0 1 2 3 4 5 6 10
9 : 0 1 3 4 5 6 7 11
10 : 0 1 2 3 4 5 7 8
11 : 1 2 3 4 5 6 7 9

Graph 31, on 12 vertices

0 : 2 3 5 6 8 9 10 11
1 : 3 4 5 6 7 8 10 11
2 : 0 4 5 7 8 9 10 11
3 : 0 1 5 6 7 9 10 11
4 : 1 2 6 7 8 9 10 11
5 : 0 1 2 3 7 8 10 11
6 : 0 1 3 4 8 9 10 11
7 : 1 2 3 4 5 9 10 11
8 : 0 1 2 4 5 6 10 11
9 : 0 2 3 4 6 7 10 11
10 : 0 1 2 3 4 5 6 7 8 9
11 : 0 1 2 3 4 5 6 7 8 9

Graph 31, on 12 vertices

0 : 2 3 5 6 8 9 10 11
Graph 32, on 12 vertices
0 : 2 3 5 6 8 9 10 11
1 : 3 4 5 6 7 8 9 10
2 : 0 4 5 7 8 9 10 11
3 : 0 1 5 6 7 9 10 11
4 : 1 2 6 7 8 9 10 11
5 : 0 1 2 3 7 8 10 11
6 : 0 1 3 4 8 9 10 11
7 : 1 2 3 4 5 9 10 11
8 : 0 1 2 4 5 6 10 11
9 : 0 1 2 3 4 6 7 11
10 : 0 2 3 4 5 6 7 8
11 : 0 1 2 3 4 5 6 7 8 9

Graph 33, on 12 vertices
0 : 2 3 5 6 8 9 10 11
1 : 3 4 5 6 7 8 9 10
2 : 0 4 5 7 8 9 10 11
3 : 0 1 5 6 7 9 10 11
4 : 1 2 6 7 8 9 10 11
5 : 0 1 2 3 7 8 10 11
6 : 0 1 3 4 8 9 10 11
7 : 1 2 3 4 5 9 10 11
8 : 0 1 2 4 5 6 10 11
9 : 0 1 2 3 4 6 7 11
10 : 0 1 2 3 4 5 6 7 8
11 : 0 2 3 4 5 6 7 8 9
Graph 34, on 12 vertices

0 : 2 3 5 6 8 9 10 11
1 : 3 4 5 6 7 8 9 10
2 : 0 4 5 7 8 9 10 11
3 : 0 1 5 6 7 8 10 11
4 : 1 2 6 7 8 9 10 11
5 : 0 1 2 3 7 8 9 10
6 : 0 1 3 4 7 8 9 10
7 : 1 2 3 4 5 9 10 11
8 : 0 1 2 3 4 5 6 11
9 : 0 1 2 3 4 5 6 7 11
10 : 0 1 2 3 4 5 6 7 8 9
11 : 0 2 3 4 6 7 8 9

Graph 35, on 12 vertices

0 : 2 3 5 6 7 8 9 10
1 : 3 4 5 6 7 8 9 10
2 : 0 4 5 7 8 9 10 11
3 : 0 1 5 6 7 8 10 11
4 : 1 2 6 7 8 9 10 11
5 : 0 1 2 3 7 8 9 10
6 : 0 1 3 4 9 10 11
7 : 1 2 3 4 5 9 10 11
8 : 0 1 2 3 4 5 6 11
9 : 0 1 2 3 4 5 6 7 11
10 : 0 1 2 3 4 5 6 7 8
11 : 0 2 3 4 5 6 7 9
A.3 Problem Graphs on 13 vertices

There are 33 problem graphs on 13 vertices, up to isomorphism.

**Graph 1, on 13 vertices**

0 : 5 6 7 8 9 10 11 12
1 : 5 6 7 8 9 10 11 12
2 : 5 6 7 8 9 10 11 12
3 : 5 6 7 8 9 10 11 12
4 : 5 6 7 8 9 10 11 12
5 : 0 1 2 3 4 10 11 12
6 : 0 1 2 3 4 10 11 12
7 : 0 1 2 3 4 10 11 12
8 : 0 1 2 3 4 10 11 12
9 : 0 1 2 3 4 10 11 12
10 : 0 1 2 3 4 5 6 7 8 9
11 : 0 1 2 3 4 5 6 7 8 9
12 : 0 1 2 3 4 5 6 7 8 9

**Graph 2, on 13 vertices**
Graph 3, on 13 vertices

0 : 5 6 7 8 9 10 11 12
1 : 5 6 7 8 9 10 11 12
2 : 5 6 7 8 9 10 11 12
3 : 5 6 7 8 9 10 11 12
4 : 5 6 7 8 9 10 11 12
5 : 0 1 2 3 4 9 11 12
6 : 0 1 2 3 4 9 11 12
7 : 0 1 2 3 4 10 11 12
8 : 0 1 2 3 4 10 11 12
9 : 0 1 2 3 4 5 6 10
10 : 0 1 2 3 4 5 7 8
11 : 0 1 2 3 4 5 6 7 8
12 : 0 1 2 3 4 5 6 7 8
Graph 4, on 13 vertices

0 : 5 6 7 8 9 10 11 12
1 : 5 6 7 8 9 10 11 12
2 : 5 6 7 8 9 10 11 12
3 : 5 6 7 8 9 10 11 12
4 : 5 6 7 8 9 10 11 12
5 : 0 1 2 3 4 8 9 10
6 : 0 1 2 3 4 9 10 11
7 : 0 1 2 3 4 10 11 12
8 : 0 1 2 3 4 5 11 12
9 : 0 1 2 3 4 5 6 12
10 : 0 1 2 3 4 5 6 7
11 : 0 1 2 3 4 6 7 8
12 : 0 1 2 3 4 7 8 9

Graph 5, on 13 vertices

0 : 4 5 6 7 9 10 11 12
1 : 4 5 6 7 9 10 11 12
2 : 4 6 7 8 9 10 11 12
3 : 5 6 7 8 9 10 11 12
4 : 0 1 2 8 9 10 11 12
5 : 0 1 3 8 9 10 11 12
6 : 0 1 2 3 9 10 11 12
7 : 0 1 2 3 9 10 11 12
8 : 2 3 4 5 9 10 11 12
9 : 0 1 2 3 4 5 6 7 8
10 : 0 1 2 3 4 5 6 7 8
11 : 0 1 2 3 4 5 6 7 8
Graph 6, on 13 vertices
0 : 3 4 5 6 9 10 11 12
1 : 4 5 6 7 9 10 11 12
2 : 5 6 7 8 9 10 11 12
3 : 6 7 8 9 10 11 12
4 : 0 1 7 8 9 10 11 12
5 : 0 1 2 8 9 10 11 12
6 : 0 1 2 3 9 10 11 12
7 : 1 2 3 4 9 10 11 12
8 : 2 3 4 5 9 10 11 12
9 : 0 1 2 3 4 5 6 7 8
10 : 0 1 2 3 4 5 6 7 8
11 : 0 1 2 3 4 5 6 7 8
12 : 0 1 2 3 4 5 6 7 8

Graph 7, on 13 vertices
0 : 3 4 6 8 9 10 11 12
1 : 4 5 6 7 9 10 11 12
2 : 5 6 7 8 9 10 11 12
3 : 5 6 7 9 10 11 12
4 : 0 1 7 8 9 10 11 12
5 : 1 2 3 8 9 10 11 12
6 : 0 1 2 3 9 10 11 12
7 : 1 2 3 4 9 10 11 12
8 : 2 4 5 9 10 11 12
9 : 0 1 2 3 4 5 6 7 8
10 : 0 1 2 3 4 5 6 7 8
11 : 0 1 2 3 4 5 6 7 8
12 : 0 1 2 3 4 5 6 7 8
Graph 8, on 13 vertices
0 : 3 4 5 6 8 9 11 12
1 : 5 6 7 8 9 10 11 12
2 : 5 6 7 8 9 10 11 12
3 : 0 4 5 7 8 9 10 12
4 : 0 3 6 7 8 10 11 12
5 : 0 1 2 3 7 9 11 12
6 : 0 1 2 4 9 10 11 12
7 : 1 2 3 4 5 10 11 12
8 : 0 1 2 3 4 9 10 11
9 : 0 1 2 3 5 6 8 10
10 : 1 2 3 4 6 7 8 9
11 : 0 1 2 4 5 6 7 8
12 : 0 1 2 3 4 5 6 7

Graph 9, on 13 vertices
0 : 3 4 6 7 8 9 11 12
1 : 4 5 6 8 9 10 11 12
2 : 4 5 6 8 9 10 11 12
3 : 0 5 6 7 9 10 11 12
4 : 0 1 2 7 8 9 10 11
5 : 1 2 3 7 8 10 11 12
6 : 0 1 2 3 7 9 10 12
7 : 0 3 4 5 6 8 10 12
8 : 0 1 2 4 5 7 11 12
9 : 0 1 2 3 4 6 10 11
Graph 10, on 13 vertices
0 : 3 4 5 6 8 9 10 11
1 : 4 5 6 7 8 10 11 12
2 : 4 5 6 7 8 10 11 12
3 : 0 5 6 7 8 9 10 12
4 : 0 1 2 7 8 9 10 12
5 : 0 1 2 3 9 10 11 12
6 : 0 1 2 3 9 10 11 12
7 : 1 2 3 4 8 9 10 12
8 : 0 1 2 3 4 7 10 11
9 : 0 3 4 5 6 7 11 12
10 : 0 1 2 3 4 5 6 8
11 : 0 1 2 5 6 7 8 9
12 : 1 2 3 4 5 6 7 9

Graph 11, on 13 vertices
0 : 3 4 5 7 8 10 11 12
1 : 4 5 6 7 8 9 10 11
2 : 5 6 7 8 9 10 11 12
3 : 0 4 6 7 9 10 11 12
4 : 0 1 3 7 8 9 11 12
5 : 0 1 2 6 8 9 10 12
6 : 1 2 3 5 8 9 10 11
7 : 0 1 2 3 4 10 11 12
8 : 0 1 2 4 5 6 9 11
Graph 12, on 13 vertices
0 : 3 4 6 7 9 10 11 12
1 : 3 5 6 8 9 10 11 12
2 : 4 5 7 8 9 10 11 12
3 : 0 1 6 7 9 10 11 12
4 : 0 2 6 8 9 10 11 12
5 : 1 2 7 8 9 10 11 12
6 : 0 1 3 4 9 10 11 12
7 : 0 2 3 5 9 10 11 12
8 : 1 2 4 5 9 10 11 12
9 : 0 1 2 3 4 5 6 7 8
10 : 0 1 2 3 4 5 6 7 8
11 : 0 1 2 3 4 5 6 7 8
12 : 0 1 2 3 4 5 6 7 8

Graph 13, on 13 vertices
0 : 3 4 6 7 9 10 11 12
1 : 3 5 6 8 9 10 11 12
2 : 4 5 7 8 9 10 11 12
3 : 0 1 6 7 9 10 11 12
4 : 0 2 6 8 9 10 11 12
5 : 1 2 7 8 9 10 11 12
6 : 0 1 3 4 8 9 11 12
7 : 0 2 3 5 9 10 11 12
Graph 14, on 13 vertices
0 : 3 4 6 8 9 10 11 12
1 : 3 5 6 7 9 10 11 12
2 : 4 5 7 8 9 10 11 12
3 : 0 1 6 7 9 10 11 12
4 : 0 2 6 8 9 10 11 12
5 : 1 2 7 8 9 10 11 12
6 : 0 1 3 4 9 10 11 12
7 : 1 2 3 5 9 10 11 12
8 : 0 2 4 5 9 10 11 12
9 : 0 1 2 3 4 5 6 7 8
10 : 0 1 2 3 4 5 6 7 8
11 : 0 1 2 3 4 5 6 7 8
12 : 0 1 2 3 4 5 6 7 8

Graph 15, on 13 vertices
0 : 3 4 7 8 9 10 11 12
1 : 3 5 6 8 9 10 11 12
2 : 4 5 6 7 9 10 11 12
3 : 0 1 6 7 9 10 11 12
4 : 0 2 6 8 9 10 11 12
5 : 1 2 7 8 9 10 11 12
6 : 1 2 3 4 9 10 11 12
Graph 16, on 13 vertices
0 : 3 4 5 6 9 10 11 12
1 : 3 5 6 7 8 9 11 12
2 : 4 5 7 8 9 10 11 12
3 : 0 1 6 7 8 10 11 12
4 : 0 2 6 8 9 10 11 12
5 : 0 1 2 7 9 10 11 12
6 : 0 1 3 4 8 9 11 12
7 : 1 2 3 5 8 10 11 12
8 : 1 2 3 4 6 7 11 12
9 : 0 1 2 4 5 6 11 12
10 : 0 2 3 4 5 7 11 12
11 : 0 1 2 3 4 5 6 7 8 9 10
12 : 0 1 2 3 4 5 6 7 8 9 10

Graph 17, on 13 vertices
0 : 3 4 5 6 8 9 11 12
1 : 3 5 6 7 9 10 11 12
2 : 4 5 7 8 9 10 11 12
3 : 0 1 6 7 9 10 11 12
4 : 0 2 7 8 9 10 11 12
5 : 0 1 2 6 8 9 11 12
Graph 18, on 13 vertices

0 : 3 4 5 7 8 9 11 12
1 : 3 6 7 8 9 10 11 12
2 : 4 5 6 7 8 10 11 12
3 : 0 1 5 6 9 10 11 12
4 : 0 2 6 7 8 9 11 12
5 : 0 2 3 7 8 10 11 12
6 : 1 2 3 4 9 10 11 12
7 : 0 1 2 4 5 9 10 11
8 : 0 1 2 4 5 9 10 12
9 : 0 1 3 4 6 7 8 10
10 : 1 2 3 5 6 7 8 9
11 : 0 1 2 3 4 5 6 7
12 : 0 1 2 3 4 5 6 8

Graph 19, on 13 vertices

0 : 3 4 6 7 8 10 11 12
1 : 3 5 6 8 9 10 11 12
2 : 4 5 7 8 9 10 11 12
3 : 0 1 5 7 8 10 11 12
4 : 0 2 6 7 9 10 11 12
5: 1 2 3 7 9 10 11 12
6: 0 1 4 8 9 10 11 12
7: 0 2 3 4 5 10 11 12
8: 0 1 2 3 6 10 11 12
9: 1 2 4 5 6 10 11 12
10: 0 1 2 3 4 5 6 7 8 9
11: 0 1 2 3 4 5 6 7 8 9
12: 0 1 2 3 4 5 6 7 8 9

Graph 20, on 13 vertices
0: 3 4 6 7 8 10 11 12
1: 3 5 6 7 9 10 11 12
2: 4 5 6 7 8 10 11 12
3: 0 1 5 8 9 10 11 12
4: 0 2 6 7 8 9 10 11
5: 1 2 3 8 9 10 11 12
6: 0 1 2 4 7 9 10 12
7: 0 1 2 4 6 9 11 12
8: 0 2 3 4 5 9 10 11
9: 1 3 4 5 6 7 8 12
10: 0 1 2 3 4 5 6 8
11: 0 1 2 3 4 5 7 8
12: 0 1 2 3 5 6 7 9

Graph 21, on 13 vertices
0: 3 4 6 7 9 10 11 12
1: 3 5 6 8 9 10 11 12
2: 4 5 6 8 9 10 11 12
3: 0 1 5 7 8 10 11 12
Graph 22, on 13 vertices

0 : 3 4 6 7 8 10 11 12
1 : 3 5 6 8 9 10 11 12
2 : 4 5 7 8 9 10 11 12
3 : 0 1 5 6 7 10 11 12
4 : 0 2 6 7 8 9 11 12
5 : 1 2 3 7 9 10 11 12
6 : 0 1 3 4 8 9 11 12
7 : 0 2 3 4 5 10 11 12
8 : 0 1 2 4 6 9 10 12
9 : 1 2 4 5 6 8 11 12
10 : 0 1 2 3 5 7 8 12
11 : 0 1 2 3 4 5 6 7 9
12 : 0 1 2 3 4 5 6 7 8 9 10

Graph 23, on 13 vertices

0 : 3 4 6 7 8 10 11 12
1 : 3 5 6 8 9 10 11 12
2 : 4 5 7 8 9 10 11 12
Graph 24, on 13 vertices

0 : 3 4 6 7 8 9 10 12
1 : 3 5 6 8 9 10 11 12
2 : 4 5 7 8 9 10 11 12
3 : 0 1 5 6 7 9 11 12
4 : 0 2 6 7 8 10 11 12
5 : 1 2 3 7 9 10 11 12
6 : 0 1 3 4 8 9 11 12
7 : 0 2 3 4 5 10 11 12
8 : 0 1 2 4 6 9 10 11
9 : 1 2 4 5 6 8 11 12
10 : 0 1 2 3 5 7 8 12
11 : 0 1 2 3 4 5 6 7 8 9
12 : 0 1 2 3 4 5 6 7 9 10

Graph 25, on 13 vertices

0 : 2 4 5 6 8 9 10 11
1 : 3 4 6 7 8 9 11 12
Graph 26, on 13 vertices
0: 2 4 5 6 8 9 11 12
1: 3 4 6 7 8 9 10 12
2: 0 4 5 7 9 10 11 12
3: 1 5 6 7 8 10 11 12
4: 0 1 2 6 8 9 10 12
5: 0 2 3 7 8 10 11 12
6: 0 1 3 4 8 9 11 12
7: 1 2 3 5 9 10 11 12
8: 0 1 3 4 5 6 10 12
9: 0 1 2 4 6 7 11 12
10: 1 2 3 4 5 7 8 12
11: 0 2 3 5 6 7 9 12
12: 0 1 2 3 4 5 6 7 8 9 10 11

Graph 27, on 13 vertices
0: 2 4 5 6 8 9 11 12
Graph 30, on 13 vertices

0 : 2 4 5 6 8 9 11 12
1 : 3 4 6 7 8 9 10 11
2 : 0 4 5 7 9 10 11 12
3 : 1 5 6 7 8 10 11 12
4 : 0 1 2 6 8 9 10 11
5 : 0 2 3 7 8 10 11 12
6 : 0 1 3 4 8 9 11 12
7 : 1 2 3 5 9 10 11 12
8 : 0 1 3 4 5 6 10 12
9 : 0 1 2 4 6 7 11 12
10 : 1 2 3 4 5 7 8 12
11 : 0 1 2 3 4 5 6 7 9
12 : 0 2 3 5 6 7 8 9 10
Graph 31, on 13 vertices

0 : 2 4 5 6 7 9 10 12
1 : 3 4 6 7 8 10 11 12
2 : 0 4 5 7 8 9 10 11
3 : 1 5 6 7 8 9 11 12
4 : 0 1 2 6 8 9 10 11
5 : 0 2 3 7 8 9 11 12
6 : 0 1 3 4 9 10 11 12
7 : 0 1 2 3 5 10 11 12
8 : 1 2 3 4 5 9 10 12
9 : 0 2 3 4 5 6 8 11
10 : 0 1 2 4 6 7 8 12
11 : 1 2 3 4 5 6 7 9
12 : 0 1 3 5 6 7 8 10

Graph 32, on 13 vertices

0 : 2 4 5 6 7 9 10 11
1 : 3 4 6 7 8 9 11 12
2 : 0 4 5 7 8 9 10 12
3 : 1 5 6 7 8 10 11 12
4 : 0 1 2 6 9 10 11 12
5 : 0 2 3 7 8 9 10 11
6 : 0 1 3 4 8 10 11 12
7 : 0 1 2 3 5 9 10 12
8 : 1 2 3 5 6 9 11 12
9 : 0 1 2 4 5 7 8 11
10 : 0 2 3 4 5 6 7 12
11 : 0 1 3 4 5 6 8 9
Graph 33, on 13 vertices

0 : 2 4 5 6 8 9 11 12
1 : 3 4 6 7 8 9 10 12
2 : 0 4 5 7 9 10 11 12
3 : 1 5 6 7 8 9 10 11
4 : 0 1 2 6 7 9 11 12
5 : 0 2 3 7 8 10 11 12
6 : 0 1 3 4 8 9 10 11
7 : 1 2 3 4 5 10 11 12
8 : 0 1 3 5 6 9 10 12
9 : 0 1 2 3 4 6 8 11
10 : 1 2 3 5 6 7 8 12
11 : 0 2 3 4 5 6 7 9
12 : 0 1 2 4 5 7 8 10

A.4 Problem Graphs on 14 vertices

There are 11 problem graphs on 14 vertices, up to isomorphism.

Graph 1, on 14 vertices

0 : 3 4 6 7 10 11 12 13
1 : 4 5 7 8 10 11 12 13
2 : 5 6 8 9 10 11 12 13
3 : 0 6 7 9 10 11 12 13
4 : 0 1 7 8 10 11 12 13
5 : 1 2 8 9 10 11 12 13
6 : 0 2 3 9 10 11 12 13
7 : 0 1 3 4 10 11 12 13
Graph 2, on 14 vertices
0 : 3 5 6 7 9 10 11 12
1 : 4 5 7 8 9 10 12 13
2 : 4 6 7 8 9 11 12 13
3 : 0 5 6 8 9 10 11 13
4 : 1 2 7 8 10 11 12 13
5 : 0 1 3 6 9 10 12 13
6 : 0 2 3 5 9 11 12 13
7 : 0 1 2 4 8 10 11 12
8 : 1 2 3 4 7 10 11 13
9 : 0 1 2 3 5 6 12 13
10 : 0 1 3 4 5 7 8 11
11 : 0 2 3 4 6 7 8 10
12 : 0 1 2 4 5 6 7 9
13 : 1 2 3 4 5 6 8 9

Graph 3, on 14 vertices
0 : 3 5 6 7 8 10 12 13
1 : 4 7 8 9 10 11 12 13
2 : 4 7 8 9 10 11 12 13
3 : 0 5 6 7 9 10 11 13
4 : 1 2 5 6 8 9 11 12
Graph 4, on 14 vertices

0 : 3 4 6 8 9 11 12 13
1 : 3 5 6 7 10 11 12 13
2 : 4 5 7 8 9 10 12 13
3 : 0 1 6 7 9 11 12 13
4 : 0 2 6 8 9 10 12 13
5 : 1 2 7 8 10 11 12 13
6 : 0 1 3 4 9 10 12 13
7 : 1 2 3 5 9 11 12 13
8 : 0 2 4 5 10 11 12 13
9 : 0 2 3 4 6 7 12 13
10 : 1 2 4 5 6 8 12 13
11 : 0 1 3 5 7 8 12 13
12 : 0 1 2 3 4 5 6 7 8 9 10 11
13 : 0 1 2 3 4 5 6 7 8 9 10 11

Graph 5, on 14 vertices

0 : 3 4 6 8 9 11 12 13
1 : 3 5 6 7 10 11 12 13
Graph 6, on 14 vertices

0 : 3 4 7 8 9 11 12 13
1 : 3 5 6 8 10 11 12 13
2 : 4 5 6 7 9 10 12 13
3 : 0 1 6 7 9 11 12 13
4 : 0 2 6 8 9 10 12 13
5 : 1 2 7 8 10 11 12 13
6 : 0 1 3 4 9 10 12 13
7 : 1 2 3 5 9 11 12 13
8 : 0 2 4 5 10 11 12 13
9 : 0 2 3 4 6 7 10 11
10 : 1 2 4 5 6 8 9 11
11 : 0 1 3 5 7 8 9 10
12 : 0 1 2 3 4 5 6 7 8
13 : 0 1 2 3 4 5 6 7 8
Graph 7, on 14 vertices
0 : 3 4 7 8 9 11 12 13
1 : 3 5 6 8 10 11 12 13
2 : 4 5 6 7 9 10 11 12
3 : 0 1 6 7 8 11 12 13
4 : 0 2 6 8 9 10 11 12
5 : 1 2 7 9 10 11 12 13
6 : 1 2 3 4 8 10 11 12
7 : 0 2 3 5 9 11 12 13
8 : 0 1 3 4 6 9 10 13
9 : 0 2 4 5 7 8 10 13
10 : 1 2 4 5 6 8 9 13
11 : 0 1 2 3 4 5 6 7
12 : 0 1 2 3 4 5 6 7
13 : 0 1 3 5 7 8 9 10

Graph 8, on 14 vertices
0 : 2 4 6 7 8 10 11 13
1 : 3 5 6 7 9 10 11 12
2 : 0 4 6 8 9 11 12 13
3 : 1 5 7 8 9 10 12 13
4 : 0 2 6 8 10 11 12 13
5 : 1 3 7 9 10 11 12 13
6 : 0 1 2 4 8 10 11 12
7 : 0 1 3 5 9 10 11 13
8 : 0 2 3 4 6 10 12 13
9 : 1 2 3 5 7 11 12 13
10 : 0 1 3 4 5 6 7 8
Graph 9, on 14 vertices

0 : 2 4 6 7 8 9 11 12
1 : 3 5 6 7 9 10 11 13
2 : 0 4 6 8 9 10 11 12
3 : 1 5 7 8 9 10 12 13
4 : 0 2 6 8 10 11 12 13
5 : 1 3 7 9 10 11 12 13
6 : 0 1 2 4 8 10 11 13
7 : 0 1 3 5 9 11 12 13
8 : 0 2 3 4 6 10 12 13
9 : 0 1 2 3 5 7 11 12
10 : 1 2 3 4 5 6 8 13
11 : 0 1 2 4 5 6 7 9
12 : 0 2 3 4 5 7 8 9
13 : 1 3 4 5 6 7 8 10

Graph 10, on 14 vertices

0 : 2 4 6 7 8 9 10 12
1 : 3 5 6 7 8 10 11 13
2 : 0 4 6 8 9 10 11 12
3 : 1 5 7 8 9 10 11 13
4 : 0 2 6 8 9 11 12 13
5 : 1 3 7 9 10 11 12 13
6 : 0 1 2 4 8 11 12 13
7 : 0 1 3 5 9 10 12 13
Graph 11, on 14 vertices

0 : 2 4 6 7 8 10 11 13
1 : 3 5 6 7 8 9 11 12
2 : 0 4 6 8 9 10 12 13
3 : 1 5 7 8 9 10 11 12
4 : 0 2 6 8 10 11 12 13
5 : 1 3 7 9 10 11 12 13
6 : 0 1 2 4 8 9 12 13
7 : 0 1 3 5 9 10 11 13
8 : 0 1 2 3 4 6 11 12
9 : 1 2 3 5 6 7 12 13
10 : 0 2 3 4 5 7 11 13
11 : 0 1 3 4 5 7 8 10
12 : 1 2 3 4 5 6 8 9
13 : 0 2 4 5 6 7 9 10

A.5 Problem Graphs on 15 vertices

There are 9 problem graphs on 15 vertices, up to isomorphism.

Graph 1, on 15 vertices

0 : 3 4 6 8 10 12 13 14
1 : 4 5 7 9 10 12 13 14
Graph 2, on 15 vertices

0 : 3 4 7 8 9 11 12 14
1 : 3 5 6 8 9 10 12 13
2 : 4 5 6 7 9 10 12 13
3 : 0 1 6 7 9 11 13 14
4 : 0 2 6 8 9 11 13 14
5 : 1 2 8 10 11 12 13
6 : 1 2 3 4 9 10 13 14
7 : 0 2 3 5 10 11 12 14
8 : 0 1 4 5 10 11 12 14
9 : 0 1 2 3 4 6 12 13
10 : 1 2 5 6 7 8 12 14
11 : 0 3 4 5 7 8 13 14
12 : 0 1 2 3 5 7 8 9 10
Graph 3, on 15 vertices

0 : 3 4 5 8 10 11 12 13
1 : 4 6 7 8 9 10 13 14
2 : 5 6 7 8 9 11 12 14
3 : 0 6 7 9 10 11 12 13
4 : 0 1 5 8 10 11 13 14
5 : 0 2 4 8 10 11 12 14
6 : 1 2 3 7 9 10 12 14
7 : 1 2 3 6 9 11 13 14
8 : 0 1 2 4 5 9 12 13
9 : 1 2 3 6 7 8 12 13
10 : 0 1 3 4 5 6 12 14
11 : 0 2 3 4 5 7 13 14
12 : 0 2 3 5 6 8 9 10
13 : 0 1 3 4 7 8 9 11
14 : 1 2 4 5 6 7 10 11

Graph 4, on 15 vertices

0 : 3 4 6 7 9 10 12 13
1 : 3 5 6 8 9 11 12 14
2 : 4 5 7 8 10 11 13 14
3 : 0 1 6 7 9 10 12 13
4 : 0 2 6 8 9 11 12 14
5 : 1 2 7 8 10 11 13 14
6 : 0 1 3 4 9 10 12 13
7 : 0 2 3 5 9 11 12 14

199
Graph 5, on 15 vertices
0 : 3 4 6 7 9 11 12 13
1 : 3 5 6 8 10 11 13 14
2 : 4 5 7 8 9 10 12 14
3 : 0 1 6 7 9 11 13 14
4 : 0 2 6 8 9 10 12 13
5 : 1 2 7 8 10 11 12 14
6 : 0 1 3 4 9 10 12 13
7 : 0 2 3 5 9 11 12 14
8 : 1 2 4 5 10 11 13 14
9 : 0 2 3 4 6 7 12 14
10 : 1 2 4 5 6 8 12 13
11 : 0 1 3 5 7 8 13 14
12 : 0 2 4 5 6 7 9 10
13 : 0 1 3 4 6 8 10 11
14 : 1 2 3 5 7 8 9 11

Graph 6, on 15 vertices
0 : 3 4 6 7 9 11 12 14
1 : 3 5 6 8 10 11 12 13
2 : 4 5 7 8 9 10 13 14

200
Graph 7, on 15 vertices

0 : 3 4 6 7 9 11 12 14
1 : 3 5 6 8 10 11 13 14
2 : 4 5 7 8 9 10 12 13
3 : 0 1 6 7 9 11 13 14
4 : 0 2 6 8 9 10 12 14
5 : 1 2 7 8 10 11 13 14
6 : 0 1 3 4 9 10 12 13
7 : 0 2 3 5 9 11 13 14
8 : 1 2 4 5 10 11 12 14
9 : 0 2 3 4 6 7 12 13
10 : 1 2 4 5 6 8 13 14
11 : 0 1 3 5 7 8 12 14
12 : 0 1 3 4 6 8 9 11
13 : 1 2 3 5 6 7 9 10
14 : 0 2 4 5 7 8 10 11
Graph 8, on 15 vertices
0 : 3 4 6 7 10 11 12 14
1 : 3 5 6 8 9 10 13 14
2 : 4 5 7 8 9 11 12 13
3 : 0 1 6 7 9 11 13 14
4 : 0 2 6 8 10 11 12 13
5 : 1 2 7 8 9 10 12 14
6 : 0 1 3 4 9 10 12 13
7 : 0 2 3 5 9 11 12 14
8 : 1 2 4 5 10 11 13 14
9 : 1 2 3 5 6 7 12 13
10 : 0 1 4 5 6 8 12 14
11 : 0 2 3 4 7 8 13 14
12 : 0 2 4 5 6 7 9 10
13 : 1 2 3 4 6 8 9 11
14 : 0 1 3 5 7 8 10 11

Graph 9, on 15 vertices
0 : 3 4 6 8 9 11 12 13
1 : 3 5 6 7 10 11 13 14
2 : 4 5 7 8 9 10 12 14
3 : 0 1 6 7 9 11 12 13
4 : 0 2 6 8 9 10 12 14
5 : 1 2 7 8 10 11 13 14
6 : 0 1 3 4 9 10 12 13
7 : 1 2 3 5 9 11 13 14
8 : 0 2 4 5 10 11 12 14

202
REFERENCES


[15] Ken ichi Kawarabayashi and Bjarne Toft. “Any 7-chromatic graphs has $K_7$ or $K_{4,4}$ as a minor”. In: *Combinatorica* 25.3 (2005), pp. 327–353.


VITA

Dantong Zhu was born on April 28, 1994 in Beijing, China. She grew up in Beijing and graduated from the High School Affiliated to Renmin University of China in June 2012. Between August 2012 and May 2016, she attended Lafayette College in Easton, PA and graduated with a B.S. in Mathematics. She started her PhD in the Algorithm, Combinatorics, and Optimization program at Georgia Tech in Atlanta, GA in August 2016. Her PhD advisors are Robin Thomas and Xingxing Yu.

Dantong plans to complete the requirements for her PhD in the fall of 2021. After graduation, she will join Uber as an Applied Scientist in San Francisco, CA.