DECAY OF ENTROPY AND INFORMATION OF
MULTIDIMENSIONAL KAC MODELS COUPLED TO LARGE
SYSTEMS

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DECAY OF ENTROPY AND INFORMATION OF MULTIDIMENSIONAL KAC MODELS COUPLED TO LARGE SYSTEMS

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Mathematics is the music of reason.

*James Joseph Sylvester*
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List of Symbols

:= defined as
N set of natural numbers (without zero)
$\mathbb{S}^k$ $k$-dimensional unit sphere in $\mathbb{R}^{k+1}$
$\langle \cdot , \cdot \rangle$ scalar product
$\mathbf{1}$ identity matrix
$L^1$ Lebesgue space of (Lebesgue-)integrable functions with finite $L^1$ norm $\| \cdot \|_1$
$H^k$ Sobolev space with $k = p = 1$
$\| \cdot \|$ operator norm
$\| \cdot \|_1$ $L^1$-norm
d$v, d\omega, dx$ Lebesgue measure
$\sigma \otimes \tau$ tensor product of vectors

g(\cdot) Gaussian distribution, see (6)
$\beta$ inverse temperature of the thermostat or heat reservoir, respectively
$N$ number of particles in the Kac system
$M$ number of particles in the heat reservoir
$\mathcal{L}, \mathcal{L}'$ time evolution operator of the (transformed) Kac master equation
$\lambda$ rate parameter for the collisions of particles within the Kac system
$\mu$ rate parameter for the collisions of particles in the Kac system with the coupled system
$K$ kinetic energy, see (26), (108)
$P$ momentum, see (27)
$S(f|\gamma)$ relative entropy of the PDF $f$ with respect to $\gamma$, see (30)
$\mathcal{I}(f)$ Fisher-information of $f$ with respect to the Lebesgue measure, see (33)
$\tilde{\mathcal{I}}(h)$ Fisher-information of $h$ with respect to the Gaussian measure $g(\mathbf{v})d\mathbf{v}$, see (34)
$P_s$ Ornstein-Uhlenbeck semigroup, see (149)
$M_\sigma, M_{\sigma}^{(i,j)}$ collision matrix, see (4)
$N_\sigma, N_\sigma^{(i)}, N_\sigma^{(j)}$ part of collision matrix, see (17)
$\mathbf{w}_\sigma^{(i)}, \mathbf{w}_\sigma^{(j)}$ collision vectors, see (17)
$\mathbf{P}$ projection onto the first $N$ variables $\mathbf{P} : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N, (v_1, \ldots, v_N, w) \mapsto (v_1, \ldots, v_N)$
Summary

We study the approach to equilibrium in relative entropy of systems of gas particles modeled via the Kac master equation in arbitrary dimensions. First we study the Kac system coupled to a thermostat, and secondly connected to a heat reservoir. The use of the Fisher-information allows simple proofs with weak regularity assumptions. As a result, we obtain exponential decay rates for the entropy and information that are essentially independent of the size of the systems.
1. Introduction

In the seminal paper [10] from 1956, Mark Kac suggested a probabilistic model to describe a spatially homogeneous system of gas particles, now called the Kac Model. Combining the velocities of the $N \in \mathbb{N}$ particles into a "master vector", he introduces energy and momentum preserving pair collisions by deriving the transformation of the master vector. The occurrence of the collisions is modeled by a Poisson-like process. This leads to a linear evolution equation for the probability density function describing the system, called the Kac master equation. The importance of the Kac model is due to the following two reasons: First, Kac was able to derive the non-linear spatially homogeneous Boltzmann equation from the Kac model. Second, the Kac model enables the investigation of concepts in kinetic gas theory that have been largely inaccessible so far with the much more involved Boltzmann equation, in particular the approach to equilibrium.

The connection to the Boltzmann equation is given through the notion of what Kac called initially the Boltzmann property, and got later known as chaotic sequences (see [10, Eq. (3.9)], [1, Eq. (7)], [3, Def. 1.4], [7, Def. 1.1]). The crucial fact proven by Kac is the Propagation of Chaos, i.e., chaotic sequences stay chaotic under the Kac evolution. It implies that the evolution of the $k$-particle marginal of the initial distribution under the non-linear spatially homogenous Boltzmann equation is equal to letting the initial distribution evolve under the Kac master equation, and then taking the $k$-particle marginal.

One way to measure the approach to equilibrium is by the gap of the generator of the
time evolution. The gap has been explicitly computed in 2000 by Carlen, Carvalho
and Loss, see [5]. The obtained waiting times for the approach to equilibrium are
of order $N$. Hence, this approach is only useful if the system is already close to
equilibrium.

A more natural approach is to consider the entropy of the system relative to the
equilibrium state. In contrary to the gap, the relative entropy is proportional to
the size of the system. In 2003, Cedric Villani proved in [14] for arbitrary initial
conditions, that the entropy production is bounded from below with a bound that
is inversely proportional to the size of the system. The result is essentially sharp as
proven 2014 by Amit Einav in [8] by giving an example with essentially that rate.
The example is based on the idea to have a few particles carrying almost all kinetic
energy of the Kac system, while the large majority is very stable. This provides
evidence - although not a proof - that we can not expect a reasonably fast entropy
decay for arbitrary initial conditions. Therefore, it is necessary for better decay rates
to consider specific classes of initial conditions.

In this paper, we consider a multidimensional Kac system first coupled to a thermo-
stat, and secondly coupled to a heat reservoir. We obtain exponential decay rates for
the entropy and information of the Kac system in both models that are essentially
independent of the size of the systems. The approach via the information provides
simple proofs and weak assumptions.

The thermostat is treated as an infinite gas at thermal equilibrium. For each collision
between a particle in the Kac system and a particle in the thermostat a scattering
angle is randomly selected by a probability measure on the $(d - 1)$-dimensional unit
sphere. The post-collisional velocity vectors are obtained by rotating the vector containing the pre-collisional velocities in $2d$ dimensions depending on the scattering angle. Since the thermostat is assumed to be infinite, the post-collisional velocity for the particle in the thermostat is discarded so that the distribution of the thermostat does not change over time. Hence, the Kac system evolves regardless of the initial state to the unique equilibrium state given by the distribution of the thermostat. Although the thermostat is an idealization, it provides realistic rates for the approach to equilibrium that are independent of the size of the system.

More realistic is the Kac system coupled to a heat reservoir. Hereby, we replace the thermostat by a system that is no longer assumed to have infinitely many particles but a number of particles $M \gg N$ that is much larger than the number of particles in the Kac system. The collision process is the same as for the thermostat with the important difference, that the post-collisional velocities of the particles in the heat reservoir do not get discarded anymore. That is, the distribution of the particles in the heat reservoir now changes over time. The fact that one does not know how the heat reservoir evolves makes this model more difficult than the one with the thermostat. However, we can obtain the velocity distribution of the Kac system by the use of marginals with respect to the heat reservoir. This leads to an excellent estimate of the entropy and information.

For one-dimensional models, our obtained decay rates are essentially already known. The decay rate for the thermostat has been computed in [4] for a one-dimensional model. However, the proof does not carry over to the multidimensional case due to technical issues.
The decay rate for a one-dimensional Kac model coupled to a heat reservoir has been proven 2018 by Bonetto, Geisinger, Loss and Ried in [1], and 2020 by Bonetto, Han and Loss in [2]. The first proof in [1] is technically very involved using Nelson’s hyper-contractive estimate for a first entropy inequality and the geometric Brascamp-Lieb inequality for a correlation inequality needed to prove the final entropy inequality. In [2] Bonetto, Han and Loss introduced the approach via the information and obtained the exact same decay rates with a much simpler proof. They also proved a similar result for the standard Kac model on the sphere $S^{N+M-1}(\sqrt{N+M})$ that could not be obtained by the methods in [1].

Therefore, the main results of this paper are on the one hand the proof of the decay rates for multidimensional models. On the other hand, we show that the information provides a natural approach to the entropy decay with simple proofs and weak assumptions.

The structure of the paper is as follows: In section 2 we consider the Kac system coupled to a thermostat. In section 3 we treat the Kac model coupled to a heat reservoir. In both sections, we first present the model with the appropriate definitions and then state the main theorems. The proofs are given in the respective following subsections. In section 4 we provide a discussion of the results and potential future work. In the Appendix A and B we provide a summary of the Ornstein-Uhlenbeck semigroup and the connection between Entropy and Information including detailed proofs for all statements.
2. The Kac Model coupled to a Thermostat

2.1. Model and Assumptions

In this section, we develop the model used for the Kac system coupled to a thermostat. The technical details are covered at the end of the subsection in remark 2.2.

Any state of the Kac system is specified by a probability distribution \( f \in L^1(\mathbb{R}^dN, dv) \).

The kinematic of the collisions is explained in great detail in [6, Appendix A]. To model the collision between two particles \( v, w \in \mathbb{R}^d \) with scattering angle \( \sigma \in S^{d-1} \) we use the reflection map

\[
\begin{pmatrix} v \\ w \end{pmatrix} \mapsto M_{\sigma} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} v - (\sigma[v-w])\sigma \\ w + (\sigma[v-w])\sigma \end{pmatrix}, \quad M_{\sigma} := \begin{pmatrix} 1 - \sigma \otimes \sigma & \sigma \otimes \sigma \\ \sigma \otimes \sigma & 1 - \sigma \otimes \sigma \end{pmatrix}. \tag{1}
\]

For each collision, a scattering angle is selected by a probability measure \( \rho \) on the sphere \( S^{d-1} \), for which we assume the symmetry condition

\[
\int_{S^{d-1}} \sigma_i \sigma_j \, d\rho(\sigma) = \frac{1}{d} \delta_{ij}, \tag{2}
\]

where \( \delta_{ij} = 1 \) if \( i = j \) and zero otherwise. The collisions among the particles in the Kac system are modeled by the operator \( Q : L^1(\mathbb{R}^{dN}, dv) \rightarrow L^1(\mathbb{R}^{dN}, dv) \) defined by \( \text{I} \)

\[
Q[f](v) := \binom{N}{2}^{-1} \sum_{1 \leq i < j \leq N} \int_{S^{d-1}} f \left( M^{(i,j)}_{\sigma}v \right) \, d\rho(\sigma), \tag{3}
\]

---

\(^1\) For \( N = 1 \) we set \( Q = \mathbb{1} \) so that \( \mathbb{1} - Q = 0 \), and the first part in the master equation vanishes.

Note, that this follows naturally, if we define the first part of the master equation via \( (\mathbb{1} - Q) := \binom{N}{2}^{-1} \sum_{i < j} (1 - \tilde{Q}_{i,j})[f], \) where \( \tilde{Q}_{i,j}[f](v) := \int_{S^{d-1}} f \left( M^{(i,j)}_{\sigma}v, t \right) \, d\rho(\sigma). \)
where for \( \sigma \in S^{d-1} \) and \( 1 \leq i < j \leq N \) we define the collision matrix \( M^{(i,j)}_\sigma \in (\mathbb{R}^d)^{N \times N} \) as the identity matrix except for the four entries \((i,i), (i,j), (j,i), (j,j)\):

\[
M^{(i,j)}_\sigma := \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 1 & \sigma \otimes \sigma \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix}.
\] (4)

Note that \( M^{(i,j)}_\sigma v = (v_1, ..., v^*_i, ..., v^*_j, ..., v_N) \) with \( (v^*_i, v^*_j)^T = M_\sigma (v_i, v_j)^T \) describes precisely the collision between the \( i \)-th and \( j \)-th particle.

We further assume, that the velocity vectors of the single particles in the thermostat are distributed according to the Maxwell distribution

\[
\tilde{g}(v) := \left( \frac{\beta}{2\pi} \right)^{\frac{d}{2}} e^{-\frac{\beta}{2} v^2}, \quad v \in \mathbb{R}^d.
\] (5)

Hereby, the mass is assumed to be \( m = 1 \) and the inverse temperature is \( \beta := \frac{1}{kT} \).

The equilibrium state for the Kac system is then given by

\[
g(v) := \left( \frac{\beta}{2\pi} \right)^{\frac{dN}{2}} e^{-\frac{\beta}{2} v^2}, \quad v \in \mathbb{R}^{dN}.
\] (6)

The thermostat is assumed to contain infinitely many particles. Therefore, the velocity distribution of the particles in the thermostat is not influenced by the collision process, i.e., the distribution of the particles in the thermostat remains constant over time. This leads to the following operator \( R : L^1(\mathbb{R}^{dN}, dv) \to L^1(\mathbb{R}^{dN}, dv) \) representing the interaction with the thermostat.
with \( R_j : L^1(\mathbb{R}^{dN}, d\nu) \to L^1(\mathbb{R}^{dN}, d\nu) \) defined by

\[
R_j[f] := \int_{\mathbb{R}^d} dw \int_{S^{d-1}} d\rho(\sigma) g \left( \left[ \mathbf{M}^{(j,N+1)}(\sigma) \right]_{N+1} \right) f \left( \mathbf{P} \mathbf{M}^{(j,N+1)}(\sigma) \right).
\] (8)

Hereby, \( P : \mathbb{R}^{N+1} \to \mathbb{R}^N, (v_1, \ldots, v_N, w) \mapsto (v_1, \ldots, v_N) \) denotes the projection onto the first \( N \) variables.

The time evolution of the Kac system is governed by the semigroup \((e^{Lt})_{t \geq 0}\) generated by the time evolution operator \( L : L^1(\mathbb{R}^{dN}, d\nu) \to L^1(\mathbb{R}^{dN}, d\nu) \) defined by

\[
L \coloneqq \lambda N(Q - 1) + \mu N(R - 1).
\] (9)

That is, for any initial PDF \( f_0 \in L^1(\mathbb{R}^{dN}, d\nu) \), the PDF \( f_t \in L^1(\mathbb{R}^{dN}, d\nu) \) describing the probability of finding the Kac system in a given state \( \nu = (v_1, \ldots, v_N) \in \mathbb{R}^{dN} \) at time \( t \) is given by

\[
f_t = e^{Lt} f_0.
\] (10)

The function \( f_t \) is a PDF if and only if \( f_0 \) is a PDF since the operators \( Q \) and \( R \) are averages over rotations. By proposition 6.2 in [9], it is equivalent to solve the Kac master equation given by the evolution equation corresponding to the abstract Cauchy problem of the generator \( L \):

\[
\frac{d}{dt} f_t = L f_t.
\] (11)

The parameter \( \lambda \) describes the rate with which the particles in the Kac collide with
each other. Similarly, the parameter $\mu$ gives the rate with particles in the Kac system collide with particles in the thermostat.

In remark 2.1 we investigate the collision mapping. In remark 2.2 the technical details of the above construction are rigorously derived.

**Remark 2.1.**

1) The linearity of the collision transformation plays an important role in the proofs of the main results. It is open, how the proofs transfer to nonlinear collision transformations such as the swapping map

\[
(v, w, \sigma) \mapsto \left(\frac{v + w}{2} + \frac{|v - w|}{2} \sigma, \frac{v + w}{2} - \frac{|v - w|}{2} \sigma\right),
\]

(12)

where $\sigma \in S^{d-1}$ denotes the scattering angle (see [6, Appendix A] for details about the collision mappings). The main problem is to prove (or disprove) that the generator of the time evolution $L$ commutes with the Ornstein-Uhlenbeck semigroup.

2) The collision transformation $(v_i, v_j) \mapsto M_\sigma(v_i, v_j)$ of two particles within the Kac system is a momentum and kinetic energy conserving rotation. It follows by straight calculation that

\[
M_\sigma = M_\sigma^T = M_\sigma^{-1},
\]

(13)

which implies that

\[
\det(M_\sigma) = 1, v_i + v_j = v_i^* + v_j^*, v_i^2 + v_j^2 = (v_i^*)^2 + (v_j^*)^2,
\]

(14)

where $(v_i^*, v_j^*)^T = M_\sigma(v_i, v_j)^T$ are the post-collisional velocity vectors. It follows, that
\[ M^{(i,j)} = M^{(i,j)^T} = M^{(i,j)^{-1}}, \quad \det(M^{(i,j)}) = 1, \]
\[ \sum_{i=1}^{N} (M^{(i,j)}v)_i = \sum_{i=1}^{N} v_i, \quad (M^{(i,j)}v)^2 = v^2. \]  \hspace{1cm} (15)

3) It is sometimes useful to write
\[ R_j[f] = \int_{\mathbb{R}^d} d\sigma \int_{S^{d-1}} d\rho(\sigma) g(N_\sigma w + \sigma \otimes \sigma[v_j]) f_t(N^{(j)}_\sigma v + w^{(j)}_\sigma), \]  \hspace{1cm} (16)
where we use the notation \( N_\sigma := 1 - \sigma \otimes \sigma \) and
\[ N^{(i)}_\sigma := \text{diag}(1, \ldots, 1, 1 - \sigma \otimes \sigma, 1, \ldots, 1), \]
\[ w^{(i)}_\sigma := (0, \ldots, 0, \sigma \otimes \sigma[v_j], 0, \ldots, 0). \]  \hspace{1cm} (17)

Note that \( 1 \) is the identity matrix in \( d \) dimensions and
\[ N^{(i)}_\sigma^2 = N^{(i)}_\sigma, \quad M^{(j,N+1)}(v,w) = \left( N^{(j)}_\sigma v + w^{(j)}_\sigma, N_\sigma w + \sigma \otimes \sigma[v_j] \right), \]  \hspace{1cm} (18)
where \( v \in \mathbb{R}^{dN}, w \in \mathbb{R}^d \).

**Remark 2.2.** In the following, we show that the operators \( Q \) and \( R \) are bounded with \( ||Q|| = 1, ||R|| = 1 \). This implies that \( \mathcal{L} \) is bounded with \( ||\mathcal{L}|| \leq 2(\lambda + \mu)N \). Therefore, the time evolution operator \( \mathcal{L} \) is by corollary 1.5 in [9] the generator of the exponential semigroup \( (e^{\mathcal{L}t})_{t \geq 0} \). In fact, we show that \( \mathcal{L} \) generates a contraction semigroup. Further, we show that \( f_0 \) is a probability density function if and only if \( f_t \) is a probability density function for all \( t \geq 0 \). This is due to the fact that the operators \( Q \) and \( R \) are averages over rotations.

i) The operator \( Q \) is well-defined with \( ||Q|| \leq 1 \), since for any \( f \in L^1(\mathbb{R}^{dN},d\nu) \),
we get with the transformation \( p := M_{\sigma}^{(i,j)} v \) that

\[
||Qf||_1 \leq \left( \frac{N}{2} \right)^{-1} \sum_{1 \leq i < j \leq N} \int_{\mathbb{R}^d} dv \int_{S^{d-1}} d\rho(\sigma) \left| f \left( M_{\sigma}^{(i,j)} v \right) \right|
\]
\[
= \left( \frac{N}{2} \right)^{-1} \sum_{1 \leq i < j \leq N} \int_{\mathbb{R}^d} dp \int_{S^{d-1}} d\rho(\sigma) \left| f (p) \right|
\]
\[
= ||f||_1. \tag{19}
\]

ii) The operator \( R \) is well-defined with \( ||R|| \leq 1 \), since for any \( f \in L^1 (\mathbb{R}^d, dv) \), we get with the transformation

\[
\begin{pmatrix} p \\ q \end{pmatrix} := M_{\sigma}^{(j,N+1)} \begin{pmatrix} v \\ w \end{pmatrix}
\]

that

\[
||R_{j}f||_1 \leq \int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} dw \int_{S^{d-1}} d\rho(\sigma) g \left( \left[ M_{\sigma}^{(j,N+1)} \begin{pmatrix} v \\ w \end{pmatrix} \right]_{N+1} \right) \left| f \left( PM_{\sigma}^{(j,N+1)} \begin{pmatrix} v \\ w \end{pmatrix} \right) \right|
\]
\[
= \int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} dw \int_{S^{d-1}} d\rho(\sigma) g(q) |f(p)|
\]
\[
= ||f||_1. \tag{20}
\]

iii) Next, we show that \( \mathcal{L} \) generates a contraction semigroup. Note that we can write \( e^{\mathcal{L}t} \) as the convex combination
\[ e^{\mathcal{L}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} (\lambda N(Q - 1) + \mu N(R - 1))^k \]
\[ = e^{-(\lambda + \mu)Nt} \sum_{k=0}^{\infty} \frac{((\lambda + \mu)Nt)^k}{k!} \left( \frac{\lambda}{\lambda + \mu} Q + \frac{\mu}{\lambda + \mu} R \right)^k. \] (21)

Since \( ||Q|| \leq 1 \) and \( ||R|| \leq 1 \) by the previous parts, we get
\[ \left\| \frac{\lambda}{\lambda + \mu} Q + \frac{\mu}{\lambda + \mu} R \right\| \leq \frac{\lambda}{\lambda + \mu} ||Q|| + \frac{\mu}{\lambda + \mu} ||R|| \leq 1. \] (22)

This proves that
\[ \left\| e^{\mathcal{L}t} \right\| \leq 1. \] (23)

iv) Let \( f_0 \in L^1(\mathbb{R}^dN, dv) \). Then, we get that similar to part [ii] and [iii] that
\[ \int_{\mathbb{R}^dN} Q f_0 \, dv = \int_{\mathbb{R}^dN} f_0 \, dv, \quad \int_{\mathbb{R}^dN} R_j f_0 \, dv = \int_{\mathbb{R}^dN} f_0 \, dv. \] (24)

Termwise integration yields
\[ \int_{\mathbb{R}^dN} e^{\mathcal{L}t} f_0 \, dv = \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_{\mathbb{R}^dN} (\lambda N(Q - 1) + \mu N(R - 1))^k f_0 \, dv \]
\[ = \int_{\mathbb{R}^dN} (\lambda N(Q - 1) + \mu N(R - 1))^0 f_0 \, dv \]
\[ = \int_{\mathbb{R}^dN} f_0 \, dv. \] (25)

Note that we can integrate term by term since \( \mathcal{L} \) is bounded, and therefore the exponential series converges uniformly.
2.2. Definitions and Ground state transformation

A first observation is that the time evolution of the total kinetic energy and the total momentum can be computed explicitly, as done in [4, p. 6] for a one-dimensional model.

For any solution \((f_t)_{t \geq 0}\) of the Kac master equation (11) with \(f_0 \in L^1(\mathbb{R}^{dN}, d\nu)\), the total kinetic energy \(K\) of the Kac system at time \(t \in [0, \infty)\) is defined by

\[
K(t) := \frac{1}{2} \int_{\mathbb{R}^{dN}} v^2 f_t(v) \, d\nu. \tag{26}
\]

The total momentum \(P\) of the Kac system is defined by

\[
P(t) := \sum_{i=1}^{N} \int_{\mathbb{R}^{dN}} v_i f_t(v) \, d\nu. \tag{27}
\]

Differentiating under the integral sign, and inserting the master equation (11) leads to a first-order linear inhomogeneous ODE, from which we obtain the following lemma.

**Lemma 2.3.** The total kinetic energy \(K\) of the Kac system for any solution \((f_t)_{t \geq 0}\) of the Kac master equation (11) with \(f_0 \in L^1(\mathbb{R}^{dN}, d\nu)\) and finite initial kinetic energy, i.e. \(K(0) < \infty\), is given by

\[
K(t) = \left( K(0) - \frac{dN}{\beta} \right) e^{-\frac{\beta}{2d}t} + \frac{dN}{\beta}. \tag{28}
\]

The total momentum \(P\) of the Kac system for any solution \((f_t)_{t \geq 0}\) of the Kac master equation (11) with \(f_0 \in L^1(\mathbb{R}^{dN}, d\nu)\) and finite initial momentum, i.e. \(P(0) < \infty\), is given by

\[
P(t) = e^{-\frac{\beta}{d}t} P(0). \tag{29}
\]
The proof is a rather tedious computation and thus left to the appendix. 

Lemma 2.3 shows that the average kinetic energy of a particle in the Kac system tends exponentially towards the average kinetic energy of a particle in the thermostat. It is interesting to note that (28) is independent of the rate parameter $\lambda$, i.e., the collisions between particles in the Kac system do not influence the time evolution of the kinetic energy of the Kac system. This explains that the rate of convergence is independent of the number of particles in the Kac system. Similarly, the total momentum tends exponentially to zero, and is not influenced by the collisions between particles in the Kac system.

To understand the approach to equilibrium, we consider the relative entropy of the system. As in [4, p. 4], we define the relative entropy of a state $f_t \in L^1(\mathbb{R}^dN, dv)$ at time $t \in [0, \infty)$ with respect to the equilibrium state $\gamma$ as

$$S(f_t|\gamma) := \int_{\mathbb{R}^dN} f_t(v) \ln \left( \frac{f_t(v)}{\gamma(v)} \right) dv. \quad (30)$$

Note, that following the standard in this field, we define the entropy with the inverted sign. Before turning to the information, it is convenient to write our model with the constant function $H \equiv 1$ as ground state instead of the Gaussian $g$, as it was also done in [4, p. 14]. For a solution $(f_t)_{t \geq 0}$ of the master equation (11) we define $(h_t)_{t \geq 0} \in L^1(\mathbb{R}^dN, g(v)dv)$ by

$$h_t(v) := \frac{f_t(v)}{g(v)}. \quad (31)$$

Note, that $h_t$ is a PDF with respect to the Gaussian measure $g(v)dv$ if and only if $f_t$ is a PDF with respect to the Lebesgue measure. This allows us to write the relative
entropy of the system in the following way

\[ S(f_t|g) = \int_{\mathbb{R}^d} h_t \ln(h_t) g(v) dv. \]  

(32)

The Fisher information of a probability density function \( f \in L^1(\mathbb{R}^dN, dv) \) with \( \sqrt{f} \in H^1(\mathbb{R}^dN, dv) \) is defined by (compare to [13, p.1])

\[ \mathcal{I}(f) := \int_{\mathbb{R}^dN} \frac{|\nabla f(v)|^2}{f(v)} dv = \int_{\mathbb{R}^dN} \left( \frac{\nabla f}{\sqrt{f}} \right)^2 dv. \]  

(33)

Note that the integral is finite since

\[ \frac{\nabla f}{\sqrt{f}} = 2 \nabla \left[ \sqrt{f} \right] \in L^2(\mathbb{R}^dN). \]

We define additionally the information of a PDF \( h \) with respect to the Gaussian measure \( g(v)dv \) and \( \sqrt{h} \in H^1(\mathbb{R}^dN, g(v)dv) \) by

\[ \tilde{\mathcal{I}}(h) := \int_{\mathbb{R}^dN} \frac{|\nabla h(v)|^2}{h(v)} g(v) dv. \]  

(34)

The expression (32) will allow us later to transfer the decay of the information of \( h_t \) to the entropy of \( f_t \) using lemma B.3.

**Remark 2.4.** Note that we can relate the information \( \mathcal{I}(f_0) \) to the information \( \tilde{\mathcal{I}}(h_0) \).

If we start with finite kinetic energy \( K(0) < \infty \), we have

\[ \sqrt{f_0} \in H^1(\mathbb{R}^dN, dv) \quad \Leftrightarrow \quad \sqrt{h_0} \in H^1(\mathbb{R}^dN, g(v)dv). \]  

(35)

If one of the statements is true, we get in particular that

\[ \tilde{\mathcal{I}}(h_t) = \mathcal{I}(f_t) + 2\beta^2 \left( K(t) - \frac{dN}{\beta} \right) \]  

(36)
Note, that \( \lim_{t \to \infty} \tilde{I}(h_t) = \lim_{t \to \infty} I(f_t) \) since \( K(t) \to \frac{dN}{\beta} \) by lemma 2.3.

**Proof.** It is clear that \( f_0 \in L^1(\mathbb{R}^dN, d\mathbf{v}) \) if and only if \( h_0 \in L^1(\mathbb{R}^dN, g(\mathbf{v})d\mathbf{v}) \). By the definition, it is also clear that \( \nabla f_0 \) exists in a weak sense if and only if \( \nabla h_0 \) exists in a weak sense. Therefore, the following computation completes the proof:

\[
\tilde{I}(h_t) = \int_{\mathbb{R}^dN} \frac{|\nabla h_t|^2}{h_t} g(\mathbf{v})d\mathbf{v} = \int_{\mathbb{R}^dN} \left| \frac{\nabla f_t \cdot g + \beta v f_t g^2}{g^2} \right|^2 \frac{g^2}{f_t} d\mathbf{v}
\]

\[
= \int_{\mathbb{R}^dN} \frac{|\nabla f_t + \beta v f_t|^2}{f_t} d\mathbf{v}
\]

\[
= \int_{\mathbb{R}^dN} \frac{|\nabla f_t|^2 + 2\beta f_t \nabla f_t + \beta^2 v^2 f_t^2}{f_t} d\mathbf{v}
\]

\[
= I(f_t) + 2\beta \int_{\mathbb{R}^dN} v \nabla f_t d\mathbf{v} + \beta^2 \int_{\mathbb{R}^dN} v^2 f_t d\mathbf{v}
\]

\[
= I(f_t) - 2\beta \int_{\mathbb{R}^dN} \nabla \cdot (v f_t) d\mathbf{v} + 2\beta^2 K(t)
\]

\[
= I(f_t) + 2\beta^2 \left( K(t) - \frac{dN}{\beta} \right). \tag{37}
\]

The transformed master equation for \( h_t \) is given in the following lemma.

**Lemma 2.5.** The functions \((f_t)_{t \geq 0} \in L^1(\mathbb{R}^dN, d\mathbf{v})\) form a solution of the Kac master equation (11) if and only if \((h_t)_{t \geq 0} \in L^1(\mathbb{R}^dN, g(\mathbf{v})d\mathbf{v})\) defined by (31) solves the transformed master equation

\[
\frac{\partial h_t}{\partial t} = L'[h_t] := \lambda N(Q - \mathbf{1})[h] + \mu N(T - \mathbf{1})[h], \tag{38}
\]

where \( T := \frac{1}{N} \sum_{j=1}^{N} T_j \) with
\[ T_j[h_t](v) := \int_{\mathbb{R}^d} g(w) dw \int_{\mathbb{S}^{d-1}} d\rho(\sigma) h_t \left( P M^{(j,N+1)}_{\sigma}(v,w)^T \right). \]  \hfill (39)

**Remark.** The transformed time evolution operator \( \mathcal{L}' \) acting on \( L^1(\mathbb{R}^d; g(v) dv) \) generates again a contraction semigroup, since \( ||Q|| \leq 1 \) and \( ||T_j|| \leq 1 \). This follows in the same fashion as for \( \mathcal{L} \) in remark 2.2.

**Proof.** To transform the master equation we note that

\[
\frac{\partial h_t}{\partial t} = \frac{1}{g(v)} \cdot \frac{\partial f_t}{\partial t} = \lambda N \frac{1}{g(v)}(Q - 1)[f_t] + \mu \frac{1}{g(v)} \sum_{j=1}^{N}(R_j - 1)[f_t]. \]  

Next, we investigate how the operators \( Q \) and \( R_j \) act in terms of \( h_t \). Since \( g(v_{i,j}(\sigma)) = g(v) \) for all \( \sigma \in \mathbb{S}^2 \), we get

\[
\frac{1}{g(v)} Q[f_t](v) = \binom{N}{2}^{-1} \sum_{i<j} \int_{\mathbb{S}^{d-1}} d\rho(\sigma) h_t \left( M_{\sigma}^{(i,j)} v \right) g \left( M_{\sigma}^{(i,j)} v \right) g(v) = Q[h_t](v), \]  \hfill (41)

since \( (M_{\sigma}^{(i,j)} v)^2 = v^2 \). Further, we get denoting \( (v,w)^* := M_{\sigma}^{(j,N+1)}(v,w)^T \), that

\[
\frac{1}{g(v)} R_j[f_t](v) = \int_{\mathbb{R}^d} dw \int_{\mathbb{S}^{d-1}} d\rho(\sigma) g( (v,w)_{N+1}^* ) g( P(v,w)^* ) h_t( P(v,w)^* )
= T_j[h_t](v), \]  \hfill (42)

since

\[
g( (v,w)_{N+1}^* ) g( P(v,w)^* ) = g( (v,w)^* ) = g( (v,w) ) = g(v) g(w). \]  \hfill (43)

This yields the transformed master equation. \( \square \)
2.3. Main results

Our first main result is the following theorem about the decay of the information of the Kac system.

**Theorem 2.6.** Let \((h_t)_{t \geq 0}\) be a solution of the transformed master equation (38) with \(\sqrt{h_0} \in H^1(\mathbb{R}^d, g(v)dv)\). Then, the information decays exponentially in time:

\[
\tilde{I}(h_t) \leq e^{-\frac{1}{\beta} \mu t} \tilde{I}(h_0).
\]

The proof is given in section 2.4.

The connection between information and entropy is established via the Ornstein-Uhlenbeck semigroup \((P_s)_{s \geq 0}\) (see definition A.1). From lemma B.3 we know that

\[
S(f|g) = \frac{1}{\beta} \int_0^\infty \tilde{I}\left(P_se^{\mathcal{L}'t}h_0\right) \, ds.
\]

This raises the question, if the Ornstein-Uhlenbeck semigroup commutes with the time evolution operator \(\mathcal{L}'\), since then the decay rate for the information

\[
\tilde{I}(h_t) \leq c(t) \tilde{I}(h_0),
\]

would imply the same decay rate for the entropy

\[
S(f_t|g) = \frac{1}{\beta} \int_0^\infty ds \tilde{I}\left(P_se^{\mathcal{L}'t}h_0\right) = \frac{1}{\beta} \int_0^\infty ds \tilde{I}\left(e^{\mathcal{L}'t}P_sh_0\right)
\leq \frac{1}{\beta} c(t) \int_0^\infty ds \tilde{I}(P_sh_0) = c(t)S(f_0|g).
\]

For the reflection map we use as collision mechanism, the Ornstein-Uhlenbeck semi-
group indeed commutes with the time evolution operator $\mathcal{L}'$:

**Theorem 2.7.** The Ornstein-Uhlenbeck semigroup commutes with the time evolution operator $\mathcal{L}'$, i.e., for all $s \geq 0$ we have

$$P_s \mathcal{L}' = \mathcal{L}' P_s.$$  \hspace{1cm} (48)

Hereby, the operators are assumed to act on $L^1(\mathbb{R}^{dN}, g(v) dv)$. The proof is given in section 2.5.

Therefore, theorem 2.7 implies by inequality (47) the following theorem about the transfer of the decay rate from the information to the entropy.

**Theorem 2.8.** Let $(f_t)_{t \geq 0}$ be a solution of the Kac master equation (11) with $\sqrt{f_0} \in H^1(\mathbb{R}^{dN}, g(v) dv)$ and $K(0) < \infty$. Assume we have for $(h_t)_{t \geq 0}$ defined by (31) an inequality for the information

$$\tilde{I}(h_t) \leq c(t) \tilde{I}(h_0),$$  \hspace{1cm} (49)

then we get an inequality for the entropy by

$$S(f_t | g) \leq c(t) S(f_0 | g).$$  \hspace{1cm} (50)

Theorem 2.6 together with theorem 2.8 prove the entropy decay as stated in the following theorem.

**Theorem 2.9.** Let $(f_t)_{t \geq 0}$ be a solution of the master equation (11) with $\sqrt{f_0} \in H^1(\mathbb{R}^{dN}, dv)$ and $K(0) < \infty$. Then
\[ S(f_t|g) \leq e^{-\frac{1}{2} \mu t} S(f_0|g). \] (51)

**Remark 2.10.** Theorem 2.7 also shows that if \( h_t \) is a solution of the master equation (38), then \( P_s h_t \) is also a solution since
\[ (P_s h)_t = P_s h_t = P_s e^{C^t} h_0 = e^{C^t} P_s h_0 = e^{C^t} (P_s h)_0. \] (52)

### 2.4. Proof of Theorem 2.6

The proof consists of two parts. First, we estimate how pair collisions decrease the information of the Kac system. This includes the collisions of either two particles in the Kac system, or a particle in the Kac system with a particle in the thermostat.

Second, we use the convexity of the information to break down the time evolution into single pair collisions. For this, we repeatedly write the solution \( h_t \) as convex combination of simpler terms and use Jensen’s inequality.

**Lemma 2.11.** The collisions of the particles in the Kac system with each other do not increase the information of the system. I.e., for all initial conditions \( \sqrt{h_0} \in H^1(\mathbb{R}^d, g(v)dv) \) the following estimate holds true:
\[ \tilde{I}(Q h_0) \leq \tilde{I}(h_0). \] (53)

**Proof.** Using the convexity of the information (see remark 3.2), we get
\[ \tilde{I}(Qh_0) = \tilde{I} \left( \frac{N}{2} \sum_{i<j}^{N} \int_{S^{d-1}} \text{d}\rho(\sigma) \ h_0 \circ M_{\sigma}^{(i,j)} \right) \]

\[ \leq \left( \frac{N}{2} \right)^{\frac{1}{2}} \sum_{i<j}^{N} \int_{S^{d-1}} \text{d}\rho(\sigma) \ \tilde{I} \left( h_0 \circ M_{\sigma}^{(i,j)} \right). \tag{54} \]

Since the matrix \( M_{\sigma} = \begin{pmatrix} 1 - \sigma \otimes \sigma & \sigma \otimes \sigma \\ \sigma \otimes \sigma & 1 - \sigma \otimes \sigma \end{pmatrix} \) fulfills \( M_{\sigma}^2 = 1 \), the only possible eigenvalues of \( M_{\sigma} \) are \( \pm 1 \). This implies

\[ \left\| M_{\sigma}^{(i,j)} \right\| = \max_{\lambda \in \sigma(M_{\sigma}^{(i,j)})} |\lambda| = 1. \tag{55} \]

Therefore, we get

\[ \tilde{I} \left( h_0 \circ M_{\sigma}^{(i,j)} \right) = \int g(v) \text{d}v \left( \frac{M_{\sigma}^{(i,j)} \cdot \nabla h_0 \left( M_{\sigma}^{(i,j)} v \right)}{h_0 \left( M_{\sigma}^{(i,j)} v \right)} \right)^2 \]

\[ = \int g(v) \text{d}v \left( \frac{\nabla h_0 \left( M_{\sigma}^{(i,j)} v \right)}{h_0 \left( M_{\sigma}^{(i,j)} v \right)} \right)^2 = \int g(p) \text{d}p \left( \frac{\nabla h_0 (p)}{h_0 (p)} \right)^2 \]

\[ = \tilde{I}(h_0). \tag{56} \]

This completes the proof. \( \square \)

**Lemma 2.12.** The collision of a particle in the Kac system with the thermostat decreases its information by at least a factor of \( \frac{1}{d} \). The information of the other particles remains unchanged. I.e., for all initial conditions \( \sqrt{h_0} \in H^1 \left( \mathbb{R}^{dN}, g(v) \text{d}v \right) \), we have the estimate
\[ \bar{I}(T_jh_0) \leq \bar{I}(h_0) - \frac{1}{d} \int_{\mathbb{R}^{dN}} g(v) dv \frac{|(\nabla h_0(v))_j|^2}{h_0(v)}, \quad (57) \]

where \((\nabla h_0(v))_j \in \mathbb{R}^d\) denotes the \(j\)-th three dimensional block component.

**Proof.** For shorter notation, we define (see remark 2.1)

\[ v^* := PM^{(j,N+1)}(v) = N^{(j)}_\sigma v + w^{(j)}_\sigma. \quad (58) \]

Using the Cauchy-Schwarz inequality we get

\[ \nabla [T_jh_0] = \int g(w)dw \int_{S^{d-1}} d\rho(\sigma) N^{(j)}_\sigma \nabla h_0(v^*) \]

\[ |\nabla [T_jh_0]|^2 = \left( \int g(w)dw \int_{S^{d-1}} d\rho(\sigma) \frac{N^{(j)}_\sigma \nabla h_0(v^*)}{\sqrt{h_0(v^*)}} \cdot \sqrt{h_0(v^*)} \right)^2 \]

\[ \leq \int g(w)dw \int_{S^{d-1}} d\rho(\sigma) \frac{N^{(j)}_\sigma \nabla h_0(v^*)}{h_0(v^*)} \cdot T_jh_0. \quad (59) \]

Further since \(N^{(j)}_\sigma^2 = N^{(j)}_\sigma\), we get

\[ \left| N^{(j)}_\sigma \nabla h_0 \right|^2 = \nabla h_0^T N^{(j)}_\sigma \nabla h_0 = |\nabla h_0|^2 - (\nabla h_0^T)_j (\sigma \otimes \sigma)(\nabla h_0)_j \]

\[ = |\nabla h_0|^2 - |\sigma(\nabla h_0)_j|^2. \quad (60) \]

Therefore, we get with the transformation (see 18)

\[ (p, q) := M^{(j,N+1)}(v, w) \leftrightarrow (v, w) := M^{(j,N+1)}(p, q) \]

that the information decreases by
\[ \tilde{I}(T_j h_0) = \int_{\mathbb{R}^d} g(v) dv \frac{\left| \nabla [T_j h_0] \right|^2}{T_j h_0} \]

\[ \leq \int_{\mathbb{R}^d} g(v) dv \int g(w) dw \int_{S^{d-1}} d\rho(\sigma) \frac{\left| N^{(j)} \nabla h_0 (v^*) \right|^2}{h_0 (v^*)} \]

\[ = \int_{\mathbb{R}^d} g(p) dp \int_{\mathbb{R}^d} g(q) dq \int_{S^{d-1}} d\rho(\sigma) \frac{\left| N^{(j)} \nabla h_0 (p) \right|^2}{h_0 (p)} \]

\[ = \int_{\mathbb{R}^d} g(p) dp \int_{\mathbb{R}^d} g(q) dq \int_{S^{d-1}} d\rho(\sigma) \frac{|\nabla h_0|^2 - |\sigma (\nabla h_0)_j|^2}{h_0} \]

\[ \leq \tilde{I}(h_0) - \frac{1}{d} \int_{\mathbb{R}^d} g(p) dp \frac{|(\nabla h_0)_j|^2}{h_0}. \]

\[ \square \]

**Remark 2.13.** It follows using the convexity of the information from the lemma 2.12 that

\[ \tilde{I}(Th_0) = \tilde{I} \left( \frac{1}{N} \sum_{j=1}^{N} T_j h_0 \right) \leq \frac{1}{N} \sum_{j=1}^{N} \tilde{I}(T_j h_0) \leq \frac{1}{N} \left( N \tilde{I}(h_0) - \frac{1}{d} \tilde{I}(h_0) \right) \]

\[ = \left( 1 - \frac{1}{dN} \right) \tilde{I}(h_0). \]  

(61)

In other words, the operator $T$ decreases the information by a factor of at least $(1 - \frac{1}{dN})$.

Next, we can prove theorem 2.6 by repeatedly applying the convexity of the information to break down the information into simpler functions.

First, we write the solution $h_t$ as a convex series
\[ h_t = \exp \left( -\lambda Nt(1 - Q) - \mu Nt(1 - T) \right) h_0 \]
\[ = \exp \left( -\left( \lambda + \mu \right) Nt + (\lambda Q + \mu T) Nt \right) h_0 \]
\[ = e^{-\left( \lambda + \mu \right) Nt} \sum_{k=0}^{\infty} \frac{\left( \left( \lambda + \mu \right) Nt \right)^k}{k!} \left( \frac{\lambda}{\lambda + \mu} Q + \frac{\mu}{\lambda + \mu} T \right)^k h_0. \tag{62} \]

Hence, we get for the information of \( h_t \) that
\[ \tilde{I}(h_t) \leq e^{-\left( \lambda + \mu \right) Nt} \sum_{k=0}^{\infty} \frac{\left( \left( \lambda + \mu \right) t \right)^k}{k!} \tilde{I} \left( \left( \frac{\lambda}{\lambda + \mu} Q + \frac{\mu}{\lambda + \mu} T \right)^k h_0 \right). \tag{63} \]

Noting that \( \frac{\lambda}{\lambda + \mu} Q + \frac{\mu}{\lambda + \mu} T \) is also a convex combination, we get using lemma\,2.11 and remark\,2.13 that
\[ \tilde{I} \left( \left( \frac{\lambda}{\lambda + \mu} Q + \frac{\mu}{\lambda + \mu} T \right) h_0 \right) \leq \frac{\lambda}{\lambda + \mu} \tilde{I}(Qh_0) + \frac{\mu}{\lambda + \mu} \tilde{I}(Th_0) \]
\[ \leq \frac{\lambda}{\lambda + \mu} \tilde{I}(h_0) + \frac{\mu}{\lambda + \mu} \left( 1 - \frac{1}{dN} \right) \tilde{I}(h_0) \]
\[ = \left( 1 - \frac{\mu}{dN(\lambda + \mu)} \right) \tilde{I}(h_0). \tag{64} \]

The claim now follows from the evaluation of the series
\[ \tilde{I}(h_t) \leq e^{-\left( \lambda + \mu \right) Nt} \sum_{k=0}^{\infty} \frac{\left( \left( \lambda + \mu \right) Nt \right)^k}{k!} \left( 1 - \frac{\mu}{dN(\lambda + \mu)} \right)^k \tilde{I}(h_0) \]
\[ = \exp \left( -\left( \lambda + \mu \right) Nt + (\lambda + \mu) N \left( 1 - \frac{\mu}{dN(\lambda + \mu)} \right) \right) \tilde{I}(h_0) \]
\[ = e^{\frac{-1}{\mu} \lambda t} \tilde{I}(h_0). \tag{65} \]
2.5. Proof of Theorem [2.7]

It is sufficient to prove, that the Ornstein-Uhlenbeck semigroup commutes with the operators $Q$ and $T_j$ for $j \in \{1, \ldots, N\}$.

**Step 1:** We prove first, that the operator $Q$ commutes with the Ornstein-Uhlenbeck semigroup, i.e., for all $s \geq 0$ we have

$$QP_s = P_s Q.$$  \hspace{1cm} (66)

We write the transformation of the collisions using the matrix $v_{i,j}(\sigma) = M^{(i,j)}_\sigma$ as described in [4] The transformation $y = M^{(i,j)}_\sigma x$ yields

$$P_s Q[h] = \left(\frac{N}{2}\right)^{-1} \int_{\mathbb{R}^{dN}} g(x) dx \sum_{i<j} \int_{S^{d-1}} d\rho(\sigma) h \left( \frac{M^{(i,j)}_\sigma (e^{-s}v + \sqrt{1 - e^{-2s}}x)}{e^{-s}M^{(i,j)}_\sigma v + \sqrt{1 - e^{-2s}}M^{(i,j)}_\sigma x} \right)$$

$$= \left(\frac{N}{2}\right)^{-1} \int_{\mathbb{R}^{dN}} g(y) dy \sum_{i<j} \int_{S^{d-1}} d\rho(\sigma) h \left( e^{-s}M^{(i,j)}_\sigma v + \sqrt{1 - e^{-2s}}y \right)$$

$$= QP_s[h].$$  \hspace{1cm} (67)

**Step 2:** Next, we prove that the operators $T_j$ commute with the Ornstein-Uhlenbeck semigroup.

We interpret any function $h^{(1)} \in L^1(\mathbb{R}^{dN}, g(v) dv)$ as a function in two variables

$$h^{(2)} : \mathbb{R}^{dN} \times \mathbb{R}^d \rightarrow \mathbb{R}, (v, w) \mapsto h^{(1)}(v).$$  \hspace{1cm} (68)

This allows us to write
\[
T_j[h^{(1)}](v) = \int_{\mathbb{R}^d} g(w)dw \int_{\mathbb{R}^{d-1}} d\rho(\sigma) h^{(2)} \left( M^{(j,1)}_{\sigma}(v, w) \right).
\] (69)

With the definitions
\[
U[h^{(2)}](v, w) := \int_{\mathbb{R}^{d-1}} d\rho(\sigma) h^{(2)} \left( M^{(j,1)}_{\sigma}(v, w) \right),
\] (70)
\[
M[h^{(2)}](v) := \int_{\mathbb{R}^d} g(w)dw h^{(2)}(v, w),
\] (71)

we get
\[
T_j[h^{(1)}] = MU[h^{(1)}].
\] (72)

Similarly, we denote with \(P_s^{(1)}\) the \(dN\)-dimensional, and with \(P_s^{(2)}\) the \((dN+1)\)-dimensional Ornstein-Uhlenbeck semigroup. Using \(\alpha := e^{-s}\) and \(\delta := \sqrt{1-e^{-2s}}\) for shorter notation, we get
\[
P_s^{(2)} h^{(2)}(v, w) = \int_{\mathbb{R}^{dN}} g(x)dx \int_{\mathbb{R}^d} g(y)dy h^{(2)}(\alpha(v, w) + \delta(x, y))
= \int_{\mathbb{R}^{dN}} g(x)dx h^{(1)}(\alpha v + \delta x)
= P_s^{(1)} h^{(1)}(v).
\] (73)

The commutation relation between the Ornstein-Uhlenbeck semigroup and the marginal \(\mathcal{M}\) is given by
\[ P_s^{(1)} \mathcal{M}[h^{(2)}](\mathbf{v}) = \int_{\mathbb{R}^d} g(\mathbf{x}) d\mathbf{x} \int_{\mathbb{R}^d} g(p) dp \ h^{(2)}(\alpha \mathbf{v} + \delta \mathbf{x}, p) \]
\[ = \int_{\mathbb{R}^d} g(\mathbf{x}) d\mathbf{x} \int_{\mathbb{R}^d} g(p) dp \ \int_{\mathbb{R}^d} g(q) dq \ h^{(2)}(\alpha \mathbf{v} + \delta \mathbf{x}, p) \]
\[ = \int_{\mathbb{R}^d} g(\mathbf{x}) d\mathbf{x} \int_{\mathbb{R}^d} g(w) dw \ \int_{\mathbb{R}^d} g(y) dy \ h^{(2)}(\alpha \mathbf{v} + \delta \mathbf{x}, \alpha w + \delta y) \]
\[ = \int_{\mathbb{R}^d} g(w) dw \ \mathcal{M} P_s^{(2)}[h^{(2)}](\mathbf{v}, w) \]
\[ = \mathcal{M} P_s^{(2)}[h^{(2)}](\mathbf{v}, w), \quad (74) \]

where we used the rotation transformation
\[ \begin{pmatrix} p \\ q \end{pmatrix} := \begin{pmatrix} \alpha & \delta \\ -\delta & \alpha \end{pmatrix} \begin{pmatrix} w \\ y \end{pmatrix}. \]

Finally, the operator \( U \) commutes with the \( d(N+1) \)-dimensional Ornstein-Uhlenbeck semigroup:

\[ U P_s^{(2)}[h^{(2)}] = \int_{\mathbb{S}^{d-1}} d\rho(\sigma) \int_{\mathbb{R}^d} g(\mathbf{x}) d\mathbf{x} \int_{\mathbb{R}^d} g(y) dy \ h^{(2)} \left( \mathbf{M}_{\sigma}^{(j,N+1)}(\mathbf{v}, w) + \delta(\mathbf{x}, y) \right) \]
\[ = \int_{\mathbb{S}^{d-1}} d\rho(\sigma) \int_{\mathbb{R}^d} g(\mathbf{x}) d\mathbf{x} \int_{\mathbb{R}^d} g(y) dy \ h^{(2)} \left( \alpha \mathbf{M}_{\sigma}^{(j,N+1)}(\mathbf{v}, w) + \delta \mathbf{M}_{\sigma}^{(j,N+1)}(\mathbf{x}, y) \right) \]
\[ = \int_{\mathbb{S}^{d-1}} d\rho(\sigma) \int_{\mathbb{R}^d} g(\mathbf{x}) d\mathbf{x} \int_{\mathbb{R}^d} g(y) dy \ h^{(2)} \left( \alpha \mathbf{M}_{\sigma}^{(j,N+1)}(\mathbf{v}, w) + \delta(\mathbf{p}, \mathbf{q}) \right) \]
\[ = P_s^{(2)} U[h^{(2)}](\mathbf{v}, w), \quad (75) \]

where we used the transformation
\[ \begin{pmatrix} P \\ q \end{pmatrix} := \mathbf{M}_{\sigma}^{(j,N+1)} \begin{pmatrix} \mathbf{x} \\ y \end{pmatrix}. \]

The claim now follows by
\[ P_s^{(1)} T_j[h^{(1)}] = P_s^{(1)} \mathcal{M} U[h^{(2)}] = \mathcal{M} P_s^{(2)} U[h^{(2)}] = \mathcal{M} U P_s^{(2)}[h^{(2)}] = T_j P_s^{(1)}[h^{(1)}]. \]
Remark 2.14. When particles of the Kac system collide with the thermostat, the order of the collisions does not matter. That is,

\[
[T_i, T_j] := T_i T_j - T_j T_i = 0 \quad (76)
\]
on \( L^1 (\mathbb{R}^d, g(v)dv) \) for all \( i, j \in \{1, \ldots, N\} \).

Proof. We write

\[
T_i[h_0](v) = \int_{\mathbb{R}^d} g(w)dw \int_{S^{d-1}} d\sigma \ h_0 \left( N_{\sigma}^{(i)} v + w^{(i)}_{\sigma} \right), \quad (77)
\]
where we use the notation defined in \( (17) \). Clearly, we have for all \( \sigma, \tau \in \mathbb{S}^2 \), that

\[
[N_{\sigma}^{(i)}, N_{\tau}^{(j)}] = 0, \quad N_{\sigma}^{(i)} w^{(j)}_{\tau} = u^{(j)}_{\tau}. \quad (78)
\]
Hence, we get

\[
T_i T_j[h_0] = \int_{\mathbb{R}^d} g(p)dp \int_{\mathbb{R}^d} g(q)dq \int_{S^{d-1}} d\sigma \int_{S^{d-1}} d\tau h_0 \left( N_{\tau}^{(j)} \left( N_{\sigma}^{(i)} v + p^{(i)}_{\sigma} \right) + q^{(j)}_{\tau} \right)
= \int_{\mathbb{R}^d} g(p)dp \int_{\mathbb{R}^d} g(q)dq \int_{S^{d-1}} d\sigma \int_{S^{d-1}} d\tau h_0 \left( N_{\tau}^{(j)} \left( N_{\sigma}^{(i)} v + q^{(j)}_{\tau} \right) + p^{(i)}_{\sigma} \right)
= T_j T_i[h_0], \quad (79)
\]
since

\[
N_{\tau}^{(j)} \left( N_{\sigma}^{(i)} v + p^{(i)}_{\sigma} \right) + q^{(j)}_{\tau} = N_{\tau}^{(j)} N_{\sigma}^{(i)} v + N_{\tau}^{(j)} p^{(i)}_{\sigma} + q^{(j)}_{\tau} \quad (79)
= N_{\sigma}^{(i)} N_{\tau}^{(j)} v + N_{\sigma}^{(i)} q^{(j)}_{\tau} + p^{(i)}_{\sigma}
= N_{\sigma}^{(i)} \left( N_{\tau}^{(j)} v + q^{(j)}_{\tau} \right) + p^{(i)}_{\sigma}. \quad (80)
\]
3. The Kac Model coupled to a Heat Reservoir

In this section, we replace the thermostat by a finite heat reservoir. The main difference is that the number of particles in the heat reservoir \( M \in \mathbb{N} \) is assumed to be finite and thus the state of the heat reservoir changes over time.

3.1. Model

The system is described by a probability distribution \( F : \mathbb{R}^dN \times \mathbb{R}^dM \to \mathbb{R} \) of the velocity vectors of all particles, normalized with respect to the Lebesgue measure. We use the notation \((v, w) = (v_1, \ldots, v_N, w_{N+1}, \ldots, w_{N+M})\) to number the particles from 1 to \( N+M \). The pair collisions are modeled by the operator \( R_{ij} : L^1(\mathbb{R}^{d(N+M)}, du) \to L^1(\mathbb{R}^{d(N+M)}, du) \) given by

\[
R_{ij}[F](u) := \int_{S_{d-1}} d\rho(\sigma) F\left(M_{(i,j)}^{(i,j)}u\right), \quad u \in \mathbb{R}^{d(N+M)}, \tag{81}
\]

where we model the collision between the \( i \)-th and \( j \)-th particle again by the reflection map, i.e. \( M_{(i,j)}^{(i,j)} \in \mathbb{R}^{(N+M)\times(N+M)} \) is defined as in (4) and \( \rho \) is a probability measure fulfilling the symmetry condition (2).

The time evolution operator \( \mathcal{L} \) is given by (compare to [2, p. 2])

\[
\mathcal{L} := \frac{\lambda_S}{N-1} \sum_{1 \leq i < j \leq N} (R_{ij} - 1) + \frac{\lambda_R}{M-1} \sum_{N < i < j \leq N+M} (R_{ij} - 1) + \frac{\mu}{M} \sum_{i=1}^{N} \sum_{j=N+1}^{N+M} (R_{ij} - 1). \tag{82}
\]

As described in [2, p. 2], the parameter \( \lambda_S \) describes the rate of collisions of one
particle within the Kac system with any other particle within the Kac system. Analogously, the parameter $\lambda_R$ gives the rate of collisions within the heat reservoir. Finally, the rate at which a particle of the Kac system collides with any particle of the heat reservoir is $\mu$.

Since the transformation $p := M_{\sigma}^{(i,j)}u$ yields

$$||R_{ij}F||_1 \leq \int_{\mathbb{R}^{d(N+M)}} du \int_{\mathbb{S}^{d-1}} d\rho(\sigma) |F(M_{\sigma}^{(i,j)}u)|$$
$$= \int_{\mathbb{R}^{d(N+M)}} dp \int_{\mathbb{S}^{d-1}} d\rho(\sigma) |F(p)|$$
$$= ||F||_1. \quad (83)$$

the pair collision operator $R_{ij}$ is bounded by $||R_{ij}|| \leq 1$. Since $\mathcal{L}$ is therefore bounded as well, the time evolution operator generates the exponential semigroup $(e^{\mathcal{L}t})_{t \geq 0}$. The time evolution of the system is hence given by

$$F_t = e^{\mathcal{L}t} F_0. \quad (84)$$

This is equivalent to the abstract Cauchy problem

$$\frac{\partial F_t}{\partial t} = \mathcal{L}F_t, \quad (85)$$

called the Kac Master Equation.

Our goal is to investigate the approach to equilibrium of the Kac system. We assume, that the heat reservoir is initially in an equilibrium state. As in [2, p. 2], the initial probability distribution of the heat reservoir is therefore assumed to be a Gaussian function. Hence, we can write
\[
F_0(\mathbf{v}, \mathbf{w}) = f_0(\mathbf{v}) \left( \frac{\beta}{2\pi} \right)^{\frac{dM}{2}} e^{-\frac{\beta}{2} \mathbf{w}^2} = f_0(\mathbf{v}) g(\mathbf{w})
\]

(86)

where \(\beta\) denotes the inverse temperature of the initial state of the heat reservoir.

We are interested in the time evolution of the probability distribution \(f_t\) describing the Kac system. We define \(f_t\) by the marginal of \(F_t\) with respect to the Lebesgue measure over the particles in the heat reservoir (see [2, eq. (6)] or [1, eq. (8)])

\[
f_t(\mathbf{v}) := \int_{\mathbb{R}^{dM}} F_t(\mathbf{v}, \mathbf{w}) \, d\mathbf{w}.
\]

(87)

Recall that the entropy \(S\) of a probability density function \(f \in L^1(\mathbb{R}^{dN}, d\mathbf{v})\) with respect to an equilibrium state \(\gamma\) is defined by

\[
S(f|\gamma) := \int_{\mathbb{R}^{dN}} f(\mathbf{v}) \ln \left( \frac{f(\mathbf{v})}{\gamma(\mathbf{v})} \right) \, d\mathbf{v},
\]

(88)

where we use the inverted sign.

In order to define the approach to equilibrium via the entropy of the Kac system, we need to specify the equilibrium state \(\gamma\) that we compare the system to. If the Kac system is initially not in a Gaussian distributed state, we can not expect the system to reach a Gaussian distribution over time. However, in [3] it is shown, that the Kac system coupled to a heat reservoir approximates the Kac system coupled to a thermostat uniformly in time in different norms for \(M \rightarrow \infty\). In other words, we can expect that the heat reservoir stays close to its initial state over time. Therefore, it is reasonable to use the initial state \(g\) of the heat reservoir as relative state for the entropy.
As in the case of the thermostat, it is convenient to consider a ground state transformation (see also [2, p. 3]). We define for \( t \in [0, \infty) \) the function
\[
h_t(v) := \frac{f_t(v)}{g(v)}.
\]
(89)

This allows us to write the entropy via
\[
S(f_t|g) = \int_{\mathbb{R}^dN} g(v) dv h_t \ln(h_t).
\]
(90)

Recall that the information of a probability density function \( h \) with respect to the Gaussian measure \( g(v)dv \) and \( \sqrt{h} \in H^1(\mathbb{R}^dN, g(v)dv) \) is defined by
\[
\tilde{I}(h) := \int_{\mathbb{R}^dN} \frac{|\nabla h(v)|^2}{h(v)} g(v)dv.
\]
(91)

Further, we can use the rotational invariance of the Gaussian \( g(v)g(w) \) to get
\[
h_t(v) \overset{\text{def}}{=} \frac{f_t(v)}{g(v)} \int_{\mathbb{R}^{dM}} \frac{1}{g(v)} F_t(v, w)dw \\
= \int_{\mathbb{R}^{dM}} \frac{1}{g(v)} e^{\mathcal{L}_t} [f_0(v)g(w)](v, w) dw \\
= \int_{\mathbb{R}^{dM}} \frac{1}{g(v)} e^{\mathcal{L}_t} [h_0(v)g(v)g(w)] dw \\
= \int_{\mathbb{R}^{dM}} e^{\mathcal{L}_t}[h_0](v, w) g(w)dw \\
= \mathcal{M} e^{\mathcal{L}_t}[h_0](v),
\]
(92)

where \( \mathcal{M} \) is the marginal with respect to the Gaussian measure \( g(w)dw \). Note that we considered the function \( h_0 : \mathbb{R}^{dN} \to \mathbb{R} \) as a function \( \bar{h}_0 : \mathbb{R}^{d(N+M)} \to \mathbb{R} \), where \( \bar{h}_0(v, w) := h_0(v) \).
Remark 3.1. We can represent the time evolution semigroup \((e^{Lt})_{t \geq 0}\) in terms of collision histories, i.e., we can write \(e^{Lt}\) as convex series over products of pair collisions. For this, we write (see [2, p. 7])

\[
\mathcal{L} = \Lambda (Q - 1)
\]  

with

\[
\Lambda := \frac{\lambda_S N}{2} + \frac{\lambda_R M}{2} + \mu N
\]

\[
Q := \frac{\lambda_S}{\Lambda (N - 1)} \sum_{1 \leq i < j \leq N} R_{ij} + \frac{\lambda_R}{\Lambda (M - 1)} \sum_{N < i < j \leq N + M} R_{ij} + \frac{\mu}{\Lambda M} \sum_{i=1}^{N} \sum_{j=N+1}^{N+M} R_{ij}.
\]

This allows us to write \(e^{Lt}\) as the convex series

\[
e^{Lt} = e^{-\Lambda t} \sum_{k=0}^{\infty} \frac{(\Lambda t)^k}{k!} Q^k,
\]  

Next, we want to express the powers \(Q^k\) in terms of collision histories, i.e., by combinations of products of pair collisions. We introduce some notation: The index set \(I\) denotes the set of collision pairs:

\[
I := \{(i, j) : 1 \leq i < j \leq N + M\}.
\]

For \(\alpha \in I\) we define the coefficients

\[
\lambda_{\alpha} := \begin{cases} 
\frac{\lambda_S}{\Lambda (N - 1)}, & 1 \leq i < j \leq N, \\
\frac{\lambda_R}{\Lambda (M - 1)}, & N < i < j \leq N + M, \\
\frac{\mu}{\Lambda M}, & 1 \leq i \leq N < j \leq N + M.
\end{cases}
\]

For \(\alpha = (\alpha_1, \ldots, \alpha_k) \in I^k\) we set \(\lambda^\alpha := \lambda_{\alpha_k} \cdots \lambda_{\alpha_1}\) and \(R^\alpha := R_{\alpha_k} \cdots R_{\alpha_1}\). Now, we can write the powers \(Q^k\) by
\[ Q^k = \sum_{\alpha \in I^k} \lambda^\alpha R^\alpha. \]  

(99)

Note that \( \sum_{\alpha \in I} \lambda^\alpha = 1 = \sum_{\alpha \in I^k} \lambda^\alpha \) is again a convex combination. Therefore, the semigroup is given by the following convex series of pair collision operators:

\[ e^{\mathcal{L}t} = e^{-\mathcal{M}t} \sum_{k=0}^{\infty} \frac{(\Lambda t)^k}{k!} \sum_{\alpha \in I^k} \lambda^\alpha R^\alpha. \]  

(100)

**Remark 3.2.** The time evolution operator \( \mathcal{L} \) generates a contraction semigroup. This is precisely what provides us the following majorant needed to exchange the order of integration and differentiation in the proofs of this chapter. For all \( t \geq 0 \) and \( u \in \mathbb{R}^{d(N+M)} \) we get

\[
\left| \frac{d}{dt} F_t(u) \right| = |\mathcal{L}e^{\mathcal{L}t}F_0(u)| \leq ||\mathcal{L}|| \ ||e^{\mathcal{L}t}|| \ |F_0(u)| \leq ||\mathcal{L}|| |F_0(u)|
\]  

(101)

**Proof.** The claim follows from the decomposition (100) of \( e^{\mathcal{L}t} \) in collision histories. Since \( ||R^\alpha|| \leq 1 \) for all \( \alpha \in I \), we get \( ||R^\alpha|| \leq 1 \) for all \( \alpha \in I^k \). This yields

\[
||e^{\mathcal{L}t}|| \leq e^{-\mathcal{M}t} \sum_{k=0}^{\infty} \frac{(\Lambda t)^k}{k!} \sum_{\alpha \in I^k} \lambda^\alpha ||R^\alpha|| \leq 1.
\]  

(102)

**Remark 3.3.** In this remark, we cover the basic regularity facts.

1. For a solution \((F_t)_{t \geq 0} \in L^1(\mathbb{R}^{d(N+M)}, du)\) of the Kac master equation (85) the following statements are equivalent:

   a) \( F_0 \in L^1(\mathbb{R}^{d(N+M)}, du) \) is a PDF.
b) \( F_t \in L^1(\mathbb{R}^{d(N+M)}, du) \) is a PDF for all \( t \geq 0 \).

c) \( f_t \in L^1(\mathbb{R}^{dN}, dv) \) is a PDF for all \( t \geq 0 \).

d) \( h_t \in L^1(\mathbb{R}^{dN}, g(v)dv) \) is a PDF with respect to the Gaussian measure \( g(v)dv \).

**Proof.** The equivalence of [c] and [d] is clear by the definition [89]. Since by definition [87]

\[
\int_{\mathbb{R}^{dN}} f_t dv = \int_{\mathbb{R}^{d(N+M)}} F_t du
\]

the statements [b] and [c] are equivalent as well.

Using the representation [100] of \( e^{\mathcal{L}t} \) in terms of collision histories we get the equivalence of the first two statements by

\[
\int_{\mathbb{R}^{d(N+M)}} F_t(u) du = e^{-\Lambda t} \sum_{k=0}^{\infty} \frac{(\Lambda t)^k}{k!} \sum_{\alpha \in I^k} \lambda^\alpha \int_{\mathbb{R}^{d(N+M)}} du Q^k[F_0](u)
\]

\[
= e^{-\Lambda t} \sum_{k=0}^{\infty} \frac{(\Lambda t)^k}{k!} \sum_{\alpha \in I^k} \lambda^\alpha \int_{\mathbb{S}^{k(d-1)}} d\rho(\sigma) \int_{\mathbb{R}^{d(N+M)}} du F_0(M^{(\alpha_k)}_{\sigma_k} \cdots M^{(\alpha_1)}_{\sigma_1} u)
\]

\[
= e^{-\Lambda t} \sum_{k=0}^{\infty} \frac{(\Lambda t)^k}{k!} \sum_{\alpha \in I^k} \lambda^\alpha \int_{\mathbb{S}^{k(d-1)}} d\rho(\sigma) \int_{\mathbb{R}^{d(N+M)}} d\sigma F_0(p)
\]

\[
= \int_{\mathbb{R}^{d(N+M)}} F_0(p) dp,
\]

where we used the transformation \( p := M^{(\alpha_k)}_{\sigma_k} \cdots M^{(\alpha_1)}_{\sigma_1} u \).

**ii)** Assuming that \( K(0) < \infty \), we get as in remark [2.4] that

\[
\sqrt{f_0} \in H^1(\mathbb{R}^{dN}, dv) \iff \sqrt{h_0} \in H^1(\mathbb{R}^{dN}, g(v)dv).
\]
In particular, if one of the statements is true, the following relates the information with respect to the different measures:

\[
\tilde{I}(h_0) = I(f_0) + 2\beta^2 \left( K(0) - \frac{dN}{\beta} \right). \tag{106}
\]

### 3.2. Kinetic Energy

Since there is a one-to-one correspondence between the inverse temperature \( \beta \) of a Gaussian distribution and the kinetic energy

\[
K(\beta) := \frac{1}{2} \int_{\mathbb{R}^{dN}} \mathbf{v}^2 \left( \frac{\beta}{2\pi} \right)^{\frac{dN}{2}} e^{-\frac{\beta}{2} \mathbf{v}^2} \, d\mathbf{v} = \frac{dN}{2\beta}, \tag{107}
\]

it is interesting to investigate the time evolution of the kinetic energy. We consider

- the total kinetic energy \( K \) including the Kac system and the heat reservoir,
- the kinetic energy of the Kac system \( K_S \), and
- the kinetic energy of the heat reservoir \( K_R \).

These are defined by

\[
K_S(t) := \frac{1}{2} \sum_{i=1}^{N} \int_{\mathbb{R}^{d(N+M)}} u_i^2 F_t(u) \, du = \frac{1}{2} \int_{\mathbb{R}^{dN}} \mathbf{v}^2 f_t(\mathbf{v}) \, d\mathbf{v},
\]

\[
K_R(t) := \frac{1}{2} \sum_{j=N+1}^{N+M} \int_{\mathbb{R}^{d(N+M)}} u_j^2 F_t(u) \, du,
\]

\[
K(t) := \frac{1}{2} \int_{\mathbb{R}^{d(N+M)}} \mathbf{u}^2 F_t(\mathbf{u}) \, d\mathbf{u} = K_S(t) + K_R(t). \tag{108}
\]

The total kinetic energy stays constant, since the collision mechanism conserves the
kinetic energy. We will therefore write \( K(t) = K \). The time evolution of the kinetic energy of the Kac system and heat reservoir can be computed by taking the derivative, deriving an ODE, and solving the latter. This is summarized in the following lemma. The computations are somewhat lengthy and therefore left to the appendix.

**Lemma 3.4.** The total kinetic energy stays constant over time and is given by

\[
K = \frac{dM}{2\beta} + \frac{1}{2} \int_{\mathbb{R}^{dN}} v^2 f_0(v) \, dv
\]  

(109)

The evolution of the kinetic energy of the Kac system and of the heat reservoir are given by

\[
K_S(t) = \left( K_S(0) - \frac{N}{N + M} K \right) e^{-\frac{\mu^2}{2\beta}(N + M)t} + \frac{N}{N + M} K, \\
K_R(t) = \left( K_R(0) - \frac{M}{N + M} K \right) e^{-\frac{\mu^2}{2\beta}(N + M)t} + \frac{M}{N + M} K.
\]  

(110) (111)

The proof is given in section C.2.

Lemma 3.4 shows that the kinetic energy of the Kac system will tend to \( \frac{N}{N + M} K \). The decay rate is proportional to the number of particles in the Kac system, and thus much faster as what we will get for the entropy decay rate. Further, in the limit \( t \to \infty \), all particles will have on average the same kinetic energy. Since the heat reservoir is assumed to be much larger than the Kac system, it is reasonable to use the inverse temperature \( \beta \) of the initial state of the heat reservoir for the equilibrium state, that we compare the Kac system to.
3.3. Main results

Our first main result is that the information decays exponentially down to a small term.

**Theorem 3.5.** Let \((F_t)_{t \geq 0} \in L^1(\mathbb{R}^{d(N+M)}, \, du)\) be a solution of the Kac master equation (85) satisfying the initial condition (86) with \(\sqrt{f_0} \in H^1(\mathbb{R}^dN, \, dv)\). Assume further, that \(K(0) < \infty\). Then, the information of \(h_t\) defined by (89) decays as follows

\[
\tilde{I}(h_t) \leq \left[ \frac{N}{N+M} + \frac{M}{N+M} e^{-\frac{\alpha N+M}{M} t} \right] \tilde{I}(h_0).
\]

(112)

The proof is given in section 3.4.

Using the connection of the entropy to the information via the Ornstein-Uhlenbeck semigroup given in lemma 3.3 we can prove that the decay rate of the information transfers to the entropy. This was similarly done in [2, Lemma 3.1]. The crucial point is to show that the time evolution operator \(\mathcal{L}\) commutes with the Ornstein-Uhlenbeck semigroup:

**Lemma 3.6.** The time evolution operator \(\mathcal{L}\) commutes with the Ornstein-Uhlenbeck semigroup on \(L^1(\mathbb{R}^{d(N+M)}, \, du)\), i.e.,

\[
[\mathcal{L}, P_s] = \mathcal{L}P_s - P_s\mathcal{L} = 0, \quad \text{for all } s \geq 0.
\]

(113)

The proof is given in section 3.5.

**Theorem 3.7.** Let \((F_t)_{t \geq 0} \in L^1(\mathbb{R}^{d(N+M)}, \, du)\) be a solution of the Kac master
equation (85) satisfying the initial condition (86) with \( \sqrt{f_0} \in H^1(\mathbb{R}^d, dv) \) and \( K(0) < \infty \). Let further \( f_t \) and \( h_t \) be defined by (87) and (89). If we have an inequality for the information

\[
\tilde{I}(h_t) \leq c(t) \tilde{I}(h_0),
\]

then we get an inequality for the entropy by

\[
S(f_t|g) \leq c(t) S(f_0|g).
\]

**Proof.** The claim follows by the connection between entropy and information given in lemma B.3 and 3.6:

\[
S(f_t|g) = \int_{\mathbb{R}^d N} g(v) dv \, h_t \ln(h_t) \leq \frac{1}{\beta} \int_0^\infty ds \, \tilde{I}(P_s h_t)
\]

\[
= \frac{1}{\beta} \int_0^\infty ds \, \tilde{I} \left( P_s \int_{\mathbb{R}^d M} g(w) dw \, e^{c t [h_0]}(v, w) \right)
\]

\[
= \frac{1}{\beta} \int_0^\infty ds \, \tilde{I} \left( \left( \int_{\mathbb{R}^d M} g(w) dw \, e^{c t [P_s h_0]}(v, w) \right) \right)
\]

\[
\leq \frac{c(t)}{\beta} \int_0^\infty ds \, \tilde{I} \left( \int_{\mathbb{R}^d M} g(w) dw \, P_s h_0 \right)
\]

\[
= \frac{c(t)}{\beta} \int_{\mathbb{R}^d N} g(v) dv \, h_0 \ln(h_0)
\]

\[
= c(t) S(f_0|g).
\]

Our next main result is now a consequence of the last two theorems.

**Theorem 3.8.** Let \((F_t)_{t \geq 0} \in L^1(\mathbb{R}^{d(N+M)}, du)\) be a solution of the Kac master
equation (85) satisfying the initial condition (86) with \( \sqrt{f_0} \in H^1(\mathbb{R}^{d(N+M)}, du) \) and \( K(0) < \infty \). Let further \( f_t \) be defined by (87). The entropy of the Kac system decays by

\[
S(f_t | g) \leq \left[ \frac{N}{N + M} + \frac{M}{N + M} e^{-\frac{\mu}{\alpha(N+M)t}} \right] S(f_0 | g). \tag{116}
\]

It is interesting to note, that for the limit \( M \to \infty \) we get the same decay rates for the entropy and information as in the case of the thermostat.

As pointed out in [1] p. 7, we can deduce the classic Kac model with \( N + M \) particles defined by

\[
\mathcal{L}_{cl} := \frac{2}{N + M - 1} \sum_{1 \leq i < j \leq N + M} (R_{ij} - 1) \tag{117}
\]

from the heat reservoir model by setting the parameters

\[
\lambda_S := \frac{2(N - 1)}{N + M - 1}, \quad \lambda_R := \frac{2(M - 1)}{N + M - 1}, \quad \lambda_S := \frac{2M}{N + M - 1}. \tag{118}
\]

This gives us the following corollary:

**Corollary 3.9.** Let \( F_0(v, w) = f_0(v)g(w) \) and

\[
f_t(v) := \int_{\mathbb{R}^{dM}} [e^{\mathcal{L}_{cl}t} F_0] dw, \quad h_t(v) := \frac{f_t(v)}{g(v)}. \tag{119}
\]

Assume that \( \sqrt{f_0} \in H^1(\mathbb{R}^{dN}, dv) \) and \( K(0) < \infty \). The decay of information and entropy is then given by
\[ \tilde{\mathcal{I}}(h_t) \leq \left[ \frac{N}{N + M} + \frac{M}{N + M} e^{-\frac{2(N+M)}{N+M-1}} \right] \tilde{\mathcal{I}}(h_0), \quad (120) \]

\[ S(f_t|g) \leq \left[ \frac{N}{N + M} + \frac{M}{N + M} e^{-\frac{2(N+M)}{N+M-1}} \right] S(f_0|g). \quad (121) \]
3.4. Proof of theorem 3.5

The proof consists of two parts. First, we use the representation of \( e^{Lt} \) in terms of collision histories together with the convexity of the information. This reduces the problem to the derivation of a sum rule for some explicitly given matrix \( K \), which we will derive in the second part.

Inserting the representation formula (100) in equation 92, we get

\[
h_t(v) = Me^{Lt}[h_0 \circ P] = e^{-\Lambda t} \sum_{k=0}^{\infty} \frac{(\Lambda t)^k}{k!} \sum_{\alpha \in I^k} \lambda^\alpha MR^\alpha[h_0 \circ P](v)
\]

\[
= e^{-\Lambda t} \sum_{k=0}^{\infty} \frac{(\Lambda t)^k}{k!} \sum_{\alpha \in I^k} \lambda^\alpha \int_{\mathbb{S}^{k(d-1)}} d\rho(\sigma) M[h_0 \circ P] (M^{\alpha_k}_{\sigma_k} \cdots M^{\alpha_1}_{\sigma_1}(v, w)^T),
\]

where \( \rho(\sigma) = (\rho(\sigma_1), \ldots, \rho(\sigma_k)) \). The convexity of the information implies by Jensen’s inequality that

\[
\mathcal{I}(h_t|g) \leq e^{-\Lambda t} \sum_{k=0}^{\infty} \frac{(\Lambda t)^k}{k!} \sum_{\alpha \in I^k} \lambda^\alpha \int_{\mathbb{S}^{k(d-1)}} d\rho(\sigma) \mathcal{I}(h_{k,\alpha,\sigma}|g).
\]

Next, we separate the action of the rotations by denoting

\[
\begin{pmatrix}
A_k(\alpha, \sigma) & B_k(\alpha, \sigma) \\
C_k(\alpha, \sigma) & D_k(\alpha, \sigma)
\end{pmatrix}
:= M^{(\alpha_k)}_{\sigma_k} \cdots M^{(\alpha_1)}_{\sigma_1},
\]

where \( A_k(\alpha, \sigma) \in \mathbb{R}^{dN \times dN}, B_k(\alpha, \sigma) \in \mathbb{R}^{dN \times dM}, C_k(\alpha, \sigma) \in \mathbb{R}^{dM \times dN}, D_k(\alpha, \sigma) \in \mathbb{R}^{dM \times dM} \). This allows us to observe, that
\[ h_{k,\alpha,\sigma}(v) = \int_{\mathbb{R}^M} g(w) dw h_0 \left( \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \right) \]

\[ = \int_{\mathbb{R}^M} g(w) dw h_0 (A_k v + B_k w) \]  

(125)

is independent of \(C_k\) and \(D_k\). Further, this representation of \(h_{k,\alpha,\sigma}\) allows an easy computation of its gradient. Using Jensen’s inequality in the first step, the information of \(h_{k,\alpha,\sigma}\) is given by

\[ \bar{I}(h_{k,\alpha,\sigma}) \leq \int_{\mathbb{R}^{dM}} g(v)g(w) dv dw \frac{\|\nabla_v h_0(A_k v + B_k w)\|^2}{h_0(A_k v + B_k w)} \]

\[ = \int_{\mathbb{R}^{dM}} g(v)g(w) dv dw \frac{\|A_k \nabla h_0(A_k v + B_k w)\|^2}{h_0(A_k v + B_k w)} \]

\[ = \int_{\mathbb{R}^{dM}} g(p)g(q) dp dq \frac{\nabla h_0(p)^T A_k^T A_k \nabla h_0(p)}{h_0(p)} \]

\[ = \int_{\mathbb{R}^N} g(p) \frac{\nabla h_0(p)^T A_k^T A_k \nabla h_0(p)}{h_0(p)}. \]  

(126)

where we used the transformation

\[ \begin{pmatrix} p \\ q \end{pmatrix} := \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}. \]  

(127)

In summary, we get the formula

\[ \bar{I}(h_t) \leq \int_{\mathbb{R}^N} g(v) dv \frac{\nabla h_0(v)^T K \nabla h_0(v)}{h_0(v)} \]  

(128)

where

\[ K := e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\Lambda t)^k}{k!} \sum_{\alpha \in I^k} \lambda^\alpha \int_{\mathbb{S}^k(d-1)} d\rho(\sigma) A_k^T(\alpha, \sigma) A_k(\alpha, \sigma). \]  

(129)

We compute the matrix \(K\) in the following lemma.
Lemma 3.10. The matrix $K$ defined by (129) is given by

$$K = \left[ \frac{N}{N + M} + \frac{M}{N + M} e^{-\frac{\mu(N + M)}{dM} t} \right] \mathbb{1}_{dN}. \quad (130)$$

Proof. First, we observe that $A_k^T A_k$ is the top left entry of the matrix

$$\begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}. \quad (131)$$

That is, we can write

$$A_k^T A_k = P M_{\sigma_1}^{(\alpha_1)} \ldots M_{\sigma_k}^{(\alpha_k)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M_{\sigma_k}^{(\alpha_k)} \ldots M_{\sigma_1}^{(\alpha_1)} P^T. \quad (132)$$

Next, we will iteratively integrate out the factors $M_{\sigma_k}^{(\alpha_k)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M_{\sigma_k}^{(\alpha_k)}$ from the inside to the outside. For this, we note that pointwise integration in the matrix yields

$$\int_{S^{d-1}} d\rho(\sigma) M_{\sigma} \begin{pmatrix} m_1 \mathbb{1}_d & 0 \\ 0 & m_2 \mathbb{1}_d \end{pmatrix} M_{\sigma}$$

$$= \int_{S^{d-1}} d\rho(\sigma) \begin{pmatrix} m_1 \mathbb{1}_d - (m_1 - m_2)(\sigma \otimes \sigma) & 0 \\ 0 & m_2 \mathbb{1}_d - (m_2 - m_1)(\sigma \otimes \sigma) \end{pmatrix}$$

$$= \begin{pmatrix} (m_1 - \frac{1}{d}(m_1 - m_2)) \mathbb{1}_d & 0 \\ 0 & (m_2 - \frac{1}{d}(m_2 - m_1)) \mathbb{1}_d \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{m}_1 \mathbb{1}_d & 0 \\ 0 & \tilde{m}_2 \mathbb{1}_d \end{pmatrix}, \quad (133)$$

with

$$\tilde{m}_1 := m_1 - \frac{1}{d}(m_1 - m_2), \quad \tilde{m}_2 := m_2 - \frac{1}{d}(m_2 - m_1). \quad (134)$$

This proves that the matrix always stays diagonal throughout the process and shows
how the coefficients develop. In the following we compute the sum over all collision pairs and integrals over all scattering angles for this expression with the general collision matrix $M_{\sigma}^{(i,j)}$. We use the notation

$$L(m_1, m_2) := \begin{pmatrix} m_1 1_{dN} & 0 \\ 0 & m_2 1_{dM} \end{pmatrix}.$$  \hfill (135)

Further, we distinguish between the three cases of collisions in the Kac system, in the heat reservoir, and the interaction between the systems:

$$\sum_{\alpha \in I} \lambda_{\alpha} = \sum_{1 \leq i < j \leq N} \frac{\lambda_S}{\Lambda(N-1)} + \sum_{N < i < j \leq N+M} \frac{\lambda_R}{\Lambda(M-1)} + \sum_{i=1}^{N} \sum_{j=N+1}^{N+M} \frac{\mu}{\Lambda M}.$$  \hfill (136)

**Step 1** The collisions between particles in the Kac system give us

$$\frac{\lambda_S}{\Lambda(N-1)} \sum_{1 \leq i < j \leq N} \int_{S^{d-1}} d\rho(\sigma) M_{\sigma}^{(i,j)} L(m_1, m_2) M_{\sigma}^{(i,j)}$$

$$= \frac{\lambda_S}{\Lambda(N-1)} \sum_{1 \leq i < j \leq N} \int_{S^{d-1}} d\rho(\sigma) \begin{pmatrix} m_1 M_{\sigma}^{(i,j)^2} & 0 \\ 0 & m_2 1 \end{pmatrix}$$

$$= \frac{\lambda_S N}{2\Lambda} L(m_1, m_2).$$  \hfill (137)

**Step 2** The collisions between particles in the heat reservoir act similarly on $L(m_1, m_2)$:
\[
\frac{\lambda_R}{\Lambda(M-1)} \sum_{N<i<j\leq N+M} \int_{S^{d-1}} d\rho(\sigma) M^{(i,j)} \sigma L(m_1, m_2) M^{(i,j)}_\sigma \\
= \frac{\lambda_R}{\Lambda(M-1)} \sum_{N<i<j\leq N+M} \int_{S^{d-1}} d\rho(\sigma) \begin{pmatrix} m_1 \mathbb{1} & 0 \\
0 & m_2 M^{(i,j)}_\sigma \end{pmatrix}^2 \\
= \lambda_R \frac{M}{2} L(m_1, m_2).
\] (138)

**Step 3** Using (133), we can compute the interaction between the two systems by

\[
\frac{\mu}{\Lambda M} \sum_{i=1}^{N} \sum_{j=N+1}^{N+M} \int_{S^{d-1}} d\rho(\sigma) M^{(i,j)} \sigma L(m_1, m_2) M^{(i,j)}_\sigma \\
= \frac{\mu}{\Lambda M} \sum_{i=1}^{N} \sum_{j=N+1}^{N+M} \text{diag}(m_1 \mathbb{1}_d, \ldots, m_1 \mathbb{1}_d, m_1 \mathbb{1}_d, \ldots, m_1 \mathbb{1}_d, m_2 \mathbb{1}_d, m_2 \mathbb{1}_d, \ldots, m_2 \mathbb{1}_d) \\
= \frac{\mu}{\Lambda M} \begin{pmatrix} \sum_{j=N+1}^{N+M+M} (N-1)m_1 + \tilde{m}_1 \mathbb{1}_d \\
0 \end{pmatrix} \mathbb{1}_{dN} \\
= \frac{\mu}{\Lambda M} \begin{pmatrix} \sum_{i=1}^{N+M} (M-1)m_2 + \tilde{m}_2 \mathbb{1}_d \\
0 \end{pmatrix} \mathbb{1}_{dM} \\
= \frac{\mu}{\Lambda M} \begin{pmatrix} N(m_1 + \frac{M}{d}(m_2 - m_1)) \mathbb{1}_d \\
0 \end{pmatrix} \mathbb{1}_{dN} \\
= \frac{\mu N}{\Lambda} L(m_1, m_2) - \frac{\mu}{d\Lambda M} L(M(m_1 - m_2), N(m_2 - m_1)).
\] (139)

From Step 1 to 3, we get the sum of the three terms to
\[
\sum_{\alpha \in I} \lambda_\alpha \int_{S_{d-1}} d\rho(\sigma) M_\sigma^{(\alpha)} L(m_1, m_2) M_\sigma^{(\alpha)}
\]

\[
= \frac{\lambda_S N}{2\Lambda} L(m_1, m_2) + \frac{\lambda_R M}{2\Lambda} L(m_1, m_2) + \frac{\mu N}{\Lambda} L(m_1, m_2)
\]

\[
- \frac{\mu}{d\Lambda M} L(M(m_1 - m_2), N(m_2 - m_1))
\]

\[
= L(m_1, m_2) - \frac{\mu}{d\Lambda M} L(M(m_1 - m_2), N(m_2 - m_1))
\]

\[
= L(m_1', m_2') \quad (140)
\]

with

\[
\begin{pmatrix}
m_1' \\
m_2'
\end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} - \frac{\mu}{d\Lambda M} \begin{pmatrix} M & -M \\ -N & N \end{pmatrix} \begin{pmatrix} m_1 \\
m_2
\end{pmatrix} = \mathcal{P} \begin{pmatrix} m_1 \\
m_2
\end{pmatrix} . \quad (141)
\]

If we start with \( m_1 = 1, m_2 = 0 \), and apply \( \mathcal{P} \) iteratively \( k \)-times, we get

\[
\sum_{\alpha \in I^k} \lambda^\alpha \int_{S_{k(d-1)}} d\rho(\sigma) A^T_k(\alpha, \sigma) A_k(\alpha, \sigma) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} PL \left( \mathcal{P}^k \begin{pmatrix} 1 \\
0
\end{pmatrix} \right) \mathcal{P}^T.
\quad (142)
\]

To compute this expression, we decompose the start vector into the eigenvectors of \( \mathcal{P} \). The eigenvalues of \( \mathcal{P} \) are given by \( \lambda_1 = 1, \lambda_2 = 1 - \frac{\mu(N+M)}{d\Lambda M} \) with corresponding eigenvectors \( v_1 = (1, 1)^T, v_2 = \frac{1}{N+M}(M, -N)^T \). Using \( (1, 0)^T = \frac{N}{N+M} v_1 + v_2 \), we get

\[
\mathcal{P}^k \begin{pmatrix} 1 \\
0
\end{pmatrix} = \frac{N}{N+M} \begin{pmatrix} 1 \\
1
\end{pmatrix} + \left( 1 - \frac{\mu(N+M)}{d\Lambda M} \right)^k \frac{1}{N+M} \begin{pmatrix} M \\
-N
\end{pmatrix} . \quad (143)
\]

Taking the first entry, we get

\[
PL \left( \mathcal{P}^k \begin{pmatrix} 1 \\
0
\end{pmatrix} \right) \mathcal{P}^T = \left( \frac{N}{N+M} + \frac{M}{N+M} \left( 1 - \frac{\mu(N+M)}{d\Lambda M} \right)^k \right) \mathbb{1}_{dN}. \quad (144)
\]

This proves the claim as follows:
\[
K = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\Delta t)^k}{k!} \left( \frac{N}{N+M} + \frac{M}{N+M} \left( 1 - \frac{\mu(N+M)}{d\lambda M} \right) \right) 1_{dN} \\
= \left[ \frac{N}{N+M} + \frac{M}{N+M} e^{-\frac{\mu(N+M)}{d\lambda M} t} \right] 1_{dN}.
\]

3.5. Proof of theorem 3.6

Recall that for any \( F \in L^1(\mathbb{R}^{d(N+M)}, g(u)du) \), the Ornstein-Uhlenbeck semigroup is defined by

\[
P_s[F](u) = \int_{\mathbb{R}^{d(N+M)}} F \left( e^{-s}u + \sqrt{1-e^{-2s}}w \right) g(x)dx.
\]  

(145)

Using the change of variables \( y := M^{(i,j)}_\sigma x \), the rotational invariance of \( g \) and the linearity of the reflection map we get

\[
P_sR_{ij}[F](u) = \int_{\mathbb{R}^{d(N+M)}} g(x)dx \int_{S^{d-1}} d\rho(\sigma) F \left( M^{(i,j)}_\sigma \left( e^{-s}u + \sqrt{1-e^{-2s}}x \right) \right)
\]

\[
= \int_{\mathbb{R}^{d(N+M)}} g(x)dx \int_{S^{d-1}} d\rho(\sigma) F \left( e^{-s}M^{(i,j)}_\sigma u + \sqrt{1-e^{-2s}}M^{(i,j)}_\sigma x \right)
\]

\[
= \int_{\mathbb{R}^{d(N+M)}} g(y)dy \int_{S^{d-1}} d\rho(\sigma) F \left( e^{-s}M^{(i,j)}_\sigma u + \sqrt{1-e^{-2s}}y \right)
\]

\[
= R_{ij}P_s[F](u),
\]  

(146)

for all \( i, j \in \{1, \ldots, N + M\}, i \neq j \). Since \( L \) is a linear combination of operators \( R_{ij} \), this proves the claim.
4. Discussion and future work

The decay rates for the entropy and information for the Kac system coupled to a thermostat or a heat reservoir obtained in the theorems 2.6, 2.9, 3.5 and 3.8 are consistent with the decay rates proved in [4], [1] and [2]. It is interesting to note, that the decay rates are quite uniform in the sense that they are independent of the collisions within the Kac system, the probability measure used to sample the scattering angle, as well as the inverse temperature of the thermostat. They are also essentially independent of the size of the Kac system.

While the information and entropy of the Kac system coupled to the thermostat tends to zero, the information and entropy of the Kac system coupled to a heat reservoir only tends to a small number. This is due to the fact, that the Kac system in the model with the heat reservoir will in general not tend towards a Gaussian distribution anymore. However, in the limit $M \to \infty$ we recover decay rate of the Kac system coupled to a thermostat. Therefore, the estimate is expected to be optimal in some sense.

In comparison to [4] and [1] the approach via the information used in this paper and [2] provides a more natural and elegant method with simpler proofs and weaker assumptions. It also allows to treat multidimensional systems as well as the spherical Kac model (see [2] for the latter).

The required regularity for the Kac model results from the use of semi-group theory. The initial state is as a probability density function already in $L^1$. The master equation is a well defined abstract Cauchy problem on the $L^1$ space and the time evolution
is governed by a contraction semigroup. For the approach via the information we only need to assume that the square root $\sqrt{f_0}$ of the initial distribution $f$ is in $H^1$ and that the initial kinetic energy is finite.

For future work, different collision mechanisms could be considered. For example, the swapping map as given in [6, Eq. (A.9)]

$$\begin{align*}
(v, w, \sigma) \mapsto & \left( \frac{v + w}{2} + \frac{|v - w|}{2} \sigma, \frac{v + w}{2} - \frac{|v - w|}{2} \sigma, \frac{|v - w|}{|v - w|} \right)
\end{align*}$$

(147)

could be used instead of the reflection map. Since this map is firstly not linear and secondly only bijective if the scattering angle is included as third argument, it us harder to handle. In particular, it is not immediately clear, how to prove that the time evolution commutes with the Ornstein-Uhlenbeck semigroup. However, we would still expect exponential decay rates with different constants. Further, the case of hard sphere collisions or Maxwellian molecules could be discussed as suggested in [1, p. 853].

In [4, p. 19] it was proposed to consider a Kac system coupled to two heat reservoirs at different temperatures. Using the information approach of this paper, it could be possible to obtain results about the decay of entropy for that model. The interesting part would be that the two big heat reservoirs would merge over time into one heat reservoir on a much larger time scale than it takes the smaller Kac system to tend towards the equilibrium state. However, the distribution of the merged reservoirs would not be a Gaussian distribution anymore. If a Gaussian state with averaged inverse temperature is used as relative state, we would expect to obtain an exponential decay for the entropy down to a small number.
Arguably even more interesting is to consider the Kac system coupled to two thermostats at different temperatures. Even after long times, there would always be heat flowing into the Kac system from one thermostat and heat flowing from the Kac system to the other thermostat. Thus, we expect that the Kac system would tend towards a steady state but not an equilibrium state anymore. Therefore, the Kac system coupled to two thermostats might not be an approximation of the Kac system coupled to two heat reservoirs.

It could promote the field even further, if one could rigorously derive the true distributions of the equilibrium or steady states, respectively. For example, we do not know, what the distribution in the limit $t \to \infty$ of two coupled heat reservoirs with different temperatures under the Kac evolution, or what the distribution of the steady state occurring in the Kac system coupled to two thermostats would be.
Appendix

A. Ornstein-Uhlenbeck semigroup

In this chapter, we summarize the relevant properties of the Ornstein-Uhlenbeck semigroup and provide proofs as well. Hereby, we always consider the Gaussian measure \( g(v)dv \), where

\[
g(v) := \left( \frac{\beta}{2\pi} \right)^{\frac{d}{2}} e^{-\frac{\beta}{2}v^2}.
\]  

(148)

**Definition A.1.** The Ornstein-Uhlenbeck semigroup is defined by (see [11], p. 444)

\[
P_s h(v) := \int_{\mathbb{R}^d} h \left( e^{-sv} + \sqrt{1 - e^{-2sv}} \right) g(x)dx, \quad s \in [0, \infty), v \in \mathbb{R}^d,
\]

(149)

where \( P_s \) acts on \( L^1(\mathbb{R}^d, g(x)dx) \).

**Lemma A.2.** The Ornstein-Uhlenbeck semigroup is a strongly continuous semigroup.

**Proof.** It is clear that \( P_0 = 1 \), since

\[
P_0 h(v) = \int h(v) g(x)dx = h(v).
\]

(150)

Further, we have by the dominated convergence theorem that

\[
\lim_{s \to 0} P_s h(v) = \lim_{s \to 0} \int h \left( e^{-sv} + \sqrt{1 - e^{-2sv}} \right) g(x)dx = \int \lim_{s \to 0} h \left( e^{-sv} + \sqrt{1 - e^{-2sv}} \right) g(x)dx = h(v).
\]

(151)
This proves, that \( \text{s-lim}_{s \to 0} P_s = 1 \). Finally, we verify that \( P_{s+t} = P_s P_t \) for all \( s, t \geq 0 \).

We observe that

\[
P_s P_t[h](v) = \int_{\mathbb{R}^d} g(x)dx \int_{\mathbb{R}^d} g(y)dy \ h \left( e^{-t} \left( e^{-s}v + \sqrt{1 - e^{-2s}x} \right) + \sqrt{1 - e^{-2t}y} \right)
\]

\[
= \int_{\mathbb{R}^d} g(x)dx \int_{\mathbb{R}^d} g(y)dy \ h \left( e^{-(s+t)}v + e^{-t} \sqrt{1 - e^{-2s}x} + \sqrt{1 - e^{-2t}y} \right). \tag{152}
\]

Defining

\[
a := \frac{e^{-t} \sqrt{1 - e^{2s}}}{\sqrt{1 - e^{-2(s+t)}}}, \quad b := \frac{\sqrt{1 - e^{-2t}}}{\sqrt{1 - e^{-2(s+t)}}},
\]

\[
p := ax + by, \quad q := -bx + ay.
\]

yields

\[
a^2 + b^2 = \frac{e^{-2t}(1 - e^{-2s}) + 1 - e^{-2t}}{1 - e^{-2(s+t)}} = \frac{1 - e^{-2(s+t)}}{1 - e^{-2(s+t)}} = 1, \tag{153}
\]

\[
p^2 + q^2 = (a^2 + b^2) \left( x^2 + y^2 \right) = x^2 + y^2.
\]

Therefore, the transformation \( \phi(x, y) := (p, q) \) is a rotation with

\[
det(\phi'(x, y)) = det \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a^2 + b^2 = 1, \tag{154}
\]

\[
g(x)g(y) = g(p)g(q).
\]

Hence, equation (152) transforms into

\[
P_s P_t[h](v) = \int_{\mathbb{R}^d} g(p)dp \int_{\mathbb{R}^d} g(q)dq \ h \left( e^{-(s+t)}v + \sqrt{1 - e^{-2(s+t)}p} \right)
\]

\[
= \int_{\mathbb{R}^d} g(p)dp \ h \left( e^{-(s+t)}v + \sqrt{1 - e^{-2(s+t)}p} \right) = P_{s+t}[h](v). \tag{156}
\]
Lemma A.3. The Ornstein-Uhlenbeck semigroup is self-adjoint (see [11]), i.e., \( P_s \) is self-adjoint on \( L^2(\mathbb{R}^d, g(x)dx) \) for all \( s \in [0, \infty) \). In particular, it follows since \( P_s[1] = 1 \) that

\[
\int_{\mathbb{R}^d} P_s h g(x) dx = \int_{\mathbb{R}^d} h g(x) dx.
\]

(157)

Proof. For all functions \( F, G \in L^2(\mathbb{R}^d, dg(v)dv) \), we have

\[
\langle P_s F, G \rangle = \int_{\mathbb{R}^d} dv g(v) P_s F(v) G(v)
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dv dw g(v) g(w) F \left( e^{-s} v + \sqrt{1 - e^{-2s} w} \right) G(v).
\]

(158)

Defining

\[
p := e^{-s} v + \sqrt{1 - e^{-2s} w} \quad \Rightarrow \quad v = e^{-s} p - \sqrt{1 - e^{-2s} q}
\]

\[
q := -\sqrt{1 - e^{-2s} u} + e^{-s} w \quad \Rightarrow \quad w = \sqrt{1 - e^{-2s} p} + e^{-s} q,
\]

(159)

yields the transformation \( \phi(v, w) := (p, q) \) with

\[
(e^{-s})^2 + \left( \sqrt{1 - e^{-2s}} \right)^2 = 1, \quad \det(\phi'(u, w)) = 1,
\]

\[
p^2 + q^2 = v^2 + w^2, \quad g(u)g(w) = g(p)g(q).
\]

(160)

Therefore, the claim follows by
\[ \langle P_s F, G \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dp dq \, g(p)g(q) F(p)G \left( e^{-s}p - \sqrt{1 - e^{-2s}q} \right) \]
\[ = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dp dq \, g(p)g(-q) F(p)G \left( e^{-s}p + \sqrt{1 - e^{-2s}q} \right) \]
\[ = \langle F, P_s G \rangle, \quad (161) \]

where we used in the last equality, that the Gaussian \( g \) is centered, i.e. \( g(-q) = g(q) \). \( \square \)

Next, we investigate the generator of the Ornstein-Uhlenbeck semigroup. In [12, Lemma 13.1.1] an explicit formula is given for the generator:

**Lemma A.4.** The generator \( L \) of the Ornstein-Uhlenbeck semigroup is given by

\[ Lh(v) = \frac{1}{\beta} \Delta h(v) - v \cdot \nabla h(v), \quad (162) \]

where we recall that \( \beta \) is the inverse temperature of the Gaussian \( g \) as given in \[(148).\]

**Proof.** The generator \( L \) is for any test function \( h \in D(L) \) in its domain given by

\[ L[h] = \lim_{t \to 0} \frac{P_t h - h}{t} = P_s \lim_{t \to 0} \frac{P_t h - h}{t} \bigg|_{s=0} = \lim_{t \to 0} \frac{P_{s+t} h - P_s h}{t} \bigg|_{s=0} = \frac{d}{ds} P_s h \bigg|_{s=0}. \quad (163) \]

Hence, we compute the derivation of \( P_s h \) with respect to \( s \). For shorter notation, we define \( \alpha(s) := e^{-s} \) and \( \delta(s) := \sqrt{1 - e^{-2s}} \). Since we do not know the derivation of \( h \), but the derivation of \( g \), we use the transformation \( p := \alpha v + \delta w \Leftrightarrow w = \frac{p - \alpha v}{\delta} \) to get

\[ P_s h(v) = \frac{1}{\delta^d} \int_{\mathbb{R}^d} dp \, g \left( \frac{p - \alpha v}{\delta} \right) h(p). \quad (164) \]

This yields
\[
\frac{d}{ds} P_s h(v) = \frac{d}{ds} \int_{\mathbb{R}^d} dw \ g(w) h(\alpha v + \delta w) \\
= \frac{d}{ds} \left( \frac{1}{\delta^d} \right) \int_{\mathbb{R}^d} dp \ g \left( \frac{p - \alpha v}{\delta} \right) h(p) \\
= - \frac{d \alpha^2}{\delta^d} \int_{\mathbb{R}^d} dp \ g(\frac{p - \alpha v}{\delta}) h(p) \\
+ \frac{1}{\delta^d} \int_{\mathbb{R}^d} dp \ \nabla g \left( \frac{p - \alpha v}{\delta} \right) \left( \frac{\alpha v}{\delta} - \frac{\alpha^2 p - \alpha v}{\delta^2} \right) h(p) \\
= \frac{1}{\delta^d} \int_{\mathbb{R}^d} dp \ g \left( \frac{p - \alpha v}{\delta} \right) h(p) \left( -d \frac{\alpha^2}{\delta^2} - \beta \frac{\alpha}{\delta} v \cdot \frac{p - \alpha v}{\delta} + \beta \frac{\alpha^2}{\delta^2} \left( \frac{p - \alpha v}{\delta} \right)^2 \right) \\
= \int_{\mathbb{R}^d} dw \ g(w) h(\alpha v + \delta w) \left( -d \frac{\alpha^2}{\delta^2} - \beta \frac{\alpha}{\delta} v \cdot w + \beta \frac{\alpha^2}{\delta^2} w^2 \right), \tag{165}
\]

where we used that

\[
\frac{d}{ds} \alpha = \frac{d}{ds} e^{-s} = e^{-s} = -\alpha \\
\frac{d}{ds} \delta = \frac{d}{ds} \sqrt{1 - e^{-2s}} = \frac{e^{-2s}}{\sqrt{1 - e^{-2s}}} = \frac{\alpha^2}{\delta} \\
\frac{d}{ds} \frac{1}{\delta^d} = - \frac{d \alpha^2}{\delta^4} = - \frac{d \alpha^2}{\delta^5} \\
\frac{d}{ds} \frac{p - \alpha v}{\delta} = \frac{\alpha v \delta - (p - \alpha v) \frac{\alpha^2}{\delta}}{\delta^2} = \frac{\alpha v}{\delta} - \frac{\alpha^2 p - \alpha v}{\delta^2}. \tag{166}
\]

Next, we compute \( \nabla P_s h \) and \( \Delta P_s h \). First, we note that

\[
\frac{\partial}{\partial v_i} g \left( \frac{p - \alpha v}{\delta} \right) = \frac{\partial}{\partial v_i} \left( \frac{\beta}{2\pi} \right)^{\frac{d}{2}} \exp \left( -\frac{\beta}{2} \left( \frac{p - \alpha v}{\delta} \right)^2 \right) \\
= g \left( \frac{p - \alpha v}{\delta} \right) \cdot (-\beta) \left( \frac{p - \alpha v}{\delta} \right) \cdot \left( -\frac{\alpha}{\delta} \right) \cdot e_i \\
= \beta \frac{\alpha}{\delta} g \left( \frac{p - \alpha v}{\delta} \right) \cdot \frac{p_i - \alpha v_i}{\delta} \tag{167}
\]

and

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\[
\frac{\partial^2}{\partial v_i^2} g \left( \frac{p - \alpha v}{\delta} \right) = \beta \frac{\alpha}{\delta} \frac{\partial}{\partial v_i} \left( g \left( \frac{p - \alpha v}{\delta} \right) \cdot \frac{p_i - \alpha v_i}{\delta} \right) \\
= \beta \frac{\alpha}{\delta} \left( \frac{p_i - \alpha v_i}{\delta} \frac{\partial}{\partial v_i} g \left( \frac{p - \alpha v}{\delta} \right) - \frac{\alpha}{\delta} g \left( \frac{p - \alpha v}{\delta} \right) \right) \\
= \beta \frac{\alpha}{\delta} \left( \beta \frac{\alpha}{\delta} \frac{p_i - \alpha v_i}{\delta} g \left( \frac{p - \alpha v}{\delta} \right) \cdot \frac{p_i - \alpha v_i}{\delta} - \frac{\alpha}{\delta} g \left( \frac{p - \alpha v}{\delta} \right) \right) \\
= \beta \frac{\alpha^2}{\delta^2} g \left( \frac{p - \alpha v}{\delta} \right) \left( \beta \left( \frac{p_i - \alpha v_i}{\delta} \right)^2 - 1 \right),
\]

where \( e_i \) is the \( i \)-th cartesian unit vector. This leads to

\[
\nabla_v g \left( \frac{p - \alpha v}{\delta} \right) = \beta \frac{\alpha}{\delta} g \left( \frac{p - \alpha v}{\delta} \right) \frac{p - \alpha v}{\delta},
\]

\[
\Delta_v g \left( \frac{p - \alpha v}{\delta} \right) = \beta \frac{\alpha^2}{\delta^2} g \left( \frac{p - \alpha v}{\delta} \right) \left( \beta \left( \frac{p - \alpha v}{\delta} \right)^2 - d \right).
\]

Using the tranformation (164), we get

\[
\nabla_v P_s h(v) = \frac{1}{\delta^d} \int_{\mathbb{R}^d} dp \nabla_v g \left( \frac{p - \alpha v}{\delta} \right) h(p) \\
= \beta \frac{\alpha}{\delta} \frac{1}{\delta^d} \int_{\mathbb{R}^d} dp g \left( \frac{p - \alpha v}{\delta} \right) h(p) \frac{p - \alpha v}{\delta} \\
= \beta \frac{\alpha}{\delta} \int_{\mathbb{R}^d} dw g(w) h(\alpha v + \delta w) \cdot w
\]

and

56
\[ \Delta_v P_s h(v) = \frac{1}{\delta^d} \int_{\mathbb{R}^d} dp \nabla_v g \left( \frac{p - \alpha v}{\delta} \right) h(p) \]
\[ = \beta \frac{\alpha^2}{\delta^2} \frac{1}{\delta^d} \int_{\mathbb{R}^d} dp \ g \left( \frac{p - \alpha v}{\delta} \right) h(p) \left( \beta \left( \frac{p - \alpha v}{\delta} \right)^2 - d \right) \]
\[ = \beta \frac{\alpha^2}{\delta^2} \int_{\mathbb{R}^d} g(w) dw \ h(\alpha v + \delta w)(\beta w^2 - d). \] (171)

Combining (165) with (170) and (171) we get
\[ \frac{d}{ds} P_s h(v) = \frac{1}{\beta} \Delta_v P_s h(v) - v \cdot \nabla_v P_s h(v). \] (172)

Evaluating at \( s = 0 \) completes the proof. \( \square \)
B. Connection between Entropy and Information

In this chapter, we provide a summary of the connection between entropy and information as outlined \cite{2} chapter 3. For all statements we provide detailed proofs.

**Definition B.1.** The Fisher-Information of a function \( h > 0 \) that is normalized with respect to the Gaussian \( g \) can be defined by\footnote{See \cite{2}, p. 4.}

\[
\tilde{I}(h) := \int \frac{|
abla h|^2}{h} g dv. \tag{173}
\]

Recall, that \( g \) is the Gaussian with inverse temperature \( \beta \)

\[
g(v) := \left( \frac{\beta}{2\pi} \right)^{\frac{d}{2}} e^{-\frac{\beta}{2}v^2}. \tag{174}
\]

**Remark B.2.** The Fisher-Information \( \tilde{I}(\cdot) \) is convex on \( H^1(\mathbb{R}^{dN}, g(v)dv) \).

**Proof.** The function \( (x, y) \mapsto \frac{|x|^2}{y} \) is convex. Since \( F \mapsto (F, \nabla F) \) and taking the integral are both linear, the information is is convex as a composition of those functions. \( \square \)

As pointed out and proven in \cite{11}, p. 446, the entropy of the state \( h \) is connected with the information of \( P_s h \) by the following lemma.

**Lemma B.3.** The entropy \( S(f|g) \) and information of \( P_s h \) are related by the following formula

\[
S(f|g) = \frac{1}{\beta} \int_0^\infty \tilde{I}(P_s h) ds. \tag{175}
\]
Proof. First, we note that since $P_0 = 1$ we have

$$\lim_{s \to \infty} P_s h(v) = \lim_{s \to \infty} \int_{\mathbb{R}^dN} h(e^{-s}v + \sqrt{1 - e^{2s}}w) g(w)dw = \int_{\mathbb{R}^dN} h(w) g(w)dw = 1,$$

and therefore $\lim_{s \to \infty} \ln(P_s h) = 0$. It follows, that

$$h \ln(h) = -\int_0^\infty \frac{d}{ds} P_s h \ln(P_s h)ds$$

$$= -\int_{\mathbb{R}^d} \frac{dP_s h}{ds} \ln(P_s h) + P_s h \cdot \frac{1}{P_s h} \frac{dP_s h}{ds} g(x)dx$$

$$= -\int_{\mathbb{R}^d} \frac{dP_s h}{ds} \ln(P_s h) g(x)dx$$

$$= -\int_{\mathbb{R}^d} LP_s h \cdot \ln(P_s h) g(x)dx,$$

where we used the self-adjointness of $P_s$ to compute

$$\int_{\mathbb{R}^d} \frac{d}{ds} P_s h g(x)dx = \frac{d}{ds} \int_{\mathbb{R}^d} P_s h g(x)dx = \frac{d}{ds} \int_{\mathbb{R}^d} h \cdot P_s[1] g(x)dx = 0$$

and the semigroup properties (in particular $LP_s = P_s L$) to compute

$$\frac{d}{ds} P_s h = \lim_{h \to 0} \frac{P_{s+h} h - P_s h}{h} = P_s \lim_{h \to 0} \frac{P_h h - h}{h} = P_s Lh = LP_s h.$$

Further, we note that for any twice differentiable functions $F, H \in L^2(\mathbb{R}^dN, g(v)dv)$ integration by parts yields
\[ \int_{\mathbb{R}^d} \nabla F \cdot \nabla H g(v) \, dv = \int_{\mathbb{R}^d} \nabla F \cdot g \nabla H \, dv = - \int_{\mathbb{R}^d} F(g \Delta H + \nabla H \cdot \nabla g) \, dv \]
\[ = - \int_{\mathbb{R}^d} F(\Delta H - \beta v \cdot \nabla H) \, dv = - \beta \int_{\mathbb{R}^d} F \cdot LH g(v) \, dv. \]

(180)

The result follows by

\[ S(h, 1) = \int_{\mathbb{R}^d} h \ln(h) g(v) \, dv \overset{[177]}{=} - \int_0^\infty ds \int_{\mathbb{R}^d} g(v) \, dv LP_s h \cdot \ln(P_s h) \]
\[ \overset{[180]}{=} \frac{1}{\beta} \int_0^\infty ds \int_{\mathbb{R}^d} g(v) \, dv \nabla P_s h \cdot \nabla \ln(P_s h) \]
\[ = \frac{1}{\beta} \int_0^\infty ds \int_{\mathbb{R}^d} g(v) \, dv \frac{|\nabla P_s h|^2}{P_s h} \]
\[ = \frac{1}{\beta} \int_0^\infty \tilde{I}(P_s h) \, ds. \]

(181)

\[ \tilde{I}(P_s h) \leq e^{-2s} \tilde{I}(h). \]

Next, we estimate how the Ornstein-Uhlenbeck semigroup influences the information of the state.

**Lemma B.4.** For any differentiable function \( h \in L^1(\mathbb{R}^d, g(v) \, dv) \), we have the inequality

\[ \tilde{I}(P_s h) \leq e^{-2s} \tilde{I}(h). \]

(181)

**Proof using Cauchy-Schwarz’s inequality.** First, we note that

\[ \nabla_v [P_s h](v) = \int_{\mathbb{R}^d} e^{-s} \nabla h \left( e^{-s} v + \sqrt{1 - e^{2s}} w \right) g(w) \, dw = e^{-s} P_s [\nabla h](v), \]

(182)

which yields
\[
\tilde{I}(P_s h) = \int_{\mathbb{R}^d} \frac{|\nabla v[P_s h]|^2}{P_s h} g(v) dv = e^{-2s} \int_{\mathbb{R}^d} \frac{|P_s [\nabla h]|^2}{P_s h} g(v) dv.
\]

(183)

Then, we use the Cauchy-Schwarz inequality to estimate \( P_s \nabla h \):

\[
|P_s [\nabla h]|^2 = \left| P_s \left[ \frac{\nabla h}{\sqrt{h}} \right] \right|^2 \\
= \left( \int_{\mathbb{R}^d} \frac{\nabla h}{\sqrt{h}} \left[ e^{-s}v + \sqrt{1 - e^{-2s}w} \right] \cdot \sqrt{h} \left[ e^{-s}v + \sqrt{1 - e^{-2s}w} \right] g(w) dw \right)^2 \\
\leq \int_{\mathbb{R}^d} \left( \frac{\nabla h}{\sqrt{h}} \right)^2 \left[ e^{-s}v \sqrt{1 - e^{-2s}w} \right] g(w) dw \cdot \int_{\mathbb{R}^d} h(e^{-s}v + \sqrt{e^{-2s}w}) g(w) dw \\
= P_s \left( \frac{|\nabla h|^2}{h} \right) \cdot P_s h.
\]

(184)

This yields

\[
\tilde{I}(P_s h) = e^{-2s} \int_{\mathbb{R}^d} \frac{|P_s [\nabla h]|^2}{P_s h} g(v) dv \leq e^{-2s} \int_{\mathbb{R}^d} P_s \left( \frac{|\nabla h|^2}{h} \right) g(v) dv \\
= e^{-2s} \int_{\mathbb{R}^d} \frac{|\nabla h|^2}{h} \cdot P_s[1] g(v) dv = e^{-2s} \tilde{I}(h).
\]

\[\square\]

**Proof using convexity.** As in the proof using the Cauchy-Schwarz inequality, we observe (182) and (183).

To use Jensen’s inequality, we make some definitions. Since the function \( h : \mathbb{R}^d \rightarrow \mathbb{R}, \ v \mapsto v^2 \) is convex, its perspective function \( \varphi(v, t) := t \cdot h \left( \frac{v}{t} \right) = \frac{v^2}{t} \) is convex on \( \mathbb{R}^d \times (0, \infty) \). Defining

\[
\psi : \mathbb{R}^d \rightarrow \mathbb{R}^2, \ w \mapsto \left( \nabla h \left( e^{-s}v + \sqrt{1 - e^{-2s}w} \right), h \left( e^{-s}v + \sqrt{1 - e^{-2s}w} \right) \right)
\]

(185)
we get \((P_{s}[\nabla h], P_{s}h) = \int_{\mathbb{R}^{dN}} \psi(w) g(w) dw\). It follows that
\[
\frac{(P_{s}[\nabla h])^2}{P_{s}h} = \varphi \left( P_{s}[\nabla h], P_{s}h \right) = \varphi \left( \int_{\mathbb{R}^{dN}} \psi(w) g(w) dw \right) \leq \int_{\mathbb{R}^{dN}} (\varphi \circ \psi)(w) g(w) dw = \psi \left( \frac{[\nabla h]^2}{h} \right). \tag{186}
\]

Using the self-adjointness of \(P_{s}\) in the second step we obtain the claim by
\[
\int_{\mathbb{R}^{dN}} \frac{(P_{s}[\nabla h])^2}{P_{s}h} g(v) dv \leq \int_{\mathbb{R}^{dN}} P_{s} \left[ \frac{|\nabla h|^2}{h} \right] g(v) dv = \int_{\mathbb{R}^{dN}} \left[ \frac{|\nabla h|^2}{h} \right]_{1}^{P_{s}[1]} g(v) dv
= \int_{\mathbb{R}^{dN}} \frac{|\nabla h|^2}{h} g(v) dv = \tilde{I}(h). \tag{187}
\]

Summarizing the last two results (lemma B.3 and lemma B.4), we get the following corollary.

**Corollary B.5.** The entropy can be estimated by the information of the state as follows
\[
S(f|g) \leq \frac{1}{2\beta} \tilde{I}(h). \tag{188}
\]
C. Proofs for the kinetic energy and momentum

C.1. Proof of lemma \[2.3\]

*Proof for the kinetic energy of the Kac system.* The derivative of the kinetic energy is given by

\[
\frac{d}{dt} K(t) = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^{dN}} v^2 f_t(v^2) \, dv = \frac{1}{2} \int_{\mathbb{R}^{dN}} v^2 \frac{d}{dt} f_t(v^2) \, dv,
\]

(189)

where we use the Leibniz rule to differentiate under the integral sign:

i) For all \( t \geq 0 \) the mapping \( v \mapsto v^2 f_t(v) \) is Lebesgue-integrable.

ii) Since \( (f_t)_{t \geq 0} \) is a classic solution of the abstract Cauchy-Problem (10), the mapping \( t \mapsto f_t \) is differentiable. Therefore, the mapping \( t \mapsto f_t(v) \) is differentiable for all \( v \in \mathbb{R}^{dN} \).

iii) Finally, we compute a majorant. Since \( (e^{\mathcal{L} t})_{t \geq 0} \) is a contraction semigroup and \( ||\mathcal{L}|| \leq 2(\lambda + \mu)N \) by remark \[2.2\] we get

\[
\left| v^2 \frac{d}{dt} f_t(v^2) \right| = \left| v^2 \mathcal{L} e^{\mathcal{L} t} f_0(v) \right| \leq 2(\lambda + \mu)N \left| v^2 f_0(v) \right|,
\]

(190)

with \( v \mapsto |v^2 f_0(v)| \) being an integrable function.

Now, we can insert the master equation (11) in (189). We get

\[
\frac{d}{dt} K(t) = \frac{1}{2} \int_{\mathbb{R}^{dN}} v^2 \left[ \lambda N (Q - 1)[f_t] + \mu \sum_{j=1}^{N} (R_j - 1)[f_t] \right] \, dv.
\]

(191)

Therefore, we investigate the second moments of \( Q[f_t] \) and \( R_j[f_t] \). First, we observe
that the collisions between particles in the Kac system do not change the kinetic energy of the Kac system. Since

\[
\int_{\mathbb{R}^d} v^2 Q[f_i] \, dv = \left( \frac{N}{2} \right) \sum_{i<j} \int_{\mathbb{R}^d} dv \int_{\mathbb{S}^{d-1}} d\rho(\sigma) \, v^2 f \left( M^{(i,j)}_\sigma v \right)
\]

\[
= \left( \frac{N}{2} \right) \sum_{i<j} \int_{\mathbb{R}^d} dp \, p^2 f(p),
\]

(192)

where we used \( p := M^{(i,j)}_\sigma v \), the first term in the integral in [191] vanishes:

\[
\int_{\mathbb{R}^d} v^2 (Q - 1)[f_i] \, dv = 0.
\]

(193)

Second, we investigate, how the collisions of a particle in the Kac with a particle in the thermostat change the kinetic energy of the Kac system. With the change of variables \((p, q) := M^{(j,N+1)}_\sigma(v, w)\), we get

\[
\int_{\mathbb{R}^d} v^2 R_j[f_i] \, dv = \int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} dw \int_{\mathbb{S}^{d-1}} d\rho(\sigma) \, v^2 g(N_{\sigma \cdot p_j} + \sigma \otimes \sigma[v_j]) f \left( N^{(j)}_\sigma v + w^{(j)}_\sigma \right)
\]

\[
= \int_{\mathbb{R}^d} dp \int_{\mathbb{R}^d} dq \int_{\mathbb{S}^{d-1}} d\rho(\sigma) \left( N^{(j)}_\sigma p + q^{(j)}_\sigma \right)^2 g(q) f_i(p).
\]

(194)

We calculate the square using

\[
\left( N^{(j)}_\sigma p \right)^2 = p^T N^{(j)}_\sigma T N^{(j)}_\sigma p = p^T N^{(j)}_\sigma p = p^2 - p_j \cdot \sigma \otimes \sigma[p_j] = p^2 - (\sigma p_j)^2,
\]

\[
N^{(j)}_\sigma p \cdot q^{(j)}_\sigma = (1 - \sigma \otimes \sigma)[p_j] \cdot \sigma \otimes \sigma[q] = (\sigma q)(\sigma p_j) - (\sigma p_j)(\sigma q) = 0,
\]

\[
\left( q^{(j)}_\sigma \right)^2 = (\sigma \otimes \sigma[q])^2 = (\sigma q)^2.
\]

(195)

This yields
\[
\left( N^{(j)}_\sigma p + q^{(j)}_\sigma \right)^2 = p^2 - (\sigma p_j)^2 + (\sigma q)^2. \quad (196)
\]

We recall that
\[
\int_{\mathbb{R}^d} q^2 g(q) \, dq = \sum_{i=1}^d \int_{\mathbb{R}^d} q^2_i \prod_{j=1}^d \sqrt{\beta/2\pi} e^{-\beta q^2_j} \, dq_j = d \int_{\mathbb{R}} x^2 \sqrt{\beta/2\pi} e^{-\beta x^2} \, dx = d/\beta.
\]

Further, the symmetry (2) implies for all vectors \( v \in \mathbb{R}^d \) that
\[
\int_{S^{d-1}} d\rho(\sigma) (\sigma v)^2 = \sum_{i=1}^d \int_{S^{d-1}} d\rho(\sigma) \sigma_i^2 v_i^2 = \frac{1}{d} \sum_{i=1}^d v_i^2 \int_{S^{d-1}} d\rho(\sigma) \sigma^2 = \frac{1}{d} v^2. \quad (197)
\]

This yields
\[
\sum_{j=1}^N \int_{\mathbb{R}^{dN}} v^2 R_j[f_i] \, dv = \int_{\mathbb{R}^{dN}} dp \int_{\mathbb{R}^d} dq \int_{S^{d-1}} \left( Np^2 - \frac{1}{d} p^2 + \frac{N}{d} q^2 \right) g(q) f_i(p) = \left( N - \frac{1}{d} \right) K(t) + \frac{N}{\beta}. \quad (198)
\]

In summary, we get
\[
\frac{d}{dt} K(t) = \frac{\mu}{2} \left( \left( N - \frac{1}{d} \right) K(t) + \frac{N}{\beta} - NK(t) \right) = -\frac{\mu}{2d} K(t) + \frac{\mu N}{2\beta}. \quad (199)
\]

Solving this first order linear inhomogenous ODE proves the claim.

---

Proof for the momentum of the Kac system. As for the kinetic energy, we observe
\[
\frac{d}{dt} P(t) = \sum_{i=1}^N \int_{\mathbb{R}^{dN}} v_i \left[ \lambda N (Q - 1)[f_i] + \mu \sum_{j=0}^N (R_j - 1)[f_i] \right] \, dv. \quad (200)
\]

We used again the Leibniz rule to differentiate under the integral sign with the ma-
The change of variables $p := M_{ij}^{(i,j)}v$ together with the momentum conserving property $\sum_{i=1}^{N} \left( M_{ij}^{(i,j)}p \right)_i = \sum_{i=1}^{N} p_i$ from (15), prove that the first part representing the collisions between particles in the Kac system vanishes:

$$\sum_{i=1}^{N} \int_{\mathbb{R}^{dN}} v_i Q_i [f_t] d\mathbf{v} = \int_{\mathbb{R}^{dN}} \sum_{i=1}^{N} \left( M_{ij}^{(i,j)}p \right)_i f_t(p) d\mathbf{p} = P(t)$$

$$\Rightarrow \sum_{i=1}^{N} \int_{\mathbb{R}^{dN}} v_i (Q - 1) [f_t] d\mathbf{v} = 0. \quad (202)$$

Now we consider the collisions between particles in the Kac system with particles in the thermostat. Using the transformation $(p, q) := M_{i,N+1}^{(j)}(v, w)$, we get

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \int_{\mathbb{R}^{dN}} v_i R_j [f_t] d\mathbf{v}$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{\mathbb{R}^{dN}} d\mathbf{v} \int_{\mathbb{R}^{dN}} d\mathbf{w} \int_{\mathbb{S}^{d-1}} d\sigma p(\sigma) v_i g(N_{\sigma} w + \sigma \otimes \sigma[v_j]) f_t \left( N_{\sigma}^{(j)} v + w_{\sigma}^{(j)} \right)$$

$$= \int_{\mathbb{R}^{dN}} d\mathbf{p} \int_{\mathbb{R}^{dN}} d\mathbf{q} \int_{\mathbb{S}^{d-1}} d\sigma \rho(\sigma) \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ N_{\sigma}^{(j)} p + q_{\sigma}^{(j)} \right]_i g(q) f_t(p). \quad (203)$$

We further compute
\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \left( \mathbf{N}^{(j)} \mathbf{p} + \mathbf{q}^{(j)} \right)_i = \sum_{j=1}^{N} \left( \sum_{i=1}^{N} \mathbf{p}_i - \sigma \otimes \sigma \mathbf{p}_j + \sigma \otimes \sigma [\mathbf{q}] \right) = N \sum_{i=1}^{N} \mathbf{p}_i - N \sum_{j=1}^{N} \sigma \otimes \sigma \mathbf{p}_j + N \sigma \otimes \sigma \mathbf{q}. \tag{204}
\]

Noting that the symmetries imply that
\[
\int_{S^{d-1}} \sigma \otimes \sigma \, d\rho(\sigma) = \frac{1}{d} \mathbf{1}_d
\tag{205}
\]
we get
\[
\frac{d}{dt} P(t) = -\mu \int_{\mathbb{R}^{dN}} d\mathbf{p} \int_{\mathbb{R}^{d}} g(q) dq \int_{S^{d-1}} d\rho(\sigma) \left( \sum_{j=1}^{N} \mathbf{p}_j - \frac{1}{d} \sum_{j=1}^{N} \mathbf{p}_j - \frac{N}{d} \mathbf{q} \right) f_t(\mathbf{p}) = -\frac{\mu}{d} P(t),
\tag{206}
\]
where the last term vanishes since \( \int_{\mathbb{R}^{d}} g(q) dq = 0 \). Solving this first-order linear homogenous ODE proves the claim. \( \square \)

### C.2. Proof of lemma 3.4

**Total Kinetic Energy:** Since our collision mechanism is conserves the kinetic energy, as shown in equation (15), we expect the total kinetic energy to stay constant. It is sufficient to prove, that \( R_{ij} \) does not change the kinetic energy, since \( \mathcal{L} \) is a linear combination of operators of the form \( (R_{ij} - 1) \) and
\[
\frac{d}{dt} K(t) = \frac{1}{2} \int_{\mathbb{R}^{d(N+M)}} \mathbf{u}^2 \frac{\partial F_t}{\partial t} = \frac{1}{2} \int_{\mathbb{R}^{d(N+M)}} \mathbf{u}^2 \mathcal{L}[F_t](\mathbf{u}). \tag{207}
\]
For all \(i \in \{1, \ldots, N\}, j \in \{N + 1, \ldots, N + M\}\), we get with the transformation 
\[
u' := M_{\sigma}^{(i,j)} \nu
\]
that
\[
\int_{\mathbb{R}^{d(N+M)}} \nu^2 R_{ij}[F_t](\nu) \, d\nu = \int_{\mathbb{R}^{d(N+M)}} \nu^2 F_t(\nu) \, d\nu
\]
\[
= \int_{\mathbb{R}^{d(N+M)}} (M_{\sigma}^{(i,j)} \nu')^2 F_t(\nu') \, d\nu'
\]
\[
= \int_{\mathbb{R}^{d(N+M)}} \nu'^2 F_t(\nu') \, d\nu'.
\] (208)

Since \(F_0(\nu, w) = f_0(\nu)g(w)\), the claim follows using formula (107).

**Kinetic Energy of the Kac System** We first compute the derivative of the kinetic energy of the Kac system in the following lemma. Using that the total kinetic energy stays constant, we then solve the ODE.

**Lemma C.1.** The derivative of the kinetic energy of the Kac system \(K_S\) is given by
\[
\frac{d}{dt} K_S(t) = \frac{1}{2} \frac{\mu}{dM} (NK_R(t) - MK_S(t)).
\] (209)

**Proof.** Using the Kac master equation (85), we get
\[
\frac{d}{dt} K_S(t) = \frac{1}{2} \int_{\mathbb{R}^{dN}} d\nu \int_{\mathbb{R}^{dM}} d\nu \, \nu^2 \frac{\partial}{\partial t} F_t(\nu, w)
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^{dN}} d\nu \int_{\mathbb{R}^{dM}} d\nu \left[ \frac{\lambda_S}{N-1} \sum_{1 \leq i < j \leq N} (R_{ij} - 1)[F_t] + \frac{\lambda_R}{M-1} \sum_{N<i<j\leq N+M} (R_{ij} - 1)[F_t] \right] + \frac{\mu}{M} \sum_{i=1}^{N} \sum_{j=N+1}^{N+M} (R_{ij} - 1)[F_t]
\] (210)
Step 1: The first sum describes the collisions within the Kac system, which do not change $K_S$. With the transformation $(p, q) := M^{(i,j)}_\sigma(v, w)$ we get

$$\int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} dw \ v^2 \ \sum_{1 \leq i < j \leq N} R_{ij}[F_t]$$

$$= \int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} dw \ v^2 \ \sum_{1 \leq i < j \leq N} \int_{\mathbb{S}^{d-1}} d\rho(\sigma) \ F_t(M^{(i,j)}_\sigma(v, w)) \$$

$$= \int_{\mathbb{R}^d} dp \int_{\mathbb{R}^d} dq \ \sum_{1 \leq i < j \leq N} \left( p_i^*(\sigma, p_j) + p_j^*(\sigma, p_i) \right) \sum_{k=1}^{N} \frac{p_k}{p_i + p_j} F_t(p, q)$$

$$= \int_{\mathbb{R}^d} dp \int_{\mathbb{R}^d} dq \ \sum_{1 \leq i < j \leq N} F_t(p, q). \quad (211)$$

Therefore,

$$\frac{1}{2} \int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} dw \ v^2 \left[ \frac{\lambda_S}{N-1} \sum_{1 \leq i < j \leq N} (R_{ij} - 1)[F_t] \right] = 0. \quad (212)$$

Step 2: The second sum describes the collisions within the heat reservoir, which do not have any effect on the kinetic energy of the Kac system. We use again the transformation $(p, q) := M^{(i,j)}_\sigma(v, w)$ to get

$$\int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} dw \ v^2 \ \sum_{N \leq i < j \leq N + M} R_{ij}[F_t]$$

$$= \int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} dw \ v^2 \ \sum_{N \leq i < j \leq N + M} \int_{\mathbb{S}^{d-1}} d\rho(\sigma) \ F_t(M^{(i,j)}_\sigma(v, w))$$

$$= \int_{\mathbb{R}^d} dv \int_{\mathbb{R}^d} dw \ v^2 \ \sum_{N \leq i < j \leq N + M} F_t(p, q). \quad (213)$$
Hence, we get

\[
\frac{1}{2} \int_{\mathbb{R}^{dN}} dv \int_{\mathbb{R}^{dM}} dw \, v^2 \left[ \frac{\lambda_R}{M - 1} \sum_{N < i < j < N + M} (R_{ij} - 1) [F_i] \right] = 0. \tag{214}
\]

**Step 3:** The collisions between the Kac system and the heat reservoir are modeled by the third sum and give the change of the kinetic energy of the Kac system. Again, we use the transformation \((p, q) := M^{(i,j)}(v, w)\):

\[
\int_{\mathbb{R}^{dN}} dv \int_{\mathbb{R}^{dM}} dw \, v^2 \sum_{i=1}^{N} \sum_{j=N+1}^{N+M} R_{ij} [F_i] \\
= \int_{\mathbb{R}^{dN}} dv \int_{\mathbb{R}^{dM}} dw \, v^2 \sum_{i=1}^{N} \sum_{j=N+1}^{N+M} \int_{\mathbb{S}^{d-1}} d\rho(\sigma) \, F_i \left( M^{(i,j)}(v, w) \right) \\
= \int_{\mathbb{R}^{dN}} dp \int_{\mathbb{R}^{dM}} dq \sum_{i=1}^{N} \sum_{j=N+1}^{N+M} \int_{\mathbb{S}^{d-1}} d\rho(\sigma) \left[ \frac{p^2 - p_{i}^2 + p_{i}^*(\sigma, q_j)^2}{(\sigma q_j - \sigma p_i)^2} \right] F_i(p, q) \\
= \int_{\mathbb{R}^{dN}} dp \int_{\mathbb{R}^{dM}} dq \left( \sum_{i=1}^{N} \sum_{j=N+1}^{N+M} p^2 \right) + \frac{1}{d} \left( N q^2 - M p^2 \right) F_i(p, q) \\
= \frac{1}{d} (NK_R(t) - MK_S(t)) + \int_{\mathbb{R}^{dN}} dp \int_{\mathbb{R}^{dM}} dq \, p^2 \sum_{i=1}^{N} \sum_{j=N+1}^{N+M} F_i(p, q). \tag{215}
\]

This yields

\[
\frac{d}{dt} K_S(t) = \frac{1}{2dM} \frac{\mu}{\lambda_R} (NK_R(t) - MK_S(t)). \tag{216}
\]
\[
\frac{d}{dt} K_S(t) = \frac{\mu}{2dM} (N(K - K_S(t)) - MK_S(t)) = \frac{\mu}{2dM} (NK - (N + M)K_S(t)). \tag{217}
\]

The solution of this first-order inhomogenous linear ODE yields the claim for \( K_S(t) \).

Inserting the result in \( K_R(t) = K - K_S(t) \) proves the result for the kinetic energy of the heat reservoir.
References


