# DUAL REPRESENTATIONS OF POLYNOMIAL MODULES WITH APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

A Dissertation
Presented to
The Academic Faculty

By

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In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in the
College of Sciences
School of Mathematics

Georgia Institute of Technology

May 2022

# DUAL REPRESENTATIONS OF POLYNOMIAL MODULES WITH APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS


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Date approved: 4/15/2022

## ACKNOWLEDGMENTS

A very special thanks goes of course to my advisor, Anton Leykin, for his words of encouragements, weekly meetings and invaluable connections in the applied algebra community. Another huge thank you goes to a very dear friend Justin Chen for in-depth discussions on both math and non-math topics. Other sources of inspiration and interesting conversations at Georgia Tech include Kisun Lee, Tim Duff, Jaewoo Jung, Jose Acevedo, Cvetelina Hill, Trevor Gunn, and many of the other algebraically oriented PhD students at Tech. Outside of Tech, I must mention Bernd Sturmfels for suggesting the PDE point of view of this dissertation, his ability to connect people, and for the post-doc opportunity at MPI.

I would like to thank all of my coauthors: Rida Ait El Manssour, Justin Chen, Yairon Cid-Ruiz, Yoshihiro Hirose, Jonas Hirsch, Benjamin Hollering, Robert Krone, Anton Leykin, Lisa Nicklasson, Bogdan Raiţă, Jose Rodriguez, Tomonari Sei, Bernd Sturmfels, and Fatemeh Tarashi Kashani. A big thanks goes to my defense committee as well.

I also gratefully acknowledge the support for this work offered by several people and institutions outside of Georgia Tech, which allowed me to travel and do research outside for extended periods of time. For this, I want to thank the organizers of the semester program in Nonlinear algebra at ICERM, Brown University, Providence, Rhode Island; Kenji Fukumizu and Satoshi Kuriki from the Institute of Statistical Mathematics, Tokyo, Japan; Elias Tsigaridas from Sorbonne Université; Bernd Sturmfels from the Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany. I also acknowledge the support of the Vilho, Yrjö and Kalle Väisälä Foundation of the Finnish Academy of Science and Letters, the Chateaubriand fellowship, and NSF grants DMS-1719968 and DMS-2001267 for allowing me to focus on research for a few semesters.

On a final, more personal note, I of course would like to thank my parents for their constant unwavering support. Rock climbing also played an important part of my graduate

student life, and I am grateful for the small intersection of math students who enjoyed climbing with me.

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# SUMMARY

In 1939, Wolfgang Gröbner proposed using differential operators to represent ideals in a polynomial ring. Using Macaulay inverse systems, he showed a one-to-one correspondence between primary ideals whose variety is a rational point, and finite dimensional vector spaces of differential operators with constant coefficients. The question for general ideals was left open. Significant progress was made in the 1960's by analysts, culminating in a deep result known as the Ehrenpreis-Palamodov fundamental principle, connecting polynomial ideals and modules to solution sets of linear, homogeneous partial differential equations with constant coefficients.

This work aims to survey classical results, and provide new constructions, applications, and insights, merging concepts from analysis and nonlinear algebra. We offer a new formulation generalizing Gröbner's duality for arbitrary polynomial ideals and modules and connect it to the analysis of PDEs. This framework is amenable to the development of symbolic and numerical algorithms. We also study some applications of algebraic methods in problems from analysis.

# CHAPTER 1

# LOCAL DUAL SPACES

The linear differential operators with polynomial coefficients form the *Weyl algebra* $W = R\langle\partial_{\mathbf{x}}\rangle = \mathbb{K}[x_1, \ldots, x_n]\langle\partial_1, \ldots, \partial_n\rangle$. We shall forget about multiplication (i.e., composition) of differential operators in $W$, but retain *two* $R$-module structures: for $f \in R$ and $D := \sum c_\alpha \partial^\alpha \in W$ we have

- the left action $fD := \sum(fc_\alpha)\partial^\alpha$, and
- the right action: $D \cdot f$ is the differential operator that multiplies the input by $f$ before applying $D$.

The relation between the two actions is given by

$$\partial_i \cdot x_i = x_i\partial_i + 1 \text{ and } \partial_i \cdot x_j = x_j\partial_i, \text{ for } i \neq j.$$

## 1.1 Definitions

For an $R$-algebra $A$, let $W_A := A \otimes_R W$. Let $\mathfrak{m}$ denote a maximal ideal in $R$, and $\kappa(\mathfrak{m}) := R/\mathfrak{m}$ the *residue field* at $\mathfrak{m}$. We call $W_{\kappa(\mathfrak{m})}$ the *local differential space* (at $\mathfrak{m}$). Elements of $W_{\kappa(\mathfrak{m})}$ will be commonly called *differential operators*. In what follows $W_{\kappa(\mathfrak{m})}$ is perceived as a left $\kappa(\mathfrak{m})$-vector space and a right $R$-module.

For each $D \in W_{\kappa(\mathfrak{m})}$, we define a map $D \bullet \_ \colon R \to \kappa(\mathfrak{m})$ as follows. We can write $D = \sum_\alpha \overline{c_\alpha}\partial^\alpha$, where $c_\alpha \in R$ and $\overline{c_\alpha}$ is the image of $c_\alpha$ in $\kappa(\mathfrak{m})$. For each $f \in R$, we define

$$D \bullet f = \sum_\alpha \overline{c_\alpha \partial^\alpha \bullet f}.$$

Note that if $\kappa(\mathfrak{m})$ corresponds to a rational point, that is $\kappa(\mathfrak{m}) = \mathbb{K}$, then each $D \in R^* = \operatorname{Hom}_{\mathbb{K}}(R, \mathbb{K})$.

For each $D = \sum_\alpha \overline{c_\alpha}\partial^\alpha \in W_{\kappa(\mathfrak{m})}$, we define the *degree* of the operator as the degree as a polynomial in the $\partial$-variables, that is $\deg(D) = \max\{|\alpha| \colon \overline{c_\alpha} \neq 0\}$.

**Definition 1.1.1.** For $I \subseteq R$, we define the *local dual space* of $I$ at $\mathfrak{m}$

$$D_{\mathfrak{m}}[I] = \{D \in W_{\kappa(\mathfrak{m})} \colon D \bullet f = 0 \text{ for all } f \in I\}$$

The natural dual definition of a local dual space is as follows.

**Definition 1.1.2.** For $\Lambda \subseteq W_{\kappa(\mathfrak{m})}$, we define

$$I_{\mathfrak{m}}[\Lambda] = \{f \in R \colon D \bullet f = 0 \text{ for all } D \in \Lambda\}$$

It is not hard to see that if $I$ is an ideal, its local dual space $D_{\mathfrak{m}}[I]$ is a $\kappa(\mathfrak{m})$-vector space and a right $R$-module. These two properties will appear many times in this paper, which warrants the following definition.

**Definition 1.1.3.** A set $\Lambda \subseteq W_{\kappa(\mathfrak{m})}$ is called a *local dual space* if it is a $\kappa(\mathfrak{m})$-vector space, and a right $R$-module.

*Remark* 1.1.4. Note that if $D \in W_{\kappa(\mathfrak{m})}$ has degree $d$, then by Leibniz' formula the commutator $x_iD - Dx_i$ has degree $d-1$. For a multi-index $\alpha \in \mathbb{N}^n$, we have $x_i\partial^\alpha - \partial^\alpha x_i = \alpha_i\partial^{\alpha-i}$, where the multi-index $\alpha - i = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1}, \ldots, \alpha_n)$. Therefore the commutator $x_iD - Dx_i$ can be thought of as the derivative of $D$ with respect to the $\partial_i$ variable, and a right $R$-module $\Lambda \subseteq W_{\kappa(\mathfrak{m})}$ will be closed under taking derivatives with respect to the $\partial_i$ variables. $\triangle$

## 1.2 Rational points

Throughout this section, we will let $p = (p_1, \ldots, p_n) \in \mathbb{K}^n$ denote a rational point, and let $\mathfrak{m} = (x_1 - p_1, \ldots, x_n - p_n) \subseteq R$ be the corresponding maximal ideal. Then $\kappa(\mathfrak{m}) = R/\mathfrak{m} \cong \mathbb{K}$, so the operators in $W_{\kappa(\mathfrak{m})}$ are differential operators with *constant* coefficients. The operator $\partial^\alpha \in W_{\kappa(\mathfrak{m})}$ induces a linear map $R \to \mathbb{K}$, where $\partial^\alpha \bullet f = \frac{\partial^\alpha f}{\partial x^\alpha}(p)$. Hence the space $W_{\kappa(\mathfrak{m})}$ is a subspace of the dual space $R^* = \mathrm{Hom}_{\mathbb{K}}(R, \mathbb{K})$, consisting of polynomials in the "variables" $\partial_1, \ldots, \partial_n$. Note that a general element in $R^*$ can be represented as a formal power series in the variables $\partial_1, \ldots, \partial_n$. Restricting to polynomials results in the theory of Gröbner duality [25, 42, 43]. We summarize the main results needed in the upcoming sections below.

**Theorem 1.2.1.** *Suppose $\mathfrak{m} \subseteq R$ is a maximal ideal corresponding to a rational point $p \in \mathbb{K}^n$. There is an inclusion reversing bijection between ideals $J$ of the local ring $R_\mathfrak{m}$ and local dual spaces $\Lambda \subseteq W_{\kappa(\mathfrak{m})}$, given by*

$$J \mapsto D_\mathfrak{m}[J \cap R]$$
$$\Lambda \mapsto (I_\mathfrak{m}[\Lambda])_\mathfrak{m}.$$

*If $I$ is an arbitrary ideal, we have*

$$D_\mathfrak{m}[I] = D_\mathfrak{m}[I_\mathfrak{m} \cap R].$$

Next, we will generalize the above duality result for arbitrary ideals and $R$-submodules of $R^k$.

## 1.3 General maximal ideals

Let $R := \mathbb{K}[x_1, \ldots, x_n]$, where $\mathbb{K}$ is not necessarily algebraically closed. Suppose $\mathfrak{m} \subseteq R$ is a maximal ideal, and $I \subseteq R$ is $\mathfrak{m}$-primary.

We start with a few results which can be shown using elementary methods. We define $W^{(s)}_{\kappa(\mathfrak{m})} := \operatorname{span}_{\kappa(\mathfrak{m})}\{\partial^\alpha \colon |\alpha| \leq s\}$, the set of differential operators of degree at most $s$.

**Lemma 1.3.1.** *For $s = 1, 2, \ldots$, we have*

$$D_{\mathfrak{m}}[\mathfrak{m}^s] = W^{(s-1)}_{\kappa(\mathfrak{m})}$$

*Proof.* Suppose $s = 1$. Then clearly $1 \in D_{\mathfrak{m}}[\mathfrak{m}]$. If $D \in W_{\kappa(\mathfrak{m})}$ is an operator of degree $d > 0$, let $c_\alpha \partial^\alpha$ be a term of $D$ of degree $d$. If $p_i$ is the minimal polynomial of $\overline{x_i} \in R/\mathfrak{m}$ over $R$, let $p^\alpha := p_1^{\alpha_1} \cdots p_n^{\alpha_n}$. Then

$$D \bullet p^\alpha = c_\alpha \partial^\alpha \bullet p^\alpha \neq 0,$$

but $p^\alpha \in \mathfrak{m}$.

Suppose next that the claim is shown up to a given $s$, and we want to show that $D_{\mathfrak{m}}[\mathfrak{m}^{s+1}] = W^s_{\kappa(\mathfrak{m})}$. We can write any $f \in \mathfrak{m}^s$ as $f = gh$, where $g \in \mathfrak{m}$, $h \in \mathfrak{m}^{s-1}$. Then, for any $|\alpha| \leq s$, we have by Leibniz' rule and the induction assumption

$$\partial^\alpha \bullet f = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta \bullet h) \cdot (\partial^{\alpha-\beta} \bullet g) = (\partial^\alpha \bullet h) \cdot (1 \bullet g) = 0,$$

which proves the $\supseteq$ inclusion. For the opposite inclusion, we can proceed as in the base case. Let $D \in W_{\kappa(\mathfrak{m})}$ whose degree is $d > s$, let $c_\alpha \partial^\alpha$ be a term of degree $d$ in $D$, then $p^\alpha \in \mathfrak{m}^{d+1}$ but $D \bullet p^\alpha \neq 0$. $\square$

**Proposition 1.3.2.** *If $I \subseteq R$ is an ideal, then $D_{\mathfrak{m}}[I]$ is a local dual space. If in addition $I$ is $\mathfrak{m}$-primary, then $D_{\mathfrak{m}}[I]$ is a finite dimensional local dual space.*

*Proof.* Clearly $D_{\mathfrak{m}}[I]$ is a $\kappa(\mathfrak{m})$-vector space. Suppose $D \in D_{\mathfrak{m}}[I]$, $f \in R$. Then for $(Df) \bullet g = D \bullet (fg) = 0$ for any $g \in I$.

If $I$ is $\mathfrak{m}$-primary, there is some integer $N$ such that $\mathfrak{m}^N \subseteq I$. This implies that

$$D_{\mathfrak{m}}[I] \subseteq D_{\mathfrak{m}}[\mathfrak{m}^N] = W_{\kappa(\mathfrak{m})}^{(N-1)}.$$

□

**Proposition 1.3.3.** *Let $\Lambda \subseteq W_{\kappa(\mathfrak{m})}$ be a nonzero local dual space. Then $I_{\mathfrak{m}}[\Lambda]$ is an ideal contained in $\mathfrak{m}$. If in addition $\Lambda$ is a finite dimensional local dual space, then $I_{\mathfrak{m}}[\Lambda]$ is $\mathfrak{m}$-primary.*

*Proof.* Let $f \in I_{\mathfrak{m}}[\Lambda]$, $g \in R$. Then $D \bullet (fg) = Dg \bullet f = 0$ for all $D \in \Lambda$. Let $0 \neq D \in \Lambda$. Since $\Lambda$ is a right $R$-module, we can find a polynomial $h \in R$ such that $\deg(Dh) = 0$. Then the operator $(D \bullet h)^{-1} Dh = 1 \in \Lambda$. Hence if $f \in I_{\mathfrak{m}}[\Lambda]$, then $1 \bullet f = 0$, which implies that $f \in \mathfrak{m}$.

If $\Lambda$ is finite-dimensional, then $\Lambda \subseteq W_{\kappa(\mathfrak{m})}^{(N)}$ for some integer $N$. Then

$$\mathfrak{m}^{N+1} \subseteq I_{\mathfrak{m}}[D_{\mathfrak{m}}[\mathfrak{m}^{N+1}]] = I_{\mathfrak{m}}[W_{\kappa(\mathfrak{m})}^{(N)}] \subseteq I_{\mathfrak{m}}[\Lambda] \subseteq \mathfrak{m}$$

Since $\mathfrak{m}$ is maximal, this implies that $I_{\mathfrak{m}}[\Lambda]$ is $\mathfrak{m}$-primary. □

Next, we will consider extending the base field so that $\mathfrak{m}$ decomposes into a union of rational points. Let $S := R/\mathfrak{m} \otimes_{\mathbb{K}} \mathbb{K}[x_1, \dots, x_n]$. Since $S\mathfrak{m}$ is no longer necessarily maximal, we denote $S\mathfrak{m} = \bigcap_{i=1}^d \mathfrak{n}_i \subseteq S$, where each $\mathfrak{n}_i$ is maximal. In what follows, we will fix some $i = 1, \dots, d$ and let $\mathfrak{n} := \mathfrak{n}_i$.

Likewise, we can write a primary decomposition $SI = \bigcap_{i=1}^d J_i$, where $\sqrt{J_i} = \mathfrak{n}_i$. For the same fixed $i$ as above, we set $J := J_i$. We have $\mathfrak{n} \cap R = \mathfrak{m}$ and $J \cap R = I$. Note that the fields $\kappa(\mathfrak{m}) = R/\mathfrak{m}$ and $\kappa(\mathfrak{n}) = S/\mathfrak{n}$ are canonically isomorphic; we will denote these fields by $\mathbb{L}$ when the distinction between $\kappa(\mathfrak{m})$ and $\kappa(\mathfrak{n})$ is not important.

Our first goal will be to relate the local dual spaces $D_{\mathfrak{m}}[I] \subseteq W_{\kappa(\mathfrak{m})}$ and $D_{\mathfrak{n}}[SI] \subseteq W_{\kappa(\mathfrak{n})}$. While $\kappa(\mathfrak{m}) \cong \kappa(\mathfrak{n})$, the distinction between the two local differential spaces $W_{\kappa(\mathfrak{m})}$

and $W_{\kappa(\mathfrak{n})}$ is crucial, as the operators in each describe different maps

$$W_{\kappa(\mathfrak{m})} \ni D \colon R \to \mathbb{L}$$
$$W_{\kappa(\mathfrak{n})} \ni D' \colon S \to \mathbb{L}$$

To emphasize the distinction, we will use the symbol $\delta$ in $W_{\kappa(\mathfrak{n})}$ instead of $\partial$ used in $W_{\kappa(\mathfrak{m})}$. Hence any operator $D \in W_{\kappa(\mathfrak{m})}$ is of the form

$$D = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \partial^\alpha,$$

where $c_\alpha \in \mathbb{L}$, and only finitely many of them are nonzero. Likewise, any operator $D' \in W_{\kappa(\mathfrak{n})}$ is of the form

$$D' = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \delta^\alpha,$$

where $c_\alpha \in \mathbb{L}$, and only finitely many of them are nonzero. The spaces $W_{\kappa(\mathfrak{m})}$ and $W_{\kappa(\mathfrak{n})}$ are therefore naturally isomorphic $\mathbb{L}$-vector spaces via the map $D \mapsto D'$ which substitutes $\partial$ by $\delta$.

**Lemma 1.3.4.** *The operators $D \in W_{\kappa(\mathfrak{m})}$ and $D' \in W_{\kappa(\mathfrak{n})}$ agree on $R$, i.e.*

$$D \bullet f = D' \bullet f, \textit{ for all } f \in R.$$

*Proof.* It suffices to show the claim for $D = \partial^\alpha$, $D' = \delta^\alpha$ for any $\alpha \in \mathbb{N}^n$. If $f \in R$, then $g = \frac{\partial^\alpha f}{\partial x^\alpha} \in R$. Note that the diagram

$$\begin{array}{ccc} R & \hookrightarrow & S \\ \downarrow & & \downarrow \\ R/\mathfrak{m} & \xrightarrow{\cong} & S/\mathfrak{n} \end{array}$$

commutes. Since $D \bullet f$ is the image of $g$ in $R/\mathfrak{m}$, and $D' \bullet f$ is the image of $g$ in $S/\mathfrak{n}$, the claim follows. $\square$

**Theorem 1.3.5.** *If $I \subseteq R$ is an $\mathfrak{m}$-primary ideal, then*

$$D_{\mathfrak{m}}[I] = D_{\mathfrak{n}}[SI],$$

*where the equality is obtained by identifying $D \in W_{\kappa(\mathfrak{m})}$ and $D' \in W_{\kappa(\mathfrak{n})}$.*

*Proof.* Let $D' \in D_{\mathfrak{n}}[SI]$. Since $I = SI \cap R$, then $0 = D' \bullet f = D \bullet f$ for all $f \in I$.

For the converse, let $\{b_1, \ldots, b_d\} \subseteq \mathbb{L}$ denote a $\mathbb{K}$-basis of $\mathbb{L}$. Then any $f \in SI$ can be written as $f = \sum_{i=1}^{d} f_i b_i$, where $f_i \in I$. If $D \in D_{\mathfrak{m}}[I]$, then $D' \bullet f = \sum_{i=1}^{d} b_i D' \bullet f_i = 0$. $\square$

Instead of starting with $I \subseteq R$, we may also consider $J \subseteq S$ and compare its dual space to the dual space of its contraction. To this end, we start with the diagram

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d] & S \arrow[d] \\ R_{\mathfrak{m}} \arrow[r, "\psi"'] & S_{\mathfrak{n}} \end{tikzcd} \tag{1.1}$$

which commutes since $\mathfrak{m} = \phi^{-1}(\mathfrak{n})$. For an ideal $I \subseteq R$, $I_{\mathfrak{m}} \subseteq R_{\mathfrak{m}}$, we denote by $\phi(I)$, $\psi(I_{\mathfrak{m}})$ the extensions of $I$, $I_{\mathfrak{m}}$ under the maps $\phi, \psi$ respectively.

**Lemma 1.3.6.** *Using the notation of Diagram 1.1, we have $\psi(\mathfrak{m}_{\mathfrak{m}}) = \mathfrak{n}_{\mathfrak{n}}$. Moreover, any $\mathfrak{n}_{\mathfrak{n}}$-primary ideal $J_{\mathfrak{n}} \subseteq S_{\mathfrak{n}}$ satisfies $J_{\mathfrak{n}} = \psi(\psi^{-1}(J_{\mathfrak{n}}))$.*

*Proof.* Since $\mathfrak{n}$ corresponds to a rational point, it is generated by linear polynomials $x_i - c_i$, for $i = 1, \ldots, n$ and $c_i \in \mathbb{L}$. Let $p_i(x_i) \in \mathbb{K}[x_i]$ be the minimal polynomial of $c_i$ over $\mathbb{K}$. Since $c_i$ is a root of $p_i$, we have $p_i \in \mathbb{L}[x_i] \cap \mathfrak{n} = (x_i - c_i)$. Therefore we can write $p_i = (x_i - c_i)q_i$, where $q_i \in \mathbb{L}[x_i]$. As the extension $\mathbb{L}/\mathbb{K}$ is separable, we have $q_i(c_i) \neq 0$, so in particular $q_i(x_i) \notin \mathfrak{n}$. Thus for any generator of $\mathfrak{n}_\mathfrak{n}$ we have

$$\frac{(x_i - c_i)}{1} = \frac{(x_i - c_i)q_i}{q_i} = \frac{1}{q_i}\psi\left(\frac{p_i}{1}\right),$$

and since $p_i \in \mathfrak{m}$, we get $\mathfrak{n}_\mathfrak{n} \subseteq \psi(\mathfrak{m}_\mathfrak{m})$. The reverse inclusion follows from the fact that $\psi(\mathfrak{m}_\mathfrak{m}) = \phi(\mathfrak{m})_\mathfrak{n} \subseteq \mathfrak{n}_\mathfrak{n}$.

Since $J_\mathfrak{n} \subseteq \mathfrak{n}_\mathfrak{n}$, all of its generators are in $\psi(I_\mathfrak{m})$ for some ideal $I_\mathfrak{m} \subseteq R_\mathfrak{m}$. Thus $J_\mathfrak{n}$ is an extended ideal, so in particular it is equal to its contraction extension $J_\mathfrak{n} = \psi(\psi^{-1}(J_\mathfrak{n}))$. □

We now have a version of Theorem 1.3.5 for ideals in $S$.

**Theorem 1.3.7.** *If $J \subseteq S$ is an $\mathfrak{n}$-primary ideal, then*

$$D_\mathfrak{m}[J \cap R] = D_\mathfrak{n}[J]$$

*Proof.* Since Diagram 1.1 commutes, the preimage of $J_\mathfrak{n}$ in $R$ using the two paths agree. Hence $\psi^{-1}(J_\mathfrak{n}) \cap R = \phi^{-1}(J_\mathfrak{n} \cap S)$. Since $J$ is $\mathfrak{n}$-primary, $J_\mathfrak{n} \cap S = J$, so the right hand side become $\phi^{-1}(J)$. This is $\mathfrak{m}$-primary, so the equality of the ideals is equivalent to the equality in the localization $R_\mathfrak{m}$, in other words $\psi^{-1}(J_\mathfrak{n}) = \phi^{-1}(J)_\mathfrak{m}$. Applying $\psi$ on both sides and using Lemma 1.3.6 and the commutativity of Diagram 1.1, we get

$$J_\mathfrak{n} = \psi(\psi^{-1}(J_\mathfrak{n})) = \psi(\phi^{-1}(J)_\mathfrak{m}) = \phi(\phi^{-1}(J))_\mathfrak{n}$$

This is equivalent to saying that $J$ and the $\mathfrak{n}$-primary component of $\phi(\phi^{-1}(J))$ are equal, hence they have the same local dual spaces over $\mathfrak{n}$. By Theorem 1.3.5, we now have

$$D_\mathfrak{m}[J \cap R] = D_\mathfrak{m}[\phi^{-1}(J)] = D_\mathfrak{n}[\phi(\phi^{-1}(J))] = D_\mathfrak{n}[J].$$

□

If $\Lambda \subseteq W_{\kappa(\mathfrak{m})}$, we denote $\Lambda' := \{D' \colon D \in \Lambda\} \subseteq W_{\kappa(\mathfrak{n})}$.

**Theorem 1.3.8.** *If $\Lambda \subseteq W_{\kappa(\mathfrak{m})}$ is a finite dimensional local dual space, then*

$$I_{\mathfrak{m}}[\Lambda] = I_{\mathfrak{n}}[\Lambda'] \cap R.$$

*Proof.* This follows directly from Lemma 1.3.4. $\square$

*Remark* 1.3.9. While similar to the above, we have

$$SI_{\mathfrak{m}}[\Lambda] \neq I_{\mathfrak{n}}[\Lambda']$$

Indeed, if we take $\Lambda = \{1\}$, then $I_{\mathfrak{m}}[\Lambda] = \mathfrak{m}$, $I_{\mathfrak{n}}[\Lambda'] = \mathfrak{n}$, but $\mathfrak{n} \neq S\mathfrak{m}$. $\triangle$

**Theorem 1.3.10.** *If $I$ is $\mathfrak{m}$-primary, then*

$$I = I_{\mathfrak{m}}[D_{\mathfrak{m}}[I]]$$

*Proof.* The inclusion $\subseteq$ is clear. Suppose $f \notin I$. We want to find some $D \in D_{\mathfrak{m}}[I]$ such that $D \bullet f \neq 0$.

Note that $f \notin SI$, so there is some $D' \in D_{\mathfrak{n}}[SI]$ such that $0 \neq D' \bullet f$. By Lemma 1.3.4, we also have $D \bullet f \neq 0$, while by Theorem 1.3.5 $D \in D_{\mathfrak{m}}[SI]$. $\square$

**Theorem 1.3.11.** *If $\Lambda$ is a finite dimensional local dual space, then*

$$\Lambda = D_{\mathfrak{m}}[I_{\mathfrak{m}}[\Lambda]]$$

*Proof.* By Theorems 1.3.7 and 1.3.10 we can write

$$D_{\mathfrak{m}}[I_{\mathfrak{m}}[\Lambda]] = D_{\mathfrak{m}}[I_{\mathfrak{n}}[\Lambda'] \cap R] = D_{\mathfrak{n}}[I_{\mathfrak{n}}[\Lambda']] = \Lambda',$$

where the last equality follows from Theorem 1.2.1, because we are working over rational point. $\square$

Combining Theorems 1.3.10 and 1.3.11, we obtain the main duality theorem.

**Theorem 1.3.12.** *There is a bijective, inclusion reversing correspondence between* $\mathfrak{m}$*-primary ideals* $I \subseteq R$ *and finite dimensional local dual spaces* $\Lambda \subseteq W_\kappa(\mathfrak{m})$.

**Example 1.3.13.** Fix $R = \mathbb{R}[x, y]$ and the maximal ideal $\mathfrak{m} = (x - y, y^2 + 1)$, so that $\kappa(\mathfrak{m}) = \mathbb{C}$. The ideal $I = (2xy - y^2 + 1, x^2 + 1)$ is $\mathfrak{m}$-primary, and its local dual space is the 2-dimensional $\mathbb{C}$-vector space $D_\mathfrak{m}[I] = \mathrm{span}_\mathbb{C}\{1, \partial_y\}$.

Let $\Lambda$ be the local dual space generated by $\partial_x^2 + \partial_x\partial_y$ and $\partial_y^2$. This is a 5-dimensional $\mathbb{C}$-vector space spanned by $1, \partial_x, \partial_y, \partial_y^2$, and $\partial_x^2+\partial_x\partial_y$. The corresponding $\mathfrak{m}$-primary ideal is

$$I_\mathfrak{m}[\Lambda] = (\ (x - y)^3,\ (y^2 + 1)^2 + 4(x - y)^2,\ (y^2 + 1)(x^2 - 2xy - 1) - 4(x - y)^2\ ).$$

$\triangle$

**Corollary 1.3.14.** *Let* $I, J \subseteq R$ *be ideals, and let* $\Lambda, \Xi$ *be local dual spaces. Then*

$$D_\mathfrak{m}[I + J] = D_\mathfrak{m}[I] \cap D_\mathfrak{m}[J].$$

$$I_\mathfrak{m}[\Lambda + \Xi] = I_\mathfrak{m}[\Lambda] \cap I_\mathfrak{m}[\Xi]$$

*If furthermore* $I, J$ *are* $\mathfrak{m}$*-primary, and* $\Lambda, \Xi$ *are finite-dimensional, then*

$$D_\mathfrak{m}[I \cap J] = D_\mathfrak{m}[I] + D_\mathfrak{m}[J]$$

$$I_\mathfrak{m}[\Lambda \cap \Xi] = I_\mathfrak{m}[\Lambda] + D_\mathfrak{m}[\Xi]$$

*Proof.* The first two statements are immediate from the definitions. For the third one, we can write

$$\begin{aligned} D_{\mathfrak{m}}[I \cap J] &= D_{\mathfrak{m}}[I_{\mathfrak{m}}[D_{\mathfrak{m}}[I]] \cap I_{\mathfrak{m}}[D_{\mathfrak{m}}[J]]] \\ &= D_{\mathfrak{m}}[I_{\mathfrak{m}}[D_{\mathfrak{m}}[I] + D_{\mathfrak{m}}[J]]] \\ &= D_{\mathfrak{m}}[I] + D_{\mathfrak{m}}[J], \end{aligned}$$

since $D_{\mathfrak{m}}[I] + D_{\mathfrak{m}}[J]$ is a finite dimensional local dual space. The fourth statement follows analogously. □

A somewhat surprising result is that we may also take *infinite* sums and intersections.

**Corollary 1.3.15.** *Let $\{I_i\}_{i \in \mathbb{N}}$ be a set of ideals in $R$, and let $\{\Lambda\}_{i \in \mathbb{N}}$ be a set of local dual spaces in $W_{\kappa(\mathfrak{m})}$. Then*

$$D_{\mathfrak{m}}\left[\sum_{i=1}^{\infty} I_i\right] = \bigcap_{i=1}^{\infty} D_{\mathfrak{m}}[I_i],$$

$$I_{\mathfrak{m}}\left[\sum_{i=1}^{\infty} \Lambda_i\right] = \bigcap_{i=1}^{\infty} I_{\mathfrak{m}}[\Lambda_i].$$

*If furthermore all $I_i$ are $\mathfrak{m}$-primary, and all $\Lambda_i$ are finite-dimensional, then*

$$D_{\mathfrak{m}}\left[\bigcap_{i=1}^{\infty} I_i\right] = \sum_{i=1}^{\infty} D_{\mathfrak{m}}[I_i]$$

$$I_{\mathfrak{m}}\left[\bigcap_{i=1}^{\infty} \Lambda_i\right] = \sum_{i=1}^{\infty} I_{\mathfrak{m}}[\Lambda_i].$$

*Proof.* Again, the first two statements are immediate from the definitions. For the third one, for any fixed $k \in \mathbb{N}$ we have $\bigcap_{i=1}^{k} I_i \supseteq \bigcap_{i=1}^{\infty} I_i$, hence

$$D_{\mathfrak{m}}\left[\bigcap_{i=1}^{\infty} I_i\right] \supseteq D_{\mathfrak{m}}\left[\bigcap_{i=1}^{k} I_i\right] = \sum_{i=1}^{k} D_{\mathfrak{m}}[I_i].$$

Letting $k \to \infty$ gives us one inclusion. For the other one, we note that for any fixed $k \in \mathbb{N}$ we have

$$D_{\mathfrak{m}}\left[\bigcap_{i=1}^{k} I_i\right] = \sum_{i=1}^{k} D_{\mathfrak{m}}[I_i] \subseteq \sum_{i=1}^{\infty} D_{\mathfrak{m}}[I_i]$$

Letting $k \to \infty$ gives us the opposite inclusion.

The fourth statements follows dually. □

### 1.4 Infinite dimensional local dual spaces

In the previous section we established the duality between $\mathfrak{m}$-primary ideals and finite dimensional local dual spaces. Corollary 1.3.15 allows us to extend this to cover potentially infinite dimensional local dual spaces. We will see that ideals corresponding to local dual spaces will always be $\mathfrak{m}$-*closed*.

**Definition 1.4.1.** Let $\mathfrak{m} \subseteq R$ be a maximal ideal. The $\mathfrak{m}$-*closure* of $I$ is the ideal $I_{\mathfrak{m}} \cap R$. An ideal $I \subseteq R$ is $\mathfrak{m}$-*closed* if $I = I_{\mathfrak{m}} \cap R$.

In other words, an $\mathfrak{m}$-closed ideal has all of its associated primes contained in $\mathfrak{m}$. Furthermore, the set of $\mathfrak{m}$-closed ideals of $R$ corresponds to the set of ideals in the local ring $R_{\mathfrak{m}}$.

**Proposition 1.4.2.** *Let $I \subseteq R$ be an ideal, $\mathfrak{m} \subseteq R$ maximal. Then the $\mathfrak{m}$-closure of $I$ is*

$$I_{\mathfrak{m}} \cap R = \bigcap_{d=1}^{\infty} (I + \mathfrak{m}^d)$$

*Proof.* If $I \not\subseteq \mathfrak{m}$, then both sides equal the unit ideal, so we may assume $I \subseteq \mathfrak{m}$. Let

$\pi : R \twoheadrightarrow R/I =: S$ be the natural surjection, and set $\mathfrak{n} := \mathfrak{m}/I$. As $I + \mathfrak{m}^d = \pi^{-1}(\mathfrak{n}^d)$ and $I_\mathfrak{m} \cap R = \pi^{-1}(\ker(S \to S_\mathfrak{n}))$, it suffices to show that $\ker(S \to S_\mathfrak{n}) = \bigcap_{d=1}^\infty \mathfrak{n}^d$.

Set $J := \bigcap_{d=1}^\infty \mathfrak{n}^d$. An application of the Artin-Rees lemma yields $J = \mathfrak{n}J$, so by Nakayama's lemma, there exists $a \in \mathfrak{n}$ such that $(1+a)J = 0$, hence $J \subseteq \ker(S \to S_\mathfrak{n})$. Conversely, consider the natural map $\varphi : S \to \prod_{d=1}^\infty S/\mathfrak{n}^d$, which has $\ker \varphi = J$. Since $\mathfrak{n}$ is maximal in $S$, every element of $S \setminus \mathfrak{n}$ acts as a unit on $S/\mathfrak{n}^d$ for all $d \geq 1$. This implies that $\varphi$ factors through the localization $S \to S_\mathfrak{n}$, hence $\ker(S \to S_\mathfrak{n}) \subseteq \ker \varphi$.

□

Dually, the local dual spaces are infinite sums of their truncations by degree, in symbols

$$\Lambda = \sum_{d=1}^\infty (\Lambda \cap W_{\kappa(\mathfrak{m})}^{(d-1)}).$$

Next we establish the main correspondence results between $\mathfrak{m}$-closed ideals and local dual spaces.

**Proposition 1.4.3.**

1. *If $I$ is an ideal, then $D_\mathfrak{m}[I]$ is a local dual space.*

2. *If $\Lambda$ is a local dual space, then $I_\mathfrak{m}[\Lambda]$ is $\mathfrak{m}$-closed.*

3. *If $I$ is $\mathfrak{m}$-closed, then $I = I_\mathfrak{m}[D_\mathfrak{m}[I]]$.*

4. *If $\Lambda$ is a local dual space, then $\Lambda = D_\mathfrak{m}[I_\mathfrak{m}[\Lambda]]$.*

*Proof.*

1. The proof is essentially identical to the one in Proposition 1.3.2.

2. Recall that $\mathfrak{m}^d = I_\mathfrak{m}[W_{\kappa(\mathfrak{m})}^{(d-1)}]$, so we have

$$\bigcap_{d=1}^\infty (I_\mathfrak{m}[\Lambda] + \mathfrak{m}^d) = \bigcap_{d=1}^\infty I_\mathfrak{m}[\Lambda \cap W_{\kappa(\mathfrak{m})}^{(d-1)}] = I_\mathfrak{m}\left[\sum_{d=1}^\infty \left(\Lambda \cap W_{\kappa(\mathfrak{m})}^{(d-1)}\right)\right] = I_\mathfrak{m}[\Lambda]$$

3. If $I$ is $\mathfrak{m}$-closed, then $I$ is an infinite intersection of $\mathfrak{m}$-primary ideals by Proposition 1.4.2. Thus by Corollary 1.3.15 we have

$$\begin{aligned}
I &= \bigcap_{d=1}^{\infty}(I+\mathfrak{m}^d) \\
&= \bigcap_{d=1}^{\infty} I_{\mathfrak{m}}[D_{\mathfrak{m}}[I+\mathfrak{m}^d]] \\
&= I_{\mathfrak{m}}\left[\sum_{d=1}^{\infty} D_{\mathfrak{m}}[I+\mathfrak{m}^d]\right] \\
&= I_{\mathfrak{m}}\left[D_{\mathfrak{m}}\left[\bigcap_{d=1}^{\infty} I+\mathfrak{m}^d\right]\right] \\
&= I_{\mathfrak{m}}\left[D_{\mathfrak{m}}\left[I\right]\right]
\end{aligned}$$

4. Since $\Lambda = \sum_{d=1}^{\infty}(\Lambda \cap W_{\kappa(\mathfrak{n})}^{(d-1)})$, we have

$$D_{\mathfrak{m}}[I_{\mathfrak{m}}[\Lambda]] = \sum_{i=1}^{\infty} D_{\mathfrak{m}}[I_{\mathfrak{m}}[\Lambda \cap W_{\kappa(\mathfrak{m})}^{d-1}]] = \sum_{i=1}^{\infty} \Lambda \cap W_{\kappa(\mathfrak{m})}^{d-1} = \Lambda$$

□

In particular we see that for an arbitrary ideal $I$, the local dual space only records the $\mathfrak{m}$-closure.

**Corollary 1.4.4.** *If $I \subseteq R$ is an arbitrary ideal, then $D_{\mathfrak{m}}[I] = D_{\mathfrak{m}}[I_{\mathfrak{m}} \cap R]$.*

*Proof.* We get the inclusion $\supseteq$ for free, since $I \subseteq I_{\mathfrak{m}} \cap R$. For the opposite inclusion, we first note that $I \subseteq I_{\mathfrak{m}}[D_{\mathfrak{m}}[I]]$, hence also $I_{\mathfrak{m}} \cap R \subseteq I_{\mathfrak{m}}[D_{\mathfrak{m}}[I]]_{\mathfrak{m}} \cap R$. Since $I_{\mathfrak{m}}[D_{\mathfrak{m}}[I]]$ is $\mathfrak{m}$-closed, we get $I_{\mathfrak{m}} \cap R \subseteq I_{\mathfrak{m}}[D_{\mathfrak{m}}[I]]$. Applying $D_{\mathfrak{m}}[\cdot]$ on both sides yields $D_{\mathfrak{m}}[I_{\mathfrak{m}} \cap R] \supseteq D_{\mathfrak{m}}[I]$. □

**Theorem 1.4.5.** *There is a bijective, inclusion reversing correspondence between $\mathfrak{m}$-closed ideals and local dual spaces.*

We saw already in Corollary 1.3.15 that $D_{\mathfrak{m}}$ and $I_{\mathfrak{m}}$ turn arbitrary sums into intersections. The converse is also true for $\mathfrak{m}$-closed ideals and local dual spaces

**Corollary 1.4.6.** *Let $I, J$ be $\mathfrak{m}$-closed, $\Lambda, \Xi$ be local dual spaces. Then*

$$D_{\mathfrak{m}}[I \cap J] = D_{\mathfrak{m}}[I] + D_{\mathfrak{m}}[J]$$

$$I_{\mathfrak{m}}[\Lambda \cap \Xi] = I_{\mathfrak{m}}[\Lambda] + I_{\mathfrak{m}}[\Xi]$$

*Proof.* If $I, J$ are $\mathfrak{m}$-closed, so is $I \cap J$; likewise if $\Lambda, \Xi$ are local dual spaces, so is $\Lambda \cap \Xi$. Therefore

$$\begin{aligned} D_{\mathfrak{m}}[I \cap J] &= D_{\mathfrak{m}}[I_{\mathfrak{m}}[D_{\mathfrak{m}}[I]] \cap I_{\mathfrak{m}}[D_{\mathfrak{m}}[J]]] \\ &= D_{\mathfrak{m}}[I_{\mathfrak{m}}[D_{\mathfrak{m}}[I] + D_{\mathfrak{m}}[J]]] \\ &= D_{\mathfrak{m}}[I] + D_{\mathfrak{m}}[J] \end{aligned}$$

The result for $\Lambda, \Xi$ follows analogously. □

Using Corollary 1.4.6, one can also take infinite intersections. The proof is omitted as it is essentially identical to the proof of Corollary 1.3.15.

**Corollary 1.4.7.** *If $\{I_i\}_i$ is a set of $\mathfrak{m}$-closed ideals, $\{\Lambda_i\}_i$ a set of local dual spaces, then*

$$D_{\mathfrak{m}}\left[\bigcap_{i=1}^{\infty} I_i\right] = \sum_{i=1}^{\infty} D_{\mathfrak{m}}[I_i]$$

$$I_{\mathfrak{m}}\left[\bigcap_{i=1}^{\infty} \Lambda_i\right] = \sum_{i=1}^{\infty} I_{\mathfrak{m}}[\Lambda_i]$$

The following is also a straightforward consequence of Corollary 1.4.4 and theorem 1.4.5.

**Corollary 1.4.8.** *The set $I \subseteq R$ is an ideal if and only if $D_{\mathfrak{m}}[I]$ is a local dual space. The set $\Lambda \subseteq W_{\kappa(\mathfrak{m})}$ is a local dual space if and only if $I_{\mathfrak{m}}[\Lambda]$ is an ideal.*

## 1.5 Non-maximal ideals

Until now we've developed the theory of local dual spaces for maximal ideals $\mathfrak{m}$. One of the key properties is that $\mathfrak{m}$-primary ideals lead to finite dimensional local dual spaces.

For a general prime ideal $\mathfrak{p}$, one could naively define a local dual space as

$$D_{\mathfrak{p}}[I] := \{D \in W_{\kappa(\mathfrak{p})} \colon D \bullet f = 0 \text{ for all } f \in I\}$$

This leads to issues however, as we lose any hope of finiteness. Take for example the prime $\mathfrak{p} = (x) \subseteq \mathbb{C}[x, y]$. Then the operators $1, \partial_y, \partial_y^2, \dots$ are all in $D_{\mathfrak{p}}[\mathfrak{p}]$, and are $\kappa(\mathfrak{p})$-linearly independent.

Instead, we will use localization to turn $\mathfrak{p}$ into a maximal ideal.

**Definition 1.5.1.** Let $\mathfrak{p} \subseteq \mathbb{K}[x_1, \dots, x_n] =: R$ be a prime ideal. A set of variables $\{x_{i_1}, \dots, x_{i_d}\}$ is said to be *independent* if their images in $R/\mathfrak{p}$ are algebraically independent. Equivalently, $\mathfrak{p} \cap \mathbb{K}[x_{i_1}, \dots, x_{i_d}] = (0)$.

The following theorem, originally by Gröbner, relates the dimension of an ideal to the size of a maximal independent set of variables.

**Theorem 1.5.2** ([42, Thm. 27.11.6])**.** *The dimension of $\mathfrak{p}$ is $d$ if and only if there exists a maximal set of independent variables of size $d$.*

Given a prime $\mathfrak{p}$ of dimension $d$, we fix a maximal set $\{x_{i_1}, \dots, x_{i_d}\}$ of independent variables. We will relabel the variables in the original polynomial ring so that $\mathbf{t} = \{t_1, \dots, t_d\} := \{x_{i_1}, \dots, x_{i_d}\}$ and $\mathbf{y} = \{y_1, \dots, y_{n-d}\}$ are the remaining variables. We denote by $\cdot^{(\mathbf{t})}$ localization at the multiplicatively closed set $\mathbb{K}[t_1, \dots, t_d] \setminus \{0\}$. Thus we have

$$R := \mathbb{K}[t_1, \dots, t_d, y_1, \dots, y_{n-d}] = \mathbb{K}[\mathbf{t}, \mathbf{y}]$$
$$R^{(\mathbf{t})} := \mathbb{K}(t_1, \dots, t_d)[y_1, \dots, y_{n-d}] = \mathbb{K}(\mathbf{t})[\mathbf{y}]$$

We note that $R^{(\mathbf{t})}$ can be thought of as an intermediate step to $R_{\mathfrak{p}}$. Indeed, recall that if $S, T \subseteq R$ are multiplicatively closed sets, and $U$ is the image of $S$ in $T^{-1}R$, then we have an isomorphism $U^{-1}(T^{-1}R) \cong (ST)^{-1}R$. If we let $S = \mathbb{K}[\mathbf{t}] \setminus \{0\}$ and $T = R \setminus \mathfrak{p}$, by independence of the $\mathbf{t}$ variables we have $S \subseteq T$, and hence $ST = T$. In our notation, this means that

$$(R^{(\mathbf{t})})_{\mathfrak{p}^{(\mathbf{t})}} \cong (R_{\mathfrak{p}})^{(\mathbf{t})} \cong R_{\mathfrak{p}} \tag{1.2}$$

This implies also that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}^{(\mathbf{t})})$, as

$$\kappa(\mathfrak{p}^{(\mathbf{t})}) = (R^{(\mathbf{t})}/\mathfrak{p}^{(\mathbf{t})})_{\mathfrak{p}^{(\mathbf{t})}} = R/\mathfrak{p} \otimes_R (R^{(\mathbf{t})})_{\mathfrak{p}^{(\mathbf{t})}} = R/\mathfrak{p} \otimes_R R_{\mathfrak{p}} = (R/\mathfrak{p})_{\mathfrak{p}} = \kappa(\mathfrak{p}).$$

**Proposition 1.5.3.** *The ideal $\mathfrak{p}^{(\mathbf{t})} \subseteq R^{(\mathbf{t})}$ is maximal.*

*Proof.* Since $\{t_1, \dots, t_d\}$ is a maximal set of independent variables, for each $y_i$ there is a polynomial

$$p_i(\mathbf{t}, y_i) = p_{i,s_i}(\mathbf{t})y_i^{s_i} + p_{i,s_i-1}(\mathbf{t})y_i^{s_i-1} + \cdots + p_{i,0}(\mathbf{t}) \in \mathfrak{p},$$

where the $p_{i,j} \in \mathbb{K}[\mathbf{t}]$. Thus $R^{(\mathbf{t})}/\mathfrak{p}^{(\mathbf{t})}$ is integral over the field $\mathbb{K}(\mathbf{t})$, so $R^{(\mathbf{t})}/\mathfrak{p}^{(\mathbf{t})}$ is also a field $\square$

Since $\mathfrak{p}^{(\mathbf{t})} \subseteq R^{(\mathbf{t})}$ is now maximal, we can study local dual spaces at $\mathfrak{p}^{(\mathbf{t})}$. We emphasize that since $R^{(\mathbf{t})}$ is a polynomial ring in the $n - d$ variables $y_1, \dots, y_{n-d}$, the differential operators in $D_{\mathfrak{p}^{(\mathbf{t})}}$ will be polynomials in $\partial_{y_1}, \dots, \partial_{y_{n-d}}$; the operators $\partial_{t_1}, \dots, \partial_{t_d}$ are not in $W_{\kappa(\mathfrak{p}^{(\mathbf{t})})}$. Thus, despite $\kappa(\mathfrak{p})$ being isomorphic to $\kappa(\mathfrak{p}^{(\mathbf{t})})$, the local differential spaces satisfy $W_{\kappa(\mathfrak{p}^{(\mathbf{t})})} \subset W_{\kappa(\mathfrak{p})}$. Thus we can, and often will, consider elements of $W_{\kappa(\mathfrak{p}^{(\mathbf{t})})}$ as elements of $W_{\kappa(\mathfrak{p})}$. This is similar to the distinction between to $W_{\kappa(\mathfrak{m})}$ and $W_{\kappa(\mathfrak{n})}$ in Section 1.3.

Observe that extending and contracting ideals via the localization map $R \to R^{(\mathbf{t})}$ preserves primaryness. Therefore given a $\mathfrak{p}$-primary ideal $I$, the extended ideal $I^{(\mathbf{t})}$ is $\mathfrak{p}^{(\mathbf{t})}$-

primary, so the local dual space $D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}]$ is a finite dimensional $\kappa(\mathfrak{p}^{(\mathbf{t})})$-vector space. Conversely, any finite dimensional local dual space corresponds to a $\mathfrak{p}^{(\mathbf{t})}$-primary ideal, whose contraction is $\mathfrak{p}$-primary. Therefore we obtain yet another duality theorem.

**Theorem 1.5.4.** *Fix a prime ideal $\mathfrak{p} \subseteq R$ and a maximal set $\mathbf{t}$ of independent variables over $\mathfrak{p}$. There is an inclusion reversing bijection between $\mathfrak{p}$-primary ideals $I \subseteq R$ and finite dimensional local dual spaces $\Lambda \subseteq W_{\kappa(\mathfrak{p}^{(\mathbf{t})})}$. The bijection is given by*

$$I \mapsto D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}]$$
$$\Lambda \mapsto I_{\mathfrak{p}^{(\mathbf{t})}}[\Lambda] \cap R.$$

The concept of $\mathfrak{p}$-closed ideals also makes sense when $\mathfrak{p}$ is non-maximal prime: these are simply ideals whose associated primes are contained in $\mathfrak{p}$. The $\mathfrak{p}$-closure of an ideal $I \subseteq R$ is $I_{\mathfrak{p}} \cap R$. Note that we don't have an analogue of Proposition 1.4.2 for non-maximal primes.

**Example 1.5.5.** Let $R = \mathbb{Q}[t, x, y]$ and $I = (x^2, y^2 - tx)$. Here $I$ is a $\mathfrak{p} = (x, y)$-primary ideal. If we fix $\mathbf{t} = \{t\}$ as a maximal set of independent variables, our dual space elements will be polynomials in $\partial_x$ and $\partial_y$ variables, with coefficients in $\kappa(\mathfrak{p}^{(\mathbf{t})}) = \mathbb{Q}(t)$. We can compute

$$D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}] = \operatorname{span}_{\kappa(\mathfrak{p})} \left\{1,\, \partial_y,\, \partial_y^2 + \frac{2}{t}\partial_x,\, \partial_y^3 + \frac{6}{t}\partial_y\partial_x\right\}.$$

By clearing denominators, we may also write the local dual space as the $\kappa(\mathfrak{p})$-span of elements with polynomial coefficients, i.e.

$$D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}] = \operatorname{span}_{\kappa(\mathfrak{p})} \left\{1,\, \partial_y,\, t\partial_y^2 + 2\partial_x,\, t\partial_y^3 + t\partial_y\partial_x\right\}.$$

$\triangle$

We will next establish the duality between $\mathfrak{p}$-closed ideals $I \subseteq R$ and local dual spaces $\Lambda \subseteq W_{\kappa(\mathfrak{p}^{(\mathbf{t})})}$. As was the case for primary ideals, we only need that $\mathfrak{p}$-closed ideals extend to $\mathfrak{p}^{(\mathbf{t})}$-closed ideals, and conversely that $\mathfrak{p}^{(\mathbf{t})}$-closed ideals contract to $\mathfrak{p}$-closed ideals.

In general, every ideal in $R^{(\mathbf{t})}$ is the extension of an ideal in $R$, but an arbitrary ideal in $R$ does not have to be a contraction of an ideal in $R^{(\mathbf{t})}$. However, if $I \subseteq R$ is $\mathfrak{p}$-closed, then it is a contraction. To see this, we note that if $I = \bigcap_{i=1}^{s} Q_i$ is a minimal irredundant primary decomposition, then $Q_i \cap (\mathbb{K}[\mathbf{t}] \setminus \{0\}) \subseteq \mathfrak{p} \cap (\mathbb{K}[\mathbf{t}] \setminus \{0\}) = \emptyset$, so $I^{(\mathbf{t})} = \bigcap_{i=1}^{s} Q_i^{(\mathbf{t})}$ is also a minimal irredundant primary decomposition. Contracting, we get $I^{(\mathbf{t})} \cap R = \bigcap_{i=1}^{s} Q_i = I$.

**Lemma 1.5.6.** *Let $\mathfrak{p} \subseteq R$ be a prime, $\mathbf{t}$ a maximal set of independent variables over $\mathfrak{p}$. The ideal $I \subseteq R$ is $\mathfrak{p}$-closed if and only if $I^{(\mathbf{t})} \subseteq R^{(\mathbf{t})}$ is $\mathfrak{p}^{(\mathbf{t})}$-closed.*

*Proof.* Suppose $I$ is $\mathfrak{p}$-closed. If $I = \bigcap_{i=1}^{s} Q_i$ is a minimal primary decomposition, then $I^{(\mathbf{t})} = \bigcap_{i=1}^{s} Q_i^{(\mathbf{t})}$ is also a minimal primary decomposition. Since $\sqrt{Q_i} \subseteq \mathfrak{p}$, then $\sqrt{Q_i^{(\mathbf{t})}} \subseteq \mathfrak{p}^{(\mathbf{t})}$.

Conversely, suppose $I^{(\mathbf{t})}$ is $\mathfrak{p}^{(\mathbf{t})}$-closed, and let $I^{(\mathbf{t})} = \bigcap_{i=1}^{s} Q_i^{(\mathbf{t})}$ be a minimal irredundant primary decomposition. Taking contractions, by the above discussion we get $I = \bigcap_{i=1}^{s} Q_i$. Since $\sqrt{Q_i^{(\mathbf{t})}} \subseteq \mathfrak{p}^{(\mathbf{t})}$, we must have $\sqrt{Q_i} \subseteq \mathfrak{p}$. □

**Theorem 1.5.7.** *Suppose $\mathfrak{p} \subseteq R$ is prime, and $\mathbf{t}$ is a maximal independent set over $\mathfrak{p}$. There is a inclusion reversing bijective correspondence between $\mathfrak{p}$-closed ideals $I \subseteq R$ and local dual spaces $\Lambda \subseteq W_{\kappa(\mathfrak{p}^{(\mathbf{t})})}$, given by*

$$I \mapsto D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}]$$

$$\Lambda \mapsto I_{\mathfrak{p}^{(\mathbf{t})}}[\Lambda] \cap R.$$

*The correspondence turns (infinite) sums into (infinite) intersections, and vice versa.*

As was the case for $\mathfrak{m}$-closed ideals in Corollary 1.4.4, the maps in Theorem 1.5.7 retains only the $\mathfrak{p}$-closure of an arbitrary ideal.

**Corollary 1.5.8.** *If $I \subseteq R$ is an arbitrary ideal, then $D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}] = D_{\mathfrak{p}^{(\mathbf{t})}}[(I_{\mathfrak{p}} \cap R)^{(\mathbf{t})}]$.*

*Proof.* By Corollary 1.4.4, we have $D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}] = D_{\mathfrak{p}^{(\mathbf{t})}}[(I^{(\mathbf{t})})_{\mathfrak{p}^{(\mathbf{t})}} \cap R^{(\mathbf{t})}]$. Since by eq. (1.2) the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_{\mathfrak{p}} = (R^{(\mathbf{t})})_{\mathfrak{p}^{(\mathbf{t})}} \\ R^{(\mathbf{t})} \arrow[ur] & \end{tikzcd}$$

commutes, so we have $(I^{(\mathbf{t})})_{\mathfrak{p}^{(\mathbf{t})}} \cap R^{(\mathbf{t})} = (I_{\mathfrak{p}} \cap R)^{(\mathbf{t})}$. □

## 1.6 Noetherian operators

Suppose $I \subseteq R = \mathbb{K}[\mathbf{t}, \mathbf{y}]$ is a $\mathfrak{p}$-primary ideal, where $\mathbf{t}$ is a maximal set of independent variables over $\mathfrak{p}$. The corresponding local dual space $D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}]$ is finite $\kappa(\mathfrak{p})$-dimensional, and consists of differential operators of the form

$$D = \sum_{\alpha} c_\alpha \partial_{\mathbf{y}}^{\alpha} \colon R^{(\mathbf{t})} \to \kappa(\mathfrak{p})$$

where $c_\alpha \in \kappa(\mathfrak{p}^{(\mathbf{t})}) = \kappa(\mathfrak{p})$, and a finite number of $c_\alpha$ are non-zero. We can restrict the domain of $D$ to $R$: by slight abuse of notation, for $f \in R$, the result of $D \bullet f$ will be defined as $D \bullet \frac{f}{1}$.

**Definition 1.6.1.** Let $\mathfrak{p} \subseteq R$ be a prime ideal, $\mathbf{t}$ a maximal set of independent variables over $\mathfrak{p}$. A *set of Noetherian operators* of the $\mathfrak{p}$-primary ideal $I$ is a finite set $\mathcal{D} \subseteq W_{\kappa(\mathfrak{p}^{(\mathbf{t})})}$ whose $\kappa(\mathfrak{p}^{(\mathbf{t})})$-span is $D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}]$.

As a direct consequence of the duality theorem in Theorem 1.5.4 we obtain a representation of $I$ via a finite number of differential constraints.

**Proposition 1.6.2.** *Let $\mathfrak{p} \subseteq R$ be a prime ideal, $\mathbf{t}$ a maximal set of independent variables over $\mathfrak{p}$, $I \subseteq R$ a $\mathfrak{p}$-primary ideal, and $\mathcal{D} \subseteq W_{\kappa(\mathfrak{p}^{(\mathbf{t})})}$ a set of Noetherian operators. Then*

$$I = \{f \in R \colon D \bullet f = 0 \textit{ for all } D \in \mathcal{D}\}.$$

*Proof.* By duality, the polynomial $f \in I$ if and only if $f$ is annihilated by all elements of $D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}]$, which is equivalent to $f$ being annihilated by a $\kappa(\mathfrak{p}^{(\mathbf{t})})$ basis of $D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}]$, i.e. a set of Noetherian operators. $\square$

We note that this dual representation of $I$ also characterizes all sets of Noetherian operators.

**Proposition 1.6.3.** *Let $\mathfrak{p} \subseteq R$ be a prime ideal, $\mathbf{t}$ a maximal set of independent variables over $\mathfrak{p}$. Suppose $I \subseteq R$ is an ideal, and $\mathcal{D} \subseteq W_{\kappa(\mathfrak{p}^{(\mathbf{t})})}$ is a finite set such that*

$$I = \{f \in R \colon D \bullet f = 0 \textit{ for all } D \in \mathcal{D}\},$$

*then $I$ is $\mathfrak{p}$-primary and $\mathcal{D}$ is a set of Noetherian operators.*

*Proof.* Let $\Lambda = \operatorname{span}_{\kappa(\mathfrak{p}^{(\mathbf{t})})} \mathcal{D}$, and $J = I_{\mathfrak{p}^{(\mathbf{t})}}[\Lambda] \cap R$. By Theorem 1.5.4, $J$ is $\mathfrak{p}$-primary. The set $\mathcal{D}$ is a set of Noetherian operators for $J$ by construction, so we have

$$J = \{f \in R \colon D \bullet f = 0 \text{ for all } D \in \mathcal{D}\} = I.$$

$\square$

We point out that our definition of Noetherian operators differs slightly from historical definitions. In 1939, Wolfgang Gröbner [25] asked the question of ideal membership via differential operators with *polynomial coefficients*. In other words, for a $\mathfrak{p}$-primary ideal $I$, he wanted to find operators $\mathcal{D} \subseteq W_R$, as opposed to $W_{\kappa(\mathfrak{p}^{(\mathbf{t})})}$ such that

$$I = \{f \in R \colon D \bullet f \in \mathfrak{p} \text{ for all } D \in \mathcal{D}\}. \tag{1.3}$$

The question was solved for zero-dimensional ideals by Gröbner using Macaulay's inverse systems [39]; this was the content of Section 1.2. The question was left open for positive dimensional ideals. Progress was made by analysts in the 1960's, culminating in the

Ehrenpreis-Palamodov fundamental principle [6, 16, 31, 51, 66], Theorem 2.7.1. A more precise study of the fundamental principle and its implications will be revisited in Section 2.7. Informally, the fundamental principle states that a system of PDE corresponding to a primary ideal $I$ has solutions that are integrals of products of polynomials and exponential functions. The polynomials appearing in these integrals would correspond precisely to operators in $W_R$ satisfying eq. (1.3). For this reason perhaps Noetherian operators have continued to be defined as differential operators with polynomial coefficients, even in recent work such as [12, 35, 48, 63].

Next, we argue that the historical definition is compatible with our definition. Suppose $\mathcal{D} = \{D_1, \ldots, D_s\} \subseteq W_{\kappa(\mathfrak{p}^{(\mathfrak{t})})}$ is a set of Noetherian operators. Each operator can be represented as a polynomial in the $\partial_y$-variables, with coefficients in $\kappa(\mathfrak{p}) = (R/\mathfrak{p})_{\mathfrak{p}}$, represented by fractions. We can multiply each $D_i \in \mathcal{D}$ by the least common multiple of all denominators to get an operator $D_i'$ of the form

$$D_i' = \sum_\alpha \frac{\overline{f_{i,\alpha}}}{1} \partial_y^\alpha,$$

where $\overline{f_{i,\alpha}}$ is the image of $f_{i,\alpha} \in R$ in $R/\mathfrak{p}$. The set $\mathcal{D}' = \{D_1', \ldots, D_s'\}$ consists of $\kappa(\mathfrak{p})$-multiples of elements of $\mathcal{D}$, and therefore is still a set of Noetherian operators for $I$. For each $i = 1, \ldots, s$, define

$$E_i := \sum_\alpha f_{i,\alpha} \partial_y^\alpha \in W_R.$$

The key observation here is that for any $f \in R$, we have $E_i \bullet f \in \mathfrak{p} \iff D_i \bullet f = 0$, so in particular by Proposition 1.6.2 we get a representation of $I$ using differential operators with polynomial coefficients, as in eq. (1.3).

We also have a partial converse to the above. Again, let $\mathfrak{p} \subseteq R$ be a prime and $\mathfrak{t}$ be a maximal set of independent variables over $\mathfrak{p}$. Suppose $E_1, \ldots, E_s \in W_R$ are differential operators with polynomial coefficients not involving the $\partial_t$-variables. Thus we can write

$$E_i = \sum_\alpha f_{i,\alpha} \partial_y^\alpha,$$

where $f_{i,\alpha} \in R$. We can send each $E_i$ to $W_{\kappa(\mathfrak{p})}$ to get the operators

$$D_i = \sum_\alpha \frac{\overline{f_{i,\alpha}}}{1} \partial_y^\alpha,$$

and we have $E_i \bullet f \in \mathfrak{p}$ if and only if $D_i \bullet f = 0$. If $I = \{f \in R\colon E_i \bullet f$ for all $i = 1, \dots, s\}$, then by Proposition 1.6.3 the set $\{D_1, \dots, D_s\}$ is a set of Noetherian operators.

*Remark* 1.6.4. At first glance it may seem like restriction to operators involving only $\partial_{\mathbf{y}}$-variables, and no $\partial_{\mathbf{t}}$-variables, is a limitation of our definition. In fact, such a caveat is also present in all other definitions of Noetherian operators, all the way from the original formulation by Ehrenpreis [16] and [51]. $\triangle$

## 1.7 Differential primary decomposition

Next, consider a $\mathfrak{p}$-closed ideal $I \subseteq R$, and fix a maximal set $\mathbf{t}$ of independent variables over $\mathfrak{p}$. We can write $I = Q \cap (I\colon \mathfrak{p}^\infty)$, where $Q$ is either $\mathfrak{p}$-primary, or $Q = R$ if $\mathfrak{p} \notin \operatorname{Ass}(I)$. The latter case is uninteresting, as it gives a decomposition $I = R \cap I$, so in what follows we will assume that $\mathfrak{p}$ is an associated prime of $I$. In that case, the ideal $Q$ will be a $\mathfrak{p}$-primary component in a primary decomposition of $I$, and $(I\colon \mathfrak{p}^\infty)$ is the intersection of all other components. Note that both $Q$ and $(I\colon \mathfrak{p}^\infty)$ are $\mathfrak{p}$-closed, so we get

$$D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}] = D_{\mathfrak{p}^{(\mathbf{t})}}[Q^{(\mathbf{t})}] + D_{\mathfrak{p}^{(\mathbf{t})}}[(I\colon \mathfrak{p}^\infty)^{(\mathbf{t})}]$$

Since $Q$ is $\mathfrak{p}$-primary, the quotient

$$\frac{D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}]}{D_{\mathfrak{p}^{(\mathbf{t})}}[(I\colon \mathfrak{p}^\infty)^{(\mathbf{t})}]} = \frac{D_{\mathfrak{p}^{(\mathbf{t})}}[Q^{(\mathbf{t})}] + D_{\mathfrak{p}^{(\mathbf{t})}}[(I\colon \mathfrak{p}^\infty)^{(\mathbf{t})}]}{D_{\mathfrak{p}^{(\mathbf{t})}}[(I\colon \mathfrak{p}^\infty)^{(\mathbf{t})}]}$$

is a finite-dimensional $\kappa(\mathfrak{p})$-vector space. Note that the quotient is still finite-dimensional even when $\mathfrak{p}$ is not an associated prime; in that case the quotient is trivial. Furthermore, since local dual spaces only see the $\mathfrak{p}$-closure of an ideal (c.f. Corollary 1.5.8), the above quotient is finite-dimensional for *arbitrary* ideals $I$. This motivates the following definition.

**Definition 1.7.1.** Let $I \subseteq R$ be an ideal, $\mathfrak{p}$ a prime, and $\mathbf{t}$ a maximal set of independent variables over $\mathfrak{p}$. The *excess dual space* is the finite dimensional $\kappa(\mathfrak{p})$-vector space

$$E_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}] := D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}]/D_{\mathfrak{p}^{(\mathbf{t})}}[(I\colon \mathfrak{p}^\infty)^{(\mathbf{t})}]$$

Pick any basis $\overline{\mathcal{D}} = \{\overline{D_1}, \dots, \overline{D_s}\}$ of the excess dual space. Let $\mathcal{D} = \{D_1, \dots, D_s\} \subseteq D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}]$ be a set of lifts of $\overline{\mathcal{D}}$.

**Example 1.7.2.** Let $I = (y^4, xy^3, x^3y^2)$. A primary decomposition yields

$$I = Q_1 \cap Q_2 = (y^2) \cap (x^3,\, y^4,\, x^3y^2),$$

where $\mathfrak{p}_1 = \sqrt{Q_1} = (y)$ and $\mathfrak{p}_2 = \sqrt{Q_2} = (x, y)$. Since the $Q_i$ are monomial ideals, the sets

$$\begin{aligned}
\mathcal{D}_1 &= \{1, \partial_y\} \\
\mathcal{D}_2 &= \{1, \partial_x, \partial_x^2, \\
&\qquad \partial_y, \partial_y\partial_x, \partial_y\partial_x^2, \\
&\qquad \partial_y^2, \partial_y^2\partial_x, \partial_y^2\partial_x^2, \partial_y^3\}
\end{aligned}$$

are sets of Noetherian operators, with maximal independent sets $\mathbf{t}_1 = \{x\}$ and $\mathbf{t}_2 = \emptyset$

respectively.

We note that while the set $\{(\mathfrak{p}_i, \mathfrak{t}_i, \mathcal{D}_i)\}_{i=1,2}$ is a differential primary decomposition of $I$, there is another one consisting of fewer operators. Indeed, the excess dual space at $\mathfrak{p}_2$ is only four dimensional, and can be spanned by the images of $\mathcal{D}_2' = \{\partial_y^2, \partial_y^2\partial_x, \partial_y^2\partial_x^2, \partial_y^3\}$. The reason why this is true becomes apparent when looking at the staircase diagram in Figure 1.1.

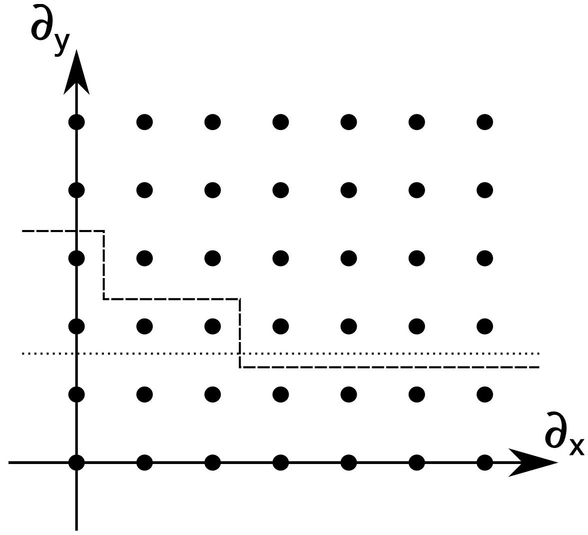


Figure 1.1: Diagram of local dual spaces in Example 1.7.2. Each lattice point corresponds to a monomial $\partial_x^i\partial_y^j$. The points under the dotted lines are a basis of $D_{\mathfrak{p}_2}[I : \mathfrak{p}_2^\infty]$, while the points under the dashed line are a basis of $D_{\mathfrak{p}_2}[I]$. The excess dual space is generated by the four points wedged between the lines.

$\triangle$

**Lemma 1.7.3.** *Let $I \subseteq R$ be a $\mathfrak{p}$-closed ideal, and suppose $f \in (I : \mathfrak{p}^\infty)$. Then $f \in I$ if and only if $D_i \bullet f = 0$ for all $i = 1, \ldots, s$.*

*Proof.* If $f \in I$, clearly $D \bullet f = 0$ for all $D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}]$. For the converse, we can dually show that $D \bullet f = 0$ for all $D \in D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}]$. Any $D \in D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}]$ can be written as a $\kappa(\mathfrak{p})$-linear combination

$$D' = \sum_{i=1}^{s} c_i D_i + cE,$$

where $E \in D_{\mathfrak{p}(\mathbf{t})}[(I\colon \mathfrak{p}^\infty)^{(\mathbf{t})}]$. By assumption $0 = E \bullet f = D_i \bullet f$, which concludes the proof. $\square$

As a consequence of Lemma 1.7.3, we can now describe an *arbitrary* ideal using only a finite number of dual space elements. Let $I \subseteq R$, and suppose $\dim(I) = d$. Let $\mathfrak{q}$ be a $d$-dimensional associated prime. Such a prime is isolated, so the $\mathfrak{q}$-closure $J = I_{\mathfrak{q}} \cap R$ is simply the $\mathfrak{q}$-primary component of $I$. The excess dual space of $J$ is equal to the local dual space, and suppose it's spanned by $\mathcal{D}_{\mathfrak{q}}$. Clearly $J$ is $\mathfrak{q}$-closed, and since $J\colon q^\infty = R$, Lemma 1.7.3 says that $f \in J$ if and only if $D' \bullet f = 0$ for all $D \in \mathcal{D}_{\mathfrak{q}}$.

We define the $k$-*dimensional hull* of $I$, denoted $I_k$, to be the intersection of all primary components of $I$ of dimension $\geq k$. Computing $\mathcal{D}_{\mathfrak{q}}$ for all associated primes $\mathfrak{q}$ of dimension $d = \dim(I)$ gives us a characterization of the $d$-dimensional hull of $I$: a polynomial $f \in I_d$ if and only if $D \bullet f = 0$ for all $D \in \mathcal{D}_{\mathfrak{q}}$, $\dim \mathfrak{q} = d$.

Next, let $\mathfrak{q}$ be a $d-1$ dimensional associated prime, and let $J = I_{\mathfrak{q}} \cap R$ be the $\mathfrak{q}$-closure of $I$. Again, let $\mathcal{D}_{\mathfrak{q}}$ be a lift of a basis of the excess dual space. Now the ideal $(J\colon \mathfrak{q}^\infty)$ contains the $d$-dimensional hull of $I$, so by Lemma 1.7.3, if $f$ is in the $d$-dimensional hull $I_d$, then $f \in J$ if and only if $D \bullet f = 0$ for all $D \in \mathcal{D}_{\mathfrak{q}}$. Note that the $d-1$ dimensional hull is the intersection of the $d$ dimensional hull and the $\mathfrak{q}$-closures of $I$ for all associated primes $\mathfrak{q}$ whose dimension is $d-1$. We conclude that $f \in I_{d-1}$ if and only if $D \bullet f = 0$ for all $D \in \mathcal{D}_{\mathfrak{q}}$, $\dim \mathfrak{q} \geq d-1$.

Clearly the $0$-dimensional hull of $I$ is just $I$ itself, so repeating this procedure dimension by dimension, we obtain a set $\{\mathcal{D}_{\mathfrak{q}}\}_{\mathfrak{q}\in \mathrm{Ass}}$ consisting of finitely many differential operators, with the property

$$f \in I \iff D \bullet f = 0 \text{ for all } D \in \mathcal{D}_{\mathfrak{q}} \text{ and } \mathfrak{q} \in \mathrm{Ass}(I).$$

We will call such a representation of the ideal $I$ a *differential primary decomposition*.

**Definition 1.7.4.** Let $I \subseteq R$ be an ideal. For each $\mathfrak{p} \in \mathrm{Ass}(I)$, let $\mathbf{t}_{\mathfrak{p}}$ be a maximal set of independent variables over $\mathfrak{p}$, and suppose that $\mathcal{D}_\mathfrak{p} \subseteq W_{\kappa(\mathfrak{p}^{(\mathbf{t}_\mathfrak{p})})}$ is a (finite) set of differential operators whose images span the excess dual space $E_{\mathfrak{p}^{(\mathbf{t}_\mathfrak{p})}}[I^{(\mathbf{t}_\mathfrak{p})}]$. Then the set of triples $\mathfrak{D} := \{(\mathfrak{p}, \mathbf{t}_\mathfrak{p}, \mathcal{D}_\mathfrak{p})\}_{\mathfrak{p} \in \operatorname{Ass}(I)}$ is called a *differential primary decomposition.*

**Example 1.7.5.** Let $I = (x(y-z), x^2z, x^3) \subseteq \mathbb{Q}[x, y, z]$. Geometrically, this corresponds to the plane $x = 0$, with the embedded line $x = y - z = 0$, with a further embedded point at the origin. If we set

$$\begin{array}{lll} \mathfrak{p}_1 = (x) & \mathfrak{p}_2 = (x, y-z) & \mathfrak{p}_3 = (x, y, z) \\ \mathbf{t}_1 = \{y, z\} & \mathbf{t}_2 = \{z\} & \mathbf{t}_3 = \emptyset \\ \mathcal{D}_1 = \{1\} & \mathcal{D}_2 = \{\partial_x\} & \mathcal{D}_3 = \{\partial_x^2\}, \end{array}$$

then the set $\mathfrak{D} = \{(\mathfrak{p}_i, \mathbf{t}_i, \mathcal{D}_i)\}_{i=1,2,3}$ is a differential primary decomposition for $I$. $\triangle$

Similarly to Noetherian operators for primary ideals, a differential primary decomposition describes an arbitrary ideal using finitely many differential conditions in $R$. The following proposition summarizes the discussion above.

**Proposition 1.7.6.** *Suppose* $\{(\mathfrak{p}, \mathbf{t}_\mathfrak{p}, \mathcal{D}_\mathfrak{p})\}_{\mathfrak{p} \in \operatorname{Ass}(I)}$ *is a differential primary decomposition. Then*

$$I = \{f \in R \colon D \bullet f \in \mathfrak{p} \textit{ for all } D \in \mathcal{D}_\mathfrak{p},\ \mathfrak{p} \in \operatorname{Ass}(I)\}$$

**Example 1.7.7.** Let $I = (x_1^2, x_1x_2) \subseteq \mathbb{Q}[x_1, x_2]$. From a differential primary decomposition, we can conclude that $I$ is the set of polynomials $f \in R$ such that $f$ vanishes on the line $x_1 = 0$ and $\frac{\partial f}{\partial x_1}$ vanishes at the origin. $\triangle$

In fact, one may use a differential primary decomposition to describe any $\mathfrak{p}$-closure of $I$. Thus we recover the definition of a differential primary decomposition given in [7].

**Proposition 1.7.8.** *Suppose* $\{(\mathfrak{p}, \mathfrak{t}_\mathfrak{p}, \mathcal{D}_\mathfrak{p})\}_{\mathfrak{p}\in\mathrm{Ass}(I)}$ *is a differential primary decomposition. Then*

$$I_\mathfrak{p} \cap R = \{f \in R \colon D \bullet f = 0 \textit{ for all } D \in \mathcal{D}_\mathfrak{q},\ \mathfrak{q} \in \mathrm{Ass}(I) \textit{ such that } \mathfrak{q} \subseteq \mathfrak{p}\}$$

*Proof.* Suppose $\mathfrak{p}$ is an isolated prime, i.e. it is minimal among associated primes. Thus it suffices to show that $\mathcal{D}_\mathfrak{p}$ is a set of Noetherian operators for the $\mathfrak{p}$-primary ideal $I_\mathfrak{p} \cap R$. In this case, the excess dual space is just the local dual space, so $\mathcal{D}_\mathfrak{p} \subseteq W_{\kappa(\mathfrak{p}^{(\mathfrak{t}_\mathfrak{p})})}$ spans $D_{\mathfrak{p}^{(\mathfrak{t}_\mathfrak{p})}}[I^{(\mathfrak{t}_\mathfrak{p})}]$, which is equal to $D_{\mathfrak{p}^{(\mathfrak{t}_\mathfrak{p})}}[(I_\mathfrak{p} \cap R)^{(\mathfrak{t}_\mathfrak{p})}]$ by Corollary 1.5.8. Thus $D_\mathfrak{p}$ is indeed a set of Noetherian operators for the $\mathfrak{p}$-closure of $I$.

Next suppose $\mathfrak{p}$ is an arbitrary associated prime, and suppose the claim is proved for all $\mathfrak{q} \subsetneq \mathfrak{p}$. Note that this implies

$$(I_\mathfrak{p} \cap R) \colon \mathfrak{p}^\infty = \{f \in R \colon D \bullet f = 0 \text{ for all } D \in \mathcal{D}_\mathfrak{q},\ \mathfrak{q} \in \mathrm{Ass}(I) \text{ such that } \mathfrak{q} \subsetneq \mathfrak{p}\}$$

Since $\mathcal{D}'_\mathfrak{p}$ spans the excess dual space of $I_\mathfrak{p} \cap R$, the claim follows from Lemma 1.7.3. □

We note that this property fully describes differential primary decompositions.

**Proposition 1.7.9.** *Let* $I \subseteq R$ *be an ideal. For each associated prime* $\mathfrak{p} \in \mathrm{Ass}(I)$*, let* $\mathfrak{t}_\mathfrak{p}$ *be a maximal set of independent variables over* $\mathfrak{p}$*, and suppose that* $\mathcal{D}_\mathfrak{p} \subseteq W_{\kappa(\mathfrak{p}^{(\mathfrak{t}_\mathfrak{p})})}$ *is a finite set of differential operators such that for each* $\mathfrak{p} \in \mathrm{Ass}(I)$

$$I_\mathfrak{p} \cap R = \{f \in R \colon D \bullet f = 0 \textit{ for all } D \in \mathcal{D}_\mathfrak{q},\ \mathfrak{q} \in \mathrm{Ass}(I) \textit{ such that } \mathfrak{q} \subseteq \mathfrak{p}\}.$$

*Then* $\{(\mathfrak{p}, \mathfrak{t}_\mathfrak{p}, \mathcal{D}_\mathfrak{p})\}_{\mathfrak{p}\in\mathrm{Ass}(I)}$ *is a differential primary decomposition of* $I$.

*Proof.* We need to show that $\mathcal{D}_\mathfrak{p}$ spans each excess dual space of $I$, which is equal to the excess dual space of the $\mathfrak{p}$-closure $I_\mathfrak{p} \cap R$.

Fix some $\mathfrak{p} \in \mathrm{Ass}(I)$. Let $\Lambda = \mathrm{span}_{\kappa(\mathfrak{p})} \mathcal{D}_\mathfrak{p}$. If we denote $J = I_\mathfrak{p} \cap R$, then we want to show that $D_{\mathfrak{p}^{(\mathfrak{t})}}[J^{(\mathfrak{t})}] = \Lambda + D_{\mathfrak{p}^{(\mathfrak{t})}}[(J \colon \mathfrak{p}^\infty)^{(\mathfrak{t})}]$.

We note that our hypothesis means that

$$J = (I_{\mathfrak{p}^{\mathfrak{t}}}[\Lambda] \cap R) \cap (J \colon \mathfrak{p}^{\infty}). \tag{1.4}$$

Take $\Lambda + D_{\mathfrak{p}^{(\mathfrak{t})}}[(J \colon \mathfrak{p}^{\infty})^{(\mathfrak{t})}]$. Dualizing this yields $J$ by eq. (1.4) and the fact that dualizing turns sums into intersections, regardless of whether $\Lambda$ is a local dual space or not. By Corollary 1.4.8, $\Lambda + D_{\mathfrak{p}^{(\mathfrak{t})}}[(J \colon \mathfrak{p}^{\infty})^{(\mathfrak{t})}]$ is a local dual space, namely $D_{\mathfrak{p}^{(\mathfrak{t})}}[J^{(\mathfrak{t})}]$, and hence $\mathcal{D}_{\mathfrak{p}}$ spans the excess dual space. $\square$

As its name suggests, a differential primary decomposition can be used to construct a primary decomposition. Special care needs to be taken, as the $\kappa(\mathfrak{p}^{(\mathfrak{t}_{\mathfrak{p}})})$-span of $\mathcal{D}_{\mathfrak{p}}$ may not be a local dual space.

**Theorem 1.7.10.** *Let $\mathcal{D} = \{(\mathfrak{p}, \mathcal{D}_{\mathfrak{p}}, \mathfrak{t}_{\mathfrak{p}})\}_{\mathfrak{p} \in \operatorname{Ass}(I)}$ be a differential primary decomposition. For each $\mathfrak{p} \in \operatorname{Ass}(I)$, let $\Lambda_{\mathfrak{p}} \subseteq W_{\kappa(\mathfrak{p}^{(\mathfrak{t}_{\mathfrak{p}})})}$ be the right $R$-module generated by $\mathcal{D}_{\mathfrak{p}}$, i.e. the smallest local dual space containing $\mathcal{D}_{\mathfrak{p}}$. If $Q_{\mathfrak{p}} = I_{\mathfrak{p}^{(\mathfrak{t}_{\mathfrak{p}})}}[\Lambda_{\mathfrak{p}}] \cap R$, then $I = \bigcap_{\mathfrak{p} \in \operatorname{Ass}(I)} Q_{\mathfrak{p}}$ is a minimal, irredundant primary decomposition.*

*Proof.* Since each $\Lambda_{\mathfrak{p}}$ is finite dimensional, each $Q_{\mathfrak{p}}$ is $\mathfrak{p}$-primary. Fix some prime $\mathfrak{p} \in \operatorname{Ass}(I)$, and let $\mathcal{D}_{\mathfrak{p}} = \{D_1, \ldots, D_s\}$. By construction, every $D \in D_{\mathfrak{p}^{(\mathfrak{t}_{\mathfrak{p}})}}[I^{(\mathfrak{t}_{\mathfrak{p}})}]$ is of the form

$$D = \sum_{i=1}^{s} c_i D_i + E,$$

where $c_i \in \kappa(\mathfrak{p})$ and $E \in D_{\mathfrak{p}^{(\mathfrak{t}_{\mathfrak{p}})}}[(I \colon \mathfrak{p}^{\infty})^{(\mathfrak{t}_{\mathfrak{p}})}]$. In particular, each $D_i \in D_{\mathfrak{p}^{(\mathfrak{t}_{\mathfrak{p}})}}[I^{(\mathfrak{t}_{\mathfrak{p}})}]$, so $\Lambda_{\mathfrak{p}}$, the right $R$-module generated by the $D_i$, is contained in $D_{\mathfrak{p}^{(\mathfrak{t}_{\mathfrak{p}})}}[I^{(\mathfrak{t}_{\mathfrak{p}})}]$. By duality, this implies that $Q_{\mathfrak{p}} \supseteq I_{\mathfrak{p}} \cap R \supseteq I$. Taking intersections, we have $\bigcap_{\mathfrak{p} \in \operatorname{Ass}(I)} Q_{\mathfrak{p}} \supseteq I$.

For the converse, we first show that for each $\mathfrak{p} \in \operatorname{Ass}(I)$

$$\bigcap_{\mathfrak{q} \subseteq \mathfrak{p}} Q_{\mathfrak{q}} \subseteq I_{\mathfrak{p}} \cap R.$$

Let $f \in \bigcap_{\mathfrak{q} \subseteq \mathfrak{p}} Q_{\mathfrak{q}}$, so by construction $D \bullet f = 0$ for all $D \in \mathcal{D}_{\mathfrak{q}}$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. By Proposition 1.7.8 $f$ lies in $I_{\mathfrak{p}} \cap R$. Finally, we conclude that

$$\bigcap_{\mathfrak{p} \in \operatorname{Ass}(I)} Q_{\mathfrak{p}} = \bigcap_{\mathfrak{p} \in \operatorname{Ass}(I)} \bigcap_{\mathfrak{q} \subseteq \mathfrak{p}} Q_{\mathfrak{q}} \subseteq \bigcap_{\mathfrak{p} \in \operatorname{Ass}(I)} (I_{\mathfrak{p}} \cap R) = I.$$

□

*Remark* 1.7.11. As was the case with Noetherian operators, we can lift a differential primary decomposition to $W_R$. More precisely, if $\{(\mathfrak{p}, \mathbf{t}_{\mathfrak{p}}, \mathcal{D}_{\mathfrak{p}})\}_{\mathfrak{p} \in \operatorname{Ass}(I)}$ is a differential primary decomposition, we can find, for each $\mathfrak{p} \in \operatorname{Ass}(I)$ and $D \in D_{\mathfrak{p}}$, an operator $E \subseteq W_R$ not involving $\partial_{\mathbf{t}_{\mathfrak{p}}}$-variables, such that $D \bullet f = 0 \iff E \bullet f \in \mathfrak{p}$. Thus we obtain a dual representation

$$I = \{f \in R \colon E \bullet f \in \mathfrak{p} \text{ for all } E \in \mathcal{E}_{\mathfrak{p}}, \mathfrak{p} \in \operatorname{Ass}(I)\}$$

in the spirit of Gröbner's representation in eq. (1.3), but for *arbitrary* ideals as opposed to primary ideals. Primary decompositions and $\mathfrak{p}$-closures can also be obtained analogously using operators in $W_R$. △

We end this section with a result by Sturmfels, Cid-Ruiz and Chen [7, 11] about the minimal size of a differential primary decomposition. The *size* of a differential primary decomposition $\{(\mathfrak{p}, \mathbf{t}_{\mathfrak{p}}, \mathcal{D}_{\mathfrak{p}})\}_{\mathfrak{p} \in \operatorname{Ass}(I)}$ is the total number of differential operators involved, i.e. the integer $\sum_{\mathfrak{p} \in \operatorname{Ass}(I)} |\mathcal{D}_{\mathfrak{p}}|$. Our algorithms in Chapter 3 will output a differential primary decomposition of minimal size.

**Definition 1.7.12.** For $\mathfrak{p} \in \operatorname{Ass}(I)$, we let the *multiplicity* of $I$ along $\mathfrak{p}$ is the positive integer

$$\operatorname{mult}_I(\mathfrak{p}) = \operatorname{length}_{R_{\mathfrak{p}}} \left( \frac{I_{\mathfrak{p}} : \mathfrak{p}_{\mathfrak{p}}^{\infty}}{I_{\mathfrak{p}}} \right),$$

The arithmetic multiplicity of $I$ is the sum of all multiplicities of $I$ along associated primes

$$\operatorname{amult}(I) = \sum_{\mathfrak{p} \in \operatorname{Ass}(I)} \operatorname{mult}_I(\mathfrak{p}).$$

*Remark* 1.7.13. The multiplicity of $I$ along $\mathfrak{p}$ can be computed in `Macaulay2` using the command

```
degree( saturate(I, P) / I) // degree( P )
```

Alternatively, a command `amult` for computing arithmetic multiplicities is included in the package `NoetherianOperators` [8]. △

Since our definition of differential primary decomposition is compatible with the one in [7, 11], the following theorem applies.

**Theorem 1.7.14** ([7, 11])**.** *Fix an ideal $I \subseteq R$. The size of a differential primary decomposition is at least $\operatorname{amult}(I)$, and this bound is tight. More precisely*

1. *If $\{(\mathfrak{p}, \mathbf{t}_\mathfrak{p}, \mathcal{D}_\mathfrak{p})\}_{\mathfrak{p} \in \operatorname{Ass}(I)}$ is a differential primary decomposition, then $|\mathcal{D}_\mathfrak{p}| \geq \operatorname{mult}_I(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Ass}(I)$.*

2. *There exists a differential primary decomposition such that $|\mathcal{D}_\mathfrak{p}| = \operatorname{mult}_I(\mathfrak{p})$.*

**Example 1.7.15.** In Example 1.7.2, the differential primary decomposition

$$\{(\mathfrak{p}_1, \mathbf{t}_1, \mathcal{D}_1), \quad (\mathfrak{p}_2, \mathbf{t}_2, \mathcal{D}_2')\}$$

has minimal size: $\operatorname{mult}_I(\mathfrak{p}_1) = 2 = |\mathcal{D}_1|$, and $\operatorname{mult}_I(\mathfrak{p}_2) = 4 = |\mathcal{D}_2'|$. △

### 1.8 Operations on ideals

In standard textbooks, many operations on ideals are described at the level of generators. For example, a sum of two ideals is generated by the union of generators for both ideals. As differential primary decompositions give us an equally valid alternative representation of an arbitrary ideal $I \subseteq R$, we investigate the effect on differential primary decompositions of certain common operations.

### 1.8.1 Intersections

Suppose $I, J \subseteq R$ are ideals with a differential primary decompositions

$$\{(\mathfrak{p}, \mathbf{t}_\mathfrak{p}, \mathcal{D}_\mathfrak{p})\}_{\mathfrak{p} \in \mathrm{Ass}(I)} \qquad\qquad \{(\mathfrak{p}, \mathbf{t}_\mathfrak{p}, \mathcal{B}_\mathfrak{p})\}_{\mathfrak{p} \in \mathrm{Ass}(J)} \tag{1.5}$$

respectively. Note that we assume that for any prime $\mathfrak{p} \in \mathrm{Ass}(I) \cap \mathrm{Ass}(J)$ we have chosen the same maximal set $\mathbf{t}_\mathfrak{p}$ of independent variables. Since local dual spaces turn intersections into sums, a differential primary decomposition for the ideal $I \cap J$ can be obtained by taking unions of the sets $\mathcal{D}_\mathfrak{p}$ and $\mathcal{B}_\mathfrak{p}$.

**Proposition 1.8.1.** *A differential primary decomposition for $I \cap J$ is given by*

$$\{(\mathfrak{p}, \mathbf{t}_\mathfrak{p}, \mathcal{D}_\mathfrak{p} \cup \mathcal{B}_\mathfrak{p})\}_{\mathfrak{p} \in \mathrm{Ass}(I) \cup \mathrm{Ass}(J)}.$$

*Here $\mathcal{B}_\mathfrak{p}, \mathcal{D}_\mathfrak{p}$ are defined to be empty sets whenever $\mathfrak{p} \notin \mathrm{Ass}(J), \mathfrak{p} \notin \mathrm{Ass}(I)$ respectively.*

*Proof.* Let $\mathfrak{p} \in \mathrm{Ass}(I)$. The excess dual space of $I \cap J$ at $\mathfrak{p}$ is

$$\frac{D_{\mathfrak{p}^{(\mathbf{t})}}[(I \cap J)^{(\mathbf{t})}]}{D_{\mathfrak{p}^{(\mathbf{t})}}[((I \cap J) \colon \mathfrak{p}^\infty)^{(\mathbf{t})}]} = \frac{D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}] + D_{\mathfrak{p}^{(\mathbf{t})}}[J^{(\mathbf{t})}]}{D_{\mathfrak{p}^{(\mathbf{t})}}[(I \colon \mathfrak{p}^\infty)^{(\mathbf{t})}] + D_{\mathfrak{p}^{(\mathbf{t})}}[(J \colon \mathfrak{p}^\infty)^{(\mathbf{t})}]}$$

Since the images of $\mathcal{D}_\mathfrak{p}$ and $\mathcal{B}_\mathfrak{p}$ span the excess dual spaces of $I, J$ at $\mathfrak{p}$, the image of their union spans the above $\kappa(\mathfrak{p})$-vector space. $\square$

*Remark* 1.8.2. Even if we start with two minimal differential primary decompositions of sizes $\mathrm{amult}(I)$ and $\mathrm{amult}(J)$, the resulting decomposition for $I \cap J$ may not be minimal: the union of basis elements may not be a basis. $\triangle$

### 1.8.2 Sums

While summing two is trivial when given generators, the computation is not so straight forward when dealing with differential primary decompositions. We keep the notation from eq. (1.5).

**Proposition 1.8.3.** *A differential primary decomposition for $I + J$ is given by*

$$\{(\mathfrak{p}, \mathfrak{t}_{\mathfrak{p}}, \mathcal{C}_{\mathfrak{p}})\}_{\mathfrak{p} \in \operatorname{Ass}(I) \cap \operatorname{Ass}(J)},$$

*where $\mathcal{C}_{\mathfrak{p}}$ spans* $\operatorname{span}_{\kappa(\mathfrak{p})} \mathcal{D}_{\mathfrak{p}} \cap \operatorname{span}_{\kappa(\mathfrak{p})} \mathcal{B}_{\mathfrak{p}}$.

*Proof.* The image of $\mathcal{C}_{\mathfrak{p}}$ spans

$$\frac{D_{\mathfrak{p}^{(\mathfrak{t})}}[I^{(\mathfrak{t})}] \cap D_{\mathfrak{p}^{(\mathfrak{t})}}[J^{(\mathfrak{t})}]}{D_{\mathfrak{p}^{(\mathfrak{t})}}[(I : \mathfrak{p}^{\infty})^{(\mathfrak{t})}] \cap D_{\mathfrak{p}^{(\mathfrak{t})}}[(J : \mathfrak{p}^{\infty})^{(\mathfrak{t})}]} = \frac{D_{\mathfrak{p}^{(\mathfrak{t})}}[(I + J)^{(\mathfrak{t})}]}{D_{\mathfrak{p}^{(\mathfrak{t})}}[(I : \mathfrak{p}^{\infty})^{(\mathfrak{t})} + (J : \mathfrak{p}^{\infty})^{(\mathfrak{t})}]},$$

from which we have a surjection onto the excess dual space of $I + J$,

$$\frac{D_{\mathfrak{p}^{(\mathfrak{t})}}[(I + J)^{(\mathfrak{t})}]}{D_{\mathfrak{p}^{(\mathfrak{t})}}[((I + J) : \mathfrak{p}^{\infty})^{(\mathfrak{t})}]},$$

□

### 1.8.3 Quotients

Let $J \subseteq R$. If $\Lambda$ is a local dual space, we define $\Lambda J$ to be the local dual space generated by elements of the form $Dg$ where $D \in \Lambda, g \in J$.

**Proposition 1.8.4.** *Let $I, J \subseteq R$ be ideals, $\mathfrak{p} \subseteq R$ a prime. Then $D_{\mathfrak{p}^{(\mathfrak{t})}}[(I : J)^{(\mathfrak{t})}] = D_{\mathfrak{p}^{(\mathfrak{t})}}[I]J$ and $D_{\mathfrak{p}^{(\mathfrak{t})}}[(I : f)^{(\mathfrak{t})}] = D_{\mathfrak{p}^{(\mathfrak{t})}}[I^{(\mathfrak{t})}]f$.*

*Proof.* Suppose that $I$ is $\mathfrak{p}$-closed; if not, replace $I$ by its $\mathfrak{p}$-closure $I_{\mathfrak{p}} \cap R$. If $D \in D_{\mathfrak{p}^{(\mathfrak{t})}}[I^{(\mathfrak{t})}]J$, then $D = \sum D_i g_i$ for some $D_i \in D_{\mathfrak{p}^{(\mathfrak{t})}}[I^{(\mathfrak{t})}]$ and $g_i \in J$. Let $f \in I : J$, then $D \bullet f = \sum D_i \bullet (g_i f) = 0$. This proves the $\supseteq$ direction.

For the opposite inclusion, we dualize. Let $f \in I_{\mathfrak{p}(\mathbf{t})}[D_{\mathfrak{p}(\mathbf{t})}[I^{(\mathbf{t})}]J]$, so $D \bullet (gf) = 0$ for all $D \in D_{\mathfrak{p}(\mathbf{t})}[I^{(\mathbf{t})}]$ and $g \in J$. Thus $gf \in I$ for all $g \in J$, implying that $f \in I \colon J$. $\square$

Naturally this also allow us to compute saturations: if $N$ is an integer so that $I \colon \mathfrak{p}^\infty = I \colon \mathfrak{p}^N$, then $D_{\mathfrak{p}(\mathbf{t})}[(I \colon \mathfrak{p}^\infty)^{(\mathbf{t})}] = D_{\mathfrak{p}(\mathbf{t})}[I^{(\mathbf{t})}]\mathfrak{p}^N$.

### 1.8.4 Ring maps

Suppose $S = \mathbb{K}[y_1, \dots, y_m]$, $R = \mathbb{K}[x_1, \dots, x_n]$ and $\phi \colon S \to R$ is a ring map. If $I \subseteq R$ is a $\mathfrak{p}$-primary ideal, then $J = \phi^{-1}(I) \subseteq S$ is a $\mathfrak{q}$-primary ideal, where $\mathfrak{q} = \phi^{-1}(\mathfrak{p})$. Thus by duality $I$ corresponds to a finite dimensional local dual space in $W_{\kappa(\mathfrak{p})}$ and $J$ corresponds to a finite dimensional local dual space in $W_{\kappa(\mathfrak{q})}$. Our goal will be to recover the latter from the former.

The map $\phi$ induces a map $\Phi \colon \kappa(\mathfrak{q}) \to \kappa(\mathfrak{p})$ on the residue fields. Thus we obtain a field extension $\kappa(\mathfrak{q}) \subseteq \kappa(\mathfrak{p})$; fix a (possibly infinite) basis $\{b_1, b_2, \dots\}$ of $\kappa(\mathfrak{p})$ over $\kappa(\mathfrak{q})$. Any element $D \in W_{\kappa(\mathfrak{p})}$ can be expressed as a polynomial in variables $\partial_{x_1}, \dots, \partial_{x_n}$ and coefficients in $\kappa(\mathfrak{p})$. Furthermore, by the multivariable chain rule (Faà di Bruno's formula) there are finitely many non-zero operators $\{E_k\}_k \in W_{\kappa(\mathfrak{q})}$ such that

$$D \bullet \phi(f) = \sum_k \Phi(E_k \bullet f) b_k$$

for all $f \in S$, and only a finite number of the $E_k$ are non-zero.

**Proposition 1.8.5.** *Let* $\mathbf{t}$*,* $\mathbf{s}$ *be maximal sets of independent variables over* $\mathfrak{p}$ *and* $\mathfrak{q}$ *respectively. Suppose* $\{D_1, \dots, D_s\}$ *is a basis of* $D_{\mathfrak{p}(\mathbf{t})}[I^{(\mathbf{t})}]$*. For each* $D_i$*, let* $\{E_{i,k}\}_k$ *be a finite set of operators in* $W_{\kappa(\mathfrak{q})}$ *such that*

$$D_i \bullet \phi(f) = \sum_k \Phi(E_{i,k} \bullet f) b_k \tag{1.6}$$

*for any* $f \in S$*. Then the set* $\bigcup_{i=1}^s \bigcup_k \{E_{i,k}\}$ *spans* $D_{\mathfrak{q}(\mathbf{s})}[J^{(\mathbf{s})}]$*.*

*Proof.* Let $\Lambda = \operatorname{span}_{\kappa(\mathfrak{q})} \bigcup_{i=1}^{s} \bigcup_k \{E_{i,k}\}$. Note that since the $D_i$ are a basis for a local dual space, $\Lambda$ must also be a local dual space.

If $f \in J$, then $\phi(f) \in I$ and $D_i \bullet \phi(f) = 0$. Since $b_k$ is a basis of $\kappa(\mathfrak{p})$ over $\kappa(\mathfrak{q})$, we must have $\Phi(E_{i,k} \bullet f) = 0$ for all $i, k$, thus $\Lambda \subseteq D_{\mathfrak{q}^{(\mathbf{s})}}[J^{(\mathbf{s})}]$.

Conversely, let $f \in I_{\mathfrak{q}^{(\mathbf{s})}}[\Lambda] \cap S$. Then $D_i \bullet \phi(f) = 0$, so $\phi(f) \in I$, or equivalently $f \in J$. Thus $I_{\mathfrak{q}^{(\mathbf{s})}}[\Lambda] \cap S \subseteq J$, and by duality $\Lambda \supseteq D_{\mathfrak{q}^{(\mathbf{s})}}[J^{(\mathbf{s})}]$. $\square$

### 1.8.5 Elimination

In general, computing the $E_{i,k}$ from a basis $D_i$ of a local dual space is difficult to do in practice, but can be done explicitly if the map $\phi\colon S \to R$ is simple enough. Elimination of variables is one such instance.

We start by eliminating a single variable. Thus we have a map

$$\begin{aligned} \phi\colon \mathbb{K}[y_1, \dots, y_{n-1}] &\to \mathbb{K}[x_1, \dots, x_n] \\ y_i &\mapsto x_i \text{ for all } i = 1, \dots, n-1 \end{aligned}$$

Suppose $I \subseteq R$ a $\mathfrak{p}$-primary ideal. The *elimination ideal* of $I$, that is the ideal obtained by eliminating the variable $x_n$, is the ideal $J = I \cap S$, which is $\mathfrak{q}$-primary, where $\mathfrak{q} = \mathfrak{p} \cap S$. Thus both the local dual spaces of $I$ and $J$ are finite dimensional. Let $\mathbf{t}$, $\mathbf{s}$ be a maximal set of independent variables over $\mathfrak{p}$ and $\mathfrak{q}$ respectively. Our goal will be to explicitly compute the local dual space corresponding to $J$ from the one corresponding to $I$, i.e. we want to obtain $D_{\mathfrak{q}^{(\mathbf{s})}}[J^{(\mathbf{s})}]$ from $D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}]$.

Let $\alpha \in \mathbb{N}^n$ such that $\alpha_n > 0$. Then clearly $\partial_x^\alpha \bullet \phi(y) = 0$, since $\phi(y)$ is not a function of $x_n$. If $\alpha_n = 0$, then $\partial_x^\alpha \bullet \phi(f) = \Phi(\partial_y^\alpha \bullet f)$, where by abuse of notation we denote $\partial_y^\alpha = \partial_{y_1}^{\alpha_1} \cdots \partial_{y_{n-1}}^{\alpha_{n-1}}$.

Note that the field extension $\kappa(\mathfrak{q}) \hookrightarrow \kappa(\mathfrak{p})$ is a simple extension, generated by the image of the missing variable $x_n$. Thus we have a basis $\{1, x_n, x_n^2, \dots\}$ of $\kappa(\mathfrak{p})$ over $\kappa(\mathfrak{q})$.

Let $D_1, \ldots, D_r$ be a basis of $D_{\mathfrak{p}^{(\mathbf{t})}}[I^{(\mathbf{t})}]$. Each $D_i \in W_{\kappa(\mathfrak{p})}$ can be written as a finite linear combination

$$D_i = \sum_k D_{i,k} x_n^k,$$

where $D_{i,k} = \sum_\alpha \Phi(c_{i,k,\alpha}) \partial_x^\alpha$ is an operator with coefficient in the subfield $\kappa(\mathfrak{q})$. We then define

$$E_{i,k} = \sum_{\alpha\colon \alpha_n = 0} c_{i,k,\alpha} \partial_y^\alpha.$$

By the discussion above, the operators $E_{i,k}$ satisfy (1.6), and thus are a basis for the local dual space of the elimination ideal $J$ by Proposition 1.8.5.

## 1.9 Extension to modules

So far we have studied ideals of $R$ and their local dual spaces. With slightly modified definitions, the theory extends to $R$-submodules of $U \subseteq R^k$. While polynomial modules are certainly less familiar than ideals, they play a major role in analysis. As we will see in Chapter 2, ideals correspond to PDE systems whose solutions are scalar functions $u(\mathbf{z})\colon \mathbb{R}^n \to \mathbb{C}$, while module correspond to PDE systems whose solutions are vector valued functions $u(\mathbf{z})\colon \mathbb{R}^n \to \mathbb{C}^k$. Many of the proofs translate to this new setting almost verbatim; we will not repeat these here.

For this section, let $k \in \mathbb{N}$, and let $e_1, \ldots, e_k$ denote the standard basis vectors, so that $R^k = \bigoplus_{i=1}^k Re_i$. Suppose $\mathfrak{p} \subseteq R$ is a prime, and let $D = (D_1, \ldots, D_k) \in W_{\kappa(\mathfrak{p})}^k$. We can also write

$$D = \sum_{i=1}^k \sum_\alpha c_{\alpha,i} \partial_\alpha e_i,$$

in which case the degree of $D$ is the largest $|\alpha|$ such that $c_{\alpha,i} \neq 0$. We define the result of applying $D$ to some $f = (f_1, \ldots, f_k) \in R^k$ to be

$$D \bullet f = \sum_{i=1}^{k} D_i \bullet f_i; \tag{1.7}$$

thus $D$ corresponds to a map $R^k \to \kappa(\mathfrak{p})$. We also have a right $R$-action defined as $(Dg) \bullet f := D \bullet (gf)$, where $g \in R$, $f \in R^k$.

**Definition 1.9.1.** Let $U \subseteq R^k$ be an $R$-module, $\mathfrak{m} \subseteq R$ a maximal ideal. The *local dual space* of $U$ at $\mathfrak{m}$ is

$$D_{\mathfrak{m}}[U] = \{D \in W^k_{\kappa(\mathfrak{m})} \colon D \bullet f = 0 \text{ for all } f \in U\}.$$

Dually, if $\Lambda \subseteq W^k_{\kappa(\mathfrak{m})}$, define

$$I_{\mathfrak{m}}[\Lambda] = \{f \in R^k \colon D \bullet f = 0 \text{ for all } D \in \Lambda\}.$$

For submodules of $R^k$ (and indeed submodules of any module), we have properties analogue to extensions and contractions of ideals under localization. We will summarize the main ones. Let $\phi \colon R^k \to R^k_{\mathfrak{p}}$ be the natural map. If $N \subseteq R^k_{\mathfrak{p}}$ is a submodule, we denote by $N \cap R^k$ the module $\phi^{-1}(N) \subseteq R^k$. Likewise if $U \subseteq R^k$ is a submodule, we denote by $U_{\mathfrak{p}}$ the module $UR^k_{\mathfrak{p}}$.

1. For any submodule $N \subseteq R^k_{\mathfrak{p}}$, we have $N = (N \cap R^k)_{\mathfrak{p}}$.

2. A submodule $U \subseteq R^k$ is of the form $N \cap R^k$ for some submodule $N \subseteq R^k_{\mathfrak{p}}$ if and only if $U = U_{\mathfrak{p}} \cap R^k$. In this case we say that $U$ is $\mathfrak{p}$-*closed*.

Breaking from standard notation in commutative algebra, we say that a prime $\mathfrak{p} \subseteq R$ is *associated* to $U$ if there exists some $u \in R^k$ such that

$$\mathfrak{p} = (U \colon u) := \{f \in R \colon fu \in U\}.$$

The set of associated primes of $U$ is denoted $\operatorname{Ass}(U)$, and since $R$ is Noetherian, it is a finite, nonempty set of prime ideals $\operatorname{Ass}(U) := \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$. The $R$-submodule $U \subseteq R^k$ is said to be $\mathfrak{p}$-primary if $s = 1$, i.e. $\operatorname{Ass}(U) = \{\mathfrak{p}\}$, or equivalently if $x \notin \mathfrak{p}$ and $xu \in U$ implies $u \in U$. A minimal irredundant primary decomposition of $U$ is a list of $R$-submodules $U_1, \dots, U_s \subseteq R^k$ such that $U = \bigcap_{i=1}^s U_s$, and each $U_i$ is $\mathfrak{p}_i$-primary.

*Remark* 1.9.2. In reference textbooks on commutative algebra, e.g. [4, 17], what we defined as $\operatorname{Ass}(U)$ would be called $\operatorname{Ass}(R^k/U)$. We motivate our change of convention by the fact that we only consider submodules of $R^k$. This is also in line with the notation used for ideals ($k = 1$), where $\operatorname{Ass}(I)$ is the set of associated primes of the $R$-module $R/I$. △

Similarly to ideals, each module $U \subseteq R^k$ comes equipped with a variety, $V(U) \subseteq \mathbb{K}^n$, referred to as the *support* of $U$ by algebraists, or the *characteristic variety* by analysts. We define it as

$$V(U) := \bigcup_{\mathfrak{p} \in \operatorname{Ass}(U)} V(\mathfrak{p});$$

see also Section 2.6 for alternative characterizations.

Next, we will translate some of our results to the language of modules. We include proofs only if they differ significantly from the respective proof for ideals.

**Proposition 1.9.3** (c.f. Theorem 1.2.1)**.** *Suppose $\mathfrak{m} \subseteq R$ is a maximal ideal corresponding to a rational point $p \in \mathbb{K}^n$. There is an inclusion reversion bijection between $R_\mathfrak{m}$ submodules $V \subseteq R_\mathfrak{m}^k$ and local dual spaces $\Lambda \subseteq W_{\kappa(\mathfrak{m})}^k$, given by $J \mapsto D_\mathfrak{m}[J \cap R^k]$ and $\Lambda \mapsto (I_\mathfrak{m}[\Lambda])_\mathfrak{m}$. If $U \subseteq R^k$ is an $R$-submodule, we have $D_\mathfrak{m}[U] = D_\mathfrak{m}[U_\mathfrak{m} \cap R^k]$.*

*Proof sketch.* While the module case was not explicitly covered, one can adapt the constructions in [42, Chapt. 28 and 31]. The key observation is that $(R^k)^* = (R^*)^k$, and that a linear functional $D = (D_1, \dots, D_k)$ acts on a $k$-tuple of polynomials $f = (f_1, \dots, f_k)$

precisely as in (1.7), that is

$$D(f) = \sum_{i=1}^{k} D_i(f_i).$$

$\square$

**Lemma 1.9.4** (c.f. Lemma 1.3.1)**.** *Suppose* $\mathfrak{m} \subseteq R$ *is a maximal ideal. For* $s = 1, 2, \ldots$ *we have*

$$D_{\mathfrak{m}}[\mathfrak{m}^s R^k] = \left(W^{(s-1)}_{\kappa(\mathfrak{m})}\right)^k,$$

*where* $\left(W^{(s-1)}_{\kappa(\mathfrak{m})}\right)^k$ *is the set of operators in* $W^k_{\kappa(\mathfrak{m})}$ *of degree at most* $s-1$*.*

**Proposition 1.9.5** (c.f. Proposition 1.3.3)**.** *Let* $\Lambda \subseteq W^k_{\kappa(\mathfrak{m})}$ *be a nonzero local dual space. Then* $I_{\mathfrak{m}}[\Lambda] \subseteq R^k$ *is an* $R$*-submodule such that* $\mathrm{Ann}_R(R^k/I_{\mathfrak{m}}[\Lambda]) \subseteq \mathfrak{m}$*. If in addition* $\Lambda$ *is finite dimensional, then* $I_{\mathfrak{m}}[\Lambda]$ *is* $\mathfrak{m}$*-primary.*

*Proof.* Let $x \in \mathrm{Ann}_R(R^k/I_{\mathfrak{m}}[\Lambda])$. Since $\Lambda$ is a local dual space, it contains an element of the form $1e_i$ for some $i = 1, \ldots, n$. Then, since $xe_i \in I_{\mathfrak{m}}[\Lambda]$, we must have $1e_i \bullet xe_i = \overline{x} = 0 \in R/\mathfrak{m}$. Hence $x \in \mathfrak{m}$.

Since $\Lambda$ is finite dimensional, we have $\Lambda \subseteq (W^{(N)}_{\kappa(\mathfrak{m})})^k$ for some integer $N$. Then

$$\mathfrak{m}^{N+1} R^k \subseteq I_{\mathfrak{m}}[D_{\mathfrak{m}}[\mathfrak{m}^{N+1} R^k]] = I_{\mathfrak{m}}[(W_{\kappa(\mathfrak{m})^{(N)}})^k] \subseteq I_{\mathfrak{m}}[\Lambda]$$

Thus $R^k/I_{\mathfrak{m}}[\Lambda]$ is annihilated by a power of $\mathfrak{m}$. Since $\mathfrak{m}$ is maximal, every element not in $\mathfrak{m}$ is a nonzerodivisor on $R^k/I_{\mathfrak{m}}[\Lambda]$. $\square$

**Lemma 1.9.6** (c.f. Lemma 1.3.6)**.** *Using the notation of Diagram 1.1, any* $\mathfrak{n}_{\mathfrak{n}}$*-primary submodule* $V_{\mathfrak{n}} \subseteq S^k_{\mathfrak{n}}$ *satisfies* $V_{\mathfrak{n}} = \psi(\psi^{-1}(V_{\mathfrak{n}}))$*, where* $\psi$ *denotes the extension of* $\psi\colon R_{\mathfrak{m}} \to S_{\mathfrak{n}}$ *to* $R^k_{\mathfrak{m}} \to S^k_{\mathfrak{n}}$*.*

*Proof.* Let $r_1, \dots, r_s$ be generators of the $\mathfrak{m}$-primary submodule $\phi^{-1}(V) \subseteq R^k$. Since the diagram (1.1) commutes, we have $\phi^{-1}(V) = \psi^{-1}(V_\mathfrak{m}) \cap R^k$. Because $\phi^{-1}(V)$ is $\mathfrak{m}$-primary, we have the same equality in $R^k_\mathfrak{m}$, i.e. $\phi^{-1}(V)_\mathfrak{m} = \psi^{-1}(V_\mathfrak{n})$. The elements $r_1, \dots, r_s$ generate the vector space $\phi^{-1}(V)_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}\phi^{-1}(V)_\mathfrak{m}$, which is equal to $\psi^{-1}(V_\mathfrak{n})/\psi^{-1}(\mathfrak{m}_\mathfrak{m})\psi^{-1}(V_\mathfrak{n})$. Therefore the images $\psi(x_i)$ generate the vector space $V_\mathfrak{n}/\mathfrak{n}_\mathfrak{n}V_\mathfrak{n}$. By Nakayama's lemma, the $\psi(x_i)$ generate $V_\mathfrak{n}$, so $V_\mathfrak{n}$ is the extension of a module under the map $\phi$, so it is equal to its contraction-extension. □

**Proposition 1.9.7** (c.f. Proposition 1.4.2)**.** *Let $U \subseteq R^k$ be an $R$-submodule, $\mathfrak{m} \subseteq R$ a maximal ideal. Then the $\mathfrak{m}$-closure of $U$ is*

$$U_\mathfrak{m} \cap R = \bigcap_{d=1}^{\infty} (U + \mathfrak{m}^d R^k).$$

*Proof.* Let $I = \operatorname{Ann}_R(R^k/U)$. If $I \not\subseteq \mathfrak{m}$, then both sides equal $R^k$, so we may assume $I \subseteq \mathfrak{m}$. Let $\pi\colon R^k \twoheadrightarrow R^k/U =: V$ be the natural surjection; note that $V$ is also an $S := R/I$-module. Set $\mathfrak{n} := \mathfrak{m}/I$. As $U + \mathfrak{m}^d R^k = \pi^{-1}(\mathfrak{n}^d V)$ and $U_\mathfrak{m} \cap R = \pi^{-1}(\ker(V \to V_\mathfrak{n}))$, it suffices to show that $\ker(V \to V_\mathfrak{n}) = \bigcap_{d=1}^\infty \mathfrak{n}^d V$. Set $W := \bigcap_{d=1}^\infty \mathfrak{n}^d V$. By Artin-Rees we have $W = \mathfrak{n}W$, so by Nakayama, there exists $a \in \mathfrak{n}$ such that $(1 + a)W = 0$, hence $W \subseteq \ker(V \to V_\mathfrak{n})$. Conversely, consider the natural map $\phi\colon V \to \prod_{d=1}^\infty V/(\mathfrak{n}^d V)$, which has $\ker \phi = W$. Since $\mathfrak{n}$ is maximal in $S$, every element of $S \setminus \mathfrak{n}$ acts by multiplication as an automorphism on $V/\mathfrak{n}^d V$ for all $d \geq 1$. This implies that $\phi$ factors through the localization $V \to V_\mathfrak{n}$, hence $\ker(V \to V_\mathfrak{n}) \subseteq \ker \phi$. □

**Theorem 1.9.8** (c.f. Theorems 1.3.12 and 1.4.5)**.** *Suppose $\mathfrak{m} \subseteq R$ is a maximal ideal.*

1. *There is a bijective, inclusion reversing correspondence between $\mathfrak{m}$-primary $R$-submodules $U \subseteq R^k$ and finite dimensional local dual spaces $\Lambda \subseteq W^k_{\kappa(\mathfrak{m})}$.*

2. *There is a bijective, inclusion reversing correspondence between $\mathfrak{m}$-closed $R$-submodules $U \subseteq R^k$ and local dual spaces $\Lambda \subseteq W^k_{\kappa(\mathfrak{m})}$.*

*Both correspondences are given by $U \mapsto D_{\mathfrak{m}}[U]$ and $\Lambda \mapsto I_{\mathfrak{m}}[\Lambda]$. The correspondence turns sums into intersections, and intersections into sums (c.f. Corollary 1.4.7).*

Since modules behave well under localizations, the theory extends to local dual spaces over non-maximal primes exactly as it does for ideals. Recall that for a prime $\mathfrak{p}$ and a maximal set of independent variables $\mathbf{t}$ over $\mathfrak{p}$, we denote by $(\cdot)^{(\mathbf{t})}$ the localization at the multiplicatively closed set $\mathbb{K}[t_1, \dots, t_d] \setminus \{0\}$. We record this result for completeness

**Theorem 1.9.9** (c.f. Theorem 1.5.7 and corollary 1.5.8)**.** *Suppose $\mathfrak{p} \subseteq R$ is a prime ideal, and let $\mathbf{t}$ be a maximal set of independent variables over $\mathfrak{p}$.*

1. *There is a bijective, inclusion reversing correspondence between $\mathfrak{p}$-primary $R$-submodules $U \subseteq R^k$ and finite dimensional local dual spaces $\Lambda \subseteq W^k_{\kappa(\mathfrak{p}^{\mathbf{t}})}$.*
2. *There is a bijective, inclusion reversing correspondence between $\mathfrak{p}$-closed $R$-submodules $U \subseteq R^k$ and local dual spaces $\Lambda \subseteq W^k_{\kappa(\mathfrak{p}^{(\mathbf{t})})}$.*

*Both correspondences are given by $U \mapsto D_{\mathfrak{p}^{(\mathbf{t})}}[U^{(\mathbf{t})}]$ and $\Lambda \mapsto I_{\mathfrak{p}^{(\mathbf{t})}}[\Lambda] \cap R^k$. The correspondence turns sums into intersections, and intersections into sums.*

Let $U \subseteq R^k$ be a $\mathfrak{p}$-primary $R$-submodule. The finite set $\mathcal{D} \subseteq W^k_{\kappa(\mathfrak{p}^{(\mathbf{t})})}$ is a set of Noetherian operators if their $\kappa(\mathfrak{p})$ span equals the local dual space $D_{\kappa(\mathfrak{p}^{(\mathbf{t})})}[U^{(\mathbf{t})}]$. As was the case for ideals, these determine the module via

$$U = \{f \in R^k \colon D \bullet f = 0 \text{ for all } D \in \mathcal{D}\}.$$

One can also lift a set of Noetherian operators to $W^k_R$ to recover the classical definition by Ehrenpreis and Palamodov.

A differential primary decomposition is a list of triples $\{(\mathfrak{p}, \mathbf{t}_{\mathfrak{p}}, \mathcal{D}_{\mathfrak{p}})\}_{\mathfrak{p} \in \mathrm{Ass}(U)}$ such that for each $\mathfrak{p} \in \mathrm{Ass}(U)$, the images of the finite number of operators in $\mathcal{D}_{\mathfrak{p}}$ span the excess dual space

$$\frac{D_{\mathfrak{p}^{(\mathbf{t}_\mathfrak{p})}}[U^{(\mathbf{t}_\mathfrak{p})}]}{D_{\mathfrak{p}^{(\mathbf{t}_\mathfrak{p})}}[(U\colon \mathfrak{p}^\infty)^{(\mathbf{t}_\mathfrak{p})}]}$$

as a $\kappa(\mathfrak{p})$-vector space. The $R$-module $(U\colon \mathfrak{p}^\infty)$ is the submodule $\{f \in R^k\colon \mathfrak{p}^N f \subseteq U,\, N \gg 0\}$. Our definition is compatible with the one in [7, 11], therefore the minimal size of a differential primary decomposition is

$$\operatorname{amult}(U) = \sum_{\mathfrak{p}\in\operatorname{Ass}(U)} \operatorname{mult}_U(\mathfrak{p}) = \sum_{\mathfrak{p}\in\operatorname{Ass}(U)} \operatorname{length}_{R_\mathfrak{p}}\left(\frac{U_\mathfrak{p}\colon \mathfrak{p}_\mathfrak{p}^\infty}{U_\mathfrak{p}}\right).$$

**Example 1.9.10.** Let $R = \mathbb{Q}[x, y]$, and suppose $U \subseteq R^2$ is the $R$-submodule generated by the columns of the matrix

$$M = \begin{bmatrix} 0 & x^2 & xy^2 \\ x & 0 & -2y^2 \end{bmatrix}.$$

Then $\{(\mathfrak{p}_i, \mathbf{t}_i, \mathcal{D}_i)\}_{i=1,2}$ is a minimal differential primary decomposition of $U$, where

$$\mathfrak{p}_1 = (x) \qquad\qquad \mathfrak{p}_2 = (x, y)$$

$$\mathbf{t}_1 = \{y\} \qquad\qquad \mathbf{t}_2 = \emptyset$$

$$\mathcal{D}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2\partial_x \\ 1 \end{bmatrix} \right\} \qquad\qquad \mathcal{D}_2 = \left\{ \begin{bmatrix} \partial_x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \partial_y \end{bmatrix} \right\}.$$

In particular, the vector of polynomials $f = (f_1, f_2)^T \in R^2$ lies in $U$ if and only if $f$ and $2\frac{\partial f}{\partial x} + g$ vanishes on the line $x = 0$ and $\frac{\partial f}{\partial x}$ and $\frac{\partial g}{\partial y}$ vanish at the origin.

$\triangle$

# CHAPTER 2
# FROM ALGEBRA TO ANALYSIS

The notion of Noetherian operators, which answered Gröbner's question of representing arbitrary ideal using differential conditions, was surprisingly answered by analysts in the context of solving linear partial differential equations with constant coefficients. The connection to analysis is less surprising once we consider a simple ordinary differential equation familiar from undergraduate calculus. Consider the homogeneous linear ordinary differential equation

$$c_m\phi^{(m)} + \cdots + c_2\phi'' + c_1\phi' + c_0\phi = 0, \tag{2.1}$$

where $c_1, \ldots, c_m \in \mathbb{C}$ are constant coefficients, and $\phi\colon \mathbb{R} \to \mathbb{C}$ is an unknown smooth function. To solve the equation, we find roots of the *characteristic polynomial*

$$p(x) := c_m x^m + \cdots + c_2 x^2 + c_1 x + c_0 = 0.$$

Suppose $\lambda_1, \ldots, \lambda_s$ are the roots of $p(x)$, so we can write $p(x) = (x-\lambda_1)^{m_1} \cdots (x-\lambda_s)^{m_s}$, where $m_i$ is the *multiplicity* of the root $\lambda_i$. Then the solutions to (2.1) is an $m$-dimensional $\mathbb{C}$-vector space spanned by the functions

$$\{x^s e^{\lambda_i x} \colon 0 \le s < m_i, 1 \le i \le s\}.$$

Thus the solution set to the ODE corresponds exactly to a factorization of the polynomial $p(x)$, and therefore represents the scheme structure of the ideal $(p(x)) \subseteq \mathbb{C}[x]$. Furthermore, we note that the set

$$\{(\mathfrak{p}, \mathfrak{t}_\mathfrak{p}, \mathcal{D}_\mathfrak{p})\}_{\mathfrak{p}\in \mathrm{Ass}(I)} = \{((x-\lambda_i), \emptyset, \{1, x, x^2, \dots, x^{m_i-1}\})\}_{i=1,\dots,s}$$

is a differential primary decomposition of the ideal $I = (p(x))$.

Similar results are true in much greater generality. In this section, we will see how to view ideals and modules as systems of linear partial differential equations. We will also discuss he duality between solutions to systems of PDE and polynomial modules, summarizing fundamental results by Oberst [47, 49, 50]. Using Noetherian operators and differential primary decompositions will allow us to describe the solution sets of these PDE using a finite sum of certain integral solutions. The connecting glue between the algebra and analysis is the celebrated Fundamental Theorem by Ehrenpreis and Palamodov [6, 16, 31, 51]. We will also improve upon the fundamental theorem by using differential primary decompositions as opposed to full sets of Noetherian operators.

## 2.1 PDE, polynomials and modules

In line with our notation from Chapter 1, let $R = \mathbb{K}[x_1, \dots, x_n]$. We will introduce a new set of variables, $z_1, \dots, z_n$, and equate $x_i$ with the operator $\frac{\partial}{\partial z_i}$. A *partial differential equation* is a $k$-vector of polynomials in $\mathbb{K}[\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}]$. By replacing the symbol $\frac{\partial^\alpha}{\partial \mathbf{z}^\alpha}$ by $x^\alpha$ we can identify a PDE with a vector $v \in R^k$. The vector $v$ corresponds to the differential equation

$$\sum_{i=1}^{k} v_i \bullet u_i(z_1, \dots, z_n) = 0,$$

where $u = (u_1, \dots, u_k)$ is an unknown $k$-tuple of functions. Here $x^\alpha \bullet u(z_1, \dots, z_n) = \frac{\partial^\alpha u}{\partial z^\alpha}$.

A *partial differential operator* with constant coefficients is a $\ell \times k$ matrix $M$ with polynomial entries, i.e. $M \in R^{\ell \times k}$. The matrix $M$ operates on $k$-tuples of functions

$u = (u_1, \ldots, u_k)\colon \mathbb{R}^n \to \mathbb{C}^k$; the result $M \bullet u\colon \mathbb{R}^n \to \mathbb{C}^\ell$ is an $\ell$-tuple of functions given by

$$(M \bullet u)_i = \sum_j M_{ij} \bullet u_j,$$

where the monomial $x^\alpha$ corresponds to the operator $\frac{\partial^\alpha}{\partial z^\alpha}$.

A *system of PDE* can thus be written as $M \bullet u = 0$. The system consists of $\ell$ equations, each of which corresponds to a row of $M$, and the solution is a $k$-tuple of functions $u$.

We illustrate our setup with a few examples.

**Example 2.1.1** ($n = 3, k = 4, \ell = 2$)**.** Let

$$M = \begin{bmatrix} x_1 & x_2 & x_3 & 0 \\ 0 & x_1 & x_2 & x_3 \end{bmatrix}.$$

A solution $u = (u_1, u_2, u_3, u_4)$ is a (sufficiently differentiable) function $\mathbb{R}^3 \to \mathbb{C}^4$ such that the following $\ell = 2$ equations are satisfied:

$$M \bullet u = 0 \iff \begin{cases} \frac{\partial u_1}{\partial z_1} + \frac{\partial u_2}{\partial z_2} + \frac{\partial u_3}{\partial z_3} = 0 \\ \frac{\partial u_2}{\partial z_1} + \frac{\partial u_3}{\partial z_2} + \frac{\partial u_4}{\partial z_3} = 0. \end{cases}$$

Without going into details quite yet, we can check that one possible family of solutions are functions of the form

$$u(z_1, z_2, z_3) = (u_1, u_2, u_3, u_4)^T = \left[0, \quad \frac{\partial^2 \phi}{\partial z_3^2}, \quad \frac{\partial^2 \phi}{\partial z_2 \partial z_3}, \quad \frac{\partial^2 \phi}{\partial z_2^2} - \frac{\partial^2 \phi}{\partial z_1 \partial z_3}\right]^T,$$

where $\phi(z_1, z_2, z_3)\colon \mathbb{R}^3 \to \mathbb{C}$ is twice differentiable. $\triangle$

**Example 2.1.2.** The gradient operator in three dimensions is

$$u(z_1, z_2, z_3) \mapsto \begin{bmatrix} \frac{\partial u}{\partial z_1} \\ \frac{\partial u}{\partial z_2} \\ \frac{\partial u}{\partial z_3} \end{bmatrix},$$

which corresponds to the differential operator $\operatorname{grad} := \left[\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right]$. The curl operator in three dimensions is

$$\begin{bmatrix} u_1(z_1, z_2, z_3) \\ u_2(z_1, z_2, z_3) \\ u_3(z_1, z_2, z_3) \end{bmatrix} \mapsto \begin{bmatrix} \frac{\partial u_3}{\partial z_2} - \frac{\partial u_2}{\partial z_3} \\ \frac{\partial u_1}{\partial z_3} - \frac{\partial u_3}{\partial z_1} \\ \frac{\partial u_2}{\partial z_1} - \frac{\partial u_1}{\partial z_2} \end{bmatrix},$$

which corresponds to the differential operator $\operatorname{curl} := \left[\begin{smallmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{smallmatrix}\right]$. We know from calculus that curl-free smooth functions are precisely gradients. In our notation, we have $\operatorname{curl} \bullet u = 0$ if and only if $u = \operatorname{grad} \bullet v$ for some smooth function $v$. Algebraically, this corresponds to the fact that the matrix $\operatorname{grad}$ is the syzygy matrix of $\operatorname{curl}$. △

The set of solutions $u$ to the system of PDE $M \bullet u = 0$ depends of course on which function space $\mathcal{F}$ we seek solutions. In the control theory literature, the space $\mathcal{F}$ is often called the space of *signals* [47]. Classical choices for $\mathcal{F}$ include the space of compactly supported smooth functions $C_0^\infty$, smooth functions $C^\infty$, distributions $\mathcal{D}'$, or even formal power series $\mathbb{C}[[z_1, \dots, z_n]]$. We require the space $\mathcal{F}$ to be an $R$-module, i.e. closed under differentiation, in which case an operator $M \in R^{\ell \times k}$ describes an $R$-module homomorphism $\mathcal{F}^k \to \mathcal{F}^\ell$. We will revisit the different choices for $\mathcal{F}$, their algebraic properties, and the consequences on the solutions spaces in the upcoming sections.

Suppose $u \in \mathcal{F}^k$ is a solution to the system of PDE $M \bullet u = 0$, and let $U$ be the $R$-submodule of $R^k$ generated by the *rows* of $M$. Recall that every element of $U \subseteq R^k$ corresponds to a PDE. It is easy to see that $u$ is also a solution to any PDE in the module

$U$. Conversely, since $R$ is Noetherian, any $R$-submodule $U \subseteq R^k$ is finitely generated, so there is a matrix $M$ whose rows generate the module $U$. Therefore, each system of PDE $M \bullet u = 0$ corresponds to an $R$-submodule of $U \subseteq R^k$. Following standard notation, we write $U = \operatorname{im}_R M^T$, i.e. the module generated by the rows of $M$.

**Definition 2.1.3.** Let $\mathcal{F}$ be an $R$-module, and $U$ be an $R$-submodule of $R^k$. The solution set of the PDE given by $U$ is the $R$-module

$$\operatorname{Sol}_{\mathcal{F}}(U) := \{u \in \mathcal{F}^k \colon v \bullet u = 0 \text{ for all } v \in U\}.$$

If $M$ is a $\ell \times k$ matrix with entries in $R$, we denote by $\operatorname{Sol}_{\mathcal{F}}(M)$ the solution set of the submodule of $R^k$ generated by the rows of $M$. An $R$-module $\Lambda \subseteq \mathcal{F}^k$ is called a *solution space* if there is a matrix $M \subseteq R^{\ell \times k}$, or an $R$-submodule $U \subseteq R^k$, such that $\Lambda = \operatorname{Sol}_{\mathcal{F}}(M)$, or $\Lambda = \operatorname{Sol}_{\mathcal{F}}(U)$.

In control theory, solution sets are often called *systems* or *behaviors*. As discussed above, the solution space of a module is fully determined by its generators: if $U$ is generated by the $\ell$ rows of the matrix $M$, we can write

$$\operatorname{Sol}_{\mathcal{F}}(U) = \operatorname{Sol}_{\mathcal{F}}(\operatorname{im}_R M^T) = \{u \in \mathcal{F}^k \colon M \bullet u = 0\}.$$

We also have a dual object: we can ask which differential equations are satisfied by a given space of solutions.

**Definition 2.1.4.** Let $\mathcal{F}$ be an $R$-module, and $\Lambda \subseteq \mathcal{F}^k$ be a solution space. The set of differential operators annihilating $\Lambda$ is

$$\operatorname{Diff}(\Lambda) := \{v \in R^k \colon v \bullet u = 0 \text{ for all } u \in \Lambda\}.$$

The following result is a straightforward consequence of the definitions, c.f. the duality between $D_{\mathfrak{m}}[\_]$ and $I_{\mathfrak{m}}[\_]$ in Chapter 1.

**Proposition 2.1.5.** *The mappings* $U \mapsto \mathrm{Sol}_{\mathcal{F}}(U)$ *and* $\Lambda \mapsto \mathrm{Diff}(\Lambda)$ *are inclusion reversing, and we have*

$$U \subseteq \mathrm{Diff}(\mathrm{Sol}(U)) \qquad \Lambda = \mathrm{Sol}(\mathrm{Diff}(\Lambda))$$

$$\mathrm{Sol}(U) = \mathrm{Sol}(\mathrm{Diff}(\mathrm{Sol}(U))) \qquad \mathrm{Diff}(\Lambda) = \mathrm{Diff}(\mathrm{Sol}(\mathrm{Diff}(\Lambda)))$$

The single inclusion above may be strict. For a concrete example, suppose $n = 1$, $k = 1$, $\mathcal{F} = C_c^\infty$, the set of compactly supported functions, and $U = (x)$. The solution set $\mathrm{Sol}_{\mathcal{F}}(U)$ is the set of compactly supported smooth functions whose first derivative vanishes. These are of course the constant functions, however only the zero function has compact support among these. Hence $\mathrm{Sol}_{\mathcal{F}}(U) = \{0\}$, so $U \subsetneq \mathrm{Diff}(\mathrm{Sol}_{\mathcal{F}}(U)) = R^k$. In Section 2.3 we will describe a condition turning the only inclusion into an equality.

## 2.2 Distribution theory

In its most general form, the Ehrenpreis-Palamodov fundamental principle can be used when the function space $\mathcal{F}$ is the space of *distributions* $\mathcal{D}'$. We will give a brief outline of the properties of the space of distributions and its $R$-module structure. A more detailed account can be found in standard analysis textbook, such as [32].

Let $\mathcal{D} := C_c^\infty(\mathbb{R}^n, \mathbb{C})$ denote the space of smooth functions $\mathbb{R}^n \to \mathbb{C}$ whose support is compact. A *distribution* is a linear functional $u\colon \mathcal{D} \to \mathbb{C}$ such that, for every compact set $K \subset \mathbb{R}^n$, there exists positive real constants $C, d$ such that

$$u(f) \leq C \sum_{|\alpha| \leq d} \sup |\partial^\alpha f| \text{ for all } f \in C_c^\infty \text{ with } \mathrm{supp}(f) \subseteq K.$$

These are the linear functionals that are continuous in when $\mathcal{D}$ is endowed with the topology of sequential convergence. Therefore we call the set of distributions $\mathcal{D}'$. It is a subspace of the vector space dual of $\mathcal{D}$.

The space of distributions contains all smooth functions as a subspace, as we can associate to any smooth function $\phi \in C^\infty(\mathbb{R}^n, \mathbb{C})$ the distribution $f \mapsto \int \phi \cdot f$ for any $f \in \mathcal{D}$. The space of distributions also contains several non-traditional, potentially discontinuous ”functions” such as Dirac delta functions and step functions.

As mentioned earlier, the space $\mathcal{D}'$ is an $R$-module, which means that we can take derivatives of distributions. If $u \in \mathcal{D}$, we define for all $i = 1, \ldots, n$, $f \in \mathcal{D}$

$$(x_i \bullet u)(f) = \left(\frac{\partial u}{\partial z_i}\right)(f) := -u\left(\frac{\partial f}{\partial z_i}\right) = -u(x_i \bullet f).$$

The negative sign ensures compatibility with derivative of smooth functions, i.e. if $\phi \in C^\infty$ corresponds to the distribution $u\colon f \mapsto \int f \cdot \phi$, then using integration by parts the function $x_i \bullet \phi$ corresponds to the distribution $x_i \bullet u$.

**Example 2.2.1.** Let $n = 1$, and consider the step function

$$u(f) := \int_0^\infty f(z)\,\mathrm{d}z$$

for all compactly supported functions $f$. Its derivative is the Dirac delta function, supported at the point $0$, since

$$(x \bullet u)(f) = -u(x \bullet f) = -\int_0^\infty \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x = -(0 - f(0)) = f(0).$$

$\triangle$

## 2.3 Algebraic properties of function families

The matrix $M \subseteq R^{\ell \times k}$ describes a morphism of free $R$ modules $M\colon R^k \to R^\ell$. Since $R$ is Noetherian, the kernel $\ker_R M \subseteq R^k$ is a finitely generated $R$-submodule, generated by $\{s_1, \ldots, s_{k'}\} \subseteq R^k$. Let $S$ be the *syzygy matrix* of $M$, i.e. the $k \times k'$ matrix whose columns are $s_1, \dots, s_{k'}$. Thus $S$ corresponds to an $R$-module morphism $S\colon R^{k'} \to R^k$ such that the sequence

$$R^{k'} \xrightarrow{S} R^k \xrightarrow{M} R^\ell \tag{2.2}$$

is exact, that is $\ker_R M = \operatorname{im}_R S$.

Let $\mathcal{F}$ be an $R$-module. Tensoring (2.2) by $\mathcal{F}$, we obtain the complex

$$\mathcal{F}^{k'} \xrightarrow{S} \mathcal{F}^k \xrightarrow{M} \mathcal{F}^\ell, \tag{2.3}$$

meaning that $\operatorname{im}_{\mathcal{F}} S \subseteq \ker_{\mathcal{F}} M = \operatorname{Sol}(M)$. Thus we can obtain a subset of the solutions fully algebraically by doing a syzygy computation, a standard procedure for computer algebra software. We call a matrix $S$ such that $\operatorname{im}_{\mathcal{F}} S \subseteq \operatorname{Sol}_{\mathcal{F}}(M)$ a *vector potential*.

An *flat* $R$-module preserves exactness after tensoring. In other words, if $\mathcal{F}$ is flat the sequence (2.3) becomes exact, implying that $\operatorname{Sol}(M) = \operatorname{im}_{\mathcal{F}} S$. Perhaps the most important flat space of functions is the space of compactly supported smooth functions $\mathcal{F} = C_c^\infty$, a property shown by Malgrange in 1960 [40]. Therefore solving PDE systems over the space of compactly supported smooth functions becomes trivial, as we only need a to perform a syzygy computation. From an analytic point of view, the existence of compactly supported solutions is of interest in the context of understanding the space of Young measures; see e.g. [34].

Unfortunately many interesting function spaces are not flat. Take for example the set of smooth functions $\mathcal{F} = C^\infty$, or the set of distributions $\mathcal{F} = \mathcal{D}'$. If $M = [x]$ is a $1 \times 1$ matrix, solving $M \bullet u = 0$ means finding a function $u(z)\colon \mathbb{R} \to \mathbb{C}$ such that $\frac{\mathrm{d}u}{\mathrm{d}z} = 0$ (or its distributional counterpart). Clearly the constant function $u(z) = 1$ is a solution, but the syzygy matrix of $M$ is the zero matrix. Instead, Oberst [47] showed that $C^\infty$ and $\mathcal{D}'$ are *injective cogenerators*, which induces a strong duality between $R$-submodules $U \subseteq R^k$ and sets of solutions $\operatorname{Sol}(U)$.

**Definition 2.3.1.** The $R$-module $\mathcal{F}$ is an injective cogenerator if the following holds: any complex of $R$-modules

$$0 \to N' \to N \to N'' \to 0$$

is exact if and only if the dual

$$0 \leftarrow \mathrm{Hom}_R(N', \mathcal{F}) \leftarrow \mathrm{Hom}_R(N, \mathcal{F}) \leftarrow \mathrm{Hom}_R(N'', \mathcal{F}) \leftarrow 0$$

is exact.

Suppose $\mathcal{F}$ is an injective cogenerator, and $U \subseteq R^k$ is a submodule. We have the exact sequence

$$0 \to U \to R^k \to R^k/U \to 0,$$

and applying $\mathrm{Hom}_R(\_, \mathcal{F})$ yields the exact sequence

$$0 \to \mathrm{Hom}_R(R^k/U, \mathcal{F}) \to \mathcal{F}^k \to \mathrm{Hom}_R(U, \mathcal{F}) \to 0.$$

To understand the map $\mathcal{F}^k = \mathrm{Hom}(R^k, \mathcal{F}) \to \mathrm{Hom}(U, \mathcal{F})$, let $e_1, \ldots, e_k$ be a free basis of $R^k$. A function $u = (u_1, \ldots, u_k) \in \mathcal{F}^k$ corresponds to the homomorphism $\phi \in \mathrm{Hom}(R^k, \mathcal{F})$ defined by $\phi(e_i) = u_i$. The map then sends $\phi$ to the restriction $\phi|_U$. The kernel of $\mathcal{F}^k \to \mathrm{Hom}_R(U, \mathcal{F})$ is thus precisely the set of functions that are annihilated by $U$, that is $\mathrm{Sol}_{\mathcal{F}}(U)$. By exactness, we get

$$\mathrm{Hom}_R(R^k/U, \mathcal{F}) \cong \mathrm{Sol}_{\mathcal{F}}(U) \text{ and } \mathcal{F}^k/\mathrm{Sol}_{\mathcal{F}}(U) \cong \mathrm{Hom}_R(U, \mathcal{F}) \tag{2.4}$$

**Proposition 2.3.2.** *Let $\mathcal{F}$ be an injective cogenerator, and $U \subseteq R^k$ and $R$-submodule. Then $U = \mathrm{Diff}(\mathrm{Sol}(U))$.*

*Proof.* The diagram

$$\begin{tikzcd}
\mathcal{F}^k/\operatorname{Sol}(U) \arrow[r, "\cong"'] \arrow[d, "\cong"] & \operatorname{Hom}(U,\mathcal{F}) \arrow[d, "\phi"'] \\
\mathcal{F}^k/\operatorname{Sol}(\operatorname{Diff}(\operatorname{Sol}(U))) \arrow[r, "\cong"] & \operatorname{Hom}(\operatorname{Diff}(\operatorname{Sol}(U)),\mathcal{F})
\end{tikzcd}$$

commutes, so $\phi$, the map induced by the inclusion $U \subseteq \operatorname{Diff}(\operatorname{Sol}(U))$, must be a bijection. Since $\mathcal{F}$ is an injective cogenerator, we must have $U = \operatorname{Diff}(\operatorname{Sol}(U))$. $\square$

**Theorem 2.3.3.** *If $\mathcal{F}$ is an injective cogenerator, the maps*

$$U \mapsto \operatorname{Sol}_{\mathcal{F}}(U)$$

$$\Lambda \mapsto \operatorname{Diff}(\Lambda)$$

*are an inclusion reversing bijection between $R$-submodules $U \subseteq R^k$ and solution spaces $\Lambda$.*

**Corollary 2.3.4.** *Let $\mathcal{F}$ be an injective cogenerator, $U, U' \subseteq R^k$ be $R$-submodules, and $\Lambda, \Lambda' \subseteq \mathcal{F}^k$ be solution spaces. Then*

$$\operatorname{Sol}_{\mathcal{F}}(U + U') = \operatorname{Sol}_{\mathcal{F}}(U) \cap \operatorname{Sol}_{\mathcal{F}}(U') \qquad \operatorname{Diff}(\Lambda + \Lambda') = \operatorname{Diff}(\Lambda) \cap \operatorname{Diff}(\Lambda')$$

$$\operatorname{Sol}_{\mathcal{F}}(U \cap U') = \operatorname{Sol}_{\mathcal{F}}(U) + \operatorname{Sol}_{\mathcal{F}}(U') \qquad \operatorname{Diff}(\Lambda \cap \Lambda') = \operatorname{Diff}(\Lambda) + \operatorname{Diff}(\Lambda')$$

*Proof.* The following are immediate from the definitions

$$\operatorname{Sol}_{\mathcal{F}}(U + U') = \operatorname{Sol}_{\mathcal{F}}(U) \cap \operatorname{Sol}_{\mathcal{F}}(U') \qquad \operatorname{Diff}(\Lambda + \Lambda') = \operatorname{Diff}(\Lambda) \cap \operatorname{Diff}(\Lambda')$$

We also have

$$\begin{aligned}\operatorname{Diff}(\Lambda\cap\Lambda') &= \operatorname{Diff}(\operatorname{Sol}(\operatorname{Diff}(\Lambda))\cap\operatorname{Sol}(\operatorname{Diff}(\Lambda')))\\ &= \operatorname{Diff}(\operatorname{Sol}(\operatorname{Diff}(\Lambda)+\operatorname{Diff}(\Lambda')))\\ &= \operatorname{Diff}(\Lambda)+\operatorname{Diff}(\Lambda'),\end{aligned}$$

since $\operatorname{Diff}(\Lambda)+\operatorname{Diff}(\Lambda')$ is an $R$-submodule of $R^k$.

For the last equation, consider the exact sequence

$$0 \to R^k/(U\cap U') \xrightarrow{\alpha} R^k/U \oplus R^k/U' \xrightarrow{\beta} R^k/(U+V) \to 0,$$

where $\alpha(\overline{x}) = (\overline{x}, -\overline{x})$, and $\beta(\overline{x},\overline{y}) = \overline{x+y}$. Applying $\operatorname{Hom}_R(\_,\mathcal{F})$, we have by (2.4)

$$0 \leftarrow \operatorname{Sol}(U\cap U') \leftarrow \operatorname{Sol}(U)\oplus\operatorname{Sol}(U) \leftarrow \operatorname{Sol}(U)\cap\operatorname{Sol}(U') \leftarrow 0.$$

Since the sequence

$$0 \leftarrow \operatorname{Sol}(U)+\operatorname{Sol}(U') \leftarrow \operatorname{Sol}(U)\oplus\operatorname{Sol}(U) \leftarrow \operatorname{Sol}(U)\cap\operatorname{Sol}(U') \leftarrow 0.$$

is also exact, we have $\operatorname{Sol}(U)+\operatorname{Sol}(U') = \operatorname{Sol}(U\cap U')$.

□

Let $U = \bigcap_{i=1}^{s} U_i$ be a minimal, irredundant primary decomposition. The duality, now between modules and solutions sets, translates the primary decomposition to a decomposition of the solution set as the sum of $R$-modules

$$\operatorname{Sol}_{\mathcal{F}}(U) = \sum_{i=1}^{s} \operatorname{Sol}_{\mathcal{F}}(U_i),$$

for any injective cogenerator $\mathcal{F}$. Thus finding the solution set for an arbitrary submodule

$U \subseteq R^k$ is reduced to the same task for a primary submodule. This observation is key in some of the earlier algorithmic work [12, 48]. We remark that the sets $\mathrm{Sol}_{\mathcal{F}}(U_i)$ may intersect nontrivially.

*Remark* 2.3.5. Primary decomposition of modules is built into Macaulay2 since version 1.17. If `U` is a submodule of $\mathtt{R}^{\mathtt{k}}$, e.g. obtained from `U = image transpose M`, where `M` is the matrix corresponding to the PDE $M \bullet u = 0$, a list of matrices $\{M_i\}_{i=1}^s$ such that $U_i = \mathrm{im}_R\, M_i^T$ can be obtained by running the commands

```
primaryDecomposition comodule U /
      (N -> image generators N + image relations N) /
      mingens /
      transpose
```

We have that $u \in \mathrm{Sol}_{\mathcal{F}}(U)$ if and only if $M_i \bullet u = 0$ for some $i = 1, \ldots, s$. $\triangle$

**Example 2.3.6.** Let $R = \mathbb{C}[x, y, z]$, and consider the PDE given by

$$M = \begin{pmatrix} 0 & -xz^2 & xy^2 \\ -x^2y^2 & x^4 & 0 \\ -xyz^2 & z^2 & x^3y - y^2 \\ -x^2z^2 & 0 & x^4 \end{pmatrix}$$

The module $U = \mathrm{im}_R\, M^\top$ has two associated primes, namely $(0)$ and $(x)$. The solution set decomposes into $\mathrm{Sol}_{C^\infty}(M) = \mathrm{Sol}_{C^\infty}(M_1) + \mathrm{Sol}_{C^\infty}(M_2)$, where

$$M_1 = \begin{pmatrix} 0 & -z^2 & y^2 \\ -z^2 & 0 & x^2 \\ -y^2 & x^2 & 0 \end{pmatrix} \qquad M_2 = \begin{pmatrix} x^2 & 0 & 0 \\ 0 & x^2 & 0 \\ 0 & 0 & x^2 \\ 0 & -xz^2 & xy^2 \\ xyz^2 & -z^2 & y^2 \end{pmatrix}$$

Using techniques described later in Section 2.7, we note that $\mathrm{Sol}_{C^\infty}(M_1) = \mathrm{im}_{C^\infty} S$, where

$$S = \begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix},$$

and $\mathrm{Sol}_{C^\infty} M_2$ consists of functions of the form

$$\begin{pmatrix} \phi(b,c) \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a\psi(b,c) \\ \frac{\partial\psi}{\partial b}(b,c) \\ 0 \end{pmatrix},$$

where $x, y, z$ act as $\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}$ respectively, and $\phi$, $\psi$ are smooth functions $\mathbb{R}^2 \to \mathbb{C}$. Note that $C^\infty$ can be replaced by $\mathcal{D}'$, or indeed any injective cogenerator. $\triangle$

## 2.4 Syzygies and vector potentials

We now focus on solutions that are represented by a vector potentials, that is solutions $u \in \mathrm{Sol}(M)$ such that $u = S \bullet v$ for some $k$-tuple of functions $v \in \mathcal{F}^k$. Our goal is to compute the subspace of solutions to $M$ that are derived from vector potentials.

To this end, let $S$ be the syzygy matrix of $M$, so that

$$R^{k'} \xrightarrow{S} R^k \xrightarrow{M} R^\ell \tag{2.5}$$

is an exact sequence. The columns of $S$ are syzygies of $M$. The transpose of this sequence is

$$R^{k'} \xleftarrow{S^T} R^k \xleftarrow{M^T} R^\ell. \tag{2.6}$$

This is a complex but it is generally not exact. Applying $\operatorname{Hom}_R(\,\cdot\,, \mathcal{F})$, we get the complex

$$\mathcal{F}^{k'} \xrightarrow{S} \mathcal{F}^k \xrightarrow{M} \mathcal{F}^\ell, \tag{2.7}$$

and hence $\operatorname{im}_{\mathcal{F}}(S) \subseteq \operatorname{Sol}_{\mathcal{F}}(M)$. This means that $S \bullet \psi$ is a solution to our PDE $M$ for any $\psi \in \mathcal{F}^k$. If the equality $\operatorname{im}_{\mathcal{F}}(S) = \operatorname{Sol}_{\mathcal{F}}(M)$ holds then we say that $M$ *admits a vector potential*.

We briefly recall some definitions. An element $f$ in an $R$-module $U$ is a *torsion element* if $rf = 0$ for some $r \in R\backslash\{0\}$. The *torsion submodule* of $U$ is the module of torsion elements. The module $U$ is *torsion* if it is equal to its torsion submodule. The module $U$ is *torsion-free* if its torsion submodule is zero.

**Theorem 2.4.1.** *Let $\mathcal{F}$ be an injective cogenerator. Suppose that the sequence* (2.5) *is exact. Then the following are equivalent:*

*(1) The PDE $M$ admits a vector potential, i.e. the sequence (2.7) is exact.*

*(2) The sequence* (2.6) *is exact.*

*(3) The module $R^k/U = R^k/\operatorname{im}_R(M^T)$ is torsion-free.*

*(4) The module $U = \operatorname{im}_R(M^T)$ is $(0)$-primary.*

*Proof.* The equivalence of (1) and (2) holds because $\mathcal{F}$ is an injective cogenerator. The proof for the equivalence between (2) and (3) can be found in [57, Prop. 2.1]. For the equivalence between (3) and (4), let $\mathfrak{p}$ be an associated prime of $U$. By definition $\mathfrak{p} = (U\colon f)$ for some $f \in R^k$. We may also express this as

$$\mathfrak{p} = \{r \in R\colon rf = 0 \text{ in } R^k/U\}$$

Hence $\mathfrak{p}$ contains nonzero elements if and only if $\overline{f}$ is torsion in $R^k/U$. We conclude that $R^k/U$ is torsion-free if and only if $(0)$ is the only associated prime. □

If the conditions in Theorem 2.4.1 are met then we have a parametrization of all solutions:

$$\mathrm{Sol}_{\mathcal{F}}(M) \;=\; \mathrm{im}_{\mathcal{F}}(B) \;=\; \big\{B \bullet \psi\colon \psi \in \mathcal{F}^{k'}\big\}. \tag{2.8}$$

In general, $\mathrm{Sol}_{\mathcal{F}}(M)$ is strictly contained in $\mathrm{im}_{\mathcal{F}}(B)$: not all solutions of $M$ are in the image of $B$. In that case, the operator can be split into two operators $M_0$ and $M_1$, where $M_0$ admits a vector potential and $M_1$ does not, in the following strong sense: for all $B \in R^{k\times k'}$, there exists $\psi \in \mathcal{F}^{k'}$ such that $B \bullet \psi \notin \mathrm{Sol}_{\mathcal{F}}(M_1)$. This condition is equivalent to $(0) \notin \mathrm{Ass}(U_1)$ for $U_1 = \mathrm{im}_R(M_1^T)$. It is also equivalent to $R^k/U_1$ being a torsion module.

We write $U = U_0 \cap U_1$, where $U_0$ is $(0)$-primary, and $(0) \notin \mathrm{Ass}(U_1)$. This is obtained from a primary decomposition of $U$, where $U_1$ is the intersection of all primary components that are not $(0)$-primary. The solutions satisfy $\mathrm{Sol}_{\mathcal{F}}(U) = \mathrm{Sol}_{\mathcal{F}}(U_0) + \mathrm{Sol}_{\mathcal{F}}(U_1)$. This is known in control theory [57] as the *controllable-uncontrollable decomposition*, a concept we will revisit in more detail in Chapter 5, namely in Section 5.2.

**Theorem 2.4.2.** *The PDE $U$ has compactly supported solutions if and only if* $(0) \in \mathrm{Ass}(U)$.

*Proof.* This result is contained in [57, §3]. We offer a short proof. Suppose $(0) \in \mathrm{Ass}(U)$. Then $U$ is a submodule of a $(0)$-primary module $U_0 \subseteq R^k$. Write $U_0$ as the $R$-row span of a matrix $M_0$ and let $S_1$ be any column in its syzygy matrix $S$. Then for any compactly supported distribution $\psi$, the solution $S_1 \bullet \psi \in \mathrm{Sol}_{\mathcal{F}}(U)$ is also compactly supported.

For the converse, suppose $(0) \notin \mathrm{Ass}(U)$. Let $\phi \in \mathcal{F}^k \backslash \{0\}$ be a compactly supported solution. By Theorem 2.3.3, there exists $f \in R^k \backslash U$ such that $f^T \bullet \phi \neq 0$. Since $R^k/U$ is torsion, $rf \in U$ for some nonzero $r \in R$. Thus $r \bullet f^T \bullet \phi = 0$. Taking Fourier transforms, by the Paley-Wiener-Schwartz Theorem [32, Thm. 7.3.1], we get the equation $r(\xi) \cdot f(\xi)^T \hat{\phi}(\xi) = 0$ of analytic functions. Since $r(\xi) \neq 0$, we must have $f(\xi)^T \hat{\phi}(\xi) = 0$, a contradiction. $\square$

In conclusion, for any linear PDE $U$ as above, the solution space $\mathrm{Sol}_{\mathcal{F}}(U)$ decomposes into a subspace $\mathrm{Sol}_{\mathcal{F}}(U_0) = \mathrm{im}_{\mathcal{F}}(S)$ which contains all compactly supported solutions, and another subspace $\mathrm{Sol}_{\mathcal{F}}(U_1)$, with no compactly supported solutions at all.

### 2.5 Exponential solutions

The key to the correspondence between Noetherian operators and solutions to PDEs lie in the interplay between the "Fourier dual" variables $x_1, \ldots, x_n$, which we identified with $\partial_{z_1}, \ldots, \partial_{z_n}$, and $z_1, \ldots, z_n$, which we will identify correspondingly with $\partial_{x_1}, \ldots, \partial_{x_n}$. The exponential function $\mathbf{z} \mapsto \exp(\mathbf{x}^T \cdot \mathbf{z})$ will play a major role in this section, so we shall assume throughout this $\mathcal{F} = \mathcal{D}'$ or $C^\infty$, which both are injective cogenerators containing the exponential functions.

The following simple observation will drive much of the intuition.

**Lemma 2.5.1.** *Let $p, q$ be $n$-variate polynomials. Then*

$$p(\partial_{\mathbf{z}}) \bullet (q(\mathbf{z}) \exp(\mathbf{x}^T \cdot \mathbf{z})) = q(\partial_{\mathbf{x}}) \bullet (p(\mathbf{x}) \exp(\mathbf{x}^T \cdot \mathbf{z}))$$

*Proof.* The expression $q(\mathbf{z}) \exp(\mathbf{x}^T \cdot \mathbf{z})$ is equal to $q(\partial_{\mathbf{x}}) \bullet \exp(\mathbf{x}^t \cdot \mathbf{z})$. Similarly, we can rewrite $p(\mathbf{x}) \exp(\mathbf{x}^T \cdot \mathbf{z}) = p(\partial_{\mathbf{z}}) \bullet \exp(\mathbf{x}^T \cdot \mathbf{z})$. Since $\partial_{\mathbf{z}}$ and $\partial_{\mathbf{x}}$ commute, the result follows. $\square$

Let $\mathfrak{p} \subseteq R$ be a prime ideal. Recall that we can write elements $D \in W^k_{\kappa(\mathfrak{p})}$ as polynomials in $\partial_{\mathbf{x}}$-variables, with coefficients in $\kappa(\mathfrak{p})^k$, i.e.

$$D = D(\mathbf{x}, \partial_{\mathbf{x}}) = \sum_{\alpha} \frac{\overline{a_\alpha}}{b_\alpha} \partial_{\mathbf{x}}^{\alpha},$$

where $\overline{a_\alpha} \in (R/P)^k$ is the image of the vector $a_\alpha \in R^k$, and $b_\alpha \notin \mathfrak{p}$. If we let $W \subseteq \mathbb{C}^n$ denote the variety where the $b_\alpha$ vanish, the Zariski open set $V(\mathfrak{p}) \setminus W$ is non-empty.

Replacing the symbol $\partial_{\mathbf{x}}$ by $\mathbf{z}$, we can interpret $D(\mathbf{x}, \mathbf{z})$ as a $k$-tuple of rational functions, whose value on $(V(\mathfrak{p}) \setminus W) \times \mathbb{C}^k$ does not depend on the choice of representatives $a_\alpha$. If we fix a point $\mathbf{x}_0 \in V(\mathfrak{p}) \setminus W$, the function

$$u(\mathbf{z}) := D(\mathbf{x}_0, \mathbf{z}) \exp(\mathbf{x}_0^T \cdot \mathbf{z}) \tag{2.9}$$

is a function in $\mathcal{F}^k$. Given an $R$-submodule $U \subseteq R^k$, we can now ask which functions of the form (2.9) belong to $\operatorname{Sol}(U)$. The answer to this question reveals the connection between Noetherian operators and solution sets.

**Theorem 2.5.2.** *Let $\mathfrak{p} \subseteq R$ be a prime ideal, and $U \subseteq R^k$ be an $R$-submodule. Suppose that $\mathbf{t}, \mathbf{y}$ is a partition of the variables $x_1, \dots, x_n$ such that $\mathbf{t}$ is a maximal set of independent variables over $\mathfrak{p}$. Let $\mathcal{N} = \{\alpha \in \mathbb{N}^n \colon \alpha_i = 0$ if $x_i \in \mathbf{t}\}$ be the set of multi-indices indexing monomials in the dependent variables only. Denote by $\mathcal{B}$ be the set of polynomials in the variables $z_i$ for which $i$ is such that $x_i \in \mathbf{y}$, whose coefficients are $k$-tuples of rational functions, i.e. functions $D(\mathbf{x}, \mathbf{z})$ of the form*

$$D(\mathbf{x}, \mathbf{z}) = \sum_{\alpha \in \mathcal{N}} \frac{a_\alpha(\mathbf{x})}{b_\alpha(\mathbf{x})} \mathbf{z}^{\alpha},$$

*where $a_\alpha(\mathbf{x}) \in R^k$, $b_\alpha(\mathbf{x}) \in R \setminus \mathfrak{p}$. Then there is a one-to-one correspondence between* *elements of the local dual space* $D_{\mathfrak{p}^{(\mathbf{t})}}[U^{(\mathbf{t})}]$ *and the set*

$$\{D(\mathbf{x},\mathbf{z}) \in \mathcal{B}\colon\, D(\mathbf{x}_0,\mathbf{z})\exp(\mathbf{x}_0^T\cdot\mathbf{z}) \in \mathrm{Sol}(U) \textit{ for all } \mathbf{x}_0 \in V(\mathfrak{p})\}.$$

*The correspondence sends* $D(\mathbf{x},\partial_{\mathbf{x}})$ *to* $D(\mathbf{x},\mathbf{z})$ *and vice-versa.*

*Proof.* Suppose $D \in D_{\mathfrak{p}^{(\mathbf{t})}}[U^{(\mathbf{t})}]$, so we have

$$D = D(\mathbf{x},\partial_{\mathbf{x}}) = \sum_{\alpha\in\mathcal{N}} \frac{a_\alpha(\mathbf{x})}{b_\alpha(\mathbf{x})}\partial_{\mathbf{x}}^{\alpha},$$

Fix any $\mathbf{x}_0 \in V(\mathfrak{p})$ and consider the function $D(\mathbf{x}_0,\mathbf{z})\exp(\mathbf{x}_0^T\cdot\mathbf{z})$. If $u(\mathbf{x}) \in U$, by applying Lemma 2.5.1

$$\begin{aligned} u(\mathbf{x}) \bullet (D(\mathbf{x}_0,\mathbf{z})\exp(\mathbf{x}_0^T\cdot\mathbf{z})) &= u(\partial_{\mathbf{z}}) \bullet (D(\mathbf{x}_0,\mathbf{z})\exp(\mathbf{x}_0^T\cdot\mathbf{z})) \\ &= \left[D(\mathbf{x},\partial_{\mathbf{x}}) \bullet (u(\mathbf{x})\exp(\mathbf{x}^T\bullet\mathbf{z}))\right]_{\mathbf{x}=\mathbf{x}_0} \end{aligned}$$

Note that since $D(\mathbf{x},\partial_{\mathbf{x}}) \in D_{\mathfrak{p}^{(\mathbf{t})}}[U^{(\mathbf{t})}]$, for any polynomial $r \in R$, the evaluation of $D(\mathbf{x},\partial_{\mathbf{x}}) \bullet (u(\mathbf{x})r(\mathbf{x}))$ at any $\mathbf{x}_0 \in V(\mathfrak{p})$ is zero. That is, we have

$$[D(\mathbf{x},\partial_{\mathbf{x}}) \bullet (u(\mathbf{x})r(\mathbf{x}))]_{\mathbf{x}=\mathbf{x}_0} = 0 \text{ for all univariate polynomials } r \in R.$$

Since polynomials are dense in the space of entire functions [31, p.245], we have that $D(\mathbf{x}_0,\mathbf{z})\exp(\mathbf{x}_0^T\cdot\mathbf{z})$ is a solution to the PDE $U$ for all $\mathbf{x}_0 \in V(\mathfrak{p})$.

For the converse, we can use the same argument in reverse. Let $D = D(\mathbf{x},\mathbf{z}) \in \mathcal{B}$ such that $D(\mathbf{x}_0,\mathbf{z})\exp(\mathbf{x}_0^T\cdot\mathbf{z})$ is a solution. Then, if $h(\mathbf{x})$ is any linear combination of exponential functions $\mathbf{x} \mapsto \exp(\mathbf{x}^T\cdot\mathbf{z})$, the evaluation of

$$D(\mathbf{x},\partial_{\mathbf{x}}) \bullet (u(\mathbf{x})h(\mathbf{x}))$$

at any $\mathbf{x} = \mathbf{x}_0 \in V(\mathfrak{p})$ is zero for all $u \in U$. Since linear combinations of exponential functions $\mathbf{x} \mapsto \exp(\mathbf{x}^T \cdot \mathbf{z})$ are dense in the space of entire functions, the same is true if $h(\mathbf{x})$ is a polynomial. If we substitute the symbol $\mathbf{z}$ by $\partial_{\mathbf{x}}$ in $D(\mathbf{x}, \mathbf{z})$, and consider it as an element of $W_{\kappa(\mathfrak{p}^{(\mathbf{t})})}$, we have $D \bullet u = 0$ for all $u \in U$, hence $D \in D_{\mathfrak{p}^{(\mathbf{t})}}[U^{(\mathbf{t})}]$. $\square$

**Example 2.5.3.** The prototypical example for Theorem 2.5.2 is familiar from undergraduate differential calculus. Let $I \subseteq \mathbb{Q}[x]$ be the principal ideal generated by $p(x) = x^3 - 3x^2 + 4$. This corresponds to the ODE $v'''(z) - 3v''(z) + 4 = 0$. The *characteristic polynomial* of the ODE is $p$, and by factoring $p(x) = (x-2)^2(x+1)$ we obtain a *fundamental set of solutions* $e^{2z}, ze^{2z}, e^{-z}$. These solutions characterize the dual spaces

$$D_{(x-2)}[I] = \operatorname{span}_{\mathbb{Q}}\{1, \partial_x\} \qquad\qquad D_{(x+1)}[I] = \operatorname{span}_{\mathbb{Q}}\{1\},$$

obtained by replacing the symbol $z$ by $\partial_x$. $\triangle$

## 2.6 Modules and Varieties

Let $U \subseteq R^k$ be an $R$-submodule generated by the rows of the $\ell \times k$ matrix $M$, and suppose $U = \bigcap_{i=1}^s U_i$ is a primary decomposition, where $U_i$ is $\mathfrak{p}_i$-primary. As we saw in Section 1.9, the support of $U$ is the variety

$$V(U) = \bigcup_{i=1}^s V(\mathfrak{p}_i).$$

If $k = 1$, so that $U = I \subseteq R$ is an ideal, then the support $V(U)$ coincides with the variety $V(I)$ attached as usual to an ideal $I$, namely the common zero set in $\mathbb{C}^n$ of all polynomials in $I$.

The support is generally reducible, with $\leq s$ irreducible components. For instance, the module $M$ in Example 2.7.5 has six associated primes, and an explicit primary decomposition was given in (2.14). However, the support $V(U)$ has only four irreducible components in $\mathbb{C}^4$, namely one hyperplane, two 2-dimensional planes, and one nonlinear surface (twisted cubic).

The relationship between modules and ideals mirrors the relationship between PDE for vector-valued functions and related PDE for scalar-valued functions. To pursue this a bit further, we now define two ideals that are naturally associated with a given module $M \subseteq R^k$.

The first ideal is the *annihilator* of the quotient module $R^k/U = \mathrm{coker}_R(M^T)$, which is

$$I \;:=\; \mathrm{Ann}_R(R^k/U) \;=\; \big\{\, f \in R \,:\, fm \in M \text{ for all } m \in R^k \big\}.$$

The second is the zeroth *Fitting ideal* of $R^k/U$, which is the ideal in $R$ generated by the $k \times k$ minors of the presentation matrix $M^T$. It is independent of the choice of $M$, and we write

$$J \;:=\; \mathrm{Fitt}_0(R^k/U) \;=\; \big(\, k \times k \text{ subdeterminants of } M \,\big).$$

We are interested in the affine varieties in $\mathbb{C}^n$ defined by these ideals. They are denoted by $V(I)$ and $V(J)$ respectively. The following is a standard result in commutative algebra.

**Proposition 2.6.1.** *The three varieties above are equal for every submodule $U$ of $R^k$, that is,*

$$V(U) \;=\; V(I) \;=\; V(J) \;\subseteq\; \mathbb{C}^n. \tag{2.10}$$

*Proof.* This follows from [17, Proposition 20.7]. □

*Remark* 2.6.2. It can happen that $\mathrm{rank}(M) < k$, for instance when $k > l$. In that case, $I = J = \{0\}$ and $V(U) = \mathbb{C}^n$. Geometrically, the module $U$ furnishes a coherent sheaf that is supported on the entire space $\mathbb{C}^n$. For instance, let $k = n = 2, l = 1$ and $M = [\, \partial_1 \ {-\partial_2} \,]$. The PDE asks for pairs $(\psi_1, \psi_2)$ such that $\partial\psi_1/\partial z_1 \;=\; \partial\psi_2/\partial z_2$. We see that $\mathrm{Sol}(M)$ consists of all pairs $\big(\partial\alpha/\partial z_2 \,, \partial\alpha/\partial z_1\big)$, where $\alpha \;=\; \alpha(z_1, z_2)$ runs over functions in two variables. Indeed, the module $U \;=\; \mathrm{im}_R(M^T)$ is $(0)$-primary, so the solution set admits a vector potential given by $\left[\begin{smallmatrix}\partial_2\\ \partial_2\end{smallmatrix}\right]$, the syzygy matrix of $M$. $\triangle$

The following example shows that (2.10) is not true at the level of schemes.

**Example 2.6.3** ($n = k = 3, l = 5$)**.** Let $R = \mathbb{C}[\partial_1, \partial_2, \partial_3]$ and $U$ the submodule of $R^3$ generated by rows of

$$M = \begin{bmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_1^2 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_1 \\ 0 & 0 & \partial_3 \end{bmatrix}$$

We find $I = \langle \partial_1^2, \partial_1\partial_2 \rangle \supset J = \langle \partial_1^4, \partial_1^3\partial_3, \partial_1^2\partial_2, \partial_1\partial_2\partial_3 \rangle$. The sets of associated primes are

$$\begin{array}{rrcll} & \mathrm{Ass}(I) & = & \big\{\langle\partial_1\rangle, \langle\partial_1,\partial_2\rangle\big\} & \text{with } \mathrm{amult}(I) = 2 \\ \subset & \mathrm{Ass}(U) & = & \big\{\langle\partial_1\rangle, \langle\partial_1,\partial_2\rangle, \langle\partial_1,\partial_3\rangle\big\} & \text{with } \mathrm{amult}(U) = 4 \\ \subset & \mathrm{Ass}(J) & = & \big\{\langle\partial_1\rangle, \langle\partial_1,\partial_2\rangle, \langle\partial_1,\partial_3\rangle, \langle\partial_1,\partial_2,\partial_3\rangle\big\} & \text{with } \mathrm{amult}(J) = 5 \end{array}$$

The support $V(U)$ is a plane in 3-space, on which $I$ and $J$ define different scheme structures. Our module $M$ defines a coherent sheaf on that plane that lives between these two schemes. We consider the PDE in each of the three cases, and from this we derive the solution sets. To begin with, functions in $\mathrm{Sol}(J)$ have the form

$$\alpha(z_2, z_3) \,+\, z_1\beta(z_3) \,+\, z_1^2\gamma(z_3) \,+\, z_1\delta(z_2) \,+\, c \cdot z_1^3,$$

for any functions $\alpha, \beta, \gamma, \delta$. The first two terms give functions in the subspace $\mathrm{Sol}(I)$.

Elements in $\mathrm{Sol}(U)$ are vectors

$$\begin{bmatrix} \rho(z_2, z_3) \\ \sigma(z_3) + z_1\tau(z_3) \\ \omega(z_2) \end{bmatrix},$$

for any functions $\rho, \sigma, \tau, \omega$. These represent all functions $\mathbb{C}^3 \to \mathbb{C}^3$ that satisfy the five PDE given by the matrix $M$. $\triangle$

*Remark* 2.6.4. The quotient $R/I$ embeds naturally into the direct sum of $k$ copies of $R^k/U$, via $1 \mapsto e_j$. This implies $\mathrm{Ass}(I) \subseteq \mathrm{Ass}(U)$. It would be worthwhile to understand how the differential primary decompositions of $I$, $J$ and $U$ are related, and to study implications for the solution spaces $\mathrm{Sol}(I)$, $\mathrm{Sol}(J)$ and $\mathrm{Sol}(U)$. What relationships hold between these? $\triangle$

**Lemma 2.6.5.** *Fix a $\ell \times k$ matrix $M$ with entries in $R$ and its module $U = \mathrm{im}_R\, M^T \subseteq R^k$. A point $\mathbf{u} \in \mathbb{C}^n$ lies in $V(U)$ if and only if there exist constants $c_1, \ldots, c_k \in \mathbb{C}$, not all zero, such that*

$$\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} \exp(u_1 z_1 + \cdots + u_n z_n) \;\in\; \mathrm{Sol}(U). \tag{2.11}$$

*More precisely,* (2.11) *holds if and only if* $M(\mathbf{u})(c_1, \ldots, c_k)^T = 0$.

*Proof.* Let $m_{ij}(\partial)$ denote the entries of the matrix $M(\partial)$. Then (2.11) holds if and only if

$$\sum_{i=1}^{k} m_{ij}(\partial_{\mathbf{z}}) \bullet (c_i \exp(u_1 z_1 + \cdots + u_n z_n)) \;=\; 0 \quad \text{ for all } j = 1, \ldots, l.$$

This is equivalent to

$$\sum_{i=1}^{k} c_i\, m_{ij}(\mathbf{u}) \exp(u_1 z_1 + \cdots + u_n z_n) \;=\; 0 \quad \text{ for all } j = 1, \ldots, l.$$

This condition holds if and only if $M(\mathbf{u})(c_1, \ldots, c_k)^T$ is the zero vector in $\mathbb{C}^l$. We conclude that, for any given $\mathbf{u} \in \mathbb{C}^n$, the previous condition is satisfied for some $c \in \mathbb{C}^k \backslash \{0\}$ if and only if $\operatorname{rank}(M(\mathbf{u})) < k$ if and only if $\mathbf{u} \in V(U) = V(I)$. Here we use Proposition 2.6.1. $\square$

Here is an alternative way to interpret the characteristic variety of a system of PDE:

**Proposition 2.6.6.** *The solution space* $\operatorname{Sol}(U)$ *contains a polynomial-exponential solution* $q(\mathbf{z}) \cdot \exp(\mathbf{u}^T \cdot \mathbf{z})$ *if and only if* $\mathbf{u} \in V(U)$. *Here* $q$ *is some vector of* $k$ *polynomials in* $n$ *unknowns.*

*Proof.* One direction is clear from Lemma 2.6.5. Next, suppose $q(\mathbf{z}) \exp(\mathbf{u}^T \cdot \mathbf{z}) \in \operatorname{Sol}(U)$. The partial derivative of this function with respect to any unknown $z_i$ is also in $\operatorname{Sol}(U)$. Hence

$$\partial_i \bullet (q(\mathbf{z}) \exp(\mathbf{u}^t \mathbf{z})) \;=\; (\partial_i \bullet q(\mathbf{z})) \exp(\mathbf{u}^t \mathbf{z}) \;+\; u_i q(\mathbf{z}) \exp(\mathbf{u}^t \mathbf{z}) \;\in\; \operatorname{Sol}(U) \quad \text{for } i = 1, \ldots, n.$$

Hence the exponential function $(\partial_i \bullet q(\mathbf{z})) \exp(\mathbf{u}^t \mathbf{z})$ is in $\operatorname{Sol}(U)$. Since the degree of $\partial_i \bullet q(\mathbf{z})$ is less than that of $q(\mathbf{z})$, we can find a sequence $D = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_s}$ such that $D \bullet q$ is a nonzero constant vector and $(D \bullet q) \exp(\mathbf{u}^t \mathbf{z}) \in \operatorname{Sol}(U)$. Lemma 2.6.5 now implies that $\mathbf{u} \in V(U)$. $\square$

The solution space $\operatorname{Sol}(U)$ to a submodule $U \subseteq R^k$ is a vector space over $\mathbb{C}$. It is infinite-dimensional whenever $V(U)$ is a variety of positive dimension. This follows from Lemma 2.6.5 because there are infinitely many points $\mathbf{u}$ in $V(U)$. However, if $V(U)$ is a finite subset of $\mathbb{C}^n$, then $\operatorname{Sol}(U)$ is finite-dimensional. This is the content of the next theorem.

**Theorem 2.6.7.** *Consider a module* $U \subseteq R^k$. *Its solution space* $\operatorname{Sol}(U)$ *is finite-dimensional over* $\mathbb{C}$ *if and only if* $V(U)$ *has dimension* $0$. *In this case,* $\dim_{\mathbb{C}} \operatorname{Sol}(U) = \dim_{\mathbb{K}}(R^k/U) = \operatorname{amult}(U)$. *There is a basis of* $\operatorname{Sol}(U)$ *given by vectors* $q(\mathbf{z}) \exp(\mathbf{u}^T \cdot \mathbf{z})$, *where* $\mathbf{u} \in V(U)$

*and $q(\mathbf{z})$ runs over a finite set of polynomial vectors, whose cardinality is the length of $U$ along the maximal ideal $\langle x_1 - u_1, \ldots, x_n - u_n\rangle$. There exist polynomial solutions if and only if $\mathfrak{m} = \langle x_1, \ldots, x_n\rangle$ is an associated prime of $U$. The polynomial solutions are found by solving the PDE given by the $\mathfrak{m}$-primary component of $M$.*

*Proof.* This is the main result in Oberst's article [46], proved in the setting of injective cogenerators $\mathcal{F}$. The same statement for $\mathcal{F} = C^\infty$ appears in [6, Ch. 8, Theorem 7.1]. The scalar case $(k = 1)$ is found in [41, Theorem 3.27]. The proof given there uses solutions in the power series ring, which is an injective cogenerator, and it generalizes to modules. $\square$

By a *polynomial solution* we mean a vector $q(\mathbf{z})$ whose coordinates are polynomials. The $\mathfrak{m}$-primary component in Theorem 2.6.7 is computed by a double saturation step. When $U = I$ is an ideal then this double saturation is $I : (I : \mathfrak{m}^\infty)$, as seen in [41, Theorem 3.27]. For submodules $U$ of $R^k$ with $k \geq 2$, we would compute $U : \mathrm{Ann}(R^k/(U : \mathfrak{m}^\infty))$. The inner colon $(U : \mathfrak{m}^\infty)$ is the intersection of all primary components of $U$ whose variety $V_i$ does not contain the origin $0$. It is computed as $(U : f) = \{m \in R^k : fm \in U\}$, where $f$ is a random homogeneous polynomial of large degree. The outer colon is the module $(U : g)$, where $g$ is a general polynomial in the ideal $\mathrm{Ann}(R^k/(U : f))$. See also [7, Proposition 2.2].

It is an interesting problem to identify polynomial solutions when $V(U)$ is no longer finite, and to decide whether these are dense in the infinite-dimensional space of all solutions. Here "dense" refers to the topology on $\mathcal{F}$ used by Lomadze in [38]. The following result gives an algebraic characterization of the closure in $\mathrm{Sol}(U)$ of the subspace of polynomial solutions.

**Proposition 2.6.8.** *Let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the maximal ideal corresponding to the origin. The polynomial solutions are dense in $\mathrm{Sol}(U)$ if and only if the origin $0$ lies in every associated variety $V(\mathfrak{p})$, $\mathfrak{p} \in \mathrm{Ass}(U)$, of the module $U$, i.e. if and only if $U$ is $\mathfrak{m}$-closed. If this fails then the topological closure of the space of polynomial solutions $q(\mathbf{z})$ to $U$ is the* *solution space of the* $\mathfrak{m}$*-closure of* $U$*.*

*Proof.* This proposition is our reinterpretation of Lomadze's result in [38, Theorem 3.1]. □

The result gives rise to algebraic algorithms for answering analytic questions about a system of PDE. The property in the first sentence can be decided by running the primary decomposition algorithm in [7]. For the second sentence, we need to compute a double saturation as above. This can be carried out in `Macaulay2` as well.

## 2.7 The Ehrenpreis-Palamodov fundamental principle

We saw in Theorem 2.5.2 that local dual spaces correspond precisely to exponential solutions of a system of PDE when $\mathcal{F} = \mathcal{D}'$ or $\mathbb{C}^\infty$. The celebrated Ehrenpreis-Palamodov fundamental principle asserts that in fact any solution $v \in \mathrm{Sol}_{\mathcal{F}}(U)$ is a superposition of these exponential solutions.

The fundamental principle appears in different forms in the books by Björk [6, Theorem 8.1.3], Ehrenpreis [16], Hörmander [31, Section 7.7] and Palamodov [51]. Other references with different emphases include [5, 37, 49, 66]. For a perspective from commutative algebra see [10, 11].

**Theorem 2.7.1** (Ehrenpreis-Palamodov fundamental principle)**.** *Let* $U \subseteq R^k$ *be a* $\mathfrak{p}$*-primary module,* $\mathbf{t}$ *a maximal set of independent variables over* $\mathfrak{p}$*, and let* $\mathcal{B} = \{B_1, \ldots, B_s\} \in W_R$ *be a set of operators, involving no* $\partial_{\mathbf{t}}$*-variables, such that*

$$U = \{f \in R^k \colon B_i \bullet f \in \mathfrak{p} \textit{ for all } i = 1, \ldots, s\}.$$

*If we write* $B_i = B_i(\mathbf{x}, \partial_{\mathbf{x}})$ *as a polynomial in* $2n$ *variables, then all distributional solutions*

*$u \in \mathrm{Sol}_{\mathcal{D}'}(U)$ to the PDE described by $U$ are of the form*

$$u(z_1, \ldots, z_n) = \sum_{i=1}^{s} \int_{V(\mathfrak{p})} B_i(\mathbf{x}, \mathbf{z}) \exp(\mathbf{x} \cdot \mathbf{z}) \, \mathrm{d}\mu_i(\mathbf{x})$$

*for a suitable set of measures $\mu_1, \ldots, \mu_s$.*

We remark that the statement in its original form interprets the variables $x_i$ as $-\mathrm{i}\frac{\partial}{\partial z_i}$, as is standard in Fourier analysis, in which case the representation takes the form

$$u(z_1, \ldots, z_n) = \sum_{i=1}^{s} \int_{V(\mathfrak{p})} B_i(\mathbf{x}, \mathbf{z}) \exp(\mathrm{i}\mathbf{x} \cdot \mathbf{z}) \, \mathrm{d}\mu_i(\mathbf{x}).$$

We recover the statement in Theorem 2.7.1 by doing a change of variables $\phi \colon x_i \mapsto \mathrm{i}x_i$, and absorbing the extra i into the measure, noting that if

$$U = \{f \in R^k \colon B_i(\mathbf{x}, \partial_{\mathbf{x}}) \bullet f \in \mathfrak{p} \text{ for all } i = 1, \ldots, s\},$$

then

$$\phi(U) = \{f \in R^k \colon B_i(\mathrm{i}\mathbf{x}, \partial_{\mathbf{x}}) \bullet f \in \phi(\mathfrak{p}) \text{ for all } i = 1, \ldots, s\}.$$

As was discussed in the end of Section 1.6, the image in $W_{\kappa(\mathfrak{p}^{(\mathbf{t})})}$ of the set $\mathcal{B} \subseteq W_R$ in Theorem 2.7.1 is in fact a set of Noetherian operators. We will argue that the conclusion of the Ehrenpreis-Palamodov fundamental principle holds for our definition of Noetherian operators as well. Suppose $\mathcal{D} = \{D_1, \ldots, D_s\} \subseteq W_{\kappa(\mathfrak{p}^{(\mathbf{t})})}$ is a set of Noetherian operators for the $\mathfrak{p}$-primary module $U$. As in Section 2.5, we can interpret $D_i(\mathbf{x}, \mathbf{z})$ as a $2n$ variate vector of functions, each of whose entries is a polynomial in variables $\mathbf{z}$ with rational function coefficients in variables $\mathbf{x}$, whose denominator does not vanish identically on $V(\mathfrak{p})$. Thus we view $D_i(\mathbf{x}, \mathbf{z})$ as a $2n$-variate vector of rational functions whose value is well defined on a Zariski open subset of $V(\mathfrak{p}) \times \mathbb{C}^n$.

**Corollary 2.7.2.** *Let $U \subseteq R$ be a $\mathfrak{p}$-primary ideal, $\mathfrak{t}$ a maximal set of independent variables over $\mathfrak{p}$, and let $\mathcal{D} = \{D_1, \ldots, D_s\} \subseteq W_{\kappa(\mathfrak{p}^{(\mathfrak{t})})}$ be a set of Noetherian operators. Then all distributional solutions $u \in \mathrm{Sol}_{\mathcal{D}'}(U)$ to the PDE described by $U$ are of the form*

$$u(z_1, \ldots, z_n) = \sum_{i=1}^{s} \int_{V(\mathfrak{p})} D_i(\mathbf{x}, \mathbf{z}) \exp(\mathbf{x} \cdot \mathbf{z}) \, \mathrm{d}\mu_i(\mathbf{x})$$

*for a suitable set of measures $\mu_1, \ldots, \mu_s$.*

*Proof.* We can clear all denominators by multiplying $D_i$ by some polynomial $g_i \notin \mathfrak{p}$, thus we can write $g_i D_i$ as the image of an operator with coefficients in $(R/\mathfrak{p})^k$. Taking a representative for each coefficient, we get operators $B_i \in (W_R)^k$, whose images in $(W_{\kappa(\mathfrak{p}^{(\mathfrak{t})})})^k$ are precisely the $g_i D_i$, so the operators $B_i$ satisfy the condition of Theorem 2.7.1. Furthermore, over $V(\mathfrak{p}) \times \mathbb{C}^n$, the functions $B_i(\mathbf{x}, \mathbf{z})$ and $g_i(\mathbf{x}) D_i(\mathbf{x}, \mathbf{z})$ evaluate to the same complex value. Hence for each solution $u$ we have

$$\begin{aligned} u(\mathbf{z}) &= \sum_{i=1}^{s} \int_{V(\mathfrak{p})} B_i(\mathbf{x}, \mathbf{z}) \exp(\mathbf{x} \cdot \mathbf{z}) \, \mathrm{d}\mu_i'(\mathbf{x}) \\ &= \sum_{i=1}^{s} \int_{V(\mathfrak{p})} g_i(\mathbf{x}) D_i(\mathbf{x}, \mathbf{z}) \exp(\mathbf{x} \cdot \mathbf{z}) \, \mathrm{d}\mu_i'(\mathbf{x}) \end{aligned}$$

for some measures $\mu_i'$. The claim then follows when we define $\mu_i$ to be the measure supported on $V(\mathfrak{p})$ such that $\mathrm{d}\mu_i(\mathbf{x}) = g(\mathbf{x}) \mathrm{d}\mu_i'(\mathbf{x})$. □

Another improvement we can make on the classical Ehrenpreis-Palamodov fundamental principle is by using a differential primary decomposition instead of a full set of Noetherian operators for every primary component. The classical method for dealing with non-primary ideals was to first take a primary decomposition, and then apply Theorem 2.7.1 for each primary component. This in general yields an integral representation of each solution $u$ with redundant summands. The following simple example demonstrates the problem.

**Example 2.7.3.** Let $U$ be the $R$-module generated my the rows of

$$M = \begin{bmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{bmatrix}.$$

A primary decomposition gives $U = U_0 \cap U_1$, which are generated by the rows of the matrices

$$M_0 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \text{ and } M_1 = \begin{bmatrix} x_1^2 & x_1x_2 & x_2^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1^2 & x_1x_2 & x_2^2 \end{bmatrix}^T.$$

The module $U_0$ is $(0)$-primary, with a single Noetherian operator $D_1 = [-x_2, x_1]^T$, so the solution set $\mathrm{Sol}(U_0)$ consists of functions of the form

$$u_0(z_1, z_2) = \int_{\mathbb{C}} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \exp(z_1x_1 + z_2x_2)\, \mathrm{d}\mu(x_1, x_2) = \begin{bmatrix} -\partial_{z_2} \bullet \phi(z_1, z_2) \\ \partial_{z_1} \bullet \phi(z_1, z_2) \end{bmatrix},$$

where $\phi(z_1, z_2) := \int_{\mathbb{C}} \exp(z_1x_1 + z_2x_2)\, \mathrm{d}\mu(x_1, x_2)$ for a suitable measure $\mu$.

The module $U_1$ is $(x_1, x_2)$ primary, so the measures involved in the solution set of $U_1$ will be point measures supported at the origin. The six operators

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \partial_1 \\ 0 \end{bmatrix}, \begin{bmatrix} \partial_2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \partial_1 \end{bmatrix}, \begin{bmatrix} 0 \\ \partial_2 \end{bmatrix}$$

form a set of Noetherian operators, and the Ehrenpreis-Palamodov fundamental principle yields the following solutions for $U_1$:

$$u_1(z_1,z_2) = \int_{\{(0,0)\}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \exp(x_1z_1 + x_2z_2)\,\mathrm{d}\mu_1(x_1,x_2) + \cdots$$

$$\cdots + \int_{\{(0,0)\}} \begin{bmatrix} 0 \\ z_2 \end{bmatrix} \exp(x_1z_1 + x_2z_2)\,\mathrm{d}\mu_6(x_1,x_2).$$

Since each measure is a point measure, we can rewrite $v$ as a vector of polynomials

$$v(z_1,z_2) = \begin{bmatrix} c_1 + c_2z_1 + c_3z_2 \\ c_4 + c_5z_1 + c_6z_2 \end{bmatrix},$$

for some $c_1,\dots,c_6 \in \mathbb{C}$. Hence we conclude that the solutions to the PDE $U$ are of the form

$$u(z_1,z_2) = u_0(z_1,z_2) + u_1(z_1,z_2) = \begin{bmatrix} c_1 + c_2z_1 + c_3z_2 \\ c_4 + c_5z_1 + c_6z_2 \end{bmatrix} + \begin{bmatrix} -\partial_{z_2} \bullet \phi(z_1,z_2) \\ \partial_{z_1} \bullet \phi(z_1,z_2) \end{bmatrix}.$$

Note however that some of the terms in the polynomial vector $u_1$ can be absorbed into the vector $u_0$, since

$$\begin{bmatrix} c_1 + c_2z_1 + c_3z_2 \\ c_4 + c_5z_1 + c_6z_2 \end{bmatrix} = \begin{bmatrix} (c_2 + c_6)z_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\partial_{z_2} \bullet \psi(z_1,z_2) \\ \partial_{z_1} \bullet \psi(z_1,z_2) \end{bmatrix},$$

where $\psi(z_1,z_2) = -c_1z_2 - (c_3/2)z_2^2 + c_4z_1 + (c_5/2)z_1^2 + c_6z_1z_2$. Thus we get the shorter representation of solutions in $\mathrm{Sol}(U)$ using only two of the original seven Noetherian operators

$$u(z_1,z_2) = \begin{bmatrix} cz_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\partial_{z_2} \bullet \phi(z_1,z_2) \\ \partial_{z_1} \bullet \phi(z_1,z_2) \end{bmatrix},$$

for any $c \in \mathbb{C}$, $\phi \in \mathcal{D}'$. $\triangle$

The following theorem shows that the non-redundant summands in the integral representation are precisely the ones that generate the excess dual spaces. In other words, a differential primary decomposition yields a minimal integral representation for the solution set of a PDE, which contains $\operatorname{amult}(U)$ summands.

**Theorem 2.7.4.** *Let $U \subseteq R^k$ be an $R$-submodule, and let $\{(\mathfrak{p}, \mathbf{t}_\mathfrak{p}, \mathcal{D}_\mathfrak{p})\}_{\mathfrak{p} \in \operatorname{Ass}(U)}$ be a differential primary decomposition. Let $\mathcal{D}_\mathfrak{p} = \{D_{\mathfrak{p},1}, \ldots, D_{\mathfrak{p},s_\mathfrak{p}}\}$. Then all distributional solutions $u \in \operatorname{Sol}(U)$ to the PDE described by $U$ are of the form*

$$u(z_1, \ldots, z_n) = \sum_{\mathfrak{p} \in \operatorname{Ass}(I)} \sum_{i=1}^{s_\mathfrak{p}} \int_{V(\mathfrak{p})} D_{\mathfrak{p},i}(\mathbf{x}, \mathbf{z}) \exp(\mathbf{x} \cdot \mathbf{z}) \, \mathrm{d}\mu_{\mathfrak{p},i}(\mathbf{x}), \tag{2.12}$$

*for a suitable set of measures $\{\mu_{\mathfrak{p},i}\}_{i=1,\ldots,s_\mathfrak{p}; \mathfrak{p} \in \operatorname{Ass}(U)}$.*

*Proof.* For each $\mathfrak{p} \in \operatorname{Ass}(U)$, let $\Lambda$ be the local dual space generated by $\mathcal{D}_\mathfrak{p}$. There is a $\kappa(\mathfrak{p})$-basis of $\Lambda$ of the form $\mathcal{E}_\mathfrak{p} = \mathcal{D}_\mathfrak{p} \cup \{E_1, \ldots, E_{r_\mathfrak{p}}\}$; note that $r_\mathfrak{p} > 0$ if and only if $\mathfrak{p}$ is an embedded prime. Suppose $Q_\mathfrak{p}$ is the corresponding $\mathfrak{p}$-primary module, so by Theorem 1.7.10 $U = \bigcap Q_\mathfrak{p}$ is a primary decomposition, and the set $\mathcal{E}_\mathfrak{p}$ is a set of Noetherian operators for $Q_\mathfrak{p}$. Hence by Ehrenpreis-Palamodov we have a representation

$$\begin{aligned} v(\mathbf{z}) = & \sum_{\mathfrak{p} \in \operatorname{Ass}(U)} \sum_{i=1}^{s_\mathfrak{p}} \int_{V(\mathfrak{p})} D_{\mathfrak{p},i}(\mathbf{x}, \mathbf{z}) \exp(\mathbf{x}^T \cdot \mathbf{z}) \, \mathrm{d}\mu_{\mathfrak{p},i}(\mathbf{x}) \\ & + \sum_{\mathfrak{p} \in \operatorname{Ass}(U)} \sum_{i=1}^{r_\mathfrak{p}} \int_{V(\mathfrak{p})} E_{\mathfrak{p},i}(\mathbf{x}, \mathbf{z}) \exp(\mathbf{x}^T \cdot \mathbf{z}) \, \mathrm{d}\nu_{\mathfrak{p},i}(\mathbf{x}) \end{aligned}$$

We will show that with a suitable choice of measures, the summands involving $E_{\mathfrak{p},i}$ can be written as integrals involving $D_{\mathfrak{p},i}$. More precisely, for $\mathfrak{p} \in \operatorname{Ass}(U)$, we claim that there are measures $\{\eta_{\mathfrak{q},i}\}_{i=1,\dots,s_{\mathfrak{q}};\mathfrak{q}\subseteq\mathfrak{p}}$ such that

$$\begin{aligned} w(\mathbf{z}) :=& \int_{V(\mathfrak{p})} E_{\mathfrak{p},i}(\mathbf{x},\mathbf{z}) \exp(\mathbf{x}^T\cdot\mathbf{z}) \,\mathrm{d}\nu_{\mathfrak{p},i}(\mathbf{x}) \\ =& \sum_{\mathfrak{q}\subseteq\mathfrak{p}} \sum_{j=s_{\mathfrak{q}}} \int_{V(\mathfrak{q})} D_{\mathfrak{q},j}(\mathbf{x},\mathbf{z}) \exp(\mathbf{x}^T\cdot\mathbf{z}) \,\mathrm{d}\eta_{\mathfrak{q},j}(\mathbf{x}). \end{aligned}$$

If $\mathfrak{p}$ is a minimal prime, nothing needs to be shown. Fix some non minimal $\mathfrak{p} \in \mathrm{Ass}(U)$, and suppose then by induction that the claim is true for all $\mathfrak{q} \subsetneq \mathfrak{p}$.

From the definition of a differential primary decomposition, we can write

$$E_{\mathfrak{p},i} = \sum_{j=1}^{s_{\mathfrak{p}}} c_j D_{j,\mathfrak{p}} + E',$$

where $c_j \in \kappa(\mathfrak{p})$ and $E' \in D_{\mathfrak{p}^{(\mathbf{t})}}[(U\colon \mathfrak{p}^\infty)^{(\mathbf{t})}]$. Thus we have

$$\begin{aligned} w(\mathbf{z}) =& \sum_{j=1}^{s_{\mathfrak{p}}} \int_{V(\mathfrak{p})} D_{\mathfrak{p},j}(\mathbf{x},\mathbf{z}) \exp(\mathbf{x}^T\cdot\mathbf{z}) \cdot c_j(\mathbf{x}) \,\mathrm{d}\nu_{\mathfrak{p},i} \\ &+ \int_{V(\mathfrak{p})} E'(\mathbf{x},\mathbf{z}) \exp(\mathbf{x}^T\cdot\mathbf{z}) \,\mathrm{d}\nu_{\mathfrak{p},i} \end{aligned}$$

Note however that by Theorem 2.5.2, the last term is a solution to the PDE corresponding to $U\colon \mathfrak{p}^\infty$. Therefore by Ehrenpreis-Palamodov it has a representation as a sum of integrals involving $D_{\mathfrak{q},i}$ and $E_{\mathfrak{q},i}$ for all $\mathfrak{q} \subsetneq \mathfrak{p}$. However by the induction assumption, we don't need summands involving $E_{\mathfrak{q},i}$, so we can write

$$\begin{aligned} w(\mathbf{z}) =& \sum_{j=1}^{s_{\mathfrak{p}}} \int_{V(\mathfrak{p})} D_{\mathfrak{p},j}(\mathbf{x},\mathbf{z}) \exp(\mathbf{x}^T\cdot\mathbf{z}) \cdot c_j(\mathbf{x}) \,\mathrm{d}\nu_{\mathfrak{p},i} \\ &+ \sum_{\mathfrak{q}\subsetneq\mathfrak{p}} \sum_{j=1}^{s_{\mathfrak{q}}} \int_{V(\mathfrak{q})} D_{\mathfrak{q},j}(\mathbf{x},\mathbf{z}) \exp(\mathbf{x}^T\cdot\mathbf{z}) \,\mathrm{d}\eta_{\mathfrak{q},j}, \end{aligned}$$

hence the claim. □

**Example 2.7.5** ($n = 4, k = 2, l = 3$)**.** Let $U \subset R^2$ be the module generated by the rows of

$$M = \begin{bmatrix} \partial_1\partial_3 & \partial_1^2 \\ \partial_1\partial_2 & \partial_2^2 \\ \partial_1^2\partial_2 & \partial_1^2\partial_4 \end{bmatrix} \tag{2.13}$$

Computing $\mathrm{Sol}(M)$ means solving $\frac{\partial^2\psi_1}{\partial z_1\partial z_3} + \frac{\partial^2\psi_2}{\partial z_1^2} = \frac{\partial^2\psi_1}{\partial z_1\partial z_2} + \frac{\partial^2\psi_2}{\partial z_2^2} = \frac{\partial^3\psi_1}{\partial z_1^2\partial z_2} + \frac{\partial^3\psi_2}{\partial z_1^2\partial z_4} = 0.$ Two solutions are $\psi(\mathbf{z}) = \big(\phi(z_2, z_3, z_4), 0\big)$ and $\psi(\mathbf{z}) = \exp(s^2tz_1 + st^2z_2 + s^3z_3 + t^3z_4)\cdot \big(t, -s\big)$.

We apply Theorem 2.7.4 to derive the general solution to (2.13). The module $M$ has six associated primes, namely $\mathfrak{p}_1 = (\partial_1)$, $\mathfrak{p}_2 = (\partial_2, \partial_4)$, $\mathfrak{p}_3 = (\partial_2, \partial_3)$, $\mathfrak{p}_4 = (\partial_1, \partial_3)$, $\mathfrak{p}_5 = (\partial_1, \partial_2)$, and $\mathfrak{p}_6 = (\partial_1^2 - \partial_2\partial_3, \partial_1\partial_2 - \partial_3\partial_4, \partial_2^2 - \partial_1\partial_4)$. Four of them are minimal and two are embedded. If we let $m_i = \mathrm{mult}_I(\mathfrak{p}_i)$, we find that $m_1 = m_2 = m_3 = m_4 = m_6 = 1$ and $m_5 = 4$. A minimal primary decomposition

$$M \;=\; M_1 \cap M_2 \cap M_3 \cap M_4 \cap M_5 \cap M_6 \tag{2.14}$$

is given by the following primary submodules of $R^4$, each of which contains $M$:

$$M_1 = \mathrm{im}_R \begin{bmatrix} \partial_1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_2 = \mathrm{im}_R \begin{bmatrix} \partial_2 & \partial_4 & 0 & 0 & \partial_3 \\ 0 & 0 & \partial_2 & \partial_4 & \partial_1 \end{bmatrix}, \quad M_3 = \mathrm{im}_R \begin{bmatrix} \partial_2 & \partial_3 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$M_4 = \mathrm{im}_R \begin{bmatrix} \partial_3^5 & \partial_1 & 0 & 0 \\ 0 & \partial_2 & \partial_1 & \partial_3 \end{bmatrix}, \quad M_5 = \mathrm{im}_R \begin{bmatrix} \partial_1 & \partial_2^5 & 0 & 0 \\ 0 & 0 & \partial_1^2 & \partial_2^2 \end{bmatrix}, \quad M_6 = \mathrm{im}_R \begin{bmatrix} \partial_1 & \partial_2 & \partial_3 \\ \partial_2 & \partial_4 & \partial_1 \end{bmatrix}.$$

The number of differential operators $D_{i,j}(\mathbf{x}, \partial_\mathbf{x})$ needed in a minimal differential primary decomposition is $\sum_{i=1}^6 m_i = 9$. We choose them to be

$$B_{1,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; B_{2,1} = \begin{bmatrix} x_1 \\ -x_3 \end{bmatrix}, \; B_{3,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; B_{4,1} = \begin{bmatrix} x_2z_1 \\ -1 \end{bmatrix},$$

$$B_{5,i} = \begin{bmatrix} 0 \\ z_1 z_2 \end{bmatrix}, \begin{bmatrix} 0 \\ z_1 \end{bmatrix}, \begin{bmatrix} 0 \\ z_2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; B_{6,1} = \begin{bmatrix} x_4 \\ -x_2 \end{bmatrix}.$$

These nine vectors describe all solutions to our PDE. For instance, $B_{3,1}$ gives the solutions $\begin{bmatrix} \alpha(z_1, z_4) \\ 0 \end{bmatrix}$, and $B_{5,1}$ gives the solutions $\begin{bmatrix} 0 \\ z_1 z_2 \beta(z_3, z_4) \end{bmatrix}$, where $\alpha, \beta$ are bivariate functions. Furthermore $B_{1,1}$ and $B_{6,1}$ encode the two families of solutions mentioned after (2.13).

For the latter, we note that $V(\mathfrak{p}_6)$ is the surface in $\mathbb{C}^4$ with parametric representation $(x_1, x_2, x_3, x_4) = (s^2 t, st^2, s^3, t^3)$ for $s, t \in \mathbb{C}$. This surface is the cone over the twisted cubic curve, in the same notation as in [10, Section 1]. The kernel under the integral in (2.12) equals

$$\begin{bmatrix} x_4 \\ -x_2 \end{bmatrix} \exp\bigl(x_1 z_1 + x_2 z_2 + x_3 z_3 + x_4 z_4\bigr) = t^2 \begin{bmatrix} t \\ -s \end{bmatrix} \exp\bigl(s^2 t z_1 + st^2 z_2 + s^3 z_3 + t^3 z_4\bigr).$$

This is a solution to $M_6$, and hence to $M$, for any values of $s$ and $t$. Integrating the left hand side over $\mathbf{x} \in V(\mathfrak{p}_6)$ amounts to integrating the right hand side over $(s, t) \in \mathbb{C}^2$. Any such integral is also a solution. Ehrenpreis–Palamodov tells us that these are all the solutions. $\triangle$

# CHAPTER 3

# ALGORITHMS AND COMPUTATIONS

A fundamental problem in computer algebra is *primary decomposition*. Most currently implemented algorithms in computer algebra systems [15, 19, 22, 58] perform primary decomposition of submodules $U \subseteq R^k$ by producing a set of generators for each primary component.

In applications, one often does not have access to an exact representation of a problem, but rather some approximation with possible errors introduced by measurements or finite-precision arithmetic. The last decade of developments in *numerical algebraic geometry* [61, 62] provides tools for the numerical treatment of such polynomial models. In that paradigm, a prime ideal $\mathfrak{p} \subset \mathbb{C}[\mathbf{x}]$ is represented by a *witness set*, i.e. a set of $\deg(\mathfrak{p})$ points approximately on $V(\mathfrak{p}) \cap L$, where $L$ is a generic affine-linear space of dimension $c = \operatorname{codim}(\mathfrak{p})$. Similarly, radical ideals are collections of witness sets corresponding to irreducible components. Dealing with general, non-prime ideals and modules is much more subtle, since these have embedded primes that cannot be detected by witness sets. One idea, pioneered by Leykin [36], is to consider *deflations* of ideals. Deflation has the effect of exposing embedded and non-reduced components as isolated components, which can subsequently be represented using witness sets. One drawback is that the deflated ideal lies in a polynomial ring with many new variables.

Representing an ideal via Noetherian operators is particularly well suited to the framework of numerical algebraic geometry. Suppose $I \subseteq R$ is an unmixed ideal, i.e. an ideal with no embedded primes. A *numerical irreducible decomposition* [60] can be used to obtain witness sets for each irreducible component. We thus have a numerical representation of each associated prime, which together with a set of Noetherian operators—or a numerical representation thereof—provides an ideal membership test.

In this chapter, we present algorithms to compute Noetherian operators and differential primary decomposition. As was established in Chapter 1, the data of a differential primary decomposition fully determines an $R$-submodule $U \subseteq R^k$ and can be used to create a primary decomposition. Thus we represent a primary decomposition not by exhibiting module generators, but by simply returning a differential primary decomposition. Furthermore, a differential primary decomposition also describes all solution to the system of PDE represented by $U$, so our algorithms can also be thought of as PDE solvers.

The algorithms presented here are based on methods for ideals given in [8, 9, 10], and for modules in [7]. The first algorithm for computing Noetherian operators was developed by Oberst [48], and a subsequent developments by Damiano, Sabadini and Struppa [12] resulted in a Gröbner basis based implementation in the computer algebra system `CoCoA` [1], an implementation that has since become inaccessible. On drawback of the algorithm from Damiano et al. is that it has the restrictive assumption that the characteristic variety corresponds to the origin, which may not always be the case if the base field is not algebraically closed. Furthermore, as these early algorithms rely on the module being primary, using them in practice requires a primary decomposition as a pre-processing step. The output would then be a set of Noetherian operators for each primary component, which may not be a minimal representation of a module using differential operators.

This chapter will consist of two sections. In Section 3.1 we present a universally applicable, symbolic algorithm outputting a minimal differential primary decomposition for any $R$-submodule $U \subseteq R^k$.

Our second suite of algorithms is built with numerical computation in mind. Because of this, since numerical irreducible decompositions only sees the radical of an ideal, the second method is restricted to primary ideals $I$, or primary components $Q \supseteq I$ corresponding to isolated primes of $I$. We will describe a symbolic algorithm for computing Noetherian operators which can be turned into a numerical one by evaluating all polynomials at a generic point in $V(I)$. The output of the numerical algorithm will then be a set of "evaluated Noetherian operators", which by Theorem 2.5.2 correspond to instances of exponential solutions to the system of PDE. The numerical information obtained from the point evaluations of Noetherian operators can then be used as an ansatz to speed up the symbolic algorithm, thus yielding a "hybrid" algorithm.

All algorithms in this section, as well as other tools to compute dual spaces both numerically and symbolically, are implemented in the `Macaulay2` package `NoetherianOperators` [8, 35], distributed with the main `Macaulay2` distribution as of version `1.19`.

## 3.1 A general purpose algorithm

We present the algorithm for computing a differential primary decomposition developed in [7, 10]. The interpretation via solution sets of PDE is from [2]. There are no restrictions on the input and the output is minimal. The input is a submodule $U$ of $R^k$, where $R = K[x_1, \dots, x_n]$. The output is a differential primary decomposition $\{(\mathfrak{p}, \mathbf{t}_\mathfrak{p}, \mathcal{D}_\mathfrak{p})\}_{\mathfrak{p} \in \mathrm{Ass}(U)}$ of size $\mathrm{amult}(U)$ as in Theorem 1.7.14. A first step is to find $\mathrm{Ass}(U) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$. For each associated prime $\mathfrak{p}_i$, the elements $D(\mathbf{x}, \partial_\mathbf{x})$ in the finite set $\mathcal{D}_\mathfrak{p} \subset (W_{\kappa(\mathfrak{p}(\mathbf{t}_\mathfrak{p}))})^k$ are rewritten as rational functions $D(\mathbf{x}, \mathbf{z})$, using the identification of Theorem 2.5.2. Only the $\mathrm{codim}(\mathfrak{p}_i)$ many variables $z_i$ with $x_i \notin \mathbf{t}_\mathfrak{p}$ appear in these rational functions.

Our implementation of the algorithm is contained in the package `NoetherianOperators` in `Macaulay2`. The routine can be called using the command `solvePDE`, or alternatively via the command `differentialPrimaryDecomposition`. The user begins by fixing a polynomial ring $R = \mathbb{K}[x_1, \dots, x_n]$. Here $\mathbb{K}$ is usually the rational numbers `QQ`. Fairly arbitrary variable names $x_i$ are allowed. The argument of `solvePDE` is an ideal in $R$ or a submodule of $R^k$. The output is a list of pairs $\big\{(\mathfrak{p}_i, \{D_{i1}, \dots, D_{i,m_i}\})\big\}$ for $i = 1, \dots, s$, where $\mathfrak{p}_i$ is a prime ideal given by generators in $R$, and each $D_{ij}$ is a vector over a newly created polynomial ring $\mathbb{K}[x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n}]$. The set of independent variables $\mathbf{t}_{\mathfrak{p}_i}$ for each $i$ is implicit, but can be recovered in `Macaulay2` using the command `firstindependentSets`$\mathfrak{p}_i$. In `Macaulay2`, the variables $\partial_{x_i}$ are represented by the symbol $\mathtt{d}x_i$. To be precise, each new variable is created from an old variable by prepending the character $\mathtt{d}$. Substituting the symbol $\mathtt{d}x_i$ by a new variable $z_i$ in each $D_{ij}$ produces the multipliers $D_{ij}(\mathbf{x}, \mathbf{z})$ used in the Ehrenpreis-Palamodov fundamental theorem, Theorem 2.7.4. Thus the output can be thought of as the set of solutions to the associated PDE.

Each $D_{ij}$ in the output of `solvePDE` encodes an exponential solution $D_{ij}(\mathbf{x}, \mathbf{z})\exp(\mathbf{x}^T \cdot \mathbf{z})$ to $U$. Here $\mathbf{x}$ are the old variables chosen by the user, and $\mathbf{x}$ denotes points in the irreducible variety $V(\mathfrak{p}_i) \subseteq \mathbb{C}^n$. The solution is a function in the new unknowns $\mathbf{z} = (\mathtt{d}x_1, \ldots, \mathtt{d}x_n)$. For instance, if $n = 3$ and the input is in the ring $\mathtt{QQ}[\mathtt{u}, \mathtt{v}, \mathtt{w}]$ then the output lives in the ring $\mathtt{QQ}[\mathtt{u}, \mathtt{v}, \mathtt{w}, \mathtt{du}, \mathtt{dv}, \mathtt{dw}]$. Each solution to the PDE is a function $\psi(\mathtt{du}, \mathtt{dv}, \mathtt{dw})$ and these functions are parametrized by a variety $V(\mathfrak{p}_i)$ in a 3-space whose coordinates are $(\mathtt{u}, \mathtt{v}, \mathtt{w})$.

We now demonstrate how this works for three examples.

**Example 3.1.1.** Consider the ODE $u''' + 3u'' - 9u' + 5u = 0$, where $u = u(z)$ is some unknown function. This translates to the differential operator $p(x) = x^3 + 3x^2 - 9x + 5 = (x-1)^2(x+5)$. The ideal $I = (p)$ has two primary components, one of which has multiplicity 2 and the other has multiplicity 1. We solve the PDE as follows:

```
R = QQ[x];
I = ideal( x^3 + 3*x^2 - 9*x + 5 );
solvePDE(I)
  {{ideal(x - 1), {| 1 |, | dx |}}, {ideal(x + 5), {| 1 |}}}
```

The first three lines are the input. The last line is the output created by `solvePDE`. This list of $s = 2$ pairs encodes the general solution $u(z)$, which we can write as

$$\begin{aligned} u(z) =& \int_{x=1} e^{xz}\,\mathrm{d}\mu_1(x) + \int_{x=1} ze^{xz}\,\mathrm{d}\mu_2(x) + \int_{x=-5} e^{xz}\,\mathrm{d}\mu_3(x) \\ =& c_1 e^z + c_2 z e^z + c_3 e^{-5z}, \end{aligned}$$

for some constants $c_1, c_2, c_3 \in \mathbb{C}$. $\triangle$

**Example 3.1.2.** Consider the one-dimensional wave equation $\phi_{tt}(z,t) = c^2\phi_{zz}(z,t)$. If we identify $\partial_z \leftrightarrow u$ and $\partial_t \leftrightarrow v$, we encode the PDE as the ideal $I = (v^2 - c^2u^2) \subseteq \mathbb{C}[u,v]$. For $c = 3$, we solve the wave equation as follows:

```
R = QQ[u,v]; c = 3; I = ideal(v^2-c^2*u^2); solvePDE(I)
  {{ideal(3u - v), {| 1 |}}, {ideal(3u + v), {| 1 |}}}
```

Applying Theorem 2.7.4, we get the wave

$$\phi(z,t) = \phi_1(3t+z) + \phi_2(3t-z),$$

as a general solution, where $\phi_1, \phi_2$ are univariate smooth functions or distributions. More general waves will be revisited in Chapter 4. $\triangle$

**Example 3.1.3.** Consider the linear PDE

$$\alpha_{xx} + \beta_{xy} = \alpha_{yz} + \beta_{zz} = \alpha_{xxz} + \beta_{xyw} = 0.$$

We wish to find the unknown function pair $(\alpha, \beta)\colon \mathbb{R}^4 \to \mathbb{C}^2$. The PDE can be encoded as the $(\ell \times k) = (3 \times 2)$ matrix

$$\begin{bmatrix} \partial_x^2 & \partial_x\partial_y \\ \partial_x\partial_y & \partial_z^2 \\ \partial_x^2\partial_z & \partial_x\partial_y\partial_w \end{bmatrix}, \tag{3.1}$$

whose entries are in the polynomial ring $R = \mathbb{C}[\partial_x, \partial_y, \partial_z, \partial_w]$. The corresponding module $U \subseteq R^2$ is generated by the three rows of the matrix. We solve the PDE by defining our module $U$, and running `solvePDE`. Here the variable names `x1`, `x2`, `x3`, `x4` will represent $\partial_x, \ldots, \partial_z$.

```
R = QQ[x1,x2,x3,x4];
U = image matrix {{x1^2,x2*x3,x1^2*x3},{x1*x2,x3^2,x1*x2*x4}};
solvePDE(U)
```

Note that in `Macaulay2`, the command `image` constructs the module given by the *columns* of a matrix, so we need to transpose our matrix in (3.1). The output is a list of six pairs of associated primes along with generators of their excess dual spaces. Interpreting the output through the fundamental principle, we obtain a general solution with nine summands, labelled $a, b, \dots, h$ and $(\tilde{\alpha}, \tilde{\beta})$:

$$\begin{aligned}\alpha \;&=\; a_z(y,z,w) - b_y(x,y) + c(y,w) + xd(y,w) + xg(z,w) - xyh_z(z,w) + \tilde{\alpha}(x,y,z,w),\\ \beta \;&=\; \quad -a_y(y,z,w) \,+\, b_x(x,y) \,+\, e(x,w) \,+\, zf(x,w) \,+\, xh(z,w) \,+\tilde{\beta}(x,y,z,w).\end{aligned} \tag{3.2}$$

Here, $a$ is any function in three variables, $b, c, d, e, f, g, h$ are functions in two variables, and $\tilde{\psi} = (\tilde{\alpha}, \tilde{\beta})$ is any function $\mathbb{R}^4 \to \mathbb{C}^2$ that satisfies the following four linear PDE of first order:

$$\tilde{\alpha}_x + \tilde{\beta}_y \;=\; \tilde{\alpha}_y + \tilde{\beta}_z \;=\; \tilde{\alpha}_z - \tilde{\alpha}_w \;=\; \tilde{\beta}_z - \tilde{\beta}_w \;=\; 0. \tag{3.3}$$

We note that all solutions to (3.3) also belong to $\mathrm{Sol}(U)$, and they admit the integral representation

$$\tilde{\alpha} = \int t(\exp(s^2x + sty + t^2(z+w)))d\mu(s,t)\,, \;\; \tilde{\beta} = -\int s(\exp(s^2x + sty + t^2(z+w)))d\mu(s,t),$$

where $\mu$ is a suitably chosen measure on $\mathbb{C}^2$. $\triangle$

The method in `solvePDE` is described in Algorithm 1 below. A key ingredient is a translation map. We now explain this in the simplest case, when the module is supported in one point. Suppose $V(U) = \{\mathbf{u}\}$ for some $\mathbf{u} \in K^n$. We set $\mathfrak{m}_{\mathbf{u}} = \langle x_1 - u_1, \dots, x_n - u_n \rangle$. Noetherian operators for such modules can always be chosen with constant coefficients in the base field, since $\kappa(\mathfrak{m}_{\mathbf{u}}) = \mathbb{K}$. Let

$$\gamma_{\mathbf{u}} : R \to R\,, \;\; x_i \mapsto x_i + u_i \quad \text{for } \; i = 1, \ldots, n. \tag{3.4}$$

The following two results are straightforward. We will later use them when $U$ is any primary module, $\mathbf{u}$ is the generic point of $V(U)$, and $\mathbb{F} = K(\mathbf{u})$ is the associated field extension of $\mathbb{K}$.

**Proposition 3.1.4.** *A constant coefficient operator $D(\partial_{\mathbf{x}})$ is a Noetherian operator for the $\mathfrak{m}_{\mathbf{u}}$-primary module $U$ if and only if $D(\partial_{\mathbf{x}})$ is a Noetherian operator for the $\mathfrak{m}_0$-primary module $\hat{U} := \gamma_{\mathbf{u}}(U)$. Dually, $D(\mathbf{z})\exp(\mathbf{u}^t\mathbf{z})$ is in $\mathrm{Sol}(U)$ if and only if the polynomial $D(\mathbf{z})$ is in $\mathrm{Sol}(\hat{U})$.*

This observation reduces the computation of solutions for a primary module to finding the polynomial solutions of the translated module. Next, we bound the degrees of these polynomials.

**Proposition 3.1.5.** *Let $\hat{U} \subseteq R^k$ be an $\mathfrak{m}_0$-primary module. There exists an integer $r$ such that $\mathfrak{m}_0^{r+1} R^k \subseteq \hat{U}$. The space $\mathrm{Sol}(\hat{U})$ consists of $k$-tuples of polynomials of degree $\leq r$.*

Propositions 3.1.4 and 3.1.5 furnish a method for computing solutions of an $\mathfrak{m}_{\mathbf{u}}$-primary module $U$. We start by translating $U$ so that it becomes the $\mathfrak{m}_0$-primary module $\hat{U}$. The integer $r$ provides an ansatz $\sum_{j=1}^{k} \sum_{|\alpha| \leq r} c_{\alpha,j}\, \mathbf{z}^\alpha e_j$ for the polynomial solutions, where $e_1, \ldots, e_k$ are the standard basis vectors. The coefficients $c_{\alpha,j}$ are computed by linear algebra over the ground field $\mathbb{K}$. Here are the steps:

1. Let $r$ be the smallest integer such that $\mathfrak{m}_0^{r+1} R^k \subseteq \hat{U}$.

2. Let $\mathrm{M}(\hat{U})$ be the matrix whose entries are the polynomials $\hat{m}_i \bullet (\mathbf{z}^\alpha e_j) \in R$. The row labels are the generators $\hat{m}_1, \ldots, \hat{m}_l$ of $\hat{U}$, and the column labels are the $\mathbf{z}^\alpha e_j$.

3. Let $\ker_K(\mathrm{M}(\hat{U}))$ denote the $K$-linear subspace of the $R$-module $\ker_R(\mathrm{M}(\hat{U}))$ consisting of vectors $(v_{\alpha,j})$ with all entries in $K$. Every such vector gives a solution

$$\sum_{j=1}^{k} \sum_{|\alpha|\leq r} v_{\alpha,j}\, \mathbf{z}^{\alpha} \exp(\mathbf{u}^t \mathbf{z})\, e_j \;\in\; \mathrm{Sol}(U). \tag{3.5}$$

**Example 3.1.6.** $[n = k = r = 2, \ell = 3]$ The following module is $\mathfrak{m}_0$-primary of multiplicity three:

$$U \;=\; \mathrm{image}_R \begin{bmatrix} \partial_1^3 & \partial_2 - c_1\partial_1^2 - c_2\partial_1 & c_3\partial_1^2 + c_4\partial_1 + c_5 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.6}$$

Here $c_1, c_2, c_3, c_4, c_5$ are arbitrary constants in $K$. The matrix $\mathrm{M}(U)$ has three rows, one for each generator of $U$, and it has $12$ columns, indexed by $e_1, z_1e_1, \ldots, z_2^2e_1, e_2, z_1e_2, \ldots, z_2^2e_2$. The space $\ker_K(\mathrm{M}(U))$ is 3-dimensional. A basis furnishes the three polynomial solutions

$$\begin{bmatrix} -1 \\ c_5, \end{bmatrix}, \begin{bmatrix} -(z_1 + c_2z_2) \\ c_5z_1 + c_2c_5z_2 + c_4 \end{bmatrix}, \begin{bmatrix} -((z_1 + c_2z_2)^2 + 2c_1z_2) \\ c_5(z_1{+}c_2z_2)^2 + 2c_4z_1 + 2(c_1c_5{+}c_2c_4)z_2 + 2c_3 \end{bmatrix}. \tag{3.7}$$

△

We now turn to Algorithm 1. The input is a polynomial module $U \subseteq R^k$, and the output is a differential primary decomposition, representing all solutions to the PDE $U$ by Theorem 2.7.4. The method was introduced in [7, Algorithm 4.6]; we explain it here in the context of solving PDE. It is implemented in `Macaulay2` under the command `solvePDE`. In our discussion, the line numbers refer to the corresponding lines of pseudocode in Algorithm 1.

**Line 1** We begin by finding all associated primes of $U$. By [19, Theorem 1.1], the associated primes of codimension $i$ coincide with the minimal primes of $\mathrm{Ann}\,\mathrm{Ext}^i_R(U, R)$. This reduces the problem of finding associated primes of a module to the more fa-

**Algorithm 1** SolvePDE

**Input:** An arbitrary submodule $U$ of $R^k$

**Output:** List of associated primes with corresponding Noetherian multipliers.

**for** each associated prime ideal $\mathfrak{p}$ of $U$ **do**

  $W \leftarrow UR^k_{\mathfrak{p}} \cap R^k$

  $V \leftarrow (W : \mathfrak{p}^\infty)$

  $r \leftarrow$ the smallest number such that $V \cap \mathfrak{p}^{r+1}R^k$ is a subset of $W$

  $\mathfrak{t}_{\mathfrak{p}} \leftarrow$ a maximal set of independent variables modulo $\mathfrak{p}$

  $\mathbb{F} \leftarrow \mathrm{Frac}(R/\mathfrak{p}) = \kappa(\mathfrak{p})$

  $T \leftarrow \mathbb{F}[y_i \colon x_i \notin \mathfrak{t}_{\mathfrak{p}}]$

  $\gamma \leftarrow$ the map defined in (3.8)

  $\mathfrak{m} \leftarrow$ the irrelevant ideal in $T$

  $\hat{W} \leftarrow \gamma(W) + \mathfrak{m}^{r+1}T^k$

  $\hat{V} \leftarrow \gamma(V) + \mathfrak{m}^{r+1}T^k$

  $N \leftarrow$ a $\mathbb{F}$-vector space basis of the space of $k$-tuples of polynomials of degree $\leq r$

  $\mathrm{M}(\hat{W}) \leftarrow$ the matrix given by the $\bullet$-product of generators of $\hat{W}$ with elements of $N$

  $\mathrm{M}(\hat{V}) \leftarrow$ the matrix given by the $\bullet$-product of generators of $\hat{V}$ with elements of $N$

  $\mathcal{K} \leftarrow \ker_{\mathbb{F}}(\mathrm{M}(\hat{W}))/\ker_{\mathbb{F}}(\mathrm{M}(\hat{V}))$

  $\mathcal{A} \leftarrow$ a $\mathbb{F}$-vector space basis of $\mathcal{K}$

  $\mathcal{D}_{\mathfrak{p}} \leftarrow$ lifts of the vectors in $\mathcal{A}$ to vectors in $\ker_{\mathbb{F}}(M(\hat{W}))$

  **return** the pair $(\mathfrak{p}, \mathcal{D}_{\mathfrak{p}})$

miliar problem of finding minimal primes of a polynomial ideal. This method is implemented and distributed with `Macaulay2` starting from version 1.17 via the command `associatedPrimes R^k/U`. See [7, Section 2].

The remaining steps are repeated for each $\mathfrak{p} \in \operatorname{Ass}(U)$. For a fixed associated prime $\mathfrak{p}$, our goal is to identify the contribution to $\operatorname{Sol}(U)$ of the $\mathfrak{p}$-primary component of $U$.

**Lines 2–3** To achieve this goal, we study solutions for two different $R$-submodules of $R^k$. The first one, denoted $W$, is the intersection of all $\mathfrak{p}_i$-primary components of $U$, where $\mathfrak{p}_i$ are the associated primes contained in $\mathfrak{p}$. Thus $W = UR^k_{\mathfrak{p}} \cap R^k$, which is the extension-contraction module of $U$ under localization at $\mathfrak{p}$, i.e. the $\mathfrak{p}_i$-closure of $U$. It is computed as $W = (U : f^\infty)$, where $f \in R$ is contained in every associated prime $\mathfrak{p}_j$ *not* contained in $\mathfrak{p}$.

The second module, denoted $V$, is the intersection of all $\mathfrak{p}_i$-primary components of $U$, where $\mathfrak{p}_i$ is *strictly* contained in $\mathfrak{p}$. Hence $V = (W : \mathfrak{p}^\infty)$ is the saturation of $W$ at $\mathfrak{p}$. We have $W = V \cap Q$, where $Q$ is a $\mathfrak{p}$-primary component of $U$. Thus the difference between the solution spaces $\operatorname{Sol}(W)$ and $\operatorname{Sol}(V)$ is caused by the primary module $Q$, and this is captured by the excess dual space.

When $\mathfrak{p}$ is a minimal prime, $W$ is the unique $\mathfrak{p}$-primary component of $U$, and $V = R^k$.

**Line 4** The integer $r$ bounds the degree of the dual space elements associated to $W$ but not to $V$. This means that if the function $\phi(\mathbf{z}) = D(\mathbf{x}, \mathbf{z}) \exp(\mathbf{x}^T \cdot \mathbf{z})$ lies in $\operatorname{Sol}(W) \setminus \operatorname{Sol}(V)$ for all $\mathbf{x} \in V(\mathfrak{p})$, then the $\mathbf{z}$-degree of the polynomial $D(\mathbf{x}, \mathbf{z})$ is at most $r$. This will lead to an ansatz for the exponential solutions responsible for the difference between $\operatorname{Sol}(W)$ and $\operatorname{Sol}(V)$.

**Lines 5–8** The modules $W$ and $V$ are reduced to simpler modules $\hat{W}$ and $\hat{V}$ with similar properties. Namely, $\hat{W}$ and $\hat{V}$ are primary and their characteristic varieties are the origin. This reduction involves two new ingredients: a new polynomial ring $T$ in fewer variables over the field $\mathbb{F} = \kappa(\mathfrak{p})$, a finite extension of $\mathbb{K}$, and a ring map $\gamma\colon R \to T$.

Fix a maximal set $\mathtt{t}_\mathfrak{p} = \{x_{i_1}, \dots, x_{i_{n-c}}\}$ with $\mathfrak{p} \cap K[x_{i_1}, \dots, x_{i_{n-c}}] = \{0\}$. We define $T := \mathbb{F}[y_i : x_i \notin \mathtt{t}_\mathfrak{p}]$. This is a polynomial ring in $n - |\mathtt{t}_\mathfrak{p}| = c$ new variables $y_i$, corresponding to the $x_i$ not in the set $\mathtt{t}_\mathfrak{p}$ of independent variables. Writing $u_i$ for the image of $x_i$ in $\mathbb{F}$, the ring map $\gamma$ is defined as follows:

$$\gamma\colon R \to T, \quad x_i \mapsto \begin{cases} y_i + u_i, & \text{if } x_i \notin S, \\ u_i, & \text{if } x_i \in S. \end{cases} \tag{3.8}$$

By abuse of notation, we denote by $\gamma$ the extension of (3.8) to a map $R^k \to T^k$.

**Lines 9–11** Let $\mathfrak{m} := \langle y_i \colon x_i \notin \mathtt{t}_\mathfrak{p} \rangle$ be the irrelevant ideal of $T$. We define the $T$-submodules

$$\hat{W} := \gamma(W) + \mathfrak{m}^{r+1}T^k \quad \text{and} \quad \hat{V} := \gamma(W) + \mathfrak{m}^{r+1}T^k \quad \text{of } T^k.$$

These modules are $\mathfrak{m}$-primary: their solutions are finite-dimensional $\mathbb{F}$-vector spaces consisting of polynomials of degree $\leq r$. The polynomials in $\operatorname{Sol}(\hat{W}) \setminus \operatorname{Sol}(\hat{V})$ capture the difference between $\hat{W}$ and $\hat{V}$, and also the difference between $W$ and $V$ after lifting.

**Lines 12–14** We construct matrices $\mathrm{M}(\hat{W})$ and $\mathrm{M}(\hat{V})$ with entries in $\mathbb{F}[z_i \colon x_i \notin \mathtt{t}_\mathfrak{p}]$. As in (3.5), their kernels over $\mathbb{K}$ correspond to polynomial solutions of $\hat{W}$ and $\hat{V}$. The set $N = \{\mathbf{z}^\alpha e_j \colon |\alpha| \leq r, j = 1, \dots, k\}$ is a $\mathbb{F}$-basis for elements of degree $\leq r$ in $\mathbb{F}[z_i \colon x_i \notin \mathtt{t}_\mathfrak{p}]^k$. The $y_i$-variables act on the $z_i$ variables as partial derivatives, i.e. $y_i = \frac{\partial}{\partial z_i}$. We define the matrix $\mathrm{M}(\hat{W})$ as follows. Let $\hat{W}_1, \dots, \hat{W}_\ell$ be generators of $\hat{W}$. The rows of $\mathrm{M}(\hat{W})$ are indexed by these generators, the columns are indexed by $N$, and the entries are the polynomials $\hat{W}_i \bullet \mathbf{z}^\alpha e_j$. In the same way we construct $\mathrm{M}(\hat{V})$.

**Lines 15–16** Let $\ker_{\mathbb{F}}(\mathrm{M}(\hat{W}))$ be the space of vectors in the kernel of $\mathrm{M}(\hat{W})$ whose entries are in $\mathbb{F}$. The $\mathbb{F}$-vector space $\ker_{\mathbb{F}}(\mathrm{M}(\hat{W}))$ parametrizes the polynomial solutions

$$\sum_{j=1}^{k} \sum_{|\alpha| \le r} v_{\alpha,j} \, \mathbf{z}^\alpha e_j \ \in \ \mathrm{Sol}(\hat{W}).$$

The same holds for $\hat{V}$. The quotient space $\mathcal{K} := \ker_{\mathbb{F}}(\mathrm{M}(\hat{W}))/\ker_{\mathbb{F}}(\mathrm{M}(\hat{V}))$ characterizes excess solutions in $\mathrm{Sol}(\hat{W})$ relative to $\mathrm{Sol}(\hat{V})$. Under the duality in Theorem 2.5.2, the space $\mathcal{K}$ is precisely the excess dual space. Write $\mathcal{A}$ for a $\mathbb{F}$-basis of $\mathcal{K}$.

**Lines 17–18** We interpret $\mathcal{A}$ as a set of differential operators for $U$ as follows. For each element $\bar{\mathbf{v}} \in \mathcal{A}$, we choose a representative $\mathbf{v} \in \ker_{\mathbb{F}}(\mathrm{M}(\hat{W}))$. The entries of $\mathbf{v}$ are in $\mathbb{F} = \kappa(\mathfrak{p})$. The differential operator in $W^k_{\kappa(\mathfrak{p}^{(\mathbf{t}_\mathfrak{p})})}$ corresponding to $\mathbf{v}$ is the following vector

$$D(\mathbf{x}, \partial_{\mathbf{x}}) \ = \ \sum_{j=1}^{k} \sum_{|\alpha| \le r} u_{\alpha,j}(\mathbf{x}) \, \partial_{\mathbf{x}}^\alpha e_j.$$

Applying the above procedure to each each $\bar{\mathbf{v}} \in \mathcal{A}$ yields a set $\mathcal{D}_\mathfrak{p}$ of differential operators which spans the excess dual space. By Theorem 2.7.4, these operators describe the contribution of a $\mathfrak{p}$-primary component of $U$ to $\mathrm{Sol}(U)$.

The output of Algorithm 1 is a list of pairs $(\mathfrak{p}, \mathcal{D}_\mathfrak{p})$, where $\mathfrak{p}$ ranges over $\mathrm{Ass}(U)$ and $\mathcal{D}_\mathfrak{p} = \{D_1, \ldots, D_m\}$ is a subset of $W_{\kappa(\mathfrak{p})}$. The cardinality $m$ is the multiplicity of $U$ along $\mathfrak{p}$. The output describes the solutions to the PDE given by $U$. Consider the functions

$$\phi_\mathfrak{p}(z_1, \ldots, z_n) \ = \ \sum_{i=1}^{m} \int_{V(\mathfrak{p})} D_i(\mathbf{x}, \mathbf{z}) \exp(x_1 \, z_1 + \cdots + x_n z_n) \, d\mu_i(\mathbf{x}).$$

Then the space of solutions to $U$ consists of all functions

$$\sum_{\mathfrak{p}\in \mathrm{Ass}(U)} \phi_{\mathfrak{p}}(z_1,\dots,z_n).$$

The output also describes a differential primary decomposition. Indeed, the command `differentialPrimaryDecomposition` described in [7] is identical to our command `solvePDE`. All examples in [7, Section 6] can be interpreted as solving PDE.

## 3.2 Primary ideals via Macaulay matrices

### 3.2.1 Symbolic version

In this subsection, we present algorithms to symbolically compute bases for local dual spaces. The method is an adaptation of the classical theory of Macaulay inverse systems involving *Macaulay matrices*. The method outlined below will only work for finite dimensional local dual spaces, i.e. for ideals $I$ that are either $\mathfrak{p}$-primary, or for which $\mathfrak{p}$ is an isolated prime. This is because if $\mathfrak{p}$ is a minimal prime of $I$, then the local dual space of $I$ is equal to the local dual space of $I_{\mathfrak{p}} \cap R$, which is just the $\mathfrak{p}$-primary component of $I$.

We will assume that $I \subseteq \mathbb{K}[x_1,\dots,x_n]$ is a $\mathfrak{p}$-primary ideal, where $\mathfrak{p}$ is a zero-dimensional prime; if not, substitute $I, \mathfrak{p}$ by $I^{(\mathbf{t})}$ and $\mathfrak{p}^{(\mathbf{t})}$ as in Section 1.5. We define the *degree $d$ truncated local dual spaces* as

$$D_{\mathfrak{p}}^{(d)}[I] := \{D \in D_{\mathfrak{p}}[I] \mid \deg(D) \le d\} = D_{\mathfrak{p}}[I] \cap W_{\kappa(\mathfrak{p})}^{(d)} = D_{\mathfrak{p}}[I + \mathfrak{p}^{d+1}],$$

where the last equality follows from Lemma 1.3.1.

Fix a degree $d$, and let $C := \{\partial^\beta \mid |\beta| \le d\} \subseteq W_{\kappa(\mathfrak{p})}$, the set of all $\partial$-monomials of degree at most $d$. Pick a generating set $\{f_1,\dots,f_r\}$ for $I$, and let $F := \{\mathbf{x}^\alpha f_i \mid i = 1,\dots,r,\ |\alpha| < d\}$. For a fixed total ordering $\prec$ on $\partial$-monomials, we define the *degree $d$ Macaulay matrix $M$* of dimension $|F| \times |C|$, where the rows are indexed by $F$, and the columns are indexed by $C$ and ordered according to $\prec$. The entry corresponding to the row $\mathbf{x}^\alpha f_i$ and column $\partial^\beta$ of the Macaulay matrix is the value

$$M_{\alpha,i;\beta} = \partial^\beta \bullet (\mathbf{x}^\alpha f_i) \in \kappa(\mathfrak{p}).$$

Any $D = \sum_{|\beta|\leq d} v_\beta \partial^\beta \in W_{\kappa(\mathfrak{p})}$ is specified by its coefficient (column) vector $v = (v_\beta)_\beta$. Every entry of $Mv$ is of the form $D \bullet g$ for some $g \in I$, so every element in the truncated local dual space $D_{\mathfrak{p}}^{(d)}[I]$ corresponds to a vector in the kernel of the Macaulay matrix.

**Proposition 3.2.1.** *With notation as above, let $\{v^{(k)}\}_k$ be a basis of the kernel of the degree $d$ Macaulay matrix, and let $D_k := \sum_\beta v_\beta^{(k)} \partial^\beta$. Then $\{D_k\}_k$ is a basis for the truncated local dual space $D_{\mathfrak{p}}^{(d)}[I]$, i.e. a set of Noetherian operators for $I + \mathfrak{p}^{d+1}$.*

*Proof.* Let $D \in D_{\mathfrak{p}}^{(d)}[I]$. We can write $D = \sum_{|\beta|\leq d} v_\beta \partial^\beta$ for some vector $v = (v_\beta)_\beta$. Clearly $v \in \ker M$, so $v = \sum_k c_k v^{(k)}$, which implies $D = \sum_k c_k D_k$.

Conversely, we must show that $D_k$ is in $D_P^{(d)}[I]$ for each $k$. As in Section 1.3, let $S = \kappa(\mathfrak{p}) \otimes_{\mathbb{K}} R$, and $\mathfrak{n} \subseteq S$ be a maximal ideal corresponding to a rational point $p = (p_1, \ldots, p_n) \in \kappa(\mathfrak{p})^n$ such that $\mathfrak{n} \cap R = \mathfrak{p}$. We use Lemma 1.3.4 and Theorem 1.3.5 to relate $D_{\mathfrak{p}}[I]$ and $D_{\mathfrak{n}}[IS]$. The set

$$\{\mathbf{x}^\alpha f_i \mid |\alpha| < d,\ i = 1, \ldots, r\} \cup \{(\mathbf{x} - p)^\beta f_i \mid |\beta| \geq d,\ i = 1, \ldots, r\}$$

spans $IS$ as a $\kappa(\mathfrak{p})$-vector space. By construction, $D_k \bullet (\mathbf{x}^\alpha f_i) = 0$ for all $|\alpha| < d$. Note that each $f_i$ vanishes at $p$, so $f_i \in \mathfrak{n}$. For each $j$, the term $x_j - p_j$ is also in $\mathfrak{n}$. If $|\beta| \geq d$ then $(\mathbf{x} - p)^\beta f_i \in \mathfrak{n}^{d+1}$. Since the $\partial$-degree of $D_k$ is at most $d$, $D_k \in D_{\mathfrak{n}}[\mathfrak{n}^{d+1}]$, so $D_k \bullet ((\mathbf{x} - p)^\beta f_i) = 0$. Therefore $D_k \in D_{\mathfrak{n}}[IS]$, so $D_k \in D_{\mathfrak{p}}[I]$. □

It is clear that $D_{\mathfrak{p}}^{(1)}[I] \subseteq D_{\mathfrak{p}}^{(2)}[I] \subseteq \cdots$, and since the local dual space is finite dimensional, this chain will stabilize to $D_{\mathfrak{p}}[I]$ after a finite number of steps, namely at the step when $\mathfrak{p}^{d+1} \subseteq I$. Furthermore, as the $D_p^{(d)}[I]$ are right $R$-modules, the chain stabilizes as soon as the dimension stops increasing, that is when $\dim_{\kappa(P)} D_P^{(d)}[I] = \dim_{\kappa(P)} D_P^{(d+1)}[I]$. Thus we get a termination criterion which doesn't require *a priori* computation of the power $d$ such that $\mathfrak{p}^d \subseteq I$.

The procedure above is summarized in Algorithm 2, which computes Noetherian operators for the $\mathfrak{p}$-primary component of $I$ via kernels of successively larger Macaulay matrices. Each kernel element is represented by a vector with entries in $\kappa(\mathfrak{p})$, which can be converted to an operator in $W_{\kappa(\mathfrak{p})}$. The algorithm computes the local dual space, and then constructs Noetherian operators from a basis thereof, so the output Noetherian operators will depend on a choice of basis of the local dual space. In our `Macaulay2` implementation, we always choose a basis in reduced column echelon form.

**Algorithm 2** Compute Noetherian operators symbolically in dimension zero

**Require:** $I = (f_1, \dots, f_r)$ a zero-dimensional ideal, $\mathfrak{p}$ a minimal prime of $I$, $\prec$ an ordering on monomials $\partial^\beta$

**Ensure:** A set of Noetherian operators for the $\mathfrak{p}$-primary component of $I$

```
procedure NoetherianOperatorsZero(I, p)
    K ← ∅
    d ← 0                                          ▷ d corresponds to the degree bound
    repeat
        d ← d + 1
        F ← vector with entries x^α f_i, where |α| < d, i = 1, 2, ..., r
        C ← vector with entries ∂^β = ∂_{x_1}^{β_1} ··· ∂_{x_n}^{β_n}, where |β| ≤ d, in the order ≺
        M ← the Macaulay matrix with entries ∂^β • (x^α f_i) (rows indexed by F, columns by C)
        K_d ← ker M
    until dim K_d = dim K_{d−1}                  ▷ Stop when the dimension of the kernel stabilizes
    K ← ColReduce(K_d)              ▷ Rewrites the generators of K_d in a reduced column echelon form
    return C^T K, a row vector of Noetherian operators in W_{κ(p)}
```

For the general case, if $I$ is positive-dimensional, we reduce to the zero dimensional case by choosing a maximal set of $\mathfrak{t}$ of independent variables over $\mathfrak{p}$, yielding Algorithm 3.

*Remark* 3.2.2. Algorithm 2 describes how to find dual spaces (and therefore Noetherian operators) using Macaulay matrices. This is not the only dual space algorithm. We present it here because it is the most general and simplest to describe. The algorithm of [43]

**Algorithm 3** Compute Noetherian operators symbolically in positive dimension

**Require:** $I \subseteq \mathbb{K}[\mathbf{t}, \mathbf{y}]$ an ideal, where $\mathbf{t}, \mathbf{y}$ are independent and dependent variables for $I$ respectively, $\mathfrak{p}$ a minimal prime of $I$, $\prec$ an ordering on monomials $\partial_{\mathbf{y}}^{\beta}$

**Ensure:** A set of Noetherian operators for the $\mathfrak{p}$-primary component of $I$

1: **procedure** NOETHERIANOPERATORS($I, \mathfrak{p}$)

2: **return** NOETHERIANOPERATORSZERO($I^{(\mathbf{t})}, \mathfrak{p}^{(\mathbf{t})}$).

instead uses antidifferentiation to find dual space basis elements of each degree from the previous degree elements, and it has better run time when the dimension of the dual space in each degree is low. That paper focuses on the case when the coefficient field is $\mathbb{C}$ but the algorithm can be applied any time the prime $\mathfrak{p}$ is a rational point. We do not know of a way to generalize it to nonrational points. In our code in the `NoetherianOperators` package, the default strategy is antidifferentiation when the point is rational and Macaulay matrices when it is not. △

**Example 3.2.3.** Consider the 1-dimensional primary ideal $Q = ((x_1^2 - x_3)^2, x_2 - x_3(x_1^2 - x_3)) \subseteq R = \mathbb{Q}[x_1, x_2, x_3]$. Its radical is $\mathfrak{p} = (x_1^2 - x_3, x_2)$, and we may choose $x_1, x_2$ as the dependent variables and $x_3$ as the independent variable. Thus in $R^{(\mathbf{t})} = \mathbb{Q}(x_3)[x_1, x_2]$, $Q^{(\mathbf{t})}$ is a zero-dimensional primary ideal whose radical is $\mathfrak{p}^{(\mathbf{t})}$. In degree 1, the Macaulay matrix has a 2-dimensional kernel. In degree 2, the Macaulay matrix is

$$M = \begin{array}{r} \\ {\scriptstyle (x_1^2-x_3)^2} \\ {\scriptstyle (x_2-x_3(x_1^2-x_3))} \\ {\scriptstyle x_1(x_1^2-x_3)^2} \\ {\scriptstyle x_1(x_2-x_3(x_1^2-x_3))} \\ {\scriptstyle x_2(x_1^2-x_3)^2} \\ {\scriptstyle x_2(x_2-x_3(x_1^2-x_3))} \end{array} \begin{array}{c} \begin{array}{cccccc} {\scriptstyle 1} & {\scriptstyle \partial_{x_1}} & {\scriptstyle \partial_{x_2}} & {\scriptstyle \partial_{x_1}^2} & {\scriptstyle \partial_{x_1}\partial_{x_2}} & {\scriptstyle \partial_{x_2}^2} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 8x_3 & 0 & 0 \\ 0 & -2x_3x_1 & 1 & -2x_3 & 0 & 0 \\ 0 & 0 & 0 & 8x_3x_1 & 0 & 0 \\ 0 & -2x_3^2 & x_1 & -6x_3x_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2x_3x_1 & 2 \end{bmatrix} \end{array}$$

with entries in $\kappa(\mathfrak{p}^{(\mathfrak{t})}) = \kappa(\mathfrak{p})$. Performing linear algebra in the field $\kappa(\mathfrak{p})$, we see that the kernel of $M$ is generated by $(1,0,0,0,0,0)^T$ and $(0,1,2x_1x_3,0,0,0)^T$. Since the dimension of the kernel did not increase, we terminate the loop in Algorithm 2 and conclude that $\{1, \partial_{x_1} + 2x_1x_3\partial_{x_2}\}$ is a set of Noetherian operators for $Q$.

Contrary to the algorithm in [10], our algorithm does not go through the *punctual Hilbert scheme*. To make this clear, we perform a parallel computation following [10, Algorithm 3.8]. Write $\mathbb{F} := \kappa(P)$. The point in the punctual Hilbert scheme corresponding to $Q$ is the ideal

$$I = \langle y_1, y_2 \rangle^2 + \gamma(Q) \cdot \mathbb{F}[y_1, y_2],$$

where $\gamma$ is the inclusion map

$$\gamma\colon R \hookrightarrow \mathbb{F}[y_1,y_2], \quad \begin{aligned} x_1 &\mapsto y_1 + x_1 \\ x_2 &\mapsto y_2 + x_2 \\ x_3 &\mapsto x_3 \end{aligned}$$

Here $I = (y_1 - 1/(2x_1x_3)y_2, y_2^2)$. A basis for $I^\perp$ can be computed using e.g. the classical Macaulay matrix method. The degree 2 Macaulay matrix is

$$\begin{array}{c|cccccc} & 1 & \partial_{x_1} & \partial_{x_2} & \partial_{x_1}^2 & \partial_{x_1}\partial_{x_2} & \partial_{x_2}^2 \\ \hline (y_1 - 1/(2x_1x_3)y_2) & 0 & 1 & \frac{-1}{2\,x_1x_3} & 0 & 0 & 0 \\ y_2^2 & 0 & 0 & 0 & 0 & 0 & 2 \\ y_1(y_1 - 1/(2x_1x_3)y_2) & 0 & 0 & 0 & 2 & \frac{-1}{2\,x_1x_3} & 0 \\ y_1y_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_2(y_1 - 1/(2x_1x_3)y_2) & 0 & 0 & 0 & 0 & 1 & \frac{-1}{x_1x_3} \\ y_2y_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

with entries in $\mathbb{F}$, and, as expected, its kernel corresponds to the Noetherian operators

$\{1, \partial_{x_1} + 2x_1x_3\partial_{x_2}\}$. △

**Example 3.2.4.** We compute a primary decomposition using our symbolic algorithm. Consider the rational normal scroll $S(2,2) \subseteq \mathbb{P}^5$ given by the prime ideal

$$\mathfrak{p} := I\left(2 \times 2 \text{ minors of } \begin{bmatrix} x_0 & x_1 & x_3 & x_4 \\ x_1 & x_2 & x_4 & x_5 \end{bmatrix}\right) \subseteq \mathbb{K}[x_0, \ldots, x_5]$$

which has codimension 3 and degree 4. We can take $x_1, x_3, x_4$ as the dependent variables, and $x_0, x_2, x_5$ as independent variables.

Consider the ideal $I$ generated by the following three polynomials:

$$f_1 := x_1^4 - 2x_0x_1^2x_2 + x_0^2x_2^2 + x_1x_2x_3x_4 - x_0x_2x_4^2 - x_1^2x_3x_5 + x_0x_1x_4x_5$$
$$f_2 := x_1^4 - 2x_0x_1^2x_2 + x_0^2x_2^2 + x_1x_2x_3x_4 - x_1^2x_4^2 - x_0x_2x_3x_5 + x_0x_1x_4x_5$$
$$f_3 := x_2^2x_3x_4 - x_1x_2x_4^2 + x_4^4 - x_1x_2x_3x_5 + x_1^2x_4x_5 - 2x_3x_4^2x_5 + x_3^2x_5^2$$

This ideal was constructed to be a complete intersection defined by suitable linear combinations of generators of $\mathfrak{p}^2$. Our goal is to compute a primary decomposition of $I$. Using Macaulay2 v1.15 on an Intel® Core™ i7-1065G7 CPU @ 1.30GHz, the command `primaryDecomposition I` did not terminate within 9 hours. On the other hand, `minimalPrimes I` quickly returns the primes

$\mathfrak{p}_1 = (x_5, x_4, x_1^2 - x_0x_2)$,

$\mathfrak{p}_2 = (x_4, x_3, x_1^2 - x_0x_2)$,

$\mathfrak{p}_3 = (x_2, x_1, x_4^2 - x_3x_5)$,

$\mathfrak{p}_4 = (x_1, x_0, x_2^2x_3x_4 + x_4^4 - 2x_3x_4^2x_5 + x_3^2x_5^2)$,

$\mathfrak{p}_5 = (x_4^2 - x_3x_5, x_2x_4 - x_1x_5, x_1x_4 - x_0x_5, x_2x_3 - x_0x_5, x_1x_3 - x_0x_4, x_1^2 - x_0x_2)$

Note that $\mathfrak{p}_5 = \mathfrak{p}$ is the prime ideal of the original rational normal scroll. The primes $\mathfrak{p}_i$

have dimension 3 and degrees (2, 2, 2, 4, 4) respectively. We then run Algorithm 3 for the ideal $I$ and each minimal prime $\mathfrak{p}_i$. Noetherian operators for the $\mathfrak{p}_1$-primary component of $I$ are

$$D_{1,1} = 1$$

$$D_{1,2} = x_1\partial_{x_4} + x_2\partial_{x_5}$$

$$D_{1,3} = \partial_{x_1}$$

$$D_{1,4} = x_1x_3^2\partial_{x_1}^2 + 4x_0^2x_2\partial_{x_4}^2 + 8x_0x_1x_2\partial_{x_4}\partial_{x_5} + 4x_0x_2^2\partial_{x_5}^2 - 8x_0x_3\partial_{x_4}$$

For the $\mathfrak{p}_2$-primary component, we get Noetherian operators

$$D_{2,1} = 1$$

$$D_{2,2} = x_1\partial_{x_3} + x_2\partial_{x_4}$$

$$D_{2,3} = \partial_{x_1}$$

$$D_{2,4} = x_1x_5^2\partial_{x_1}^2 + 4x_0x_2^2\partial_{x_3}^2 + 8x_1x_2^2\partial_{x_3}\partial_{x_4} + 4x_2^3\partial_{x_4}^2 + 8x_1x_5\partial_{x_3}$$

For the $\mathfrak{p}_3$-primary component, we get Noetherian operators

$$
\begin{aligned}
D_{3,1} &= 1 \\
D_{3,2} &= \partial_{x_4} \\
D_{3,3} &= x_4\partial_{x_1} + x_5\partial_{x_2} \\
D_{3,4} &= x_3^2x_5\partial_{x_1}^2 + 2x_3x_4x_5\partial_{x_1}\partial_{x_2} + x_3x_5^2\partial_{x_2}^2 - 2x_0x_4\partial_{x_1} \\
D_{3,5} &= x_3^2x_4x_5\partial_{x_1}^3 + 3x_3^2x_5^2\partial_{x_1}^2\partial_{x_2} + 3x_3x_4x_5^2\partial_{x_1}\partial_{x_2}^2 + x_3x_5^3\partial_{x_2}^3 \\
&\quad - 6x_0x_3x_5\partial_{x_1}^2 - 6x_0x_4x_5\partial_{x_1}\partial_{x_2} + 6x_3x_4\partial_{x_1} \\
D_{3,6} &= -27x_3^3x_4x_5\partial_{x_1}^4 - 108x_3^3x_5^2\partial_{x_1}^3\partial_{x_2} - 162x_3^2x_4x_5^2\partial_{x_1}^2\partial_{x_2}^2 - 108x_3^2x_5^3\partial_{x_1}\partial_{x_2}^3 - 27x_3x_4x_5^3\partial_{x_2}^4 \\
&\quad + 324x_0x_3^2x_5\partial_{x_1}^3 + 648x_0x_3x_4x_5\partial_{x_1}^2\partial_{x_2} + 324x_0x_3x_5^2\partial_{x_1}\partial_{x_2}^2 + (-324x_0^2x_4 - 648x_3^2x_4)\partial_{x_1}^2 \\
&\quad - 648x_3^2x_5\partial_{x_1}\partial_{x_2} + 81x_0^2x_5\partial_{x_4}^2 + 1944x_0x_3\partial_{x_1}
\end{aligned}
$$

For the $\mathfrak{p}_4$-primary component, we get Noetherian operators

$$D_{4,1} = 1$$

Let $Q$ denote the $\mathfrak{p}$-primary component. We get Noetherian operators

$$
\begin{aligned}
D_{5,1} &= 1, \\
D_{5,2} &= \partial_{x_4}, \\
D_{5,3} &= \partial_{x_3}, \\
D_{5,4} &= \partial_{x_1}, \\
D_{5,5} &= 2x_0x_5\partial_{x_1}\partial_{x_3} + x_0x_2\partial_{x_3}^2 + x_2x_4\partial_{x_1}\partial_{x_4}, \\
D_{5,6} &= x_2^4x_5^2\partial_{x_1}^2 + (-8x_2^5x_4 + 4x_2^3x_4x_5^2 + 32x_0x_2^2x_5^3 - 8x_0x_5^5)\partial_{x_1}\partial_{x_3} \\
&\quad + (-4x_2^5x_5 + 16x_2^3x_4x_5^2 + 2x_2^3x_5^3 - 4x_2x_4x_5^4)\partial_{x_1}\partial_{x_4} + (4x_2^4x_5^2 - x_2^2x_5^4)\partial_{x_4}^2, \\
D_{5,7} &= -x_2^4x_4x_5\partial_{x_1}^2 + (8x_0x_2^4x_5 - 32x_0x_2^2x_4x_5^2 - 4x_0x_2^2x_5^3 + 8x_0x_4x_5^4)\partial_{x_1}\partial_{x_3} \\
&\quad + (4x_2^5x_4 - 2x_2^3x_4x_5^2 - 16x_0x_2^2x_5^3 + 4x_0x_5^5)\partial_{x_1}\partial_{x_4} + (8x_0x_2^3x_5^2 - 2x_0x_2x_5^4)\partial_{x_3}\partial_{x_4}, \\
D_{5,8} &= (-8x_2^{11}x_4x_5^3 - 8x_0x_2^8x_5^6 + 6x_0x_2^6x_5^8 - x_0x_2^4x_5^{10})\partial_{x_1}^3 \\
&\quad + (96x_0x_2^{11}x_5^3 + 96x_0x_2^9x_4x_5^4 - 48x_0x_2^9x_5^5 - 120x_0x_2^7x_4x_5^6 + 48x_0x_2^5x_4x_5^8 - 6x_0x_2^3x_4x_5^{10})\partial_{x_1}^2\partial_{x_3} \\
&\quad + (384x_0x_2^{10}x_4x_5^3 - 96x_0x_2^8x_4x_5^5 + 384x_0^2x_2^7x_5^6 - 384x_0^2x_2^5x_5^8 + 120x_0^2x_2^3x_5^{10} - 12x_0^2x_2x_5^{12})\partial_{x_1}\partial_{x_3}^2 \\
&\quad + \cdots
\end{aligned}
$$

From this we deduce that the multiplicity of $Q$ over $\mathfrak{p}$ is 8. Note that this is consistent with the fact that $2(4) + 2(4) + 2(6) + 4(1) + 4(8) = \deg I = 4^3$. Furthermore, as the set of Noetherian operators of $Q$ contains the set of Noetherian operators of $\mathfrak{p}^2$, namely $\{1, \partial_{x_1}, \partial_{x_3}, \partial_{x_4}\}$, we see that $Q$ is strictly contained in $\mathfrak{p}^2$. We can also see that the $\mathfrak{p}_2$-primary component is radical. $\triangle$

### 3.2.2 A numerical algorithm

Let $I \subseteq \mathbb{K}[\mathbf{t}, \mathbf{y}]$ be a primary ideal of dimension $d$, where $\mathbf{t}$ is a maximal set of independent variables over $\mathfrak{p} = \sqrt{I}$. Let $\{D_1, \dots, D_m\}$ be a set of Noetherian operators for $I$ and write

$$D_i := \sum_{\alpha} c_{\alpha,i}(\mathbf{t}, \mathbf{y}) \partial_{\mathbf{y}}^{\alpha}, \qquad c_{\alpha,i} \in \kappa(\mathfrak{p})$$

Fix a generic point $(\mathbf{t}_0, \mathbf{y}_0) \in \mathbb{V}(I)$ on the variety of $I$. We denote by $D_i(\mathbf{t}_0, \mathbf{y}_0)$ the *specialized Noetherian operator*

$$D_i(\mathbf{t}_0, \mathbf{y}_0) = \sum_{\alpha} c_{\alpha,i}(\mathbf{t}_0, \mathbf{y}_0) \partial_{\mathbf{y}}^{\alpha} \in W_{\kappa(\mathfrak{p}(\mathbf{t}))}.$$

Recall that by Theorem 2.5.2 a specialized Noetherian operator corresponds to an exponential solution by substituting variables $\mathbf{z}$ in place of the $\partial_{\mathbf{y}}$ to obtain the function

$$u_i(\mathbf{z}) = D_i(\mathbf{t}_0, \mathbf{y}_0, \mathbf{z}) \exp((\mathbf{t}_0, \mathbf{y}_0)^T \cdot \mathbf{z})$$

**Theorem 3.2.5.** *Assume that $\mathbb{K}$ is algebraically closed. Let $\{D_1, \dots, D_m\}$ be a set of Noetherian operators of a primary ideal $I$, and let $(\mathbf{t}_0, \mathbf{y}_0) \in \mathbb{V}(I)$. If $\mathbf{t}_0$ is general, then*

$$\operatorname{span}_{\mathbb{K}}\{D_1(\mathbf{t}_0, \mathbf{x}_0), \dots, D_m(\mathbf{t}_0, \mathbf{x}_0)\} = D_{\mathfrak{m}}[I + (\mathbf{t} - \mathbf{t}_0)],$$

*where $\mathfrak{m}$ is the maximal ideal corresponding to the point $(\mathbf{t}_0, \mathbf{y}_0)$.*

A proof can be found in [9, Theorem 4.1]. We explain the intuition behind the theorem. Since $\mathbf{t}$ is a set of independent variables, the ideal $I + (\mathbf{t} - \mathbf{t}_0)$ is zero-dimensional, i.e. a union of points, one of which is the point $(\mathbf{t}_0, \mathbf{y}_0)$. Thus the local dual space of $I + (\mathbf{t} - \mathbf{t}_0)$ at the point $(\mathbf{t}_0, \mathbf{y}_0)$ is finite dimensional, and consists of differential operators with coefficients in the ground field $\mathbb{K}$. On the other hand, starting with a set of Noetherian operators for $I$, and evaluating their coefficients at $(\mathbf{t}_0, \mathbf{y}_0)$ also yields a set of differential operators with coefficients in the ground field. Theorem 3.2.5 says that these two sets of differential operators span the same $\mathbb{K}$-vector space.

Theorem 3.2.5 acts as a shortcut for the computation of evaluated Noetherian operators. To compute evaluations of Noetherian operators, we adapt Algorithm 2 to the ideal $I + (\mathbf{t} - \mathbf{t}_0)$. This results in Algorithm 4, in which the only difference with Algorithm 2 is that the Macaulay matrix is evaluated at the point $(\mathbf{t}_0, \mathbf{y}_0)$. The column reduction in step 11 is used to construct a basis consistent with the one computed in the symbolic algorithm. More precisely, for a fixed ordering $\prec$, the numerical matrix $K(p)$ in Algorithm 4 is precisely the symbolic matrix $K$ in Algorithm 2 evaluated at the point $p$. Thus if the output of NOETHERIANOPERATORS$(I, \mathfrak{p})$ is $\{D_1(\mathbf{t}, \mathbf{y}), \dots, D_m(\mathbf{t}, \mathbf{y})\}$, then the output of NOETHERIANOPERATORSATPOINT$(I, (\mathbf{t}_0, \mathbf{y}_0))$ would be $\{D_1(\mathbf{t}_0, \mathbf{y}_0), \dots, D_m(\mathbf{t}_0, \mathbf{y}_0)\}$. In general, Algorithm 4 will be faster than Algorithm 3, as computations in the former are done in the base field $\mathbb{K}$ rather than in $\kappa(\mathfrak{p})$, which is an extension of the rational function field in $\mathbf{t}$.

**Algorithm 4** Compute specializations of Noetherian operators at a point

**Require:** $I \subseteq \mathbb{K}[\mathbf{t}, \mathbf{y}]$ an ideal, where $\mathbf{t}, \mathbf{y}$ are independent and dependent variables for $I$ respectively, $\mathfrak{p}$ a minimal prime of $I$, $\prec$ an ordering on monomials $\partial_{\mathbf{y}}^{\gamma}$, and $p \in \mathbb{V}(\mathfrak{p})$ a generic point.

**Ensure:** A set of Noetherian operators for the $\mathfrak{p}$-primary component of $I$, specialized at $p$

1: **procedure** NOETHERIANOPERATORSATPOINT$(I, p)$
2: $\quad K \leftarrow \emptyset$
3: $\quad d \leftarrow 0$ $\qquad \triangleright$ $d$ corresponds to the degree bound
4: $\quad$ **repeat**
5: $\quad\quad d \leftarrow d + 1$
6: $\quad\quad F \leftarrow$ vector with entries $\mathbf{y}^{\alpha}\mathbf{t}^{\beta} f_i$, where $|\alpha + \beta| < d$, $i = 1, 2, \dots, r$
7: $\quad\quad C \leftarrow$ vector with entries $\partial_{\mathbf{y}}^{\gamma}$, where $|\gamma| \leq d$, in the order given by $\prec$
8: $\quad\quad M \leftarrow$ the Macaulay matrix with entries $(\partial_{\mathbf{y}}^{\gamma} \bullet (\mathbf{y}^{\alpha}\mathbf{t}^{\beta} f_i))(p)$ (rows indexed by $F$, columns by $C$)
9: $\quad\quad K_d \leftarrow \ker M$
10: $\quad$ **until** $\dim K_d = \dim K_{d-1}$ $\qquad \triangleright$ Stop when the dimension of the kernel stabilizes
11: $\quad K(p) \leftarrow$ COLREDUCE$(K_d)$ $\qquad \triangleright$ Rewrites generators of $K_d$ in reduced column echelon form
12: $\quad$ **return** $C^T K(p)$

Algorithm 4 is implemented in the package `NoetherianOperators` [35] under the command `specializedNoetherianOperators(I, pt)`. The input `I` is an ideal and `pt` is a point, and the output is a list of evaluated Noetherian operators for the primary component on which the point `pt` lies approximately.

A set of evaluated Noetherian operators also translates to a probabilistic, numerical algorithm for testing ideal membership. Suppose $I$ is an unmixed ideal, and that for each $\mathfrak{p} \in \operatorname{Ass}(\mathfrak{p})$ we have a generic point $p_\mathfrak{p} = (\mathbf{t}_\mathfrak{p}, \mathbf{y}_\mathfrak{p})$, and a set $\mathcal{D}_\mathfrak{p}(p_\mathfrak{p})$ of evaluated Noetherian operators for each $\mathfrak{p}$-primary component at the points $p_\mathfrak{p}$. Then a polynomial $f \in R$ belongs to the ideal $I$ with high probability if $D(\mathbf{t}_\mathfrak{p}, \mathbf{y}_\mathfrak{p}) \bullet f = 0$ for all $D \in \mathcal{D}_\mathfrak{p}(p_\mathfrak{p})$ and $\mathfrak{p} \in \operatorname{Ass}(I)$. This is because $D(\mathbf{t}_\mathfrak{p}, \mathbf{y}_\mathfrak{p}) \bullet f$ is equal to the evaluation of the rational function $D \bullet f \in \kappa(\mathfrak{p})$ at the point $p_\mathfrak{p}$.

**Example 3.2.6.** Let $I = (x^2, y^2 - tx)$ be an ideal in $\mathbb{C}[t, x, y]$. Here $t$ is an independent variable, and $x, y$ are dependent. We sample four points $(1, 0, 0), (2, 0, 0), (3, 0, 0), (4, 0, 0)$ on the variety $\mathbb{V}(I)$. Running Algorithm 4 gives four differential operators with constant coefficients for each point, shown in Table 3.1.

Table 3.1: Specialized Noetherian operators at different points

| $(t, x, y)$ | Operator 1 | Operator 2 | Operator 3 | Operator 4 |
|---|---|---|---|---|
| $(1, 0, 0)$ | $1$ | $\partial_y$ | $\frac{1}{2}\partial_y^2 + \partial_x$ | $\frac{1}{6}\partial_y^3 + \partial_x\partial_y$ |
| $(2, 0, 0)$ | $1$ | $\partial_y$ | $\frac{1}{2}\partial_y^2 + \frac{1}{2}\partial_x$ | $\frac{1}{6}\partial_y^3 + \frac{1}{2}\partial_x\partial_y$ |
| $(3, 0, 0)$ | $1$ | $\partial_y$ | $\frac{1}{2}\partial_y^2 + \frac{1}{3}\partial_x$ | $\frac{1}{6}\partial_y^3 + \frac{1}{3}\partial_x\partial_y$ |
| $(4, 0, 0)$ | $1$ | $\partial_y$ | $\frac{1}{2}\partial_y^2 + \frac{1}{4}\partial_x$ | $\frac{1}{6}\partial_y^3 + \frac{1}{4}\partial_x\partial_y$ |

Note that these computations also work when using floating point approximations. After loading the package `NoetherianOperators`, we can use the following `Macaulay2` snippet to compute numerically compute the first row of Table 3.1. The last line is the output.

```
R = CC[x,y,t]; I = ideal(x^2, y^2 - t*x);
pt = matrix{{1.0, 0, 0}}
```

```
specializedNoetherianOperators(I, pt)
  {| 1 |, | dy |, | .5dy^2+.5dx |, | .166667dy^3+.5dxdy |}
```

Running Algorithm 3, the symbolic version of Algorithm 4, on $I$, we obtain a set of four Noetherian operators

$$1, \quad \partial_y, \quad \frac{1}{2}\partial_y^2 + \frac{1}{t}\partial_y, \quad \frac{1}{6}\partial_y^3 + \frac{1}{t}\partial_x\partial_y. \tag{3.9}$$

As predicted by Theorem 3.2.5, the specialized Noetherian operators are simply the Noetherian operators evaluated at the corresponding points.

We can check that the polynomial $xy^2 \in I$ numerically. Suppose we are given the point $(1,0,0)$ and the corresponding evaluated Noetherian operators. Then we check

$$\begin{aligned}
f_1(t,x,y) &= 1 \bullet xy^2 = xy^2 && \implies f_1(1,0,0) = 0 \\
f_2(t,x,y) &= \partial_y \bullet xy^2 = 2xy && \implies f_2(1,0,0) = 0 \\
f_3(t,x,y) &= \left(\frac{1}{2}\partial_y^2 + \partial_x\right) \bullet xy^2 = 2x + y^2 && \implies f_3(1,0,0) = 0 \\
f_4(t,x,y) &= \left(\frac{1}{6}\partial_y^3 + \partial_x\partial_y\right) \bullet xy^2 = 2y && \implies f_4(1,0,0) = 0.
\end{aligned}$$

Similarly, we observe that the polynomial $xy \notin I$, since

$$\left(\frac{1}{6}\partial_y^3 + \partial_x\partial_y\right) \bullet xy = 1 \neq 0.$$

△

### 3.2.3 A hybrid approach

One of the bottlenecks in the performance of Algorithm 3 is working with a large Macaulay matrix over the field $\kappa(\mathfrak{p})$. On the other hand, Algorithm 4 performs the same computations over $\mathbb{C}$, which is much faster. In particular, Algorithm 4 computes an evaluated set of

Noetherian operators, which reveals the monomial support in $\partial_{\mathbf{y}}$ of a valid set of Noetherian operators. This information can then be used to reduce the size of the Macaulay matrix and symbolically produce a set of Noetherian operators in a single iteration. This approach often takes a fraction of the time taken by Algorithm 3.

Let $I = (f_1, \ldots, f_r) \subseteq \mathbb{K}[\mathbf{t}, \mathbf{y}]$ be unmixed, $\mathfrak{p}$ a minimal associated prime of $I$, and $p \in \mathbb{V}(\mathfrak{p}) \subseteq \mathbb{K}^n$ be a generic rational point on the variety of $\mathfrak{p}$. Let $\mathcal{D}' = \{D_1(p), \ldots, D_m(p)\}$ be the output of Algorithm 4, i.e. a reduced set of specialized Noetherian operators. Let $D_i(p) = \sum_{\beta \in B} c_{i,\beta}(p)\partial_{\mathbf{y}}^{\beta}$, where $B \subset \mathbb{N}^n$ is finite and let $d_i = \deg D_i$ be the $\partial_{\mathbf{y}}$-degree of the operator. Here $c_{i,\beta}(p)$ is the evaluation of some element $c_{i,\beta}(\mathbf{y}, \mathbf{t}) \in \kappa(\mathfrak{p})$, where the vector $(c_\beta(\mathbf{y}, \mathbf{t}))_{|\beta| \le d}$ is in the kernel of the degree $d$ Macaulay matrix $M_d$.

Let $M$ be the submatrix of $M_d$ obtained by keeping only columns corresponding to $\partial_{\mathbf{y}}^{\beta}$ with $\beta \in B$. The vector $(c_\beta(\mathbf{y}, \mathbf{t}))_{\beta \in B}$ is in the kernel of $M$, and because the operators are reduced, the kernel is one-dimensional. Thus in order to find a symbolic representation of the operator $D_i(\mathbf{y}, \mathbf{t})$, it suffices to find the kernel of the matrix $M$ over $\kappa(\mathfrak{p})$.

One can further optimize the procedure by starting with fewer rows than necessary, and adding rows until the kernel becomes one-dimensional. Since rows are indexed by $\mathbf{y}^\alpha f_j$ for all $|\alpha| < d_i$ and $j = 1, 2, \ldots, r$, one could for example run the algorithm for $|\alpha| < 0, 1, \ldots, d_i$ until the dimension of the kernel is 1. This method, described in pseudocode in Algorithm 5, is implemented in the package `NoetherianOperators` under the command `hybridNoetherianOperators`.

**Example 3.2.7.** Consider the primary ideal $I = (x^2, y^2 - xt) \subseteq \mathbb{C}[t, x, y]$ from Example 3.2.6. Let $p = (1, 0, 0)$. Algorithm 4 reveals that the reduced set of Noetherian operators specialized at $p$ are $\{1, \partial_y, \frac{1}{2}\partial_y^2 + \partial_x, \frac{1}{6}\partial_y^3 + \partial_x\partial_y\}$. To find the unevaluated operator corresponding to $\frac{1}{6}\partial_y^3 + \partial_x\partial_y$ for example, it suffices to find the kernel of the following

**Algorithm 5** Hybrid computation of Noetherian operators

**Require:** $I = (f_1, \ldots, f_r)$ an unmixed ideal, $\mathfrak{p}$ a minimal prime of $I$, a generic point $p \in \mathbb{V}(\mathfrak{p})$

**Ensure:** A set of Noetherian operators for the $\mathfrak{p}$-primary component of $I$

```
procedure HybridNoetherianOperators(I, p, p)
    N' ← NoetherianOperatorsAtPoint(I, p)
    N ← ∅
    for all D' ∈ N' do
        C ← vector with entries ∂_x^β = ∂_{x_1}^{β_1} ··· ∂_{x_n}^{β_n} for each ∂_x^β appearing in D'.
        d ← 0
        repeat
            d ← d + 1
            R ← vector with entries x^α f_i, where |α| < d, i = 1, 2, ..., r
            M ← the matrix with entries ∂_x^β • (x^α f_i) (rows indexed by F, columns by C)
            K ← ker M
        until dim K = 1
        D ← C^T K
        N ← N ∪ {D}
    return lift of N in W_R
```

submatrix of the Macaulay matrix

$$\begin{array}{c} \\ x^2 \\ y^2-xt \\ x(x^2) \\ x(y^2-xt) \\ y(x^2) \\ y(y^2-xt) \end{array} \begin{array}{c} \begin{array}{cc} \partial_x\partial_y & \partial_y^3 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -t & 6 \end{bmatrix} \end{array}$$

over $\kappa(\sqrt{I})$. The kernel is 1-dimensional and generated by $(1, t/6)$, so we conclude that $\frac{1}{6}\partial_y^3 + \partial_x\partial_y$ is the Noetherian operator $\frac{t}{6}\partial_y^3 + \partial_x\partial_y$ evaluated at the point $(0, 0, 1)$. We repeat the procedure with all other operators to obtain the complete set of Noetherian operators $\{1, \partial_y, \frac{t}{2}\partial_y^2 + \partial_x, \frac{t}{6}\partial_y^3 + \partial_x\partial_y\}$.

If we had used Algorithm 3 we would have had to compute the kernel of the degree 4 Macaulay matrix, which has size $(40 \times 15)$. $\triangle$

**Example 3.2.8.** Consider the Noetherian operators $D_{5,1}, \ldots, D_{5,8}$ for the $P$-primary component in Example 3.2.4. The largest Noetherian operator has degree 3, which means that we have to compute the kernel of the degree 4 Macaulay matrix, which has dimensions $(252 \times 35)$. Over the field $\kappa(\mathfrak{p})$ this takes about 2 minutes. In contrast, computing the kernel of the evaluated Macaulay matrix over $\mathbb{C}$ takes about 0.4 seconds.

Following Algorithm 5, we note that the largest matrix we need to deal with has dimensions $(12 \times 13)$, which allows us to obtain, symbolic Noetherian operators in about 1 second. $\triangle$

*Remark* 3.2.9. Another way of obtaining symbolic Noetherian operators from numerical data is to reconstruct them from point evaluations, as described in [9]. Since each coefficient can be represented by a rational function, we can run an interpolation routine on each coefficient. This yields a valid set of Noetherian operators given enough point evaluations. This is implemented in the command `numericalNoetherianOperators` in the `NoetherianOperators` package. $\triangle$

# CHAPTER 4

# WAVES

Waves are special solutions to the PDE system, in the spirit of the wave equation in Example 3.1.2. We develop geometric theory and algebraic algorithms for finding them. Our point of departure is the *simple wave*

$$\phi_{\xi,u}(\mathbf{z}) = \exp(\mathrm{i}\xi^T \cdot z) \cdot u. \tag{4.1}$$

Here $\xi \in \mathbb{R}^n$, $u \in \mathbb{C}^k$ and $\mathrm{i} = \sqrt{-1}$. The exponential function is applied to the dot product of $\mathbf{z} = (z_1, \ldots, z_n)$ with the purely imaginary vector $\mathrm{i}\xi$, resulting in trigonometric functions. The real vector $\xi$ is the *frequency*, while the complex vector $u$ is the *amplitude*. If the matrix $M \in R^{\ell \times k}$ describes a PDE, its simple wave solutions are characterized by a system of $\ell$ polynomial equations in their $n + k$ coordinates:

$$\phi_{\xi,u} \in \mathrm{Sol}(M) = 0 \quad \text{if and only if} \quad M(\xi) \cdot u = 0. \tag{4.2}$$

Our standing assumption is that all entries of the matrix $M$ are homogeneous polynomials in $x_1, \ldots, x_n$ of the same degree $d$. Recall that $x_i$ acts as $\partial_{z_i} = \frac{\partial}{\partial z_i}$ on distributions $\phi \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C})$. Since $M$ has homogeneous entries, this implies that $M$ is elliptic—therefore smoothing—if and only if there are no nontrivial wave solutions. Therefore, the existence of wave solutions has a major impact on the analytical properties of the operator, c.f. [56, Chapter 2.1].

More general wave solutions are obtained from superpositions of simple waves and taking limits. In the superpositions we allow here, the amplitude $u$ is fixed, whereas the frequency $\xi$ runs over linear subspaces of $\mathbb{R}^n$ all of whose points satisfy (4.2). Taking limits of such superpositions leads to waves that are distributions with small support. This construction will be explained in detail in Section 4.1.

In Section 4.2 we turn to algebraic geometry, and we introduce projective varieties that parametrize wave solutions. These generalize the determinantal varieties of matrices of linear forms. In Section 4.3 we examine the analytic meaning of wave varieties and obstruction varieties, and discuss the analytic implications of working algebraically in complex projective spaces. In Section 4.4 we introduce varieties of wave pairs. These generalize Fano varieties [30, Example 6.19] inside Grassmannians. We present methods for computing wave pairs and wave varieties, and we illustrate these on several examples. In the context of a given PDE $M$, these scenarios give interesting distributional solutions to $M$ with low-dimensional support.

The material in this chapter, based on the paper [28], arose from a desire to understand the hierarchy of wave cones in [3]. These are subsets of $\mathbb{C}^k$ which play an important role in the regularity theory of PDE. Our presentation connects this thread of analysis with nonlinear algebra via Noetherian operators and differential primary decompositions.

We close the introduction with a well-known equation from the theory of elasticity [21, 45].

**Example 4.0.1** (Saint-Venant's tensor)**.** Set $d = 2, k = \binom{n+1}{2}, \ell = k^2$, and identify $\mathbb{C}^k$ with the space of symmetric $n \times n$ matrices. We consider matrix-valued distributions $\phi : \mathbb{R}^n \to \mathbb{C}^k$. The *Saint-Venant operator* $M$ characterizes the kernel of the 2-dimensional X-ray transform:

$$M \bullet \phi \;=\; \big(\partial_i\partial_j\phi_{ab} + \partial_a\partial_b\phi_{ij} - \partial_i\partial_a\phi_{jb} - \partial_j\partial_b\phi_{ia}\big)_{i,j,a,b=1,\ldots,n}. \tag{4.3}$$

In our notation, $M$ is an $\ell \times k$ matrix whose nonzero entries are quadratic monomials $\partial_i\partial_j$. By removing redundant rows, using [21], the number of rows of $M$ can be reduced to

$\ell = \frac{1}{6}\binom{n^2}{2}$. The PDE $M$ has a vector potential $S$ given by the symmetric gradient:

$$B \bullet \psi \;=\; \big(\,\partial_i \psi_j + \partial_j \psi_i\,\big)_{i,j=1,\ldots,n}.$$

The wave pair variety $\mathcal{P}_A^{n-1}$ of Section 4.4 lives in the space $\mathbb{P}^{n-1} \times \mathbb{P}^{k-1}$, and it coincides with the incidence variety $\mathcal{I}_A$ in (4.13). It is defined by the $3 \times 3$ minors of the $(n+1) \times (n+1)$-matrix

$$\begin{bmatrix} 0 & y_1 & y_2 & \cdots & y_n \\ y_1 & z_{11} & z_{12} & \cdots & z_{1n} \\ y_2 & z_{12} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_n & z_{1n} & z_{2n} & \cdots & z_{nn} \end{bmatrix}. \tag{4.4}$$

The wave variety $\mathcal{W}_A \subset \mathbb{P}^{k-1}$ of Section 4.2 is given by the $3 \times 3$ minors of the $n \times n$ matrix $(z_{ij})$. This example is a variant of the curl operator in Proposition 4.4.6, with (4.4) playing the role of (4.23). It underscores the relevance of nonlinear algebra [41] for the physical sciences. $\triangle$

## 4.1 Spaces and Waves

The notion of waves arises from superpositions of the simple waves (4.1),

$$\phi(x) \;=\; \sum_{j=1}^{p} \lambda_j \phi_{\xi_j,u}(x). \tag{4.5}$$

Here the amplitude $u \in \mathbb{C}^k$ is fixed but the frequencies $\xi_1, \ldots, \xi_p$ vary in $\mathbb{R}^n$. The coefficients $\lambda_1, \ldots, \lambda_p$ are complex numbers. If each summand in (4.5) satisfies the PDE $M$ then so does $\phi(x)$.

Waves are thus simply linear combinations of exponential functions, which play a special role. We introduce the *Schwartz space* $\mathcal{S} = \mathcal{S}(\mathbb{R}^n, \mathbb{C}^k)$ whose elements are smooth functions $f$ such that $||x^\beta \partial^\alpha f||_\infty$ is finite for all $\alpha, \beta \in \mathbb{N}^n$. This space includes the simple waves (4.1), since the coordinates of $\xi$ are real. However, many nice functions, such as polynomials, are not in $\mathcal{S}$.

Most relevant for us is that $\mathcal{D} = C_c^\infty(\mathbb{R}^n, \mathbb{C}^k)$ is a subspace of $\mathcal{S} = \mathcal{S}(\mathbb{R}^n, \mathbb{C}^k)$. The key feature of the Schwartz space is the endomorphism known as the *Fourier transform* $\hat{} : \mathcal{S} \to \mathcal{S}$. By applying $\hat{}$ twice, we see that every function in $\mathcal{S}$ admits an integral representation

$$f(x) \;\; = \;\; \int_{\mathbb{R}^n} \exp(\, 2\pi i \, \xi \cdot x) \, \hat{f}(\xi) \, d\xi. \tag{4.6}$$

The dual to the Schwarz space consists of the *tempered distributions*. We have inclusions

$$\mathcal{D} \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{S}' \hookrightarrow \mathcal{D}'. \tag{4.7}$$

All of these spaces are $R$-modules because the linear map $\partial^\alpha : \mathcal{D} \to \mathcal{D}$ is continuous, so we get a dual $(\partial^\alpha)^* : \mathcal{D}' \to \mathcal{D}'$ which restricts to $\mathcal{S}'$ and $\mathcal{S}$.

The integral representation (4.6) of Schwartz functions by the Fourier transform implies that every distribution $\delta \in \mathcal{D}'$ can be approximated by a sequence of waves $\phi^{(1)}, \phi^{(2)}, \ldots$ of the form (4.5).

**Lemma 4.1.1.** *The linear span of the exponential functions* $x \mapsto \exp(i\xi \cdot x)$ *is dense in* $\mathcal{D}'$.

Our aim is to create interesting distributions by taking limits of waves (4.5) in $\mathcal{D}'$. To this end, suppose that $\xi_1, \ldots, \xi_p$ span a linear subspace $\pi$ of $\mathbb{R}^n$ such that $M \bullet \phi_{\xi,u} = 0$ for all $\xi \in \pi$. We then consider the closure in $\mathcal{D}'$ of the space of all waves (4.5) whose frequencies $\xi_j$ are in $\pi$. Each element in that closure satisfies the PDE $M$, and the closure contains distributional solutions with small support. This motivates the following definitions. As before, $M \in R^{\ell \times k}$ is a matrix whose entries are homogeneous of degree $d$. A *wave pair* for $M$ is a pair $(u, \pi)$, where $u \in \mathbb{C}^k$ and $\pi$ is a linear subspace of $\mathbb{R}^n$, such that $M(\xi)u = 0$ for all $\xi \in \pi$. If $(u, \pi)$ is a wave pair then any superposition (4.5) with $\xi_1, \ldots, \xi_p \in \pi$ is a *classical wave solution* of $M$. A *wave solution* to $M$ is any distribution in the closure in $\mathcal{D}'$

of the classical wave solutions.

**Proposition 4.1.2.** *Consider any wave pair* $(u, \pi)$ *for the operator* $M$ *and set* $r = \operatorname{codim}(\pi)$. *The associated wave solutions are precisely the distributions of the form*

$$\phi(z) \;=\; \delta\big(L_1(z), \ldots, L_{n-r}(z)\big) \cdot u, \tag{4.8}$$

*where* $L_1, \ldots, L_{n-r}$ *are linear form satisfying* $\pi^\perp = \{z \in \mathbb{R}^n : L_1(z) = \cdots = L_{n-r}(z) = 0\}$ *and* $\delta$ *is any distribution in* $\mathcal{D}'(\mathbb{R}^{n-r}, \mathbb{C})$. *Thus, equation* (4.8) *characterizes wave pairs as follows: if* $\phi(z)$ *is a solution to the PDE* $M$ *for all* $\delta \in \mathcal{D}'(\mathbb{R}^{n-r}, \mathbb{C})$ *then* $(u, \pi)$ *is a wave pair.*

*Remark* 4.1.3. The notation $\delta(L\cdot) := \delta(L_1(z), \ldots, L_{n-r}(z))$ refers to an extension from smooth functions to distributions. Following [32, Chapter 6], one can define it as follows. Given a real matrix $L \in \mathbb{R}^{(n-r)\times r}$, fix an orthonormal basis $v_1, \ldots, v_r$ for $\ker(L)$ and let $L' \in \mathbb{R}^{r\times n}$ be the matrix with the $v_i$ as rows. The matrix $H = \begin{bmatrix} L \\ L' \end{bmatrix}$ defines an endomorphism $\mathbb{R}^n \to \mathbb{R}^n$, $z \mapsto (y', y'')$, where $y' \in \mathbb{R}^{n-r}$, $y'' \in \mathbb{R}^r$. Its inverse is $H^{-1} = \begin{bmatrix} L^T(LL^T)^{-1} & L'^T \end{bmatrix}$. If $\delta\colon \mathbb{R}^{n-r} \to \mathbb{C}$ is smooth, then, by a change of variables, for any test function $f \in \mathcal{D}(\mathbb{R}^n, \mathbb{C})$,

$$\begin{aligned} \delta(L\cdot)(f) \;&=\; \textstyle\int_{\mathbb{R}^n} \delta(L(x)) f(x)\, dx \;=\; \int_{\mathbb{R}^n} \delta(y') f(H^{-1}y) |\det(H^{-1})|\, dy \\ &=\; \tfrac{1}{\sqrt{\det(LL^T)}} \textstyle\int_{\mathbb{R}^n} \delta(y') f(H^{-1}y)\, dy \;=\; \tfrac{1}{\sqrt{\det(LL^T)}} \int_{\mathbb{R}^{n-r}} \delta(y') \int_{\mathbb{R}^r} f(H^{-1}y)\, dy''\, dy'. \end{aligned}$$

We write this as $\delta(L\cdot)(f) = \delta(1(F))$, with the constant function $1 : \mathbb{R}^r \to \mathbb{C}$ and $F(y) = \frac{1}{\sqrt{\det(LL^T)}} f(H^{-1}y)$. There exists a unique distribution $\delta\otimes 1$ such that $(\delta\otimes 1)(F) = \delta(1(F)) = 1(\delta(F))$ for all $F \in \mathcal{D}(\mathbb{R}^n, \mathbb{C})$. Now define $\delta(L\cdot)(f) := (\delta\otimes 1)(F)$ for *arbitrary* distributions $\delta$. $\triangle$

*Proof of Proposition 4.1.2.* Write $z = (z_1, \ldots, z_n)$ and $y = (y_1, \ldots, y_{n-r})$ for the coordinates of $\mathbb{R}^n$ and $\mathbb{R}^{n-r}$, and let $L$ denote the $(n-r)\times n$ matrix of coefficients of $L_1, \ldots, L_{n-r}$.

For $\eta \in \mathbb{R}^{n-r}$, consider the wave function $z \mapsto \delta_\eta(Lz) \cdot u$ associated with the exponential function $\delta_\eta(y) = \exp(\mathrm{i}\eta \cdot y)$. Applying the differential operator $M$ to that wave function yields

$$M \bullet (\delta_\eta(Lz) \cdot u) \;=\; M \bullet (\exp(\mathrm{i}\eta Lz) \cdot u) \;=\; \mathrm{i}^d \exp(\mathrm{i}\eta Lz) \cdot M(\eta L)u. \qquad (4.9)$$

This vector of length $\ell$ is zero for all $\eta \in \mathbb{R}^{n-r}$ if and only if $(u, \pi)$ is a wave pair. Since the space spanned by the exponential functions $\delta_\eta$ for $\eta \in \mathbb{R}^{n-r}$ is dense in the space of all distributions, by Lemma 4.1.1, the first assertion follows.

The second statement follows from the fact that $M \bullet (\delta_\eta(Lz) \cdot u) = 0$ for all $\eta$ if and only if $(u, \pi)$ is a wave pair, together with the simple observation that $\delta_\eta \in \mathcal{D}'(\mathbb{R}^{n-r}, \mathbb{C})$. $\square$

We seek wave pairs $(u, \pi)$ where $r$ is as small as possible. Indeed, if $r$ is small then we can build distributional solutions with small support. What follows is the standard construction.

*Remark* 4.1.4. Let $\delta$ be the Dirac delta distribution at the origin in $\mathbb{R}^{n-r}$. The distribution $\phi$ in (4.8) is supported on the $r$-dimensional subspace $\pi^\perp$ of $\mathbb{R}^n$. If $(u, \pi)$ is a wave pair then $\phi$ satisfies the PDE $M$. Such $M$-*free measures* are important in the calculus of variations [3, 14]. $\triangle$

**Example 4.1.5.** Fix the matrix $M$ in Example 2.1.1. For all $\xi \in \mathbb{C}^3 \backslash \{0\}$, the linear space $\ker M(\xi)$ has dimension 2. It consists of all vectors $u \in \mathbb{C}^4$ such that

$$\begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & 0 \\ 0 & \xi_1 & \xi_2 & \xi_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \;=\; \begin{bmatrix} u_1 & u_2 & u_3 \\ u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (4.10)$$

This equation characterizes simple waves $\phi_{\xi,u}$ that satisfy $M$. With $r = 2$ in (4.8) we can take

$$L_1(z) \;=\; (u_3^2 - u_2u_4)z_1 + (u_1u_4 - u_2u_3)z_2 + (u_2^2 - u_1u_3)z_3 \qquad \text{for any } u \in \mathbb{R}^4. \tag{4.11}$$

In other words, for any choice of $u \in \mathbb{C}^4$, the distribution $\phi(z) = \delta(L_1(z)) \cdot u$ satisfies $M \bullet \phi = 0$.

To obtain waves with $r = 1$, $u$ must be chosen such that the three coefficients in (4.11) vanish. This means that $u$ lies in the cone over the *twisted cubic curve*:

$$(u_1, u_2, u_3, u_4) \;=\; (s^3, s^2t, st^2, t^3). \tag{4.12}$$

This is the wave variety $\mathcal{W}_M^1 \subset \mathbb{P}^3$ in Example 4.2.3. We obtain wave pairs $(u, \pi)$ with $\operatorname{codim}(\pi) = 2$, and thus solutions supported on a plane in $\mathbb{R}^4$, by taking the two linear forms

$$L_1(z) \;=\; tz_1 - sz_2 \quad \text{and} \quad L_2(z) \;=\; tz_2 - sz_3.$$

Indeed, $\phi(z) = \delta\big(L_1(z), L_2(z)\big) \cdot u$ is a wave solution of $M$, for any $\delta \in \mathcal{D}'(\mathbb{R}^2, \mathbb{C})$. $\triangle$

## 4.2 Varieties

In this section we introduce several algebraic varieties that are naturally associated with $M$ and its system of PDE. As is customary in algebraic geometry, we work in complex projective spaces rather than in real affine spaces. Every subvariety of $\mathbb{P}^{k-1}$ corresponds to a cone in $\mathbb{C}^k$, which is a complex variety defined by homogeneous equations, and by restricting to $\mathbb{R}^k$ one obtains a real cone. Among such cones are the wave cones from [3] which motivated our study. We shall return to the analytic perspective in the next section. In what follows, however, we stick to algebra. This means working in the projective spaces $\mathbb{P}^{k-1}$ and $\mathbb{P}^{n-1}$ over the complex numbers $\mathbb{C}$.

As always, we let $R = \mathbb{C}[x_1, \ldots, x_n]$, or $R = \mathbb{C}[\partial_1, \ldots, \partial_n]$, where $\partial_i = \frac{\partial}{\partial z_i}$. For any point $y \in \mathbb{P}^{n-1}$ we write $M(y)$ for the complex $\ell \times k$ matrix that is obtained from $M$ by replacing each $\partial_i$ with the coordinate $y_i$. The matrix $M(y)$ is well-defined up to scale. We view it as a point in the projective space $\mathbb{P}^{\ell k-1}$. We write $w$ for points in $\mathbb{P}^{k-1}$, and we set

$$\mathcal{I}_M \;=\; \left\{ (y,w) \in \mathbb{P}^{n-1} \times \mathbb{P}^{k-1} \,:\, M(y) \cdot w = 0 \right\}. \tag{4.13}$$

This is our algebro-geometric representation of the relation between frequencies and amplitudes seen in (4.2). The projection of the incidence variety $\mathcal{I}_M$ onto the first factor equals

$$\mathcal{S}_M \;=\; \{ y \in \mathbb{P}^{n-1} \,:\, \operatorname{rank}(M(y)) \leq k-1 \}. \tag{4.14}$$

This projective variety is the support of our PDE $M$. The variety $V(\operatorname{im}_R M^T)$ is the affine cone over $\mathcal{S}_M$. The role of the support for simple waves was highlighted in Lemma 2.6.5. A natural set of polynomials that define $\mathcal{S}_M$ set-theoretically is the $k \times k$ minors of $M$. However, these minors usually do not suffice to generate the radical ideal of $\mathcal{S}_M$. There are two interesting extreme cases, namely $\mathcal{S}_M = \mathbb{P}^{n-1}$ and $\mathcal{S}_M = \emptyset$. The former identifies PDE with compactly supported solutions (c.f. Theorem 2.4.2), while the latter identifies PDE whose only solutions are polynomials (c.f. Proposition 2.6.8).

We next consider the projection of the incidence variety $\mathcal{I}_M$ onto the second factor $\mathbb{P}^{k-1}$. The resulting projective variety is called the *wave variety* of $M$, and we write it as follows:

$$\mathcal{W}_M \;:=\; \bigcup_{y \in \mathbb{P}^{n-1}} \ker M(y).$$

The kernel in this definition is a linear subspace of $\mathbb{P}^{k-1}$, so $\mathcal{W}_M$ is a projective variety in $\mathbb{P}^{k-1}$. This is the algebraic variant of the *wave cone* considered in analysis; see [14, Theorem 1.1] and surrounding references. We shall return to this in Section 4.3 where it is denoted $\mathcal{W}_{M,\mathbb{R}}$.

**Example 4.2.1** ($n = k = \ell = 3, d = 2$)**.** Consider the second order PDE given by the matrix

$$M = \begin{bmatrix} \partial_1^2 & \partial_2^2 & \partial_3^2 \\ -\partial_2^2 & \partial_3^2 & \partial_1^2 \\ -\partial_3^2 & -\partial_1^2 & \partial_2^2 \end{bmatrix}.$$

Its support $\mathcal{S}_M$ is the smooth sextic curve in $\mathbb{P}^2$ defined by $\det(M(y)) = y_1^6 + y_2^6 + y_3^6 + y_1^2 y_2^2 y_3^2$. The wave variety $\mathcal{W}_M$ is the smooth cubic curve in $\mathbb{P}^2$ defined by $z_1^3 - z_2^3 + z_3^3 - z_1 z_2 z_3$. These two plane curves are linked by their incidence curve $\mathcal{I}_M \subset \mathbb{P}^2 \times \mathbb{P}^2$. If the entries of $M$ are replaced by random quadrics in $\partial_1, \partial_2, \partial_3$, then $\mathcal{W}_M$ is a singular curve of degree $12$ in $\mathbb{P}^2$. $\triangle$

The article [3] extended the results in [14] by introducing two refined notions of wave cones. We now recast these as algebraic varieties. For $r \in \{0, \ldots, n-1\}$, the *level $r$ wave variety* is

$$\mathcal{W}_M^r := \bigcup_{\pi \in \mathrm{Gr}(n-r,n)} \bigcap_{y \in \pi} \ker M(y). \tag{4.15}$$

The union is over the Grassmannian $\mathrm{Gr}(n-r,n)$ of linear subspaces $\pi$ of codimension $r$ in $\mathbb{P}^{n-1}$. For basics on Grassmannians and their projective embeddings see [41, Chapter 5]. For $r = n-1$, the inner intersection in (4.15) goes away, the outer union is over $y \in \mathbb{P}^{n-1}$, and we obtain the wave variety $\mathcal{W}_M$. At the other end of the spectrum, the level $0$ wave variety $\mathcal{W}_M^0 = \bigcap_{y \in \mathbb{P}^{n-1}} \ker M(y)$ is often empty. For the in-between levels $r$, we obtain a hierarchy

$$\mathcal{W}_M^0 \subseteq \mathcal{W}_M^1 \subseteq \cdots \subseteq \mathcal{W}_M^{n-1} = \mathcal{W}_M \subseteq \mathbb{P}^{k-1}. \tag{4.16}$$

We now define a second hierarchy in $\mathbb{P}^{k-1}$ by switching the intersections and the union. Namely, for any integer $r \in \{1, \ldots, n\}$, we define the *level $r$ obstruction variety* to be

$$\mathcal{O}_M^r := \bigcap_{\sigma \in \mathrm{Gr}(r,n)} \bigcup_{y \in \sigma} \ker M(y). \tag{4.17}$$

This intersection is over the Grassmannian of $(r-1)$-dimensional subspaces in $\mathbb{P}^{n-1}$. The smallest and the largest obstruction variety coincides with the corresponding wave variety.

**Lemma 4.2.2.** *We have the inclusions $\mathcal{W}_M^r \subseteq \mathcal{O}_M^{r+1}$ for all $r$, with $\mathcal{W}_M^0 = \mathcal{O}_M^1$ and $\mathcal{W}_M^{n-1} = \mathcal{O}_M^n$.*

*Proof.* Fix $w \in \mathcal{W}_M^r$ and a codimension $r$ subspace $\pi$ of $\mathbb{P}^{n-1}$ such that $M(y)w = 0$ for all $y \in \pi$. Consider any $r$-dimensional subspace $\sigma$ of $\mathbb{P}^{n-1}$. Pick a point $y'$ in the intersection $\pi \cap \sigma$. Since $M(y')w = 0$, we have $w \in \bigcup_{y\in\sigma} \ker M(y)$, and hence $w \in \mathcal{O}_M^{r+1}$. Equality holds for $r = 0$ because $\mathcal{W}_M^0 = \bigcap_{y\in\mathbb{P}^{n-1}} \ker M(y) = \mathcal{O}_M^1$, and for $r = n-1$ because $\mathcal{W}_M^{n-1} = \bigcup_{y\in\mathbb{P}^{n-1}} \ker M(y) = \mathcal{O}_M^n$. $\square$

In analogy to the wave varieties in (4.16), there is also a hierarchy of obstruction varieties:

$$\mathcal{W}_M^0 \;=\; \mathcal{O}_M^1 \;\subseteq\; \mathcal{O}_M^2 \;\subseteq \cdots \subseteq\; \mathcal{O}_M^n \;=\; \mathcal{W}_M \;\subseteq\; \mathbb{P}^{k-1}. \tag{4.18}$$

**Example 4.2.3** ($n = 3, k = 4, r = 2$)**.** Fix the matrix $M$ in Example 2.1.1 and 4.1.5. For every $w \in \mathbb{P}^3$, there exists $y \in \mathbb{P}^2$ with $M(y)w = 0$, and hence $\mathcal{W}_M^2 = \mathcal{O}_M^3 = \mathbb{P}^3$. But, for every $w$, there also exists $y \in \mathbb{P}^2$ with $M(y)w \neq 0$, and hence $\mathcal{W}_M^0 = \mathcal{O}_M^1 = \emptyset$. The variety in the middle of (4.16) and (4.18) satisfies $\mathcal{W}_M^1 = \mathcal{O}_M^2 \subset \mathbb{P}^3$. This is the twisted cubic curve $w = (s^3, s^2t, st^2, t^3)$. Indeed, the matrix $\left(\begin{smallmatrix} w_1 & w_2 & w_3 \\ w_2 & w_3 & w_4 \end{smallmatrix}\right)$ has rank $1$, with kernel $\pi = \{y \in \mathbb{P}^2 \,:\, s^2y_1 + sty_2 + t^2y_3 = 0\} \in \mathrm{Gr}(2,3)$. Every other line $\sigma \in \mathrm{Gr}(2,3)$ in the projective plane $\mathbb{P}^2$ intersects the line $\pi$. $\triangle$

We next recall a basic construction from algebraic geometry; see [30, Example 6.19]. Fix a projective variety $X \subset \mathbb{P}^{n-1}$. The *Fano variety* $\mathrm{Fano}_r(X)$ is the subvariety of the Grassmannian $\mathrm{Gr}(n-r,n)$ whose points are the linear spaces $\pi$ of codimension $r$ in $\mathbb{P}^{n-1}$ that lie in $X$. We use Fano varieties to argue that the inclusion in Lemma 4.2.2 can be strict.

**Example 4.2.4** ($k = \ell = 1, n \geq 3$)**.** A subvariety of $\mathbb{P}^0$ is either empty or a point. Let $M = [a]$ where $a$ is irreducible of degree $d$. Then $\mathrm{Fano}_1(X) = \emptyset$. Our varieties in (4.15) and (4.17) are

$$\mathcal{W}_M^r = \begin{cases} \emptyset & \text{if } \mathrm{Fano}_r(X) = \emptyset, \\ \mathbb{P}^0 & \text{if } \mathrm{Fano}_r(X) \neq \emptyset, \end{cases} \qquad \text{and} \qquad \mathcal{O}_M^{r+1} = \begin{cases} \emptyset & \text{if } r = 0, \\ \mathbb{P}^0 & \text{if } r \geq 1. \end{cases}$$

If $d \geq 2$ then $\mathrm{Fano}_1(X) = \emptyset$, so $\mathcal{W}_M^1$ is strictly contained in $\mathcal{O}_M^2$. Equality holds for $d = 1$. △

Returning to arbitrary $k$ and $\ell$, we now show that equality always holds for first order PDE. The main point for $d = 1$ is this: we can write the product $M(y)w$ as $C(w)y$ where $C(w)$ is an $\ell \times n$-matrix whose entries are linear forms in $w_1, \ldots, w_k$. We did this in (4.10).

**Proposition 4.2.5.** *If $d = 1$ then $\mathcal{W}_M^r = \mathcal{O}_M^{r+1} = \{w \in \mathbb{P}^{k-1} : \mathrm{rank}(C(w)) \leq r\}$ for all $r$.*

*Proof.* Fix $w \in \mathbb{P}^{k-1}$. The condition $w \in \mathcal{W}_M^r$ says that the kernel of the matrix $C(w)$ contains a subspace $\pi$ of codimension $r$. The condition $w \in \mathcal{O}_M^{r+1}$ says that the kernel of $C(w)$ meets every $r$-dimensional subspace $\sigma$ of $\mathbb{P}^{n-1}$. Both conditions are equivalent to $\mathrm{rank}(C(w)) \leq r$. □

Thus, the wave varieties of first order PDE are easy to write down: they are the determinantal varieties of the auxiliary matrix $C(z)$. For $d \geq 2$, elimination methods from nonlinear algebra (e.g. Gröbner bases) are needed to compute the defining equations of these varieties.

**Proposition 4.2.6.** *The wave varieties $\mathcal{W}_M^r$ and the obstruction variety $\mathcal{O}_M^r$ are indeed varieties in the projective space $\mathbb{P}^{k-1}$, i.e. they are zero sets of homogeneous polynomials in $k$ variables.*

*Proof.* The following incidence variety is closed in its ambient product space:

$$\mathcal{I}_M^r \;=\; \big\{ (y, w, \pi) \in \mathbb{P}^{n-1} \times \mathbb{P}^{k-1} \times \mathrm{Gr}(n-r, n) \,:\, M(y)w = 0 \;\text{ and }\; y \in \pi \big\}. \quad (4.19)$$

The sets we defined in (4.15) and (4.17) are derived from this variety by quantifier elimination:

$$\mathcal{W}_M^r = \{ w : \exists \pi \; \forall y \; (y, w, \pi) \in \mathcal{I}_M^r \} \quad \text{and} \quad \mathcal{O}_M^r = \{ w : \forall \pi \; \exists y \; (y, w, \pi) \in \mathcal{I}_M^r \}.$$

These two sets are closed in $\mathbb{P}^{k-1}$ because all their defining equations are homogeneous in each group of variables. For the existential quantifier this follows from the Main Theorem of Elimination Theory [41, Theorem 4.22]. For the universal quantifier once checks it directly.

We compute ideals for $\mathcal{W}_M^r$ and $\mathcal{O}_M^r$ as follows. The equations $M(y)w = 0$ are bihomogeneous of degree $(d, 1)$. The condition $y \in \pi$ translates into bilinear equations in $(y, p)$, where $p$ is the vector of Plücker coordinates of $\pi$. We view these as equations in $y$ with coefficients in $(w, p)$, and we form the ideal of all coefficient polynomials. The zero set of this ideal is the subvariety $\bigcap_{y \in \pi} \ker M(y)$, which lies in $\mathrm{Gr}(r, n) \times \mathbb{P}^{k-1}$. We now project that variety onto the second factor to obtain $\mathcal{W}_M^r$. This amounts to saturating and then eliminating the Plücker coordinates $p$. What arises is an ideal in the unknowns $w$ whose zero set is $\mathcal{W}_M^r$.

To get the ideal of $\mathcal{O}_M^r$, we modify the argument as follows. Again, we consider a fixed but unknown Plücker vector $p$ and we consider the equations for $y \in \pi$ along with $M(y)w = 0$. From these equations we eliminate $y$ to obtain polynomials in $(p, w)$ whose zero set is $\bigcup_{y \in \pi} \ker M(y)$. We now vary $p$ and we view this as a subvariety of $\mathrm{Gr}(r, n) \times \mathbb{P}^{k-1}$. We consider the defining equations of this subvariety, and we write them as polynomials in $p$ whose coefficients are polynomials in $w$. The collection of all such coefficient polynomials defines a subvariety of $\mathbb{P}^{k-1}$. By construction, that subvariety equals the desired set $\mathcal{O}_M^r$. □

## 4.3 Back to Analysis

We now return to the setting of waves $\phi : \mathbb{R}^n \to \mathbb{C}^k$ that was introduced in Section 4.1. The projective varieties $\mathcal{W}_M^r$ and $\mathcal{O}_M^r$ in $\mathbb{P}^{k-1}$ are to be viewed as affine cones in $\mathbb{C}^k$. We write

$$\begin{aligned} \mathcal{W}_{M,\mathbb{R}}^r &:= \bigcup_{\pi \in \mathrm{Gr}_{\mathbb{R}}(n-r,n)} \bigcap_{y \in \pi \setminus \{0\}} \ker M(y), \\ \mathcal{O}_{M,\mathbb{R}}^r &:= \bigcap_{\sigma \in \mathrm{Gr}_{\mathbb{R}}(r,n)} \bigcup_{y \in \sigma \setminus \{0\}} \ker M(y), \end{aligned}$$

where $\mathrm{Gr}_{\mathbb{R}}(r,n)$ is the Grassmannian of $r$-dimensional subspaces in $\mathbb{R}^n$. In these definitions, the kernel of $M(y)$ is over the complex numbers, but $\pi$ and $\sigma$ are required to be real. Hence $\mathcal{W}_{M,\mathbb{R}}^r$ and $\mathcal{O}_{M,\mathbb{R}}^r$ are subsets in $\mathbb{C}^k$, closely related to the projective varieties in (4.15) and (4.17).

Readers of [3] will note that we changed notation and nomenclature. The $\ell$-wave cone $\Lambda_{\mathcal{M}}^\ell$ from [3, Definition 1.2] is the obstruction cone $\mathcal{O}_{M,\mathbb{R}}^r$ here, while the cone $\mathcal{N}_{\mathcal{M}}^\ell$ defined later in [3, eqn (1.8)] is our wave cone $\mathcal{W}_{M,\mathbb{R}}^r$. The coming results will motivate these choices.

Proposition 4.1.2 shows why $\mathcal{W}_{M,\mathbb{R}}^r$ serves as the $r$th wave cone. The distribution in (4.8) has the form $\mathbb{R}^n \to \mathbb{C}^k \colon z \mapsto \delta(Lz) \cdot u$ where $L$ is the $(n-r) \times n$ matrix whose rows are the coefficients of $L_1, \ldots, L_{n-r}$. Recall Remark 4.1.3 for the definition of $\delta(Lz)$ as a distribution.

**Proposition 4.3.1.** *A vector $u \in \mathbb{C}^k$ lies in the wave cone $\mathcal{W}_{M,\mathbb{R}}^r$ if and only if there is a matrix $L \in \mathbb{R}^{(n-r)\times n}$ such that $z \mapsto \delta(Lz) \cdot u$ is a solution to $M$ for all distributions $\delta \in \mathcal{D}'(\mathbb{R}^{n-r}, \mathbb{C})$.*

*Proof.* By definition, a complex vector $u$ lies in the wave cone $\mathcal{W}_{M,\mathbb{R}}^r$ if and only if there exists a real subspace $\pi \in \mathrm{Gr}_{\mathbb{R}}(n-r,n)$ such that $M(\xi)u = 0$ for all $\xi \in \pi \subseteq \mathbb{R}^n$. This is equivalent to saying that $(u, \pi)$ is a wave pair for $M$. If we identify $\pi$ with the rowspace of $L$, then the result follows from Proposition 4.1.2. $\square$

We next present an analogous statement for the obstruction cones $\mathcal{O}^r_{M,\mathbb{R}}$.

**Proposition 4.3.2.** *A vector $u \in \mathbb{C}^k$ lies in $\mathcal{O}^r_{M,\mathbb{R}}$ if and only if, for all $S \in \mathbb{R}^{r\times n}$ of rank $r$, the PDE $M$ has a wave solution $z \mapsto \delta(Sz)\cdot u$ where $\delta$ is nonconstant and bounded.*

*Proof.* Suppose $u \in \mathcal{O}^r_{M,\mathbb{R}}$ and let $\sigma \in \mathrm{Gr}_{\mathbb{R}}(r, n)$ be the real rowspan of the real matrix $S$. Fix a nonzero vector $\xi \in \sigma$ such that $M(\xi)u = 0$, and let $\eta \in \mathbb{R}^r$ such that $\xi = \eta S$. The exponential function $\delta_\eta(t) = \exp(\mathrm{i}\eta^T \cdot t)$ is nonconstant and bounded. Moreover, the function $\delta_\eta(Sz)\cdot u$ is a wave solution to the PDE $M$, by the same calculation as in the proof of Proposition 4.3.1. This proves the only-if direction.

For the if-direction, let $u \notin \mathcal{O}^r_{M,\mathbb{R}}$. There exists $\sigma \in \mathrm{Gr}_{\mathbb{R}}(r, n)$ such that $M(\xi)\cdot u \neq 0$ for all $\xi \in \sigma\backslash\{0\}$. Let $S$ be as before the real matrix with rowspan $\sigma$. Now suppose $\delta(Sz)\cdot u$ is a bounded solution of $M$. By the proof of Proposition 4.1.2, this implies that $\delta(y)$ is a bounded solution of the operator $\alpha(\partial_y) = M(\partial_y S)\cdot u$. This operator is elliptic by our assumption. By classical theory (c.f. [56, Theorem 2-7]), every solution to $\alpha \bullet v = f$ with $f \in C^\infty$ is in $C^\infty$. Therefore a Liouville theorem holds: one can use the Closed Graph Theorem to deduce that there is a constant $C > 0$ such that for any solution of $\alpha \bullet v = 0$ in the unit ball $B_1$ one has

$$\|Dv\|_{L^\infty(B_{1/2})} \leq C\|v\|_{L^\infty(B_1)}\,.$$

Since the operator $\alpha$ is of homogeneous degree $d$, we can use scaling to obtain

$$\|Dv\|_{L^\infty(B_R)} \leq \frac{C}{R}\|v\|_{L^\infty(B_{2R})}.$$

Hence, every bounded solution on $\mathbb{R}^{n-r}$ is constant (c.f. [56, Chapter 2]). So, $\delta$ is constant. $\square$

We used the term "obstruction" for the variety $\mathcal{O}^r_M$ and the cone $\mathcal{O}^r_{M,\mathbb{R}}$ not because their elements are obstructions. Rather, our choice of name refers to role played by the cone $\mathcal{O}^r_{M,\mathbb{R}}$ in the paper [3] which motivated us. Since $\mathcal{O}^r_{M,\mathbb{R}}$ contains the wave cone $\mathcal{W}^{r-1}_{M,\mathbb{R}}$, the latter is empty if the former is empty. Thus, the cone $\mathcal{O}^r_{M,\mathbb{R}}$ being empty is an obstruction to the existence of wave solutions. That obstruction is a key for the "dimensional estimates" in [3, 53].

In this chapter we often transition between real numbers and complex numbers. This occurs at multiple mathematical levels, including trigonometry and projective geometry. The complex numbers represent waves in Section 4.1 and they serve as an algebraically closed field in Section 4.2. However, the argument $z$ of our solutions $\phi(z)$ are real vectors. The spaces (4.7) belong to the field real analysis, as does the study of $M$-free Radon measures in [3, 14, 33]. Recall that a Radon measure is a distribution that admits an integral representation, and one is interested in rectifiability of such measures that satisfy the PDE constraint given by $M$.

This raises the question of how complex analysis fits in. From a purely algebraic point of view, we can certainly consider solutions in the space of holomorphic functions $\phi : \mathbb{C}^n \to \mathbb{C}^k$. All our formal results extend gracefully to that setting. For instance, we can certainly take $\delta$ in (4.8) to be a holomorphic function on $\mathbb{C}^{n-r}$ to get a holomorphic solution $\phi$ to our PDE. However, from an analytic point of view, there are no meaningful waves in complex analysis. The following example is meant to illustrate the importance of reality for making waves.

**Example 4.3.3** ($n = 2, k = \ell = 1,\ d = 1, 2$)**.** We consider PDE for scalar-valued functions in two variables. The *transport equation* $M = \partial_1+\partial_2$ has the solutions $\delta(z_1-z_2)$. These are waves and $\delta$ can be any distribution. The *Cauchy-Riemann equation* $M' = \partial_1 + \mathrm{i}\partial_2$ looks very similar, and we can write its solutions formally as $\delta(z_1 + \mathrm{i}z_2)$. But, these solutions do not come from the wave cone $\mathcal{W}^r_{M,\mathbb{R}}$, since here $\pi = V(x_1 + \mathrm{i}x_2)$ is not real, and these do not give waves. If we allow solutions of the form $\delta(z_1 + \mathrm{i}z_2)$, we must add in the condition that $\delta$ be complex differentiable, therefore smooth. This violates Proposition 4.1.2, which says that $\delta$ can be chosen to be any distribution.

Passing to second order equations, one might compare $\partial_1^2 - \partial_2^2$ and $\partial_1^2 + \partial_2^2$. These two PDE look indistinguishable to the eyes of algebraist, while an analyst will see a *hyperbolic PDE* and an *elliptic PDE*. These two classes have vastly different properties for their solutions. In particular, the latter can only admit smooth solutions. $\triangle$

The affine cones $\mathcal{W}^r_{M,\mathbb{R}}$ and $\mathcal{O}^r_{M,\mathbb{R}}$ can be quite different from the complex varieties $\mathcal{W}^r_M$ and $\mathcal{O}^r_M$. In general we have $\mathcal{W}^r_M \supseteq \mathcal{W}^r_{M,\mathbb{R}}$. Indeed, if $w \in \mathcal{W}^r_M$, there is a linear subspace $\pi \in \mathrm{Gr}(n-r,n)$ such that $M(y)w = 0$ for all $y \in \pi$. For $w \in \mathbb{C}^k$ to lie in $\mathcal{W}^r_{M,\mathbb{R}}$, we must impose the additional condition that the dimension of $\pi \cap \mathbb{R}^n$ is also $n-r$. The inclusions for the obstruction cones are reversed: $\mathcal{O}^r_M \subseteq \mathcal{O}^r_{M,\mathbb{R}}$. The point $w \in \mathbb{C}^k$ lies in $\mathcal{O}^r_M$ if and only if for all $\sigma \in \mathrm{Gr}(r,n)$ there exists $y \in \sigma \setminus \{0\}$ such that $M(y)w = 0$. This condition is relaxed in $\mathcal{O}^r_{M,\mathbb{R}}$, where it suffices to consider those $\sigma$ whose real part $\sigma \cap \mathbb{R}^n$ also has dimension $r$.

### 4.4 Computing Wave Pairs

Our aim is to find wave solutions of a PDE, given by an $\ell \times k$ matrix $M$ whose entries are homogeneous polynomials of degree $d$ in $R$. Each wave (4.8) arises from a wave pair $(z,\pi)$, which serves as a blueprint for creating solutions to the PDE. Our approach allows complete freedom in making waves with desirable analytic properties, by choosing the distribution $\delta$ in Proposition 4.3.1. Inspired by Proposition 4.1.2, we define the *wave pair variety*

$$\mathcal{P}^r_M \;=\; \big\{(w,\pi) \in \mathbb{P}^{k-1} \times \mathrm{Gr}(n-r,n) : M(y)w = 0 \;\text{ for all }\; y \in \pi\big\}.$$

This is a smaller version of the incidence variety $\mathcal{I}^r_A$ we saw in (4.19). The wave variety $\mathcal{W}^r_M$ introduced in (4.15) is the projection of the wave pair variety $\mathcal{P}^r_M$ onto the first factor $\mathbb{P}^{k-1}$. For $r = n-1$ the wave pair variety coindices with the incidence variety in (4.13).

In symbols,

$$\mathcal{P}_M^{n-1} \; = \; \mathcal{I}_M. \tag{4.20}$$

It is instructive to start with the case $k = 1$. Here $\mathcal{P}_M^r$ lives in $\mathbb{P}^0 \times \mathrm{Gr}(n-r, n)$, which we identify with $\mathrm{Gr}(n-r, n)$. Consider the subvariety $\mathcal{S}_M$ of $\mathbb{P}^{n-1}$ that is defined by the $\ell$ entries of the $\ell \times 1$ matrix $M$. This is the support of our PDE, as seen in (4.14). The condition $M(y)w = 0$ for $w \in \mathbb{P}^0$ simply means that $y \in \mathcal{S}_M$. From this we conclude the following fact.

**Corollary 4.4.1.** *If $k = 1$ then $\mathcal{P}_M^r = \mathrm{Fano}_r(\mathcal{S}_M)$ is the Fano variety of the support $\mathcal{S}_M$. The points of $\mathcal{P}_M^r$ are the linear spaces of codimension $r$ in $\mathbb{P}^{n-1}$ that are contained in $\mathcal{S}_M$.*

The software `Macaulay2` has a built-in command `Fano` for computing the ideal of the Fano variety $\mathrm{Fano}_r(\mathcal{S}_M)$ from the entries of $M$. Our results in this section extend this method. We shall describe an algorithm for computing $\mathcal{P}_M^r$ and all the varieties introduced in Section 4.2.

Each of our varieties lies in a projective space or product of projective spaces. What we seek is its *saturated ideal*. To explain what this means, consider the variety $\mathcal{I}_M$ in $\mathbb{P}^{n-1} \times \mathbb{P}^{k-1}$. Its description in (4.13) is easy. The $\ell$ coordinates of $M(y)w$ are polynomials of bidegree $(d, 1)$ in

$$\mathbb{C}[y, w] \; = \; \mathbb{C}[y_1, \ldots, y_n, w_1, \ldots, w_k].$$

However, these $\ell$ polynomials do not suffice. The saturated ideal of the variety $\mathcal{I}_M$ equals

$$\big( \big( \langle \, M(y)w \, \rangle \, : \langle y_1, \ldots, y_n \rangle^\infty \big) : \langle w_1, \ldots, w_k \rangle^\infty \big). \tag{4.21}$$

Saturation is a built-in command in `Macaulay2` [24], but its execution often takes a long time. This crucial step removes extraneous contributions by the irrelevant ideals of $\mathbb{P}^{n-1}$ and $\mathbb{P}^{k-1}$.

**Example 4.4.2** ($k = \ell = n = d = 2$)**.** Suppose $M$ is a $2 \times 2$ matrix whose entries are general quadrics in $\mathbb{C}[y_1, y_2]$. The variety $\mathcal{I}_M$ consists four points in $\mathbb{P}^1 \times \mathbb{P}^1$. Its ideal (4.21) is generated by six polynomials of bidegrees $(0,4)$,$(1,2)$,$(1,2)$,$(2,1)$,$(2,1)$,$(4,0)$. The first and last equation are binary quartics that define the projections $\mathcal{S}_M$ and $\mathcal{W}_M$ into $\mathbb{P}^1$. These data encode the general solution to the PDE $M$. For a concrete example, consider

$$M \;=\; \begin{bmatrix} \partial_1^2 + 4\partial_2^2 & 17\partial_1\partial_2 \\ 2\partial_1\partial_2 & 4\partial_1^2 + \partial_2^2 \end{bmatrix}.$$

Here the general solution $\phi : \mathbb{R}^2 \to \mathbb{C}^2$ is given by the following superposition of waves

$$\phi(z_1, z_2) = \begin{bmatrix} -17 \\ 4 \end{bmatrix} \alpha(2z_1 + z_2) + \begin{bmatrix} 17 \\ 4 \end{bmatrix} \beta(-2z_1 + z_2) + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \gamma(z_1 + 2z_2) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \delta(z_1 - 2z_2),$$

where $\alpha, \beta, \gamma, \delta \in \mathcal{D}'$. This can be also found using a differential primary decomposition: the associated primes are $(x_1 - 2x_2)$, $(x_1 + 2x + 2)$, $(2x_1 - x_2)$, $(2x_1 + x_2)$, and the generators of their respective excess dual spaces are $\left[\begin{smallmatrix} -17 \\ 4 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 17 \\ 4 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$. △

The points $\pi$ in the Grassmannian $\mathrm{Gr}(n-r, n)$ will be represented as in [41, Section 5.1]. We write $\pi$ as the rowspace of an $(n-r) \times n$ matrix $S = (s_{ij})$, that is, $\pi = \{vS \,:\, v \in \mathbb{C}^{n-r}\}$. For a subset $I$ of cardinality $n-r$ in $\{1, \ldots, n\}$, the corresponding subdeterminant of $S$ is denoted $p_I$. Then $p = (p_I) \in \mathbb{C}^{\binom{n}{r}}$ is the vector of *Plücker coordinates* of $\pi$. The resulting embedding of $\mathrm{Gr}(n-r, n)$ into $\mathbb{P}^{\binom{n}{r}-1}$ is defined by the ideal $G$ of *quadratic Plücker relations* [41, Section 5.2]. Subvarieties of $\mathrm{Gr}(n-r, n)$ are represented by saturated ideals in $\mathbb{C}[p]/G$. In the special case $r = n-1$, we identify the Plücker coordinates $p$ with $y = (y_1, \ldots, y_n)$.

The wave pair variety $\mathcal{P}_M^r$ lives in $\mathbb{P}^{k-1} \times \mathbb{P}^{\binom{n}{r}-1}$. We shall compute its saturated ideal in the polynomial ring $\mathbb{C}[w, p]/G$. A pair $(w, \pi)$ lies in $\mathcal{P}_M^r$ if and only if $M(vS)w = 0$ for all $v \in \mathbb{C}^{n-r}$. To express this in Plücker coordinates, we proceed as follows. Write the $\ell$ entries of $M(vS)w$ as linear combinations of the monomials $v^\alpha$, $\alpha \in \mathbb{N}^{n-r}$, with coefficients in

$\mathbb{C}[w, S]$. Let $\mathcal{J}$ be the ideal generated by these coefficients, and consider the ring map $\psi : \mathbb{C}[w,p]/G \to \mathbb{C}[w,S]/\mathcal{J}$ which fixes each $w_i$ and maps $p_I$ to the corresponding minor of $S$.

**Algorithm 6** The ideal of the wave pair variety in Plücker coordinates.

**Require:** A matrix $M \in \mathbb{C}[x_1, \ldots, x_n]_d^{\ell \times k}$ and an integer $r \in \{0, 1, \ldots, n-1\}$
**Ensure:** The saturated ideal in $\mathbb{C}[w,p]/G$ that defines $\mathcal{P}_M^r$ as a subvariety of $\mathbb{P}^{k-1} \times \mathbb{P}^{\binom{n}{r}-1}$
  $S \leftarrow (s_{ij})$, an $(n-r) \times n$ matrix whose entries are variables
  $\mathcal{J} \leftarrow$ the ideal in $\mathbb{C}[w, S]$ generated by the coefficients of the monomials $v^\alpha$ in $M(vS)w$
  $G \leftarrow$ the ideal of quadratic Plücker relations in $\mathbb{C}[p]$, as described in [41, Section 5.1]
  $T \leftarrow \mathbb{C}[w,p]/G$, the coordinate ring of the ambient space $\mathbb{P}^{k-1} \times \mathrm{Gr}(n-r, n)$
  $\psi \leftarrow$ the map from $T$ to $\mathbb{C}[w,S]/\mathcal{J}$ that sends $p_I \mapsto \det(S_I)$ and $w_i \mapsto w_i$
  Compute $\mathcal{I} = \ker \psi$ and write its generators in the polynomial ring $\mathbb{C}[w,p]$
  **return** the ideal saturation $((\mathcal{I} : \langle w \rangle^\infty) : \langle p \rangle^\infty)$, as in (4.21).

To compute the ideal of the wave variety $\mathcal{W}_M^r$, one can now eliminate the Plücker variables from the output of Algorithm 6. This corresponds to projecting onto the first factor of $\mathcal{P}_M^r$.

We implemented Algorithm 6 in `Macaulay2`. For the code and its documentation see

https://mathrepo.mis.mpg.de/makingWaves.

Our command `wavePairs(M,r)` returns generators of the saturated ideal of $\mathcal{P}_M^r$ in $\mathbb{Q}[w,p]/G$, where $G$ is the Plücker ideal, given by the built-in command `Grassmannian(n-r-1, n-1)`. Since Algorithm 6 generalizes the computation of Fano varieties, running it can be slow. A common method for speeding this up is to restrict to an affine patch of the Grassmannian. The optional argument `Patch => ...` implements this. If `Patch` is set to `true`, then the leftmost $(n-r) \times (n-r)$ submatrix of $S$ is the identity, as in [41, eqn (5.2)]. The user can also select other charts by specifying a list of indices.

We now come to the special case of first-order PDE ($d = 1$). These are ubiquitous in applications, and computing the corresponding wave pair varieties is easier. Here we use the $\ell \times n$-matrix $C(w)$ given by $M(y)w = C(w)y$. The $\mathcal{W}_M^r$ are the determinantal varieties of $C(w)$.

**Corollary 4.4.3.** *Let $d = 1$, with notation as in Proposition 4.2.5. The wave pair variety equals*

$$\mathcal{P}_M^r \;=\; \big\{\,(w,\pi) \in \mathbb{P}^{k-1} \times \mathrm{Gr}(n-r,n) \;:\; \pi \subseteq \mathrm{kernel}(C(w))\,\big\}.$$

If $\pi$ is given as the row space of an $(n-r)\times n$ matrix $S$ then $\pi \subseteq \mathrm{kernel}(C(w))$ means that $C(w)\cdot S^T$ is the zero matrix of format $\ell \times (n-r)$. Thus, $\mathcal{P}_M^r$ is a vector bundle over the wave variety $\mathcal{W}_M^r$. We shall explore these determinantal varieties for some scenarios of geometric origin. These specify PDE which admit interesting wave solutions $x \mapsto \delta(Lx)\cdot u$.

**Example 4.4.4** (Cubic Surfaces)**.** Every smooth cubic surface in $\mathbb{P}^3$ is the determinant of a $3\times 3$ matrix of linear forms. The surface contains 27 lines, but that number can drop for special cubics. We here present an example with nine lines, namely Cayley's cubic surface:

$$M \;=\; \begin{bmatrix} \partial_1 & \partial_2 & \partial_3 \\ \partial_2 & \partial_1 & \partial_4 \\ \partial_3 & \partial_4 & \partial_1 \end{bmatrix}, \qquad C \;=\; \begin{bmatrix} w_1 & w_2 & w_3 & 0 \\ w_2 & w_1 & 0 & w_3 \\ w_3 & 0 & w_1 & w_2 \end{bmatrix} \qquad (n=4, k=\ell=3).$$

The only nontrivial wave variety consists of the six points in $\mathbb{P}^2$ where $C(w)$ has rank 2:

$$\mathcal{W}_M^2 \;=\; \mathcal{O}_M^3 \;=\; \big\{\,(1:1:0),(1:-1:0),(1:0:1),(1:0:-1),(0:1:1),(0:1:-1)\,\big\}. \tag{4.22}$$

The cubic surface $\mathcal{S}_M = \{\, y \in \mathbb{P}^3 \,:\, \det(M(y)) = 0 \,\}$ has four singular points. It is shown in [41, Figure 1.1].

The wave pair variety $\mathcal{P}_M^2$ lives in $\mathbb{P}^2 \times \mathbb{P}^5$. Its ideal is the output computed by Algorithm 6:

$$\langle w_1, w_2{-}w_3, p_{14}, p_{23}, p_{24}{+}p_{34}, p_{13}{-}p_{34}, p_{12}{+}p_{34}\rangle \cap \langle w_1, w_2{+}w_3, p_{14}, p_{23}, p_{24}{-}p_{34}, p_{13}{+}p_{34}, p_{12}{+}p_{34}\rangle \cap$$
$$\langle w_2, w_1{-}w_3, p_{13}, p_{24}, p_{14}{+}p_{34}, p_{23}{-}p_{34}, p_{12}{-}p_{34}\rangle \cap \langle w_2, w_1{+}w_3, p_{13}, p_{24}, p_{14}{-}p_{34}, p_{23}{+}p_{34}, p_{12}{-}p_{34}\rangle \cap$$
$$\langle w_3, w_1{-}w_2, p_{12}, p_{34}, p_{14}{+}p_{24}, p_{23}{+}p_{24}, p_{13}{-}p_{24}\rangle \cap \langle w_3, w_1{+}w_2, p_{12}, p_{34}, p_{14}{-}p_{24}, p_{23}{-}p_{24}, p_{13}{-}p_{24}\rangle.$$

Its projection to $\mathbb{P}^2$ is $\mathcal{W}_M^2$, while that to $\mathbb{P}^5$ yields six of the nine points in $\mathrm{Fano}_2(\mathcal{S}_M)$. $\triangle$

We conclude by explicitly computing the wave pair varieties of certain operators that are prominent in the calculus of variations. Such operators are built from div, curl, and grad. We refer to [57, Example 2.1] for a warm-up from the perspective of control theory.

Determinantal varieties are given by imposing rank constraints on matrices [30, Lecture 9]. The following construction realizes such varieties as wave cones of certain natural PDEs.

**Example 4.4.5** (Generic Determinantal Varieties)**.** Let $\mathsf{div} = (\partial_1, \partial_2, \ldots, \partial_n)$, fix $p \geq 2$, and set $k = pn$, $\ell = p$. By taking the $p$-fold direct sum of div, we obtain the first order PDE

$$M \quad = \quad \begin{bmatrix} \mathsf{div} & 0 & \cdots & 0 \\ 0 & \mathsf{div} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \mathsf{div} \end{bmatrix}$$

for distributions $\phi : \mathbb{R}^n \to \mathbb{C}^{p\times n}$ with coordinates $\phi_{ij}$, where $i = 1, \ldots, p$ and $j = 1, \ldots, n$. The matrix $C(w)$ defined by the bilinear equation $M(y)w = C(w)y$ has format $p \times n$. Its entries are distinct variables $w_{ij}$. The wave variety $\mathcal{W}_M^r \subset \mathbb{P}^{pn-1}$ is the determinantal variety of all $p\times n$ matrices $w$ of rank $\leq r$. The wave pair variety $\mathcal{P}_M^r \subset \mathbb{P}^{pn-1}\times \mathrm{Gr}(n{-}r, n)$ consists of pairs $(w, \pi)$ where $\pi$ is in the kernel of $w$. This is a *resolution of singularities* for the determinantal variety $\mathcal{W}_M^r$. We refer to Examples 12.1 and 16.18 in Harris' textbook [30]. $\triangle$

We next come to the curl operator, with its action on matrices as in [33, Example

1.16 (c)]. Fix any integer $n \geq 2$. We write curl for the $\binom{n}{2} \times n$ matrix whose rows are vectors $\partial_i e_j - \partial_j e_i$. We take $M$ to be the $p$-fold direct sum of curl. This matrix has $\ell = p\binom{n}{2}$ rows and $k = pn$ columns. The following holds for this matrix $M$.

**Proposition 4.4.6.** *Let $M$ be the curl operator for distributions $\phi : \mathbb{R}^n \to \mathbb{C}^{p \times n}$. The ideal of its wave pair variety $\mathcal{P}_M^{n-1} \subseteq \mathbb{P}^{pn-1} \times \mathbb{P}^{n-1}$ is generated by the $2 \times 2$ minors of the $(p{+}1) \times n$ matrix*

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{p1} & w_{p2} & \cdots & w_{pn} \end{bmatrix}. \tag{4.23}$$

*The wave variety $\mathcal{W}_M$ is similarly defined by the $2 \times 2$ minors of the $p \times n$ matrix $(w_{ij})$. All other wave pair varieties $\mathcal{P}_M^r$ and wave varieties $\mathcal{W}_M^r$, indexed by $r \leq n-2$, are empty.*

*Proof.* The ideal of the incidence variety $\mathcal{I}_M = \mathcal{P}_M^{n-1}$ is computed by the saturation (4.21) from

$$\langle\, M(y)w \,\rangle \;=\; \big\langle\, y_i w_{kj} - y_j w_{ki} \;:\; k = 1, \ldots, p \;\text{ and }\; 1 \leq i < j \leq n \,\big\rangle.$$

This step removes contributions from the irrelevant maximal ideal. Every $2 \times 2$ minor of (4.23) lies in this saturated ideal. Therefore, that ideal equals the prime ideal generated by all the $2 \times 2$ minors of (4.23). For $r \leq n-2$, we note that $M(y)w = C(w)y$ hands us the matrix

$$C(w) \;=\; - \begin{bmatrix} \mathrm{curl}(w_{11}, \ldots, w_{1n}) \\ \mathrm{curl}(w_{21}, \ldots, w_{2n}) \\ \cdots \quad \cdots \\ \mathrm{curl}(w_{p1}, \ldots, w_{pn}) \end{bmatrix}.$$

One checks that this $p\binom{n}{2} \times n$ matrix cannot have rank $\leq n-2$ unless $w_{ij} = 0$ for all $i, j$. $\square$

# CHAPTER 5

# CONSTANT RANK OPERATORS

Another angle from which we would like to investigate linear systems of PDE comes from the analysis of continuum mechanics problems, where one often studies a nonlinear relation without derivatives, coupled with a linear PDE [65]; this is the so called theory of compensated compactness [20, 27, 44, 64]. In the framework of compensated compactness, the class of linear PDEs that was studied most is that of (real) constant rank operators, as it gives rise to good integral estimates. We say that an operator $M \in R^{\ell \times k}$ has $\mathbb{K}$-constant rank, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, if $\operatorname{rank} M(x)$ is constant for all $0 \neq x \in \mathbb{K}^n$, also see Definition 5.1.3. In a sense, the class of real constant rank operators is the largest class where we can hope for standard harmonic analysis estimates [26]. However, the existing study of this pointwise condition on the evaluations $M(x)$ falls well under the limitations of linear algebra [20, 52, 54].

Thus, one of the main questions we address in this chapter is how to link the nonlinear algebra concepts described in Chapters 1 and 2 to a condition that was, so far, viewed only through a linear lens. One somewhat surprising fact is that there exists real or complex constant rank operators that do not admit a vector potential in $C^\infty(\mathbb{R}^n)$; such operators are as simple as the Laplacian $x_1^2 + x_2^2$ or the gradient operator. On the other hand, any real constant rank operator admits a vector potential in the space $C^\infty_\#(Q) = \{f \in C^\infty(\mathbb{R}^n)\colon f \text{ is } Q\text{-periodic}, \int_Q f = 0\}$ of periodic functions of zero average on the cube $Q = (0,1)^n$ [54]. Interestingly, the vector potential constructed in [54] is not necessarily given by the syzygy matrix $S$, but can be replaced by it, see Corollary 5.3.11 and the discussion thereafter.

Another idea used in analysis [20, 44, 54] formally gives a decomposition that looks very similar to the controllable-uncontrollable decomposition (5.2) below: If $M$ has real constant rank and we are looking for $L^p(\Omega)$ solutions to $Mv = 0$ in a bounded domain $\Omega$, we can write

$$v = Su + \texttt{error} \tag{5.1}$$

where $u \in C_c^\infty(\Omega)$ and the $\texttt{error}$ is negligible in $L^p$. This is very similar, but fundamentally different to the more algebraic controllable-uncontrollable decomposition

$$\mathrm{Sol}_{C^\infty} M = \mathrm{im}_{C^\infty} S + \mathrm{Sol}_{C^\infty} M_u, \tag{5.2}$$

where $M_u$ is an operator with trivial syzygy matrix, and $S$ is the syzygy matrix of $M$. Although the similarity between (5.1) and (5.2) is striking, not much has been done to explore possible connections. In Theorem 5.0.1, we make a first step in this direction, by proving that the constant rank condition of $M$ implies the ellipticity of $M_u$.

**Theorem 5.0.1.** *Let $M$ have real (resp. complex) constant rank. Then $M$ has a controllable-uncontrollable decomposition as in* (5.2) *with real (resp. complex) elliptic $M_u$.*

The converse is not true, as can be seen from Examples 5.3.3 and 5.3.4. Details on the decomposition (5.2) can be found in Section 5.2; the relevant definitions are in Section 5.1.

This result bridges the gap between (5.1) and (5.2) in the following sense: if the relation in (5.2) would extend, say by approximation, to locally integrable vector fields $v$, we would have that

$$v = Su + f, \quad \text{where } M_u f = 0.$$

By ellipticity of $M_u$, $f$ is real analytic, so all the roughness of $v$ is carried by the potential part, $Su$. This is also the phenomenon we encounter in (5.1).

Our approach to prove Theorem 5.0.1 consists of linking the controllable-uncontrollable decomposition (5.2) to properties of point evaluations of $M$. The following is the main novelty in this direction:

**Theorem 5.0.2.** *Let $M$ be a polynomial matrix with complex coefficients that has a controllable-uncontrollable decomposition as in* (5.2)*. Then*

$$\ker_{\mathbb{C}} M_u(\xi) = \{0\} \quad \textit{for all } \xi \in \mathbb{C}^n \textit{ such that } \operatorname{rank}_{\mathbb{C}} M(\xi) = \operatorname{rank}_R M.$$

We prove this result using tools from commutative algebra in Section 5.3, see Theorem 5.3.1. Using the same tools, we can also derive the complex version that improves the real result proved recently in [52] using linear algebra techniques:

**Theorem 5.0.3.** *Let $M$ be a polynomial matrix with complex coefficients and syzygy matrix $S$. Then*

$$\ker_{\mathbb{C}} M(\xi) = \mathrm{im}_{\mathbb{C}} S(\xi) \quad \textit{for all } \xi \in \mathbb{C}^n \textit{ such that } \operatorname{rank}_{\mathbb{C}} M(\xi) = \operatorname{rank}_R M.$$

*If* $\operatorname{rank}_{\mathbb{C}} M(\xi) < \operatorname{rank}_R M$*, we have* $\ker_{\mathbb{C}} M(\xi) \supsetneq \mathrm{im}_{\mathbb{C}} S(\xi)$.

As a simple consequence, we note that if $M$ has complex constant rank, then the pointwise exact relation $\ker_{\mathbb{C}} M(\xi) = \mathrm{im}_{\mathbb{C}} S(\xi)$ holds for all nonzero $\xi \in \mathbb{C}^n$. More generally Theorem 5.0.3 tells us precisely for which points the exact relation

$$\mathrm{Sol}_{C_c^\infty} M = \mathrm{im}_{C_c^\infty} S$$

translates to an exact relation for the evaluations of $M$ and $S$. Theorem 5.0.3 also implies the main results in [52, 54].

Another notable consequence of Theorem 5.0.3 is the characterization of the *uncontrollable* operators, $M_u$ from (5.2), which are characterized by the fact that they have trivial syzygy matrix, $\ker_R M_u = \{0\}$. This is in turn equivalent with $\ker_{\mathbb{C}} M_u(\xi) = \{0\}$ for $\xi$ outside a proper real/complex variety, see also Corollary 5.2.4.

The material in this chapter is based on the paper [29].

## 5.1 Preliminaries

Suppose $R = \mathbb{K}[x_1, \ldots, x_n] = \mathbb{K}[\partial_1, \ldots, \partial_n]$, where $x_i$ is identified with $\partial_i = \frac{\partial}{\partial z_i}$, and let $M \in R^{\ell \times k}$ be a differential operator describing the PDE $M \bullet v = 0$. In some of the more analytically oriented results and definitions in this chapter, we will make the following homogeneity assumption:

*Assumption* 5.1.1 (Homogeneity). We say that $M$ is (row-)homogeneous if for each $i = 1, \ldots, k$ there exist integers $d_i$ such that $M_{ij}$ is homogeneous of degree $d_i$ for each $j = 1, \ldots, \ell$.

Our general goal is to convert algebraic properties of the polynomial matrix $M$ into analytic properties of the system of PDEs $M \bullet v = 0$. To this end, we will only focus on homogeneous systems, i.e. for the remainder of this subsection, operators $M$ are assumed to satisfy Assumption 5.1.1.

The following ellipticity conditions are well understood analytically:

**Definition 5.1.2.** Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. We say that $M$ is $\mathbb{K}$-*elliptic* if $\ker_{\mathbb{C}} M(\xi) = \{0\}$ for all $\xi \in \mathbb{K}^n \setminus \{0\}$.

It is a classical result, see e.g., [32], that $\mathbb{R}$-ellipticity of $M$ is equivalent to analyticity of all distributional solutions of $M \bullet v = 0$. $\mathbb{C}$-ellipticity is also well understood [2, 23, 47, 59] and is equivalent to the fact that all solutions of $M \bullet v = 0$ are not only analytic, but actually polynomials, c.f. Theorem 2.6.7. We will revisit aspects of these results later, in Section 5.3.

Another important property is that of constant rank, which is particularly relevant in the study of compensated compactness [20, 27, 44].

**Definition 5.1.3.** An operator $M$ is said to be of $\mathbb{K}$-*constant rank* if there exists an integer $r$ such that $\operatorname{rank}_{\mathbb{C}} M(\xi) = r$ for all $\xi \in \mathbb{K}^n \setminus \{0\}$.

For $M$ to be $\mathbb{F}$-elliptic it is necessary that $k \leq \ell$. If $M$ is $\mathbb{F}$-elliptic then $M$ has $\mathbb{F}$-constant rank $k$.

The class of $\mathbb{R}$-constant rank operators is, roughly speaking, the largest class where standard harmonic analysis results hold, see [20, 26]. The $\mathbb{C}$-constant rank condition is not as widely used in the analysis literature, but it is algebraically more natural to handle than the $\mathbb{R}$-constant rank condition, as $\mathbb{C}$ is algebraically closed.

We will conclude this subsection with a few examples that illustrate the differences between these conditions. Part of the aim of this chapter will be to compare these pointwise conditions on the evaluations, that come from the "analysis with estimates" of the linear PDE systems, with natural conditions that come from the algebraic geometry angle; for instance, the notions of *controllability* and *uncontrollability*, to be defined in Section 5.2 will play a crucial role.

**Example 5.1.4.** The operators

$$M_1 = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}, \quad M_2 = x_1^2 - x_2^2$$

do not have $\mathbb{R}$-constant rank. Additional examples can be found in Examples 5.3.3 and 5.3.4. △

**Example 5.1.5.** The operators

$$M_3 = x_1^2 + x_2^2, \quad M_4 = \begin{pmatrix} x_1^2 + x_2^2 & 0 & -x_1^2 - x_3^2 \\ 0 & -x_2^2 - x_3^2 & x_1^2 + x_3^2 \\ -x_1^2 - x_2^2 & x_2^2 + x_3^2 & 0 \end{pmatrix}$$

have $\mathbb{R}$-constant rank but fail to have $\mathbb{C}$-constant rank. In fact, $M_3$ is $\mathbb{R}$-elliptic but not $\mathbb{C}$-elliptic. △

**Example 5.1.6.** The operator

$$M_5 = \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}$$

is of $\mathbb{C}$-constant rank but not $\mathbb{R}$-elliptic. △

**Example 5.1.7.** The operators

$$M_6 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad M_7 = \begin{pmatrix} x_1^2 + x_2^2 \\ x_1^2 - x_2^2 \end{pmatrix}, \quad M_8 = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \\ x_2 & x_1 \end{pmatrix}$$

are $\mathbb{C}$-elliptic. △

## 5.2 Controllable–Uncontrollable Decomposition

As we saw in Theorem 2.4.1 and the discussion thereafter, the syzygy matrix $S$ describes all smooth solutions to the PDE $M \bullet v = 0$ if and only if the corresponding quotient module $R^k / \operatorname{im}_R M^\top$ is torsion-free, since the space of smooth functions $C^\infty$ is an injective cogenerator.

Let $\mathcal{F}$ be an injective cogenerator. In control theory, a system $\operatorname{Sol}_\mathcal{F} M$ satisfying the conditions of Theorem 2.4.1, e.g. that $R^k / \operatorname{im}_R M^T$ is torsion-free, is said to be *controllable*. At the opposite end of the spectrum, the system $\operatorname{Sol}_\mathcal{F} A$ is said to be *uncontrollable* when the quotient module is torsion [57]. We record this result in the following simple proposition.

**Proposition 5.2.1.** *Let $U \subseteq R^k$ be an $R$-submodule. The following are equivalent*

1. *The system $\operatorname{Sol}_\mathcal{F}(U)$ is uncontrollable.*

2. *The module $R^k/U$ is torsion.*

3. *The ideal $(0)$ is not an associated prime of $U$.*

We remark that "uncontrollable" is not the same as "not controllable", since an $R$-module $M$ can have a set of torsion elements that is a nontrivial, strict subset of $M$. By exploiting the primary decomposition, we can decompose any solution space into two subspaces, one of which is controllable and the other one uncontrollable.

**Proposition 5.2.2** (Controllable-uncontrollable decomposition)**.** *Let $\mathcal{F} = C^\infty$ or $\mathcal{D}'$ (or any injective cogenerator), and $M \in R^{\ell \times k}$. There exist polynomial matrices $M_c, M_u, S$ such that we have a decomposition*

$$\operatorname{Sol}_{\mathcal{F}} M = \operatorname{im}_{\mathcal{F}} S + \operatorname{Sol}_{\mathcal{F}} M_u,$$

*where*

1. $\operatorname{Sol}_{\mathcal{F}} M_c = \operatorname{im}_{\mathcal{F}} S$*,*

2. $\operatorname{im} M_c$ *is either* $(0)$*-primary or trivial,*

3. *the prime* $(0)$ *is not an associated prime of* $\operatorname{im} M_u^\top$*,*

4. $\operatorname{im}_R M^\top = \operatorname{im}_R M_c^\top \cap \operatorname{im}_R M_u^\top$ *as $R$-modules,*

5. $\ker_R M_u = 0$*, i.e.* $\operatorname{im}_R M_u$ *is a free $R$-module,*

6. $\operatorname{Ass}(\operatorname{im} M^\top) = \operatorname{Ass}(\operatorname{im} M_c^\top) \cup \operatorname{Ass}(\operatorname{im} M_u^\top)$*, and the union is disjoint.*

*In particular, the system* $\operatorname{Sol}_{\mathcal{F}} M_c$ *is controllable, and* $\operatorname{Sol}_{\mathcal{F}} M_u$ *is uncontrollable.*

*Proof.* Our point of departure is the primary decomposition. We write

$$\operatorname{im} M^\top = U_c \cap U_u,$$

where $U_c \subseteq R^k$ is the $(0)$-primary component, and $U_u \subseteq R^k$ is the intersection of all other primary components. If there are no $(0)$-primary components (or if $\operatorname{im} M^\top$ is $(0)$-primary), then $U_c$ (or $U_u$) is equal to $R^k$. Let $M_c^\top, M_u^\top$ be polynomial matrices such that $U_c = \operatorname{im} M_c^\top$ and $U_u = \operatorname{im} M_u^\top$. Since $\mathcal{F}$ is an injective cogenerator, we have the decomposition

$$\operatorname{Sol}_{\mathcal{F}} M = \operatorname{Sol}_{\mathcal{F}} M_c + \operatorname{Sol}_{\mathcal{F}} M_u.$$

If we choose $S$ to be the syzygy matrix of $M_c$, we obtain the required decomposition. Properties 1, 2, 3, 4, 6 follow by construction.

It follows from 2. and Theorem 2.4.1 that $\operatorname{Sol}_{\mathcal{F}} M_c$ is controllable, and from 3. and Proposition 5.2.1 that $\operatorname{Sol}_{\mathcal{F}} M_u$ is uncontrollable.

For property 5. suppose that $\operatorname{Sol}_R M_u = \operatorname{im}_R S_u$ for some nonzero matrix $S_u$. Then for any nonzero compactly supported $w$ the function $S_u w$ is a nonzero compactly supported solution in $\operatorname{Sol}_{\mathcal{F}} M_u$. This is a contradiction, as it follows from the Paley-Wiener Theorem that uncontrollable systems don't contain compactly supported solutions, c.f. Theorem 2.4.2. $\square$

We remark that in the construction above, we chose $S$ to be the syzygy matrix of $A_c$, but in fact it coincides with the syzygy matrix of $A$ itself.

**Theorem 5.2.3.** *For $M$ and $M_c$ as in Proposition 5.2.2*

$$\ker_R M = \ker_R M_c.$$

*Proof.* Let $(\cdot)_{(0)}$ denote the localization at the prime $(0)$, and recall that $R_{(0)} = \mathbb{F}(x_1, \ldots, x_n)$ is the field of rational functions. As a first step we show that $\left(\operatorname{im}_R M_u^\top\right)_{(0)} = R_{(0)}^k$, where $M_u$ denotes the uncontrollable part in a decomposition as in Proposition 5.2.2. For any $v \in R^k$ the ideal $(\operatorname{im}_R M_u^\top : v)$ is nonzero, as $(0)$ is not an associated prime. Hence there is a nonzero $r \in R$ such that $rv \in \operatorname{im}_R M_u^\top$. When we localize at $(0)$ the element

$r$ becomes invertible, so $v = r^{-1}rv \in \operatorname{im}_R M_u^\top$. It follows that $\left(\operatorname{im}_R M_u^\top\right)_{(0)} = R_{(0)}^k$. As $\operatorname{im}_R M^\top = \operatorname{im}_R M_c^\top \cap \operatorname{im}_R M_u^\top$ we have $\left(\operatorname{im}_R M^\top\right)_{(0)} = \left(\operatorname{im}_R M_c^\top\right)_{(0)}$. In the special case when $M$ in uncontrollable we get $\left(\operatorname{im}_R M^\top\right)_{(0)} = R_{(0)}^k$.

Since $\operatorname{im}_R M^\top \subseteq \operatorname{im}_R M_c^\top$, there is a polynomial matrix $B$ such that $M^\top = M_c^\top B$. Suppose $u \in \ker_R M_c$, then $Mu = B^\top M_c u = 0$, so $u \in \ker_R M$.

For the converse, since $\left(\operatorname{im}_R M_c^\top\right)_{(0)} \subseteq \left(\operatorname{im}_R M^\top\right)_{(0)}$, there is some matrix $C$ with entries in $R_{(0)}$ such that $M_c^\top = M^\top C$. Clearing denominators, we have $gM_c^\top = M^\top C'$ for some $0 \neq g \in R$ and a matrix $C'$ with entries in $R$. If $u \in \ker_R M$, then $g(M_c u) = C'^\top M^\top u = 0$, and since $g \neq 0$, we must have $M_c u = 0$. Hence $\ker_R M_c = \ker_R M$. $\square$

While the controllable part is well understood as the image of the vector potential $S$, the uncontrollable part is less explored. Our Theorem 5.0.3 gives the following characterization of uncontrollable operators:

**Corollary 5.2.4.** *Let $M \in R^{\ell \times k}$ be a polynomial matrix and $S$ its syzygy matrix. Let $\mathcal{F}$ be an injective cogenerator, for instance $C^\infty(\mathbb{R}^n)$ or $\mathcal{D}'(\mathbb{R}^n)$. The following are equivalent:*

1. $\ker_{\mathcal{F}} M$ *is uncontrollable, i.e.* $R^k / \operatorname{im} M^\top$ *is torsion,*
2. $\ker_R M = \{0\}$*, i.e.* $S = 0$*,*
3. $\ker_{\mathbb{C}} M(\xi) = \{0\}$ *for all* $\xi \in \mathbb{R}^n$*, except on a proper real variety,*
4. $\ker_{\mathbb{C}} M(\xi) = \{0\}$ *for all* $\xi \in \mathbb{C}^n$*, except on a proper complex variety.*

*The proper variety in each case is* $\{\xi \in \mathbb{F}^n : \operatorname{rank} M(\xi)$ *is not maximal*$\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

*Proof.* Suppose $\ker_R M = \operatorname{im}_R S \neq \{0\}$. If $u$ is any compactly supported function, then $v = S \bullet u$ is a compactly supported solution to $M \bullet v = 0$, so in particular $\ker_{\mathcal{F}} M$ is not uncontrollable. If $\ker_R M = \{0\}$, we can apply the Controllable-Uncontrollable decomposition to get $\operatorname{Sol}_{\mathcal{F}}(M) = 0 + \operatorname{Sol}_{\mathcal{F}}(M_u)$, so in particular $\operatorname{Sol}_{\mathcal{F}}(M)$ is uncontrollable. This proves the equivalence between 1. and 2.

The equivalence of statements 2., 3., and 4. follow from Theorem 5.0.3 and the fact that $\mathbb{R}^n$ is not a subvariety of $\mathbb{C}^n$. $\square$

Therefore, the triviality of the syzygy module characterizes uncontrollability. In contrast, there is no condition on the syzygy matrix alone that can characterize controllability. This follows from Theorem 5.2.3, by taking an operator $M$ that is not controllable and noticing that $\operatorname{im}_R S$ is then the kernel of both the operator $M_c$ that is controllable and of $M$, which is not. We summarize the various alternative definitions of controllability and uncontrollability in Table 5.1.

| | controllable | uncontrollable |
|---|---|---|
| Torsion elements | $R^k/\operatorname{im} M^\top$ is torsion-free | $R^k/\operatorname{im} M^\top$ is torsion |
| Associated primes | $\operatorname{im} M^\top$ is $(0)$-primary | $(0) \notin \operatorname{Ass}(\operatorname{im} M^\top)$ |
| Solution sets | $\operatorname{Sol}_{C^\infty} M = \operatorname{im}_{C^\infty} S$ | $\operatorname{Sol}_{C_c^\infty} M = \{0\}$ |
| Syzygy matrix | **no** condition | $S = 0$ |

Table 5.1: A summary of the equivalent definitions of controllable and uncontrollable operators.

In the setting of Corollary 5.2.4, the nature of the solutions of the PDE $M \bullet v = 0$ can be very different, depending on the structure of the set of points where $\ker_{\mathbb{C}} M(\xi) \neq \{0\}$.

**Example 5.2.5.** Consider the examples $M_2$, $M_3$, $M_7$ from Examples 5.1.4, 5.1.5, and 5.1.7. All three operators are uncontrollable with

$$\ker_R M_i = \{0\}, \quad \operatorname{Sol}_{C_c^\infty} M_i = \{0\}, \quad \operatorname{Sol}_{C^\infty} M_i \neq \{0\}.$$

In each example we investigate the latter set. We also look at the varieties $X_{\mathbb{R}}$, resp. $X_{\mathbb{C}}$ where conditions 3., resp. 4. of Corollary 5.2.4 fail.

If $M = M_2 = \partial_1^2 - \partial_2^2$, then any function $v(z_1, z_2) = f(z_1 \pm z_2)$ for $f \in C^\infty(\mathbb{R})$ is a solution. The operator is not $\mathbb{R}$-elliptic. The varieties $X_{\mathbb{R}}, X_{\mathbb{C}}$ are both pairs of lines.

If $M = M_3 = \partial_1^2 + \partial_2^2$, the solutions are of the form $v(z_1, z_2) = g(z_1 \pm \mathrm{i} z_2)$, where

$g \in C^\infty(\mathbb{C})$. The increase in regularity is substantial, particularly since, in this case, the solutions are real analytic. The operator is $\mathbb{R}$-elliptic, but not $\mathbb{C}$-elliptic. The variety $X_\mathbb{C}$ is again a pair of lines, but now $X_\mathbb{R}$ is the origin.

If $M = M_7 = (M_2, M_3)^\top$, we have that the solutions are $v(z_1, z_2) = az_1z_2 + bz_1 + cz_2 + d$, which are polynomials. This is yet another increase in regularity from being analytic. The operator is $\mathbb{C}$-elliptic. Here both $X_\mathbb{C}$ and $X_\mathbb{R}$ are the origin. $\triangle$

In practice, many $\mathbb{R}$-constant rank operators happen to be also $\mathbb{C}$-constant rank, so the ellipticity of $M_u$ follows from the complex part of Theorem 5.0.1. If $M$ is controllable, then the conclusion of Theorem 5.0.1 is also trivial, as one can choose $M_u = 1$. We present a concrete example where the real part of Theorem 5.0.1 applies nontrivially.

**Example 5.2.6.** Let

$$M = \begin{bmatrix} x(x^2+y^2) & y(x^2+y^2) \end{bmatrix}.$$

The operator drops rank when $x^2 + y^2 = 0$, hence it has $\mathbb{R}$-constant rank, but not $\mathbb{C}$-constant rank. It is not $\mathbb{R}$-elliptic either, nor is it controllable, as we have $\mathrm{Ass}(\mathrm{im}\, M^\top) = \{(0), (x^2+y^2)\}$. The controllable part is given by the operator $M_c = \begin{bmatrix} x & y \end{bmatrix}$, so that $S = \begin{bmatrix} y & -x \end{bmatrix}^\top$. The uncontrollable part corresponds to the operator

$$M_u = \begin{bmatrix} x(x^2+y^2) & -y(x^2+y^2) \\ y(x^2+y^2) & x(x^2+y^2) \end{bmatrix},$$

whose determinant is $(x^2+y^2)^3$, so $M_u$ is indeed $\mathbb{R}$-elliptic. $\triangle$

By replacing the Laplacian $x^2+y^2$ with the wave operator $x^2-y^2$ in the example above, we obtain an example in which $M_u$ is not elliptic:

**Example 5.2.7.** Let

$$M = \begin{bmatrix} x(x^2-y^2) & y(x^2-y^2) \end{bmatrix}$$

Its rank drops whenever $x = \pm y$, so it does not have $\mathbb{R}$-constant rank. The uncontrollable part is described by the operator

$$M_u = \begin{bmatrix} x(x^2-y^2) & y(x^2-y^2) \\ y(x^2-y^2) & x(x^2-y^2), \end{bmatrix}$$

whose solutions take the form

$$v(a,b) = \begin{bmatrix} f_1(a+b) - af_3(a+b) + f_4(a-b) + af_6(a-b) \\ f_2(a+b) + af_3(a+b) + f_5(a-b) + af_6(a-b) \end{bmatrix},$$

where $f_1, \dots, f_6 \in \mathcal{D}'(\mathbb{R})$. △

## 5.3 Generic Rank and Associated Primes

As rank and ellipticity conditions require homogeneity by definition, Assumption 5.1.1 is implicit whenever $\mathbb{K}$-ellipticity/constant rank is mentioned. Let $M_u$ denote the uncontrollable component of a decomposition of a given operator $M \in R^{\ell \times k}$, as in Proposition 5.2.2. Moreover, we let $U$ denote the module $\operatorname{im} M^\top$.

The aim of this section is to prove Theorem 5.3.1 which contains the main Theorems 5.0.1 and 5.0.2.

**Theorem 5.3.1.** *If $M(\xi)$ has maximal rank for a point $\xi \in \mathbb{C}^n$, then $\ker_{\mathbb{C}} M_u(\xi) = \{0\}$. In particular, if $M$ has $\mathbb{F}$-constant rank, then $M_u$ is $\mathbb{F}$-elliptic, for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.*

Here we clarify that the rank of an evaluation $M(\xi)$ is maximal if $\operatorname{rank}_{\mathbb{C}} M(\xi)$ equals the generic rank, i.e. the maximal value of the map $\xi \mapsto \operatorname{rank}_{\mathbb{C}} M(\xi)$. We note that the first part of the result, concerning the point evaluations $M(\xi)$ and $M_u(\xi)$, also holds without any homogeneity restrictions on $M$.

One important observation when it comes to real constant rank is that if $M$ has $\mathbb{R}$-constant rank then this rank is equal to the rank of $M$ at a generic *complex* point, i.e. the maximal rank of $M$. Indeed, if the rank of $M$ were to drop for all of $\mathbb{R}^n$, it would also have to drop for all points in the (complex) Zariski closure of $\mathbb{R}^n$, namely all of $\mathbb{C}^n$.

The key to proving Theorem 5.3.1 is the following result, which we prove later in this section.

**Theorem 5.3.2.** *Let $\mathfrak{p}$ be a nonzero associated prime of $U = R^k / \operatorname{im} M^T$. Then*

$$\{\xi \in \mathbb{C}^n : \operatorname{rank} M(\xi) \textit{ is maximal}\} \cap V(\mathfrak{p}) = \emptyset.$$

*In particular, if $M$ has $\mathbb{R}$-constant rank then the variety $V(\mathfrak{p})$ contains no real nonzero points. If $M$ has $\mathbb{C}$-constant rank, then $\operatorname{Ass}(U) \subseteq \{(0), (x_1, \ldots, x_n)\}$.*

The converse implications of the two theorems are not true, as can be seen from the Euler equations as expressed in [13], presented below in two space dimensions.

**Example 5.3.3.** Let

$$M = \begin{bmatrix} x_1 & 0 & x_2 & x_3 & x_2 \\ 0 & x_1 & -x_3 & x_2 & x_3 \\ x_2 & x_3 & 0 & 0 & 0 \end{bmatrix}$$

This is a controllable operator, and we can simply take $M_u$ to be multiplication by $1$ which is trivially $\mathbb{C}$-elliptic. The generic rank of $M$ is 3, but it drops to 2 when $x_2 = x_3 = 0$, so $M$ does not have $\mathbb{R}$-constant rank. Moreover the only associated prime of the module $U$ is $(0)$.

For an example where the uncontrollable part is elliptic and nontrivial, consider the $9 \times 5$ matrix $N$ obtained from $M$ by multiplying each row by $x_1$, $x_2$, and $x_3$. Then $N$ has the same rank as $M$ in every point, and the associated primes are $(0)$ and $(x_1, x_2, x_3)$. The uncontrollable part $N_u$ is given by a $\mathbb{C}$-elliptic $24 \times 5$ matrix with entries of degree two. Its set of solutions in $C^\infty$ must therefore contain affine functions. △

Clearly any polynomial matrix $M$ has constant rank if and only if $M^\top$ does. If we consider $M^\top$ instead of $M$ in Example 5.3.3 we get the ideal $(x_2, x_3)$ as an associated prime, which describes exactly the points where the rank of $M^\top$ (and hence also $M$) drops. One might be tempted to believe that if the rank of a matrix $M$ drops from the generic rank at a point $\xi$, then $\xi$ is in the variety of an associated prime of the module defined by $M$ or $M^\top$. However this is not the case, as we see in the next example.

**Example 5.3.4.** The operator

$$M = \begin{bmatrix} x_3 & x_3 & 0 & x_3(x_3-x_4) & x_4(x_4-x_2-x_3)+x_2x_3 \\ x_1 & 0 & x_4(x_2+x_3-x_4)-x_2x_3 & 0 & 0 \\ 0 & x_2 & 0 & x_4(x_3-x_4) & 0 \\ 0 & 0 & x_2x_3 & x_1x_3 & x_1x_2 \end{bmatrix}$$

does not have constant rank, but both $R^4/\operatorname{im}_R M$ and $R^5/\operatorname{im}_R M^\top$ are $(0)$-primary. $\triangle$

Recall that the characteristic variety $V(U)$ is given by

$$V(U) = V(\operatorname{Ann}_R R^k/U) = V(\mathfrak{p}_1) \cup \cdots \cup V(\mathfrak{p}_s)$$

where $\operatorname{Ass}(U) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$. Alternatively, the characteristic variety is the vanishing set of the $k\times k$ minors of $M$. Hence $\xi \in V(U)$ if and only if $M(\xi)$ has non-trivial kernel. In this way we get a characterization of $\mathbb{C}$-elliptic and $\mathbb{R}$-elliptic operators as stated in Proposition 5.3.5 below. This may be compared with the description of $\mathbb{C}$-elliptic operators given in [23, Proposition 3.2].

**Proposition 5.3.5.** *Let* $\operatorname{Ass}(U) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$. *Then*

$$V(\mathfrak{p}_1) \cup \cdots \cup V(\mathfrak{p}_s) = \{\xi \in \mathbb{C}^n : \ M(\xi) \text{ has a nontrivial kernel}\}.$$

*In particular, the following are equivalent:*

1. *$M$ is $\mathbb{R}$-elliptic,*

2. *the varieties $V(\mathfrak{p}_i)$ contains no real nonzero points.*

3. $\mathrm{Sol}_{\mathcal{D}'} M$ *consists only of real analytic functions.*

*Moreover, the following are also equivalent:*

1. $M$ *is* $\mathbb{C}$*-elliptic,*

2. $U$ *is either trivial or* $(x_1, \ldots, x_n)$*-primary,*

3. $\mathrm{Sol}_{\mathcal{D}'} M$ *consists only of polynomials.*

That $\mathbb{R}$-ellipticity of $M$ is equivalent to analyticity of all solutions of $Mv = 0$ is well known [32].

*Proof.* The first statement is deduced in the paragraph before the proposition. It follows directly that $M$ is $\mathbb{C}$ or $\mathbb{R}$-elliptic if and only if the characteristic variety contains no nonzero complex or real points respectively. In the complex case this means that the only possible associated prime is the maximal ideal $(x_1, \ldots, x_n)$, as this is the only prime ideal whose variety is the origin.

The remaining equivalence is shown using the Ehrenpreis-Palamodov fundamental principle. If $U$ is $(x_1, \ldots, x_n)$-primary, all solutions to $M \bullet v = 0$ are of the form

$$v(z) = \sum_i \int D_i(x, z) \exp(\langle x^T \cdot z \rangle)\, d\mu_i(x),$$

where the $\mu_i$ are measures supported at the origin, and $D_i(x, z)$ are polynomials in $z$ with rational function coefficients in $x$. Therefore we must have $v(z) = \sum_i c_i B_i(0, z)$ for some constants $c_i$. This is clearly a polynomial in $z$. Conversely, if the characteristic variety contains a nonzero point $\xi \in \mathbb{C}^n$, then there is a nonpolynomial solution, namely $v(z) = u \exp(\xi^T \cdot z)$ for some constant vector $u \in \mathbb{C}^k$. $\square$

Recall from Proposition 5.2.2 that $\mathrm{Ass}(R^k / \operatorname{im} M_u^\top) = \mathrm{Ass}(U) \setminus \{(0)\}$. In combination with Proposition 5.3.5 we obtain the following.

**Corollary 5.3.6.** *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the nonzero associated primes of $U$. Then*

$$V(\mathfrak{p}_1) \cup \cdots \cup V(\mathfrak{p}_r) = \{\xi \in \mathbb{C}^n : \ \ker_{\mathbb{C}} M_u(\xi) \neq \{0\}\}.$$

*In particular $M_u$ is $\mathbb{R}$-elliptic if and only if the varieties $V(\mathfrak{p}_i)$, $i = 1, \dots, r$, contain no real nonzero points, and $M_u$ is $\mathbb{C}$-elliptic if and only if $\operatorname{Ass}(U) \subset \{(0), (x_1, \dots, x_n)\}$.*

*Proof of Theorem 5.3.1.* By Theorem 5.3.2 and Corollary 5.3.6 we have

$$\{\xi \in \mathbb{C}^n : \ \operatorname{rank} M(\xi) \text{ is maximal}\} \cap \{\xi \in \mathbb{C}^n : \ \ker_{\mathbb{C}} M_u(\xi) \neq \{0\}\} = \emptyset.$$

□

In preparation for the proof of Theorem 5.3.2, we introduce some notation and results from [17]. From this point until the end of the section, no homogeneity assumption is needed. In commutative algebra, *Fitting ideals* are important invariants of finitely generated modules.

**Definition 5.3.7.** Let $\phi\colon R^\ell \to R^k$ be a map of free modules, described by a $k \times \ell$ matrix with entries in $R$. The ideal $I_j(\phi)$ is defined as the ideal generated by the $j \times j$ minors (i.e. determinants of submatrices) of the matrix representing $\phi$. The *rank* of $\phi$, denoted $\operatorname{rank}_R \phi$ is the largest integer such that $I_j(\phi) \neq (0)$. If $R^\ell \xrightarrow{\phi} R^k \to V \to 0$ is a presentation of some $R$-module $V$, then the $r$th *Fitting ideal* is the ideal

$$\operatorname{Fitt}_r(V) := I_{k-r}(\phi)$$

We denote by $I(V)$, or $I(\phi)$, the first nonzero Fitting ideal; note that $I(V) = I_{\operatorname{rank}\phi}(\phi)$.

The ideal $I_j(\phi)$ of $j \times j$ minors is independent of the choice of matrix representing $\phi$. This fundamental fact is sometimes referred to as *Fitting's Lemma*.

Given a matrix $M \in R^{\ell \times k}$ we can write down a presentation $R^\ell \xrightarrow{M^\top} R^k \to R^k/U \to 0$, where $U = \operatorname{im} M^\top$. Here the $r$th Fitting ideal is the ideal generated by the minors of size $k-r$ of $M$. Furthermore, $\operatorname{rank}_R M^\top$ coincides with the rank of the evaluated matrix $M(\xi)$ for a generic point $\xi \in \mathbb{C}^n$. Thus it is easy to see that if $M$ is a $\mathbb{C}$-constant rank operator, then its Fitting ideals can only be either $(0)$ or $(x_1, \ldots, x_n)$. This gives us a fully algebraic characterization of PDE given by matrices $M$ of $\mathbb{C}$-constant rank: they correspond to modules $R^k/U$ where $I(R^k/U) = (x_1, \ldots, x_n)$.

We begin by recalling some standard results about exact sequences in commutative algebra.

**Lemma 5.3.8** ([17, Cor. 20.12]). *Suppose*

$$0 \to R^{k_m} \xrightarrow{\phi_m} R^{k_{m-1}} \xrightarrow{\phi_{m-1}} \cdots \xrightarrow{\phi_2} R^{k_1} \xrightarrow{\phi_1} R^{k_0}$$

*is an exact sequence of free $R$-modules. Then*

$$\sqrt{I(\phi_k)} \subseteq \sqrt{I(\phi_{k+1})} \quad \textit{for all } k \geq 1.$$

**Lemma 5.3.9** ([17, Cor. 20.14]). *Let $V$ be an $R$-module with finite free resolution*

$$0 \to R^{k_m} \xrightarrow{\phi_m} \cdots \to R^{k_1} \xrightarrow{\phi_1} R^{k_0} \to V \to 0. \tag{5.3}$$

*If $\mathfrak{p}$ is a prime of $R$ and $d = \operatorname{depth}(\mathfrak{p})$, then $\mathfrak{p} \in \operatorname{Ass} M$ if and only if $\mathfrak{p} \supseteq I(\phi_d)$.*

In our setting $\operatorname{depth}(\mathfrak{p})$ is defined as the maximal length of a regular sequence inside $\mathfrak{p}$. The only prime ideal of depth zero is the ideal $(0)$, so all associated primes of $V$ are detected by Lemma 5.3.9.

Since a morphism of $R$-modules $\phi\colon R^\ell \to R^k$ can be represented by a $k \times \ell$ matrix of polynomials, one can also study the linear map $\phi(\xi)\colon \mathbb{C}^\ell \to \mathbb{C}^k$ corresponding to the evaluation of the entries of $\phi$ for any point $\xi \in \mathbb{C}^n$. In general, while evaluations of exact sequences of $R$-modules are not necessarily exact, the set of points where the evaluated sequence fails to be exact is a proper (complex) algebraic variety. This means that a syzygy matrix $S$ of $M$ has the property that $\operatorname{im}_{\mathbb{C}} S(\xi) = \ker_{\mathbb{C}} M(\xi)$ for almost every $\xi \in \mathbb{C}^n$. It is also clear that this holds over $\mathbb{R}$. For a rigorous proof, see [18, Cor. 3.4].

Now we are ready to prove Theorem 5.3.2.

*Proof of Theorem 5.3.2.* Let $X \subset \mathbb{C}^n$ be the set of points where the rank of $M$ is maximal. Suppose $\mathfrak{p}$ is a nonzero associated prime of $U$, and let $d = \operatorname{depth}(\mathfrak{p})$. Consider a minimal free resolution of $M$, i.e. a free resolution which is also exact. With the notation in (5.3), the map $\phi_1$ is given by the matrix $M^\top$. By Lemma 5.3.9 we have $\mathfrak{p} \supseteq I(\phi_d)$, which implies $\mathfrak{p} \supseteq \sqrt{I(\phi_d)}$. It then follows from Lemma 5.3.8 that $\mathfrak{p} \supseteq \sqrt{I(\phi_1)}$, or equivalently $V(\mathfrak{p}) \subseteq V(I(\phi_1))$. But $V(I(\phi_1))$ are precisely the points $\xi$ where $\operatorname{rank}_{\mathbb{F}} M(\xi) < \operatorname{rank}_R M$. Hence $X \cap V(\mathfrak{p}) = \emptyset$. $\square$

Since the converse of Theorem 5.3.1 fails, it would also be interesting to characterize when an uncontrollable operator is $\mathbb{C}$-elliptic more precisely.

To further improve our understanding of $\mathbb{C}$-constant rank operators, we will give the complex version and sharpen the real result proved in [52]:

**Theorem 5.3.10.** *Let $M \in R^{\ell \times k}$ be a polynomial matrix with syzygy matrix $S$. Then*

$$\ker_{\mathbb{C}} M(\xi) = \operatorname{im}_{\mathbb{C}} S(\xi) \quad \textit{for all } \xi \textit{ where } \operatorname{rank} M(\xi) \textit{ is maximal.}$$

*In the set where the rank is not maximal, we have $\ker_{\mathbb{C}} M(\xi) \supsetneq \operatorname{im}_{\mathbb{C}} S(\xi)$.*

*Proof.* Take a free resolution of $V = R^\ell / \operatorname{im}_R M$ as in eq. (5.3). The maps $\phi_1, \phi_2$ correspond to $M, S$ respectively, and the equality $k = \operatorname{rank} M(\xi) + \operatorname{rank} S(\xi)$ is true almost everywhere.

In particular, the maximal ranks of $M$ and $S$ sum to $k$, so $\ker M(\xi) = \operatorname{im} S(\xi)$ whenever the ranks of $M$ and $S$ are maximal. By Lemma 5.3.8, the variety where the rank of $M$ drops contains the variety where the rank of $S$ drops, hence the rank of $S$ is maximal whenever the rank of $M$ is. $\square$

**Corollary 5.3.11.** *Let $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$. If $M$ has $\mathbb{F}$-constant rank, then the syzygy matrix $S$ has the property*

$$\ker_{\mathbb{C}} M(\xi) = \operatorname{im}_{\mathbb{C}} S(\xi) \quad \textit{for all } \xi \in \mathbb{F}^n \setminus \{0\}.$$

We conclude the section by an example illustrating Theorem 5.3.10.

**Example 5.3.12.** Let $M$ be the matrix from Example 5.3.3. A computation in Macaulay2 gives the syzygy matrix

$$S = \begin{bmatrix} x_2x_3 & 0 & x_3^2 \\ -x_2^2 & 0 & -x_2x_3 \\ -x_1x_3 & -x_2^2 + x_3^2 & -x_1x_2 \\ x_1x_2 & -2x_2x_3 & -x_1x_3 \\ -x_1x_3 & x_2^2 + x_3^2 & x_1x_2 \end{bmatrix}$$

The generic rank of $M$ is 3 and is attained for example at the point $\xi = (0, 1, 0)$. A simple computation shows that $\ker_{\mathbb{C}} M(\xi)$ is spanned by the two vectors $(0\ 1\ 0\ 0\ 0)^\top$ and $(0\ 0\ -1\ 0\ 1)^\top$, which also span $\operatorname{im}_{\mathbb{C}} S(\xi)$. If we instead take the point $\eta = (1, 0, 0)$ then rank $M(\eta)$ drops to 2. In this case $\ker_{\mathbb{C}} M(\eta)$ is a three dimensional space while $\operatorname{im}_{\mathbb{C}} S(\eta) = \{0\}$. $\triangle$

# CHAPTER 6
# CONCLUSION AND FUTURE RESEARCH

One of the *raisons d'être* of this dissertation is to advocate the interplay between concepts in analysis and non-linear algebra. Furthermore, keeping analysis concepts in mind can help build algebraic intuition. For example, while the proof of Theorem 5.2.3 is purely algebraic, the intuition came from analysis: since all compactly supported solutions of a system of PDE must come from the controllable part, which is characterized by a vector potential corresponding to a syzygy matrix, the syzygies of the module $U$ and its $(0)$-primary component should match.

Using tools discussed in this work can allow the encoding of certain algebraic and geometric objects as solutions to PDE.

**Example 6.0.1** ($n = 2, k = 3, l = 6$)**.** Given a $6 \times 3$ matrix $A$ with random complex entries, we set

$$M = \operatorname{diag}(x_1, x_1^2, x_1^3, x_2, x_2^2, x_2^3) \cdot A$$

Let $U = \operatorname{im}_R M^T$. Then $U$ is torus-fixed and $\mathfrak{m}$-primary, where $\mathfrak{m} = \langle \partial_1, \partial_2 \rangle$, and $\operatorname{amult}(U) = 10$. A basis of $\operatorname{Sol}(U)$ is given by ten polynomial solutions, namely the standard basis vectors $e_1, e_2, e_3$, four vectors that are multiples of $z_1, z_1, z_2, z_2$, and three vectors that are multiples $z_1^2, z_1 z_2, z_2^2$. The reader is invited to verify this with `Macaulay2`. Here is the input for one concrete instance:

```
R = QQ[x1,x2]
U = image matrix {{7*x1,5*x1^2,8*x1^3,5*x2,9*x2^2,5*x2^3},
                  {8*x1,9*x1^2,8*x1^3,4*x2,2*x2^2,4*x2^3},
                  {3*x1,2*x1^2,6*x1^3,4*x2,4*x2^2,7*x2^3}}
```

```
solvePDE(U)
```

By varying the matrix $A$, and by extracting the vector multipliers of $1, z_1$ and $z_1^2$, we obtain any complete flag of subspaces in $\mathbb{C}^3$. The vector multipliers of $1$, $z_2$, and $z_2^2$ give us another complete flag of subspaces in $\mathbb{C}^3$, and the multiplier of $z_1 z_2$ gives us the intersection line of the planes corresponding to the multipliers of $z_1$ and $z_2$. This is illustrated in Figure 6.1. Thus *flag varieties*, with possible additional structure, appear naturally in such families.

$$
\begin{tikzcd}
\text{dim} & & & \text{coefficient} & & \\
3 & & & 1 & & \\
2 & & z_1 \arrow[ur, hook] & & z_2 \arrow[ul, hook] & \\
1 & z_1^2 \arrow[ur, hook] & & z_1 z_2 \arrow[ul, hook] \arrow[ur, hook] & & z_2^2 \arrow[ul, hook] \\
0 & & & \{0\} \arrow[ull, hook] \arrow[u, hook] \arrow[urr, hook] & &
\end{tikzcd}
$$

Figure 6.1: The coefficient vectors of the solutions to the PDE in Example 6.0.1 correspond to the above linear spaces with the given inclusions. We obtain two complete flags in $\mathbb{C}^3$, along with one interaction between the two.

$\triangle$

While our methods are limited to submodules of free modules of polynomial rings, this is not particularly restrictive, as this is the setting ubiquitous in computer algebra systems. Representing polynomial modules using Noetherian operators or a differential primary decomposition can have some advantages over representing them using generators. The data required is a set of differential operators and a prime ideal, which can be represented e.g. numerically using witness sets. The dual representation also makes some operations trivial: for example a differential primary decomposition for the intersection of two modules, both given as a differential primary decomposition, is simply the union of the two differential primary decompositions. This is in contrast to the computation of an intersection of modules given by generators, which requires non-trivial Gröbner basis computation. On the other hand, computing a sum of modules given by generators is trivial, while computing the sum of modules given by differential primary decomposition is less so.

The results presented in this dissertation suggest many directions for future study and research. We present a few examples.

## 6.1 Linear PDE with polynomial coefficients

We discuss an application to PDE with non-constant coefficients, here taken to be polynomials. Our setting is the Weyl algebra $D = \mathbb{C}\langle z_1, \ldots, z_n, \partial_1, \ldots, \partial_n \rangle$. A linear system of PDE with polynomial coefficients is a *$D$-module*. For instance, consider a $D$-ideal $I$, that is, a left ideal in the Weyl algebra $D$. The solution space of $I$ is typically infinite-dimensional.

We propose constructing solutions to $I$ with the method of *Gröbner deformations* [55, Chapter 2]. Let $w \in \mathbb{R}^n$ be a general weight vector, and consider the initial $D$-ideal $\mathrm{in}_{(-w,w)}(I)$. This is also a $D$-ideal, and it plays the role of Gröbner bases in solving polynomial equations. We know from [55, Theorem 2.3.3] that $\mathrm{in}_{(-w,w)}(I)$ is fixed under the natural action of the $n$-dimensional algebraic torus $(\mathbb{C}^*)^n$ on the Weyl algebra $D$. This action is given in [55, equation (2.14)]. It gives rise to a Lie algebra action generated by the $n$ Euler operators

$$\theta_i = z_i \partial_i \quad \text{for } \ i = 1, 2, \ldots, n.$$

These Euler operators commute pairwise, and they generate a (commutative) polynomial subring $\mathbb{C}[\theta] = \mathbb{C}[\theta_1, \ldots, \theta_n]$ of the Weyl algebra $D$. If $J$ is any torus-fixed $D$-ideal then it is generated by operators of the form $z^a p(\theta) \partial^b$ where $a, b \in \mathbb{N}^n$. We define the falling factorial

$$[\theta_b] \;:=\; \prod_{i=1}^{n} \prod_{j=0}^{b_i-1} (\theta_i - j).$$

The *distraction* $\widetilde{J}$ is the ideal in $\mathbb{C}[\theta]$ generated by all polynomials $[\theta_b]p(\theta-b) \;=\; z^b p(\theta)\partial^b$, where $z^a p(\theta)\partial^b$ runs over a generating set of $J$. The space of classical solutions to $J$ is equal to that of $\widetilde{J}$. This was exploited in [55, Theorem 2.3.11] under the assumption that $J$ is holonomic, which means that $\widetilde{J}$ is zero-dimensional in $\mathbb{C}[\theta]$. We here drop that assumption.

Given any $D$-ideal $I$, we compute its initial $D$-ideal $J = \mathrm{in}_{(-w,w)}(I)$ for $w \in \mathbb{R}^n$ generic. Solutions to $I$ degenerate to solutions of $J$ under the Gröbner degeneration given by $w$. We can often reverse that construction: given solutions to $J$, we lift them to solutions of $I$. Now, to construct all solutions of $J$ we study the *Frobenius ideal* $F = \widetilde{J}$. This is an ideal in $\mathbb{C}[\theta]$.

We now describe all solutions to a given ideal $F$ in $\mathbb{C}[\theta]$. This was done in [55, Theorem 2.3.11] for zero-dimensional $F$. Ehrenpreis–Palamodov allows us to solve the general case. Here is our algorithm. We replace each operator $\theta_i = z_i\partial_i$ by the corresponding $\partial_i$. We then apply `solvePDE` to get the general solution to the linear PDE with constant coefficients. In that general solution, we now replace each coordinate $z_i$ by its logarithm $\log(z_i)$. In particular, each occurrence of $\exp(u_1 z_1 + \cdots + u_n z_n)$ is replaced by a formal monomial $z_1^{u_1} \cdots z_n^{u_n}$. The resulting expression represents the general solution to the Frobenius ideal $F$.

**Example 6.1.1.** As a warm-up, we note that a function in one variable $z_2$ is annihilated by the squared Euler operator $\theta_2^2 = z_2\partial_2 z_2\partial_2$ if and only if it is a $\mathbb{C}$-linear combination of $1$ and $\log(z_2)$. Consider the Frobenius ideal given by Palamodov's system [10, Example 11]:

$$F \;=\; \langle\, \theta_2^2\,,\, \theta_3^2\,,\, \theta_2 - \theta_1\theta_3 \,\rangle.$$

To find all solutions to $F$, we consider the corresponding ideal $\langle\, \partial_2^2\,,\, \partial_3^2\,,\, \partial_2 - \partial_1\partial_3 \,\rangle$ in

$\mathbb{C}[\partial_1, \partial_2, \partial_3]$. By `solvePDE`, the general solution to that constant coefficient system equals

$$\alpha(z_1) \ + \ z_2 \cdot \beta'(z_1) \ + \ z_3 \cdot \beta(z_1),$$

where $\alpha$ and $\beta$ are functions in one variable. We now replace $z_i$ by $\log(z_i)$ and we abbreviate $A(z_1) = \alpha(\log(z_1))$ and $B(z_1) = \beta(\log(z_1))$. Thus $A$ and $B$ are again arbitrary functions in one variable. We conclude that the general solution to the given Frobenius ideal $F$ equals

$$\phi(z_1, z_2, z_3) \ = \ A(z_1) \, + \, z_1 \cdot \log(z_2) \cdot B'(z_1) \, + \, \log(z_3) \cdot B(z_1).$$

This method can also be applied for $k \geq 2$, enabling us to study solutions for any $D$-module. $\triangle$

## 6.2 Socle Solutions

The solution space $\mathrm{Sol}(M)$ to a system $M$ of linear PDE is a complex vector space, typically infinite-dimensional. The fundamental principle in Theorem 2.7.4 decomposes that space into finitely many natural pieces, one for each of the integrals in (2.12). Each piece is labelled by a rational function $D_{ij}(\mathbf{x}, \mathbf{z})$ in $2n$ variables, and it is parametrized by measures $\mu_{ij}$ on the irreducible variety $V_i = V(\mathfrak{p}_i)$. This corresponds precisely to a differential primary decomposition, where the $D_{ij}$ span the excess dual space as a $\kappa(\mathfrak{p}_i)$-vector space.

This approach does not take full advantage of the fact that $\mathrm{Sol}(M)$, local dual spaces, and excess dual spaces are $R$-modules. Indeed, if $\psi(\mathbf{z})$ is any solution to $M$ then so is $(\partial_i \bullet \psi)(\mathbf{z})$. Therefore, our aim is to consider only $R$-module generators of dual spaces. In terms of solutions, we consider the quotient

$$\mathrm{Sol}(M)/\langle \partial_1, \ldots, \partial_n \rangle \mathrm{Sol}(M) \tag{6.1}$$

capturing of all solutions that cannot be obtained as derivatives of others. This quotient space is still infinite-dimensional over $\mathbb{C}$, but it often has a much smaller description than $\mathrm{Sol}(M)$. A solution to $M$ is called a *socle solution* if it is nonzero in (6.1). We pose the problem of modifying `solvePDE` so that the output is a minimal subset of differential operators which represent all the socle solutions. This would result in a modified "differential primary decomposition" $\{(\mathfrak{p}, \mathbf{t}_\mathfrak{p}, \mathcal{D}_\mathfrak{p})\}_{\mathfrak{p}\in \mathrm{Ass}(M)}$ such that the images of $\mathcal{D}_\mathfrak{p}$ generate the excess dual space as an $R$-module, as opposed to a $\kappa(\mathfrak{p})$-vector space.

It is instructive to revisit the general solutions to PDE we presented in this dissertation, and to highlight the socle solutions for each of them. For instance, in Example 3.2.6 we have $\mathrm{amult}(I) = 4$ but only the last one of the four Noetherian operators in eq. (3.9) gives a socle solution. The first three can be obtained from the last one via the right $R$-action and linear combinations.

## 6.3 Numerical Algebraic Geometry

We advocate the systematic development of numerical methods for linear PDE with constant coefficients. First steps towards the numerical encoding of affine schemes were taken in Section 3.2.2 for ideals $I$ with no embedded primes. The key observation is that the coefficients of the Noetherian operators for the $\mathfrak{p}$-primary component of $I$ can be evaluated at a point $\mathbf{u} \in V(\mathfrak{p})$ using only linear algebra over $\mathbb{C}$. This linear algebra step can be carried out purely numerically.

The next step would naturally be to represent a differential primary decomposition numerically. Along the way, one would extend the current repertoire of numerical algebraic geometry to modules and their coherent sheaves.

Inspired by this, we propose a numerical representation of an arbitrary module $U \subseteq R^k$. Let $\{(\mathfrak{p}, \mathbf{t}_\mathfrak{p}, \mathcal{D}_\mathfrak{p})\}_{\mathfrak{p}\in \mathrm{Ass}(U)}$ be a differential primary decomposition as in Definition 1.7.4. Assuming the ability to sample generic points $\mathbf{u}_\mathfrak{p} \in V(\mathfrak{p})$, we encode the sets $\mathcal{D}_\mathfrak{p}$ by their point evaluations $\mathcal{D}_\mathfrak{p}(\mathbf{u}_\mathfrak{p}) = \{D(\mathbf{u}_\mathfrak{p}, \partial_\mathbf{x})\colon A(\mathbf{x}, \partial_\mathbf{x}) \in \mathcal{D}_\mathfrak{p}\}$. Each evaluated operator $D(\mathbf{u}_\mathfrak{p}, \partial_\mathbf{x})$

gives an exponential solution $D(\mathbf{u}_\mathfrak{p}, \mathbf{z}) \exp(\mathbf{u}_\mathfrak{p}^T \cdot \mathbf{z})$ to the PDE given by $U$ via the correspondence in Theorem 2.5.2. We obtain a numerical module membership test: a polynomial vector $f \in R^k$ belongs to $U$ with high probability if $D(\mathbf{u}_\mathfrak{p}, \partial_\mathbf{x}) \bullet f$ vanishes at the point $\mathbf{u}_\mathfrak{p}$ for all $D \in \mathcal{D}_i(\mathbf{u}_\mathfrak{p})$ and $\mathfrak{p} \in \mathrm{Ass}(U)$. The exponential functions $\mathbf{z} \mapsto D(\mathbf{u}_\mathfrak{p}, \mathbf{z}) \exp(\mathbf{u}_\mathfrak{p}^t \, \mathbf{z})$, which depend on numerical parameters $\mathbf{u}_\mathfrak{p}$, serve as an encoding of the infinite-dimensional $\mathbb{C}$-vector space $\mathrm{Sol}(U)$.

Another potential research direction is the further development of hybrid algorithms, where numerical information is used to speed up symbolic computations. Assuming the numerical approximations to be accurate enough, the output of a hybrid algorithm is exact. In the case of Algorithm 5, numerical methods are used to find particular instances of exponential solutions in $\mathrm{Sol}(U)$, which can then be exploited to provide a better ansatz for the symbolic solutions. It is reasonable to expect that such a hybrid paradigm would extend the command `solvePDE`, in the full generality seen in Algorithm 1.

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