ON DIFFERENCE GRAPH COVERS AND THE LOCAL DIMENSION OF THE BOOLEAN LATTICE

A Thesis
Presented to
The Academic Faculty

By

Ariana Hall

In Partial Fulfillment
of the Requirements for the Degree
Masters of Mathematics in the
College of Sciences
School of Mathematics

Georgia Institute of Technology

May 2022

ON DIFFERENCE GRAPH COVERS AND THE LOCAL DIMENSION OF THE BOOLEAN LATTICE

Thesis committee:

Dr. Anton Bernshteyn
School of Mathematics
Georgia Institute of Technology

Dr. Grigory Blekherman
School of Mathematics
Georgia Institute of Technology

Dr. Christopher Heil
School of Mathematics
Georgia Institute of Technology

Dr. Zhiyu Wang (Advisor)
School of Mathematics
Georgia Institute of Technology

Dr. Ernest Croot
School of Mathematics
Georgia Institute of Technology

Date approved: April 28, 2022

ACKNOWLEDGMENTS

My father predicted that I would become a mathematician. My grandparents live in a cabin-style house in the mountains of North Carolina. Apparently, when I was a little girl, I would lay on the floor in the living room and observe the rafters on the ceiling. I do not recall what I was thinking at that time (probably something appropriate of a small child), but my father said that one day I would become a mathematician. Although he passed away just a few years after those moments, it seems the day he predicted has come to pass. But truly, is the term "mathematician" even appropriate? I have struggled with inadequacy since I began this journey four years ago in August 2018, but it seems that this is the path that God had for me since the beginning. So I first want to thank God for sustaining me on this road and for taking me down His path for my life, and I also want to thank my father. Although absent in my life through no fault of his own, both his life and his death have changed my trajectory; I would not be who I am now without his influence.

I also want to thank my mother who has been the best cheerleader a person could ask for. I certainly would not have accomplished all that I have without her. The support she has unwaveringly bestowed upon me has encouraged me to no end, and her kind, calming words will continue to push me forward. And of course, the fact that she homeschooled me for my entire K-12 education allowed me to see and experience the world in a way that made me fall in love with mathematics.

Lastly I would like to thank every professor and advisor who has helped me through this process, including those who have written letters of recommendation for me. I am particularly grateful to Dr. Zhiyu Wang for helping me to think through things in a different way, and also to Dr. Xingxing Yu for his advice and letter of recommendation through which I received a competitive offer for a PhD program in Mathematics.

TABLE OF CONTENTS

LIST OF FIGURES

SUMMARY

The format of this thesis is simple: We begin with the preliminary definitions and notation that are crucial to understanding the elements of this thesis. Then we introduce the idea of finding the local dimension of partially ordered sets. The following chapter explicitly discusses the local dimension of partially ordered set and introduces the concepts related to graphs. Chapter 4 presents concepts about the dimension and local dimension of the standard example, S_n, with several examples to illustrate them. We then veer away from partially ordered set and partial linear extensions and temporarily fixate on the correspondence between Young Diagrams and Difference Graphs. Additionally, a few nice results are presented regarding a specific type of covering for Young Diagrams.

Chapter 6 is about the Boolean lattice, which is one of the main topics of this thesis. It is, however, presented in several different sections. One concentrates on the bounds of the local dimension and the one afterward on suborders of the Boolean lattice. Chapter 7, being more explicit rather than abstract, presents a direct proof for the local difference cover number of the incomparability graph of the split of $2^{[3]}$, the Boolean lattice of dimension 3. It also encompasses the main results of this thesis. Chapter 8 discusses some open problems.

CHAPTER 1
INTRODUCTION

I could write a very boring introduction about the topic of this thesis. But I feel that is a waste of paper, ink, and reading time, especially since the summary already details the contents. This whole thesis is about various topics which will be presented promptly, so we shall let them come when they come. Let me write about something else.

Fields Metalist Andrei Okounko asserted that, "Understanding examples links with ability to compute...I worry that...this is a skill that is not adequately emphasized and developed" [1]. In spirit of his concern, this thesis will include a variety of examples. It is possible, in fact, that the majority of this thesis is examples. I learned the topics through creating my own illustrations of the ideas, and that is how I intend to teach it. In the words of several professors I have had in the past, "An example is worth a thousand words." Although some may be prettier than others, mathematics as a whole is a beautiful thing, and it should be presented as such.

The notation in this thesis will be as straight-forward as possible. Although there are many papers cited, each with their own notations, I have modified them to be consistent and intuitive for the ease of every reader, beginning or advanced. For instance, in this thesis, $2^{[n]}_{\ell,k}$ denotes the Boolean suborder induced by layers ℓ and k. In [2], the paper that uses this concept, the author uses the notation $Q^n_{\ell,k}$. Although slightly simpler, the Q is used in other sections to denote the split of a poset, so the prior definition is preferable although less attractive and, quite honestly, more arduous to type. I also tend to find straight-forward notation to be the best. With $2^{[n]}_{\ell,k}$, one can assume that it refers to some type of Boolean lattice, which is the case. Mathematicians have an impressive amount of information contained in their brains; in my non-professional opinion, notation should not take up too much space. This leaves room for more interesting things. Essentially, intuitive notation is best. And, ultimately, that is my philosophy of mathematics, "We should explain what we know in the simplest possible terms with minimal generality" [1]. Each passionate mathematician will be able to take the specific and apply it to the general if needed, so it is important, for the sake of each reader, to make a paper as specific as possible with as many examples as necessary.

CHAPTER 2
PRELIMINARIES

2.1 Posets

Given a set of elements X, a *partial order*, denoted $\leq$, is a binary relation between elements $x, y, z \in X$ that satisfies (1) reflexivity: $x \leq x$, (2) antisymmetry: $x \leq y$ and $y \leq x$ implies that $x = y$, and (3) transitivity: if $x \leq y$ and $y \leq z$ then $x \leq z$.

Comparability between elements could refer to elements being greater than or less than another element, divisibility between elements, or, as is mostly used in this thesis, whether or not elements are subsets of each other. Given $x, y \in X$, we say that x is comparable to y if and only if $x \leq y$ or $y \leq x$. We say that x is incomparable to y if and only if x is not comparable to y, and this is denoted $x \perp y$.

A *partially ordered set*, also called a *poset*, is denoted $P = (X, \leq)$ where X is a set of elements equipped with the partial order $\leq$.

A Hasse diagram takes the elements of a poset and visually realizes the comparabilities between the elements. Each element of the poset represents a vertex in the Hasse diagram. If $x \leq y$ in the poset, then the Hasse diagram will have an edge between vertex x and vertex y if and only if there does not exist some z such that $x \leq z \leq y$. Additionally, x will be below y in the Hasse diagram since $x \leq y$.

The next concept is that of a *powerset*. Taking any set S, the powerset of S, denoted $\mathcal{P}(S)$, is the set that contains all subsets of S. A powerset is also a poset with the comparability determined through subset relations. Formally, $\mathcal{P} = (S, \subseteq)$.

As an example of a Hasse Diagram, take the set $[3] = \{1, 2, 3\}$. Then

$$\mathcal{P}[3] = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

The elements $\{1\}$ and $\{1, 2\}$ are comparable since $\{1\} \subset \{1, 2\}$, whereas $\{3\}$ and $\{1, 2\}$ are incomparable. The Hasse diagram of $\mathcal{P}[3]$ is the following:

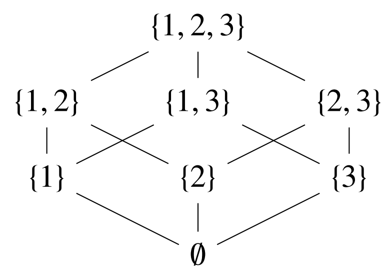


Figure 2.1: Hasse Diagram of $\mathcal{P}[3]$

This diagram explicitly shows the comparabilites and incomparabilites of $\mathcal{P}[3]$, since comparable elements have an edge between them whereas incomparable elements do not. These facts make Hasse Diagrams useful tools in the later discussions contained in this thesis.

Definition 2.1.1. *A* totally ordered set*, also called a* chain*, is a partially ordered set such that every pair of elements in the set should be comparable.*

An example of a totally ordered set would be integers with the comparability $\leq$ (not to be confused with the standard comparability notation "$\leq$"). For any distinct $a, b \in \mathbb{N}$, we know that either $a \leq b$ or $b \leq a$.

Definition 2.1.2. *Given a poset $P = (X, \leq)$, a* linear extension *of P is a chain $L = (X, \leq)$ of the elements of P such that if $x \leq y$ in P then $x \leq y$ in L.*

Definition 2.1.3. *A* realizer *of a poset P is a family of linear extensions, $\mathcal{L}$, of P such that $x \leq y$ in each $L \in \mathcal{L}$ if and only if $x \leq y$ in P.*

Definition 2.1.4. *Given a poset P, the dimension of P, denoted dim(P), is the minimum size of a realizer of P.*

A question remains, however. How do we find these linear extensions? Let us refer back to the example above with the poset $\mathcal{P}[3]$. In the Hasse diagram, $\{1\}$ is below $\{1, 2\}$

since $\{1\} \preceq \{1,2\}$. So in each linear extension, we must have $\{1\}$ below $\{1,2\}$. However, since $\{3\} \perp \{1,2\}$, we must have $\{3\}$ below $\{1,2\}$ in one linear extension and $\{1,2\}$ below $\{3\}$ in another; this will demonstrate the incomparability. Since the realizer is the family of linear extensions, $\mathcal{L}$, we should know the relationships between any two elements in the given poset based solely on $\mathcal{L}$.

Continuing with $\mathcal{P}[3]$, we hypothesize that $\dim(\mathcal{P}[3]) = 3$ since there are three pairs of incomparable elements in $\mathcal{P}[3]$ that may cause some difficulties. These are $\{1\} \perp \{2,3\}$ being the first, $\{2\} \perp \{1,3\}$ the second, and $\{3\} \perp \{1,2\}$ the third. This will imply that two PLEs will not be sufficient to realize all of the incomparabilities within the powerset.

Incomparable elements like $\{1\}$ and $\{2\}$ are easy since $\{2\}$ will always be below $\{2,3\}$, so in one linear extension we can put $\{1\}$ above $\{2,3\}$; in another, we can put $\{1\}$ below $\{2\}$, and this will create the incomparabilty between $\{1\}$ and $\{2\}$ automatically from $\{1\} \perp \{2,3\}$. An example of a family of linear extensions that gives the upper bound of the dimension is the following:

$$\underbrace{\begin{array}{ccc}
\{1,2,3\} & \{1,2,3\} & \{1,2,3\} \\
\{2,3\} & \{1,2\} & \{1,2\} \\
\{1,3\} & \{1,3\} & \{2,3\} \\
\{3\} & \{1\} & \{2\} \\
\{1,2\} & \{2,3\} & \{1,3\} \\
\{2\} & \{2\} & \{1\} \\
\{1\} & \{3\} & \{3\} \\
\emptyset & \emptyset & \emptyset \\
L_1 & L_2 & L_3
\end{array}}_{\mathcal{L}}$$

This implies that $\dim(\mathcal{P}[3]) \leq 3$, but since the dimension of $\mathcal{P}[3] \neq 2$, then we have found that $\dim(\mathcal{P}[3]) = 3$.

The next important definition is the following:

Definition 2.1.5. *Given a poset $P = (X, \preceq)$, a* partial linear extension*, or PLE, of P is a linear extension of a subposet of P. In essence, a PLE is the same as a linear extension except that a PLE does not need to include every element of the poset.*

Definition 2.1.6. *Given a poset $P = (X, \leq)$, a* local realizer *of P is a family $\mathcal{L}$ of PLEs such that (1) if $x \leq y$ in P then $x \leq y$ in $L \in \mathcal{L}$ and (2) if $x \perp y$ then there exist $L, L' \in \mathcal{L}$ such that $x \leq y$ in L and $y \leq x$ in L'.*

A local realizer is in the same vein as a realizer, except that it relies on both PLEs and and possibly linear extensions rather than only linear extensions. A local realizer is a family of PLEs and linear extensions such that every comparability in our poset is realized through the them in the family.

For example, the following local realizer of $\mathcal{P}[3]$ will have four PLEs:

$$\begin{array}{cccc}
\{1,2,3\} & & & \\
\{2,3\} & \{1,2\} & & \\
\{1,3\} & \{1,3\} & \{2\} & \\
\{1,2\} & \{2,3\} & \{3\} & \{1\} \\
\{3\} & \{1\} & \{1,2\} & \{2,3\} \\
\{2\} & \{2\} & \{1,3\} & \\
\{1\} & \{3\} & & \\
\emptyset & & & \\
L_1 & L_2 & L_3 & L_4
\end{array}$$

Hence, $\mathcal{L} = \{L_1, L_2\}$ is a local realizer rather than a realizer of $\mathcal{P}[3]$ since it contains both linear extensions and partial linear extensions.

Definition 2.1.7. *Given a poset P, a realizer $\mathcal{L}$ of P, and some $x \in P$, the frequency of x in $\mathcal{L}$, denoted $\mu(x, \mathcal{L})$ is the number of PLEs in $\mathcal{L}$ that contain x. Additionally, $\mu(\mathcal{L}) = \max_{x \in P} \mu(x, \mathcal{L})$ denotes the* maximum frequency *over all elements of P.*

The frequency will allow us to determine, in a sense, "how good" a local realizer is. For example, take the local realizer of $\mathcal{P}[3]$ from above and let $x = \{1\}$. Since $\{1\}$ appears in three PLEs, then $\mu(\{1\}, \mathcal{L}) = 3$. The maximum frequency over all elements in a poset P is denoted $\mu(\mathcal{L})$. In the case of $\mathcal{P}[3]$, the elements that occur the most times appear in three of the PLEs, and this implies that $\mu(\mathcal{L}) = 3$ for $\mathcal{P}[3]$.

Definition 2.1.8. *The* local dimension *of a poset P is the smallest $\mu(\mathcal{L})$ taken over all possible local realizers $\mathcal{L}$ of the poset P. Explicitly,* $\text{ldim}(P) = \min_{\mathcal{L}} \mu(\mathcal{L})$.

Any realizer can also be a local realizer since a linear extension is also a partial linear exention. Thus, it follows that, in general, $\operatorname{ldim}(P) \leq \dim(P)$ for a poset P.

And, as a final "hooah" for this section, Hiraguchi proved in [3] the following theorem:

Theorem 2.1.1. *[3] Given a poset $P = (X, \preceq)$ then $\dim(P) \leq \lfloor n/2 \rfloor$ where n is the number of elements in P.*

Remark 2.1.1. *The standard example S_n in Chapter 4 shows that this bound is the best possible.*

2.2 Difference Graphs

A *height-two poset* is one in which there is no chain with three elements.

Definition 2.2.1. *Given a poset $P = (X, \preceq)$ on n elements, the split of P is a height-two poset Q on $2n$ elements, with minimal elements $\{x' : x \in P\}$ and maximal elements $\{x'' : x \in P\}$ such that for all $x, y \in P$, $x' \leq y''$ in Q if and only if $x \leq y$ in P.*

Further, the set of minimal elements is denoted $A = \{x' : x \in P\}$ and the maximal elements $B = \{x'' : x \in P\}$.

Definition 2.2.2. *Given a height-two poset Q, the* incomparability graph of *Q, denoted* $\operatorname{In}(Q)$*, is a bipartite graph with partite classes $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ such that there exists an edge $a_i b_j$ if and only if $a_i \perp b_j$ in Q.*

These graphs are the visual representations of the incomparablities within the split of a poset, and they will prove extremely useful in the near future.

Definition 2.2.3. *For positive integers a and b, a* difference graph*, denoted $H = H(a, b; f)$, is a bipartite graph with $a + b$ vertices with partite sets $U = \{u_1, \ldots, u_a\}$ and $W = \{w_1, \ldots, w_b\}$ equipped with a non-increasing function $f : [a] \to [b]$ such that $f(1) = b$ and, for all $i \in [a]$, $N(v_i) = \{w_1, \ldots, w_{f(i)}\}$ if $f(i) \geq 1$.*

Difference graphs and incomparability graphs are used hand-in-hand, and we will use difference graphs to create covers of the incomparability graphs of the split of our posets.

Definition 2.2.4. *A* difference graph cover *of a graph G is a family $\mathcal{H}$ of subgraphs of G such that $E(G) = \bigcup_{H \in \mathcal{H}} E(H)$ and each H is a difference graph.*

In particular, given a height-two poset Q (which is the split of a poset P), we would want a difference graph cover to contain all edges of $\mathrm{In}(Q)$ since the edges of $\mathrm{In}(Q)$ represent the incomparabilities within the split of the poset. If not all edges of $\mathrm{In}(Q)$ are covered, then the difference graph cover will not encompass all of the incomparabilities in the split.

Definition 2.2.5. *The* total difference graph cover number*, abbreviated* tdc*, takes the smallest number of vertices in each difference graph cover of* $\mathrm{In}(Q)$*. Formally,*

$$\mathrm{tdc}(\mathrm{In}(Q)) = \min_{\mathcal{H}} \left\{ \sum_{H \in \mathcal{H}} |V(H)| \ : \ \mathcal{H} \text{ is a difference graph cover of } \mathrm{In}(Q) \right\}.$$

As an example, take the graph

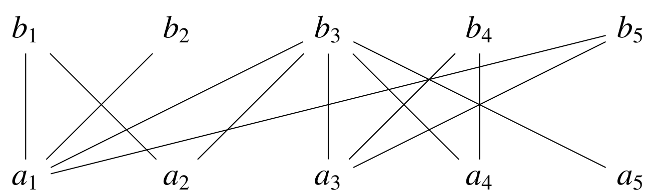


Figure 2.2: Graph $\mathrm{In}(Q)$

The key to finding a difference graph cover of $\mathrm{In}(Q)$ is to look for nested neighborhoods.

Definition 2.2.6. Nested neighborhoods *of a graph are sets of neighbors of vertices that are subsets of each other. For example, in our graph above, $N(a_2) \subset N(a_1)$, so the neighborhoods of a_2 and a_1 are nested.*

Since $N(a_2) \subset N(a_1)$ and $N(a_5) \subset N(a_4) \subset N(a_3)$, splitting these into two difference graphs will be a cover of $\mathrm{In}(Q)$. Call them H_1 and H_2 with $\mathcal{H} = \{H_1, H_2\}$. Let H_1 be the graph of the left and H_2 be the graph on the right below.

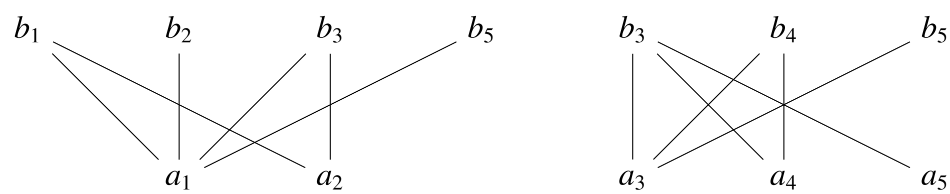


Figure 2.3: Difference graph cover of In(Q)

This is the best covering of In(Q) since the only possibility of changing the difference graphs would be to split H_1 or H_2 into two subgraphs. But this would give more occurrences of each vertex in In(Q), and in turn it would increase tdc(In(Q)). Note that $|V(H_1)| = 6$ and $|V(H_2)| = 6$, and since this is the best $\mathcal{H}$, we find that tdc(In(Q)) = 12.

Definition 2.2.7. *Given a graph G (note that this could be In(Q)), a difference graph cover $\mathcal{H}$ of G and a vertex $v \in V(G)$, the multiplicity of v in $\mathcal{H}$, denoted* $\text{mult}(v, \mathcal{H})$*, is the number of difference graphs in $\mathcal{H}$ containing v. The* local difference graph cover number *of G, or* ldc(G)*, is defined as*

$$\text{ldc}(G) = \min_{\mathcal{H}} \left\{ \max_{v_i \in V(G)} \{\text{mult}(v, \mathcal{H})\} \ : \ \mathcal{H} \text{ is a difference graph cover of } G \right\}.$$

With our graph In(Q) and difference graph cover $\mathcal{H}$ in Figure 2.3, if we look at the number of appearances of each vertex in the cover, we find that every element appears at most twice, so ldc(In(Q)) ≤ 2.

Definition 2.2.8. *A* complete bipartite graph cover *of a graph G is a set of complete bipartite graphs, $\mathcal{B}$, that cover all of the edges of G such that $\cup_{B \in \mathcal{B}} E(B) = E(G)$.*

This applies directly to the next definition, which is closely related to ldc(G).

Definition 2.2.9. *The* local complete bipartite cover number *of G, denoted lbc(G), is the least ℓ such that there is a cover of G with complete bipartite graphs in which every vertex of G appears in at most ℓ of the subgraphs in the cover.*

The following is an example of a complete biparite graph cover of Figure 2.2.

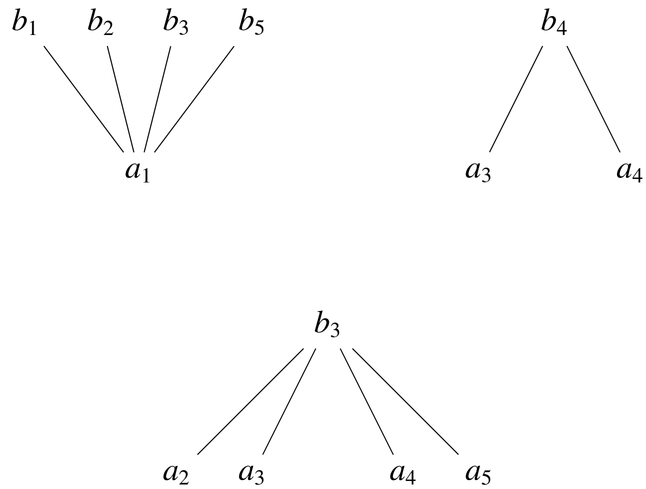


Figure 2.4: $\mathcal{B}(\mathrm{In}(Q))$

Vertices a_1, a_2, and a_5 appear once, and a_3 and a_4 appear twice. Regarding B, we have that b_1, b_2, b_4 and b_5 appear once whereas b_3 appears twice. This implies that lbc(In(Q)) is at most 2.

2.3 Difference Graphs and Partial Linear Extensions

Difference graphs and PLEs are closely linked. Given a difference graph, we can create a PLE of the split of a poset. Also, given a set of PLEs, we can create a set of difference graphs. It is important to remember that the "best" family of PLEs, the local realizer, is what gives the local dimension; similarly, a set of difference graphs realizes the local difference cover number. Hence, the local dimension is closely associated with the local difference cover number.

Given a difference graph, we can construct a PLE based on the nested neighborhoods. Again take the graph In(Q) from above with the difference graph cover in Figure 2.3. The construction of two PLEs is as follows: Starting with H_1, take the vertex with the largest neighborhood and place it at the top of the PLE. In our example, that will be a_1. Note that these could be PLEs rather than linear extensions since they may or may not include every element in the poset. The next elements in the PLE will be the neighbors of a_1 that are not neighbors of a_2, and those are b_2 and b_5. Next will come the vertex of the nested neighbor, a_2, and finally the neighbors shared by a_1 and a_2 will complete the PLE. Thus, the PLE will be $a_1, b_2, b_5, a_2, b_1, b_3$. The process for H_2 is the same. The PLE that comes from H_2 is $a_3, b_5, a_4, b_4, a_5, b_3$. Notice that these two PLEs do not, in fact, make a local realizer of the set of points. There are many incomparabilities that are not realized, and we will need at least one or two more PLEs to make a local realizer. However, in writing these PLEs, it is crucial to maintain comparabilities between elements as well. At first glance, this seems a simple undertaking. However it is quite the contrary.

Doing the opposite, if we are given a set of PLEs, we know that each PLE will be a difference graph. Noting the nested elements of the PLE, we can easily generate the associated difference graphs.

This sets the stage for an important lemma in [4], which is that the local dimension of Q of a poset is equal to the local difference graph cover number of $\mathrm{In}(Q)$ plus or minus two.

Proposition 2.3.1. *[4] If Q is a height-two poset, and $In(Q)$ is the imcomparability graph of Q, then*

$$\mathrm{ldc}(\mathrm{In}(Q)) \leq \mathrm{ldim}(Q) \leq \mathrm{ldc}(\mathrm{In}(Q)) + 2.$$

Proof. The reasoning for this theorem is intuitive. Based on the logic presented above, we know that the local dimension of the split will encompass all comparabilities and incomparabilities from Q. The difference graph cover, however, realizes the incomparabilities between the elements of A and B from Q, but it does not necessarily realize incomparabilities within A and within B. Recall that all elements in A are incomparable with each other and similarly for all elements of B. Thus, if we convert the difference graphs to PLEs, we may need at most two additional PLEs to realize all comparabilities and incomparabilities of the split Q. This implies that the difference between $\mathrm{ldim}(Q)$ and $\mathrm{ldc}(\mathrm{In}(Q))$ is at most two. □

CHAPTER 3

LOCAL DIMENSION OF POSETS

3.1 Bounds of the Local Dimension of a Poset

This chapter discusses the result of [4], which asymptotically determines the maximum local dimension of a poset on n elements. Before we introduce the main result, we state the following lemma, proven by Barrera-Cruz et. al [5], which relates the local dimension of P and its split.

Lemma 3.1.1. *[5] Let P be a poset and Q the split of P. Then,*

$$\operatorname{ldim}(Q) - 2 \leq \operatorname{ldim}(P) \leq 2\operatorname{ldim}(Q) - 1.$$

Although not proven in this thesis, the lemma above will be extremely important for the results of this thesis.

The following two theorems give upper and lower bounds on the maximum local bipartite graph cover of a graph on n vertices. First, the following result of Csirmaz, Ligeti, and Tardos [6] implies that $\operatorname{lbc}(G) \leq (1 + o(1))(n/\log_2 n)$ for any graph G with n vertices.

Theorem 3.1.2. *[6] Let $G = (V, E)$ be a graph on n vertices. The set of edges, E, can be partitioned into complete bipartite subgraphs of such that each vertex is contained in $(1 + o(1))(n/\log_2 n)$ of the bipartite subgraphs.*

The bound in Theorem 3.1.2 is best possible, up to a constant factor, by the following theorem by Chung, Erdős, and Spencer [7].

Theorem 3.1.3. *[7] There is a graph G on n vertices such that for any cover of $E(G)$ with complete bipartite graphs, there is a vertex that appears in $\Omega(n/\log n)$ graphs in the cover. In other words, $\operatorname{lbc}(G) = \Omega(n/\log n)$.*

Lemma 3.1.4. *[4] Given a poset P with n points,* $\mathrm{ldim}(P) \leq (1 + o(1))\frac{4n}{\log_2(2n)}$.

Proof. Define Q as the split of P. Showing that $\mathrm{ldim}(Q) \leq (1 + o(1))\frac{2n}{\log_2(2n)}$ will imply our lemma, since this means that $2\mathrm{ldim}(Q) \leq (1 + o(1))\frac{4n}{\log_2(2n)}$. By Lemma 3.1.1, $\mathrm{ldim}(P) + 1 \leq 2\mathrm{ldim}(Q)$, and our result will be proven.

If Q is the split of P, then Q has a minimal set, A, and a maximal set, B. We will begin with two linear extensions of Q, namely L_1 and L_2. Recall that every element of A should be comparable with every element of B, so let $A < B$ in both linear extensions. The notation "<" is used to denote elements being under other elements in the linear extension or partial linear extension. This is called a *block form*. Essentially, the minimal elements are grouped together and so are the maximal elements. Now, we need the elements of A to be incomparable with each other and similarly for B, so if some $a_i < a_j$ in L_1, then let $a_j < a_i$ in L_2, and use the same reasoning for B. We now have two linear extensions of Q where not only is $a < b$ for all $a \in A$ and $b \in B$, but also all a_i's are incomparable to each other and similarly for all b_i's. The only piece we are missing now is the incomparabilities between elements of A and B.

Let G be a graph such that its vertices are in the set $A \cup B$ with edge set E. Note that $|V(G)| = 2n$. If $ab \in E$, then a and b are incomparable in Q. By Theorem 3.1.2, the edge set of any graph G can be partitioned into complete bipartite graphs where each vertex is in at most $(1 + o(1))\frac{2n}{\log_2(2n)}$ of the complete bipartite subgraphs, $G_1, G_2, \ldots, G_m$. In this case, we use $2n$ in the place of n since $2n$ is the size of the vertex set of G. Notice that the edges must be partitioned since they represent the incomparabilities between elements of A and B. Assume that we have a family of PLEs, $\mathcal{M}$, and $M_i \in \mathcal{M}$ is one of the PLEs in the family. Also, recall that $V(G) = A \cup B$, but since M_i will not have every $a \in A$ and $b \in B$, let $V(G_i) = A_i \cup B_i$ where A_i and B_i are subsets of A and B, respectively, and G_i is one of the complete bipartite subgraphs of G. In M_i, let $B_i < A_i$. Since G_i for $i = 1, \ldots, m$ is partitioned by edges, it is necessary that the maximal elements are less than the minimal elements, because these edges represent the incomparabilities between A and

B. In our other linear extensions, L_1 and L_2, we always had $A < B$, so now, to create the incomparabilities, we must have $B_i < A_i$. Additionally, this is only true for some elements of A and B since there are some comparable elements between A and B, and hence those are realized in L_1 and L_2. With the PLEs, we are only concerned with incomparabilities between A and B. Since all of the comparabilities and incomparabilites between A and B are defined, we will say that $\mathcal{L} = \{L_1, L_2, M_1, \ldots, M_m\}$ is the local realizer for Q, and each element in Q appears at most $(1 + o(1))\frac{2n}{\log_2(2n)}$ times. This comes from the fact that each vertex appears in at most $(1+o(1))\frac{2n}{\log_2(2n)}$ of the PLEs in $\mathcal{M}$ by [4]. Adding the occurrences in L_1 and L_2 adds two more occurrences.

This implies that $\mu(x, \mathcal{L}) = (1 + o(1))\frac{2n}{\log_2(2n)}$, since any element x appears at most that number of times. Since the local dimension is the smallest $\mu(\mathcal{L})$, then $\mu(\mathcal{L}) \leq \mu(x, \mathcal{L})$, and the local dimension of Q is less than $(1 + o(1))\frac{2n}{\log_2(2n)}$, completing the proof of the upper bound. □

For the lower bound of the local dimension of a poset, we first need the following result:

Corollary 3.1.5. *[4] For some $\epsilon > 0$ and n sufficiently large, there exists a bipartite graph G such that*

$$\mathrm{ldc}(G) \geq \frac{\mathrm{tdc}(G)}{n} \geq \left(\frac{1-2\epsilon}{4e}\right)\frac{n}{\ln n}.$$

Lemma 3.1.6. *[4] There exists a poset P with n points such that* $\Omega\left(\frac{n}{\log(n)}\right) \leq \mathrm{ldim}(P)$.

Proof. For this proof, we work in two cases, the first where n is an even number and the second where n is an odd number.

Case 1: Assume that n is an even number. By Corollary 3.1.5, there is a bipartite graph $G = (A \cup B, E)$ with $|A| = |B| = n/2$ such that, for $\epsilon > 0$ and a sufficiently large n, we have

$$\mathrm{ldc}(G) \geq \left(\frac{1-2\epsilon}{4e}\right)\frac{n}{\ln n}.$$

Rewriting the right hand side, we find that

$$\mathrm{ldc}(G) \geq \Omega\left(\frac{n}{\log n}\right).$$

Now, take the split Q that is height-two with A and B the minimal and maximal elements, respectively. Further, let $a \leq b$ in Q if and only if ab is not an edge in G for $a \in A$ and $b \in B$. This means that a and b are comparable.

Next, let $\mathcal{M}$ be a local realizer of P where $\mu(\mathcal{M}) = \mathrm{ldim}(P)$. In other words, $\mathcal{M}$ has the highest concentration of the most commonly occurring element in in extensions. If L_1 and L_2 are two linear extensions of P, let $A < B$ in both linear extensions such that $a_i < a_j$ in L_1 implies that $a_j < a_i$ in L_2 and similarly for the elements of B. So not only do these two extensions maintain the comparability of elements between A and B, but also they create the incomparabilities within A and within B. The only thing we are missing is the incomparabilities between elements of A and B, so let M be a linear extension in $\mathcal{M}$ with the form $A_1 < B_1 < \cdots < A_t < B_t$ where $t \in \mathbb{N}$, and A_t and B_t could be empty. Now, denote M' to be a PLE created from the linear extension M by deleting A_1 and B_t. This is significant since all comparable pairs have already been realized in L_1 and L_2, so deleting A_1 will remove any minimal elements from the bottom of the block form, and deleting B_t will remove any maximal elements from the top of the block form. Thus, if a and b are incomparable, since we already have $a < b$ in both of the linear extensions, we need $b < a$ in M'. Then, B_1 will always have minimal elements above it, and A_t will always have maximal elements below it. Essentially, removing A_1 and B_t makes M' a more efficient form of M. Since $a < b$ for all $a \in A_1$ and $b \in B_1$, removing A_1 makes no real difference in the incomparabilities. This is the same for A_t and B_t. Since A_t and B_t are always comparable, removing B_t only makes M' more efficient. Finally, this implies that $\mu(M') \leq \mu(M) + 2$. We have the +2 since an element will appear at most two more times in M' than M. The case where n is odd follows closely. $\square$

Combining Lemma 3.1.4 and Lemma 3.1.6, we get the following theorem presented and proven in [4], which states that the maximum local dimension among all posets with n elements is $\Theta(n/\log n)$.

Theorem 3.1.7. *[4] Let P be a poset on n elements attaining the maximum local dimension among all posets on n elements. Then*

$$\Omega\left(\frac{n}{\log(n)}\right) \leq \operatorname{ldim}(P) \leq (1 + o(1))\frac{4n}{\log_2(2n)}.$$

CHAPTER 4

LOCAL DIMENSION OF THE STANDARD EXAMPLE

The Standard Example, denoted S_n, is a bipartite graph with partite classes A and B such that $|V(A)| = |V(B)| = n$. Let $a_i \in A$ for $i \in [1, n]$ and $b_j \in B$ for $j \in [1, n]$. There exists an edge $a_i b_j \in E(G)$ if and only if $i \neq j$. See S_7 below:

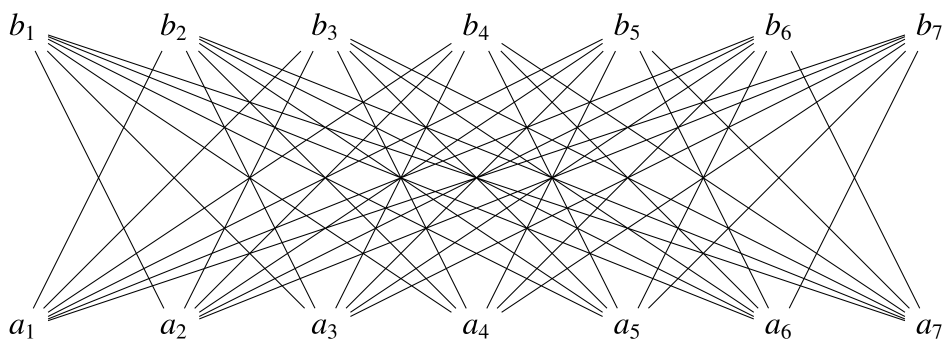


Figure 4.1: S_7

Dushnik and Miller prove in [8] that $\dim(S_n) = n$ whereas $\mathrm{ldim}(S_n) = 3$ for $n \geq 3$. Take, for example, S_5. The graph is

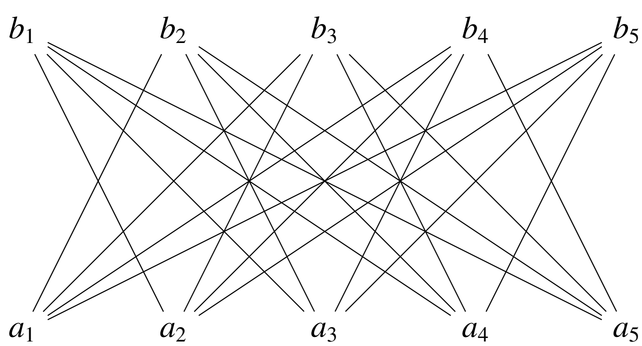


Figure 4.2: S_5

The incomparability graph of S_5, denoted $\mathrm{In}(S_5)$, is the following:

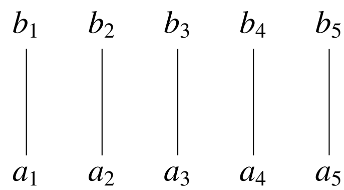


Figure 4.3: In(S_5)

Splitting these into difference graphs will help us calculate the dimension and local dimension, but since none of these are nested, we will have the difference graphs

Figure 4.4: H_i for $i = 1, 2, 3, 4, 5$

This gives us five difference graphs. Now we need to find the linear extensions and PLEs that will create the local realizer of S_5. All elements of B are incomparable to each other and similarly for the elements of A, so we know that in one PLE we should have $a_i < a_j$ for some $i, j = 1, 2, \ldots, 5$ and in another PLE we will have $a_j < a_i$; the same will occur for the elements of B.

Each of our difference graphs above will give one count per vertex of S_5, but since we will not be combining a_i and b_j for $i \neq j$ in a PLE, then first we need two linear extensions of S_5 that realize the incomparabilities within B and within A. Two examples are the following:

b_5	b_1
b_4	b_2
b_3	b_3
b_2	b_4
b_1	b_5
a_5	a_1
a_4	a_2
a_3	a_3
a_2	a_4
a_1	a_5
L_1	L_2

These two linear extensions realize all incomparabilities within the minimal and maximal elements without realizing the incomparabilities between elements of A and B together, since those will be handled in the PLEs from the difference graphs. Those PLEs will be the following:

a_1	a_2	a_3	a_4	a_5
b_1	b_2	b_3	b_4	b_5
L_3	L_4	L_5	L_6	L_7

We require many small PLEs because each element b_i is comparable to each a_j as long as $i \neq j$. To create other longer PLEs could create incomparabilities between elements that are, in fact, comparable in the standard example.

In conclusion, the local realizer of S_5 is $\mathcal{L} = \{L_1, L_2, L_3, L_4, L_5, L_6, L_7\}$ where each a_i and b_i for $i = 1, 2, \ldots, 5$ appears at most three times. Therefore, $\text{ldim}(S_5) = 3$.

Regarding the dimension of S_5, we need to have linear extensions that realize the comparabilities and incomparabilities of S_5. By [8], we know that $\dim(S_5) = 5$, so we will need five linear extensions. Recall that linear extensions, as opposed to partial linear extensions, must contain all of the elements of S_5; this is how we know that we need exactly five linear extensions. Keeping L_1 and L_2 from above, we immediately have a problem. Since those two only realize incomparabilities within A and within B, that only leaves us three more linear extensions to get our realizer. We could do this, but since these are linear extensions, that would give $\dim(S_5) \leq 7$ which, although legitimate, is unnecessarily large. The incomparabilities between A and B are as follows:

$$b_1 \perp a_1, \quad b_2 \perp a_2, \quad b_3 \perp a_3, \quad b_4 \perp a_4, \quad b_5 \perp a_5.$$

Then, we also need the incomparabilities within A and within B. The following are five linear extensions that realize everything we need:

b_5	b_1	b_2	b_5	b_4
b_4	b_3	b_1	b_3	b_3
b_3	b_4	b_5	b_2	b_2
b_2	b_5	b_4	b_1	b_1
a_1	a_2	a_3	a_4	a_5
b_1	b_2	b_3	b_4	b_5
a_2	a_5	a_4	a_1	a_1
a_3	a_4	a_5	a_2	a_2
a_4	a_3	a_1	a_3	a_3
a_5	a_1	a_2	a_5	a_4
L_1	L_2	L_3	L_4	L_5

We cannot do this in any less than five linear extensions because we must be careful to keep a_i comparable with b_j when $i \neq j$. If we tried to create more incomparabilities in each linear extension, we would arrive at a contradiction since, in some place, there would be $b_j < a_i$ even though they are meant to be comparable.

CHAPTER 5

CORRESPONDENCE BETWEEN YOUNG DIAGRAMS AND DIFFERENCE GRAPHS

In a previous chapter, we presented the association between PLEs and difference graphs. Now, we will take this a step further and make the connections between difference graphs and Young Diagrams. This will be crucial for future problems since Young Diagrams are easier to generate computationally, and the process of coloring them will give us the local difference graph cover number of the associated graph covered by the set of difference graphs.

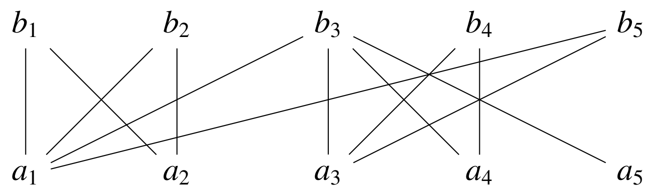
Definition 5.0.1. *A* Young Diagram *(also called a Ferrers diagram, particularly when represented using dots) is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths in non-increasing order.*

In this thesis, a Young Diagram is represented as a chart that illustrates the incomparabilities between elements of the split of a poset, and they are of the form:

	b_1	b_2	$\cdots$	b_m
a_1				
a_2				
$\vdots$				
a_n				

where squares are colored based on the incomparabilities between elements. For instance, if $a_1 \perp b_2$, then the square at the intersection of those two vertices would be colored.

Let us discuss the conversion between a difference graph and a Young diagram. The split, Q, of a poset gives a bipartite incomparability graph containing two partite classes, one of which is the minimal elements of Q and the other is the maximal elements of Q. Say, for example, we have $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$, and let us assume, for the sake of example, that we have the incomparability graph $\text{In}(Q)$:



To create Young Diagrams, we first need to split $\text{In}(Q)$ into a difference graph cover:

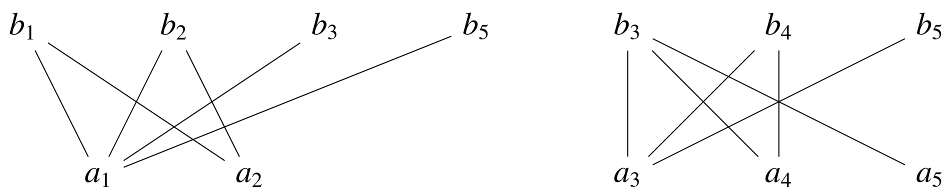


Figure 5.1: Difference graph cover of $\text{In}(Q)$

Let H_1 be the graph of the left and H_2 the graph on the right. Each of these graphs will give a Young Diagram, which will be denoted $\text{YD}(H_i)$ for $i = 1, 2$, and each edge will represent a square of the Young Diagram; therefore, the Young Diagram will show the incomparabilities between the elements. The two Young diagrams will be:

	b_1	b_2	b_3	b_5
a_1	X	X	X	X
a_2	X	X		

$\text{YD}(H_1)$

	b_3	b_4	b_5
a_3	X	X	X
a_4	X	X	
a_5	X		

$\text{YD}(H_2)$

Similarly, we can use Young Diagrams to generate the difference graph cover of the incomparability graph of the split of a poset.

CHAPTER 6

THE BOOLEAN LATTICE

6.1 Bounds of Local Dimension of the Boolean Lattice

For the most part, the rest of this thesis will show results regarding the Boolean Lattice. Note that the ground set is simply the set containing the elements of the poset.

Definition 6.1.1. *The* Boolean Lattice*, denoted* $2^{[n]}$*, is a poset with ground set* $\mathcal{P}[n]$ *and comparability "*$\subseteq$*". Formally,* $2^{[n]} = (\mathcal{P}[n], \subseteq)$.

In the form of a diagram, the lattice of $2^{[3]}$ happens to be the same as the Hasse Diagram of $\mathcal{P}[3]$, which is the following:

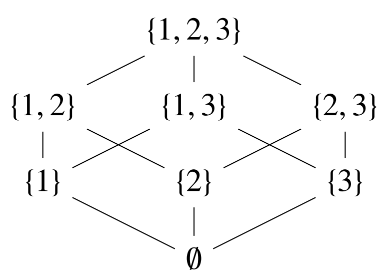


Figure 6.1: $2^{[3]}$

In [4], Kim et. al proved the following theorem regarding the local dimension of the Boolean Lattice:

Theorem 6.1.1. *[4] For n a positive integer,*

$$\frac{n}{2e\log n} \leq \operatorname{ldim}\left(2^{[n]}\right) \leq n.$$

The proof for this theorem is involved, so first we will discuss some of the important ideas used therein.

Following the notation of [4], let P be an *induced* subposet of $2^{[n]}$. If we have $0 \leq s \leq n$, then $\binom{[n]}{s}$ denotes the subsets of the set $[n]$ containing s elements. We call $\binom{[n]}{s}$ the s^{th} layer. Layers can induce subposets of $2^{[n]}$, so we will denote this subposet $P_n(s,t) = P_n$ where s and t are the layers. Then $\text{ldim}(P_n)$ is the local dimension of the induced subposet. Since P_n has fewer elements than $2^{[n]}$, clearly $\text{ldim}(P_n) \leq \text{ldim}(2^{[n]})$ and similarly for the dimension.

Proof. Let n be a sufficiently large integer, and let $k = \lceil n/e \rceil$. If we can show that $\text{ldim}(P_n(1, n-k)) = \Omega(n/\log n)$, then we will get the lower bound since $\text{ldim}(P_n(1, n-k)) \leq \text{ldim}(2^{[n]})$. Take the auxiliary bipartite graph $G_n(1, n-k) = (\mathcal{V}, \mathcal{S})$ where n is the set of edges. The set $\mathcal{V}$ will have a vertex for every element in the 1^{st} layer, $\binom{[n]}{1}$, and $\mathcal{S}$ will have a vertex for each element in the $(n-k)^{\text{th}}$ layer, $\binom{[n]}{n-k}$. Notice that these elements will be sets rather than singletons. Let $\{i\} \in \mathcal{V}$ and $S \in \mathcal{S}$, so $\{i\}$ is a singleton and S is a subset. We will now define edges between these elements. If $\{i\}$ is adjacent to some S, then $\{i\} \notin S$. As an example, let $n = 20$. Then $k = 8$ and $n - k = 12$. We find that $\binom{[n]}{n-k} = \{\{1, 2, \ldots, 12\}, \{2, 3, \ldots, 13\}, \ldots, \{9, 10, \ldots, 20\}\}$ and $\binom{[n]}{1} = \{\{1\}, \{2\}, \ldots, \{20\}\}$. Then, we know that $\{1\}$ is adjacent to all of the sets in $\mathcal{S}$ that do not contain 1, so $\{1\}$ is adjacent to $\{2, 3, \ldots, 13\}$ and the rest of the sets that come after it. The set $\{1\}$, however, is not adjacent to $\{1, 2, \ldots, 12\}$.

By [7], there exists some bipartite graph G_n such that $\text{lbc}(G_n) = \Omega(n/\log n)$. Then, since P_n is a subposet, $\text{ldim}(2^{[n]}) \geq \text{ldim}(P_n) \geq \Omega(n/\log n) = \text{lbc}(G_n)$. Now, let H be a difference subgraph of G_n. A difference subgraph is a difference graph that is also a subgraph of G_n.

We call H *small* in $S \in \mathcal{S}$ if there are less than $b = 2\ln n$ edges incident to S in H. In other words, H is small in the set S if there are less than b edges in H that share the vertex S. Otherwise, we say that H is *big* is S. Note also that if H is big in S, then H is also big in $\mathcal{S}$ as a whole. Let $\mathcal{H}$ be a difference graph cover of G_n such that $\ell = \text{ldc}(G_n)$ is realized in $\mathcal{H}$..

Case 1: Assume that all $H \in \mathcal{H}$ are small in $S \in \mathcal{S}$. We know that each S has degree

k, and this implies that there are k vertices $\{v_i\} \in \mathcal{V}$ such that $\{v_i\} \notin S$. Thus, if each H is small, then all k edges incident with S are covered by difference graphs containing at most $b-1$ of those edges; otherwise H would be big. Then, each S appears in at least

$$\frac{k}{b-1} \geq \frac{k}{b} = \frac{\lceil n/e \rceil}{b} \geq \frac{\frac{n}{e}}{b} = \frac{n}{eb} = \frac{n}{2e \ln n}$$

difference graphs $H \in \mathcal{H}$. We have $\frac{k}{b-1}$ since the k edges incident to S are divided among the maximum number of edges for each H. This implies that $\mathrm{ldc}(G_n) \geq \frac{n}{2e \ln n}$ since $\mathrm{ldc}(G_n) = \frac{k}{b-1}$.

Case 2: Assume that for each $S \in \mathcal{S}$ that there is at least one big difference graph H. By definition, the neighborhoods of $\mathcal{S}$ are nested in H. Thus, if H is big in $S_1, S_2, \ldots, S_t$, then there must be b singletons in $\mathcal{V}$ that are adjacent to each of those sets in H. So none of the b singletons are in $S_1, S_2, \ldots, S_t$. This implies that $t \leq \binom{n-b}{n-k} = \binom{n-b}{k-b}$.

Since there are at most ℓ difference graphs containing any singleton $\{v_i\} \in \mathcal{V}$, but each big difference graph contains at least b singletons, then there must be at most $\ell n/b$ big difference graphs. There are n singletons for each H, and there are ℓ difference graphs H, which is how we get ℓn. Let (v, H_{big}) denote the big difference graph containing some fixed vertex v. Not only do we know that it has at least b singletons since it is big, but also it has at most ℓn singletons since each of the n vertices in $\mathcal{V}$ is in at most ℓ of the difference graphs. This implies that the number of big difference graphs is less than or equal to $\ell n/b$. Since each $S \in \mathcal{S}$ has at least one big difference graph, then we get that

$$\frac{\ell n}{b}\binom{n-b}{k-b} \geq \binom{n}{k}$$

where $\binom{n}{k}$ is the size of S since $\binom{n}{n-k} = \binom{n}{k}$ and $\binom{n-b}{k-b} \geq t$ where t is the number of sets in which there exists at least one big H.

Now, plugging in $k = \lceil n/e \rceil$ and $b = 2\ln n$, we have

$$\ell \geq \frac{b}{n}\binom{n}{k}\binom{n-b}{k-b}^{-1} \geq \frac{b}{n}\binom{n}{k}^{b} \geq \frac{\ln n}{n}e^{2\ln n} = n\ln n \geq n\frac{1}{\ln n} \geq \frac{n}{2e\ln n}.$$

Since $\ell = \mathrm{ldc}(G_n)$ where G_n is the bipartite graph associated with P_n, then $\mathrm{ldim}(2^{[n]}) \geq \mathrm{ldim}(P_n) \geq \frac{n}{2e\ln n}$. This completes the proof of the lower bound.

The upper bound is proven in [9]. They show that $\dim(2^{[n]}) = n$, and since the local dimension of a poset is less than or equal to the dimension of a poset, then we get the upper bound as desired. $\square$

6.2 Suborders of the Boolean Lattice

Before beginning this section officially, I will first introduce a few additional notations. Since this chapter will frequently refer to layers that induce a suborder of the Boolean Lattice, denote $2^{[n]}_{l,k}$ to be the suborder of the Boolean Lattice induced by the layers l and k. If $l = k$, it is simply denoted $2^{[n]}_{k}$.

Now, if $P = (X, \leq)$, take the poset $2^{[n]}_{x}$ for each $x \in X$ with the ground set Y_x.

Definition 6.2.1. *Given two sets, the* lexicographic sum *is defined as the set of ordered pairs with one element in X and the other element in Y_x such that the first elements of the ordered pairs are comparable and the second elements in the pair are comparable. Formally, we define*

$$\sum_{x\in X} 2^{[n]}_{x} = \{(x, y) \ : \ x \in X, \ y \in Y_x\}$$

such that two elements (x, y) and (w, z) are comparable if and only if we have either $x \prec z$ or we have both $y \leq w$ and $x = z$.

The dimension of the lexicographic sum was was proven by Hiraguchi in [3], and it states that

$$\dim\left(\sum_{x\in X} 2^{[n]}_{x}\right) = \max\left\{\dim(P), \ \max\{\dim\left(2^{[n]}_{x}\right) \ : \ x \in X\}\right\}.$$

Lewis proves in [2] the following inequalities regarding local dimension of the lexicographic sums.

First,

$$\operatorname{ldim}\left(\sum_{x\in X} 2_x^{[n]}\right) \geq \max\left\{\operatorname{ldim}(P), \max\left\{\operatorname{ldim}\left(2_x^{[n]}\right) \;:\; x \in X\right\}\right\},$$

Proof. This proof is the most straight-forward of the three. Note that $\sum 2_x^{[n]}$ will have suborders equivalent to both P and $2_x^{[n]}$. This implies that the local dimension of the lexicographic sum is greater than or equal to the maximum of the local dimension of P and the local dimension of $2_x^{[n]}$. □

The next inequality is the following:

$$\operatorname{ldim}\left(\sum_{x\in X} 2_x^{[n]}\right) \leq \max\left\{\operatorname{ldim}(P), \max\left\{\dim\left(2_x^{[n]}\right) \;:\; x \in X\right\}\right\},$$

Proof. If $\mathcal{L}$ is a local realizer of P and $\mathcal{M}$ is a realizer of $2_x^{[n]}$, then Lewis [2] constructs a local realizer of $\sum 2_x^{[n]}$ in the following way:

The first case is when $|\mathcal{M}_x| \leq \mu(x, \mathcal{L})$, which means that the number of linear extensions $M \in \mathcal{M}$ is less than or equal to the number of times each x appears in the PLEs $L \in \mathcal{L}$. In this instance, replace the x in each PLE with $x \times M$ for $M \in \mathcal{M}_x$. Each M will be used at least once since there are $|\mathcal{M}_x|$ linear extensions, and this number is less than or equal to the number of PLEs containing x.

The second case is when $|\mathcal{M}_x| > \mu(x, \mathcal{L})$. For these x's, again replace the x in each PLE with $x \times M$. The only stipulation, however, is that the M's need to be unique for each PLE. Then, some of the linear extensions M may not be used since there are more linear extensions than there are PLEs containing x.

Next, let $\mathcal{N}$ denote the set of all of these new posets we have created. Essentially, using the above two cases, we can make new posets for each $x \in X$ based on the linear extensions $M \in \mathcal{M}_x$ and the PLEs in $\mathcal{L}$. Note also that each $M \in \mathcal{M}_x$ is filled with the elements $y \in Y_x$. Now, if $x \in X$ and $y \in Y_x$, then $\mu((x, y), \mathcal{N}) = \max\left\{\mu(x, \mathcal{L}), |\mathcal{M}_x|\right\}$. For instance, assume that for some given x we have $|\mathcal{M}_x| > \mu(x, \mathcal{L})$. Then the number of times (x, y) appears in $\mathcal{N}$ should be equal to $\mu(x, \mathcal{L})$ since we do not necessarily use each $M \in \mathcal{M}_x$.

The last step is to prove that $\mathcal{N}$ is a local realizer. Our set $\mathcal{N}$ will already contain the comparabilities from the posets from which it is derived, so we only need to focus on incomparabilities. Assume that we have two incomparable ordered pairs in $\mathcal{N}$, say $(w, z) \not\leq (x, y)$. Then we know that either $w \not< x$ or both $z \not\leq y$ and $x = w$. In the first case, this implies that there exists some PLE in $\mathcal{L}$ such that $x \prec w$. But when x and w are later replaced by $x \times M$ and $w \times M$, this gives us the elements (x, y) and (w, z) with $(x, y) \leq (w, z)$, which means that the incomparability is realized. In the second situation, there would exist some $M \in \mathcal{M}_x$ with $y \leq z$, but then the z would be replaced by $z \times M$ in $\mathcal{N}$. This also would realize the incomparability. Since these two options encompass all incomarabilities, $\mathcal{N}$ is a local realizer. $\square$

The last inequality is

$$\operatorname{ldim}\left(\sum_{x \in X} 2_x^{[n]}\right) \leq \operatorname{ldim}(P) + \max\left\{\operatorname{ldim}\left(2_x^{[n]}\right) \,:\, x \in X\right\}.$$

Proof. Again take $\mathcal{L}$ a local realizer of P and $\mathcal{M}$ a realizer of $2_x^{[n]}$. Take an arbitrary linear extension of $2_x^{[n]}$, call it M_x. Similarly to a previous proof, replace each x in $\mathcal{L}$ and each $x \times M$ with $x \times M_x$, and call $\mathcal{N}$ the set containing these new elements. From the same reasoning as the proof above, $\mathcal{N}$ is a local realizer of $\sum_{x \in X} 2_x^{[n]}$. Additionally, we have the fact that $\mu((x, y), \mathcal{N}) = \mu(x, \mathcal{L}) + \mu(y, \mathcal{M}_x)$ since the element (x, y) will appear for each instance of $x \in \mathcal{L}$ and also for each instance of $y \in \mathcal{M}_x$. $\square$

Proposition 6.2.1. *The three of these inequalities together form a proposition in [2].*

Definition 6.2.2. *Let X be a discrete random variable from the set $\{x_1, x_2, \ldots, x_n\}$. If $\mathbb{P}\{X =$*

$x_i\}$ *is the probability of having* $X = x_i$*, then the entropy of* X *is*

$$H(X) = -\sum_{\substack{i\in[1,n],\\ \mathbb{P}(x_i)\neq 0}} \mathbb{P}(x_i)\log\mathbb{P}(x_i).$$

The entropy of X *can be loosely considered as the approximate amount of information we obtain based on the value of* X*. Additionally, since* log *is a concave function, we have that* $H(X) \leq \log n$ *when* X *is uniformly distributed.*

An *alphabet* is a finite set containing two or more elements. Each of these elements is called a *symbol*, and a *word* is a finite sequence of said symbols. Lastly, if $\mathscr{A}$ is an alphabet, then we say that $\mathscr{A}^*$ is all possible words over $\mathscr{A}$.

Definition 6.2.3. *Let* Σ *be a finite set with* $\mathscr{A}$ *an alphabet. A prefix-free code is defined as a map* $\mathscr{C}$ *from* Σ *to* $\mathscr{A}^*$ *such that for* $x, y \in \Sigma$*, if* x *and* y *are not equal, then* $\mathscr{C}(x)$ *is not in the beginning of the word* $\mathscr{C}(y)$*, following the standard definition of a prefix.*

Using the prior definitions, we can prove the following theorem presented and proven in [2]:

Theorem 6.2.2. *[2] If* X *is a random variable that receives values from a set* Σ*, and if* $\mathscr{A}$ *is an alphabet, then the expected length of a prefix-free code* $\mathscr{C}$*, denoted* $\mathscr{C}(X)$*, is, at minimum,* $\frac{H(X)}{\log|\mathscr{A}|}$.

Crespelle in reference [10] shows the existence of what is called a *Crespelle codeword* for a given poset P.

Definition 6.2.4. *If* $P = (X, \leq)$*, then we have the ability to encode the poset as a word with an alphabet* $\mathscr{A}$ *with* $3n$ *symbols. First, let* $\mathscr{A} = \{x_i, x_m, x_f \ : \ x \in X\}$*. Let* $\mathcal{L}$ *be a local realizer of* P*. Each nontrivial PLE in* $\mathcal{L}$ *is a list of elements of* X *with the form* x_i *in the beginning and* x_m *in all other places. The trivial PLEs are simply one element of* X*, and hence the local realizer stays the same when we remove them. The next step is to* *concatenate all of these lists and replace all ending symbols, which are x_m, with x_f. This new list is called the* Crespelle codeword *for poset P.*

Let, for instance, $d = \text{ldim}(P)$. This means that, in the local realizer $\mathcal{L}$, any element $x \in X$ appears in at most d of the PLEs. However, in the Crespelle codeword, each element will appear at most dn times, which implies that the local dimension of the Crespelle codeword for P is dn.

With the definition of the Crespelle codeword, we can prove the following theorem from [2]:

Theorem 6.2.3. *[2] The local dimension of a uniformly chosen poset from a set of all n-element posets, $[n]$, is at a minimum*

$$\left(\frac{1}{4} - o(1)\right)\frac{n}{\log n}.$$

Proof. Let $n \geq 2$ since the case where $n = 1$ is trivial. Assume that P is a poset satisfying the standards given in the theorem. We can assign an equal probability to each outcome of choosing this poset. The number of partial orders will be, at minimum, the number of minimal and maximal elements, which is equal to $2^{\frac{1}{4}n^2}$. This implies, then, that $H(P) \geq \frac{1}{4}n^2$.

Now, let $d = \mathbb{E}[\text{ldim}(P)]$. Note that $\mathbb{E}$ denotes the expected value, which is a generalization of a weighted average. Then the smallest Crespelle codeword will have length dn as shown above, and by Theorem 6.2.2, $H(P) \leq dn\log(3n)$. This implies that

$$d \geq \frac{n}{4\log(3n)}$$

since $H(P) \geq \frac{1}{4}n^2$. Because $\frac{n}{4\log(3n)}$ is equivalent to $\left(\frac{1}{4} - o(1)\right)\frac{n}{\log n}$, we have proven the result. $\square$

The next theorem involves pairs of layers from the Boolean lattice.

Theorem 6.2.4. *[2] For $\ell, k < n$, a lower bound for the local dimension of layers from the Boolean lattice is given as follows:*

$$ldim\left(2^{[n]}_{\ell,k}\right) \geq \left(\frac{1}{\log n} - O\left(\frac{\log\log n}{(\log n)^2}\right)\right) \log \binom{n}{k}.$$

Proof. Without loss of generality, assume that $\ell < k$ where P is a random height-two poset with elements defined as follows. Let $A = [n]^{\ell}$ and $B = [m]$ for $n, m \in \mathcal{N}$. For every $b \in B$ let X_b be a random subset of $[n]$ with size k. Define $P = A \dot{\cup} B$. Recall that $A \dot{\cup} B$ is a disjoint union of sets A and B. Also, let every b be above $a \in A$ if $a \subseteq X_b$. Note that taking the disjoint union between A and B gives a joint distribution of random variables that are also mutually independent. This implies, then, that adding their entropies together will be at least $m \log \binom{n}{k}$. If we again define $d = \mathbb{E}[\mathrm{ldim}(P)]$, then by Theorem 6.2.2 we have

$$H(P) \leq d\left(\binom{n}{\ell} + m\right)\left(\log\left(3\binom{n}{\ell} + 3m\right)\right)$$

since n in Theorem 6.2.2 is equivalent to $\binom{n}{\ell} + m$ in this case. With some algebraic manipulation, we find that

$$d \geq \frac{m \log \binom{n}{k}}{\left(\binom{n}{\ell} + m\right)\left(\log\left(3\binom{n}{\ell} + 3m\right)\right)}.$$

Letting $m = \left\lfloor \binom{n}{\ell}\left(\log\binom{n}{\ell} - 1\right)\right\rfloor$, we get

$$d \geq \frac{log\binom{n}{k}}{log\binom{n}{\ell}} - \frac{log\binom{n}{k}}{\left(log\binom{n}{\ell}\right)^2}\left(\log\log\binom{n}{\ell} + \log 6 + \binom{n}{\ell}^{-1}\right).$$

Since it is possible for $\mathrm{ldim}(P)$ to be greater than or equal to d, let $\mathrm{ldim}(P) \geq d$. We will modify P to get a new poset, $\tilde{P}$, as follows. Let $S \subseteq A$, and for each such S, if there two or more vertices in B with a neighborhood in S, then we will delete all of those vertices except for one. Recall that P is a lexicographic sum, and, therefore, it is a sum of antichains over $\tilde{P}$. Note that an antichain is a subset of a poset that contains elements that are all incomparable to each other. Additionally, $\tilde{P}$ is not a chain. By Proposition 6.2.1, P and $\tilde{P}$ have the same local dimension, but since $\tilde{P}$ embeds into $2^{[n]}_{\ell,k}$, then we have

$$\operatorname{ldim}\left(2^{[n]}_{\ell,k}\right) \geq \operatorname{ldim}(\tilde{P}) \geq d,$$

completing the proof. □

The next theorem is also important in the discussion of this topic, and the proof is detailed in [11]:

Theorem 6.2.5. *[11]*

$$\operatorname{ldim}\left(2^{[n]}_{1,2}\right) \geq \log\log n - O(\log\log\log n)$$

as $n \to \infty$.

Now, due to the embedding of the Boolean lattice, we know that

$$\operatorname{ldim}\left(2^{[n]}_{\ell,k}\right) \geq (1 - o_{\ell,k}(1)) \log\log n.$$

for $\ell < k$ fixed in $\mathbb{N}$ and as $n \to \infty$. This inequality also follows from the fact that $2^{[n-k+1]}_{1,2}$ embeds into $2^{[n]}_{k,k+1}$ and also $2^{[n]}_{\ell,k}$ embeds into $2^{[n+1]}_{\ell,k+1}$ by [2].

To show the upper bound of the local dimension of suborders of the Boolean lattice, we have the following theorem from [2]:

Theorem 6.2.6. *[2] For* $1 \leq \ell < k \leq n$ *and* $\ell \leq \frac{n}{\log n}$,

$$\operatorname{ldim}\left(2^{[n]}_{\ell,k}\right) \leq (1 + o_\ell(1)) \frac{n}{\log n}.$$

Proof. Assume that G is a bipartite graph such that $G = (E, A \cup B)$ where $|E| = n$. With $\Gamma(v)$ denoting the set of neighbors of vertex v, for every $v \in A$, let $X = \{u \in B : vu \in S\} \subseteq \Gamma(v)$

where S is the k-sets. Recall that a k-set is some subset with k points of a ground set; specifically, it can be strictly separated from the rest of the points in the ground set. Now, let $\mathcal{L}$ be a local realizer of $2^{[n]}_{\ell,k}$ defined with k-subsets and ℓ-subsets of $E(G)$. If L is a PLE in $\mathcal{L}$, let L first contain the k-sets satisfying set X followed by the ℓ-sets containing some edge vu but with $u \notin X$. In other words, L first has the k-sets such that for a vertex in B there exists an edge in the k-set, and it is followed by the ℓ-sets with an edge between A and B but for edge vu not in the k-set. Let π_0 be the list of all ℓ-sets of edges followed by all k-sets of edges. The order within the ℓ- and k-sets is unimportant since we will now let π_1 have all ℓ-sets in opposite order followed by all k-sets in opposite order. This implies that $\mathcal{L} = \{\pi_0, \pi_1\} \cup \{L : v \in [A], X \subset [B]\}$ is a local realizer of $2^{[n]}_{\ell,k}$. Additionally, according to [2], each ℓ-set has multiplicity $\ell * 2^{\Delta-1} + 2$, where Δ is the maximum degree of any given vertex in A, and each k-set has a multiplicity of at most $|A| + 2$.

To complete this proof, if G is a bipartite graph, we can take A and B such that $|A| = \left\lceil \frac{n}{\log n - \log\log n - \log \ell} \right\rceil$ and $|B| = \lceil \log n - \log\log n - \log \ell \rceil$ with $1 \leq \ell < k \leq n$ and $\ell < \frac{n}{\log n}$ to be the partite classes of G. Then we find that

$$2^{\Delta-1} + 2\ell \leq 2^{|B|-1}\ell + 2 < \frac{n}{\log n} + 2$$

which implies that

$$|A| + 2 \leq \frac{n}{\log n} + \frac{2n}{(\log n)^2}(\log\log n + \log \ell) + 3.$$

Finally, this implies that

$$2^{[n]}_{\ell,k} \leq \frac{n}{\log n} + \frac{2n}{(\log n)^2}(\log\log n + \log \ell) + 3 = (1 + o_\ell(1))\frac{n}{\log n}$$

and the proof is complete. $\square$

CHAPTER 7

RESULTS ABOUT THE LOCAL DIFFERENCE COVER NUMBER OF THE BOOLEAN LATTICE

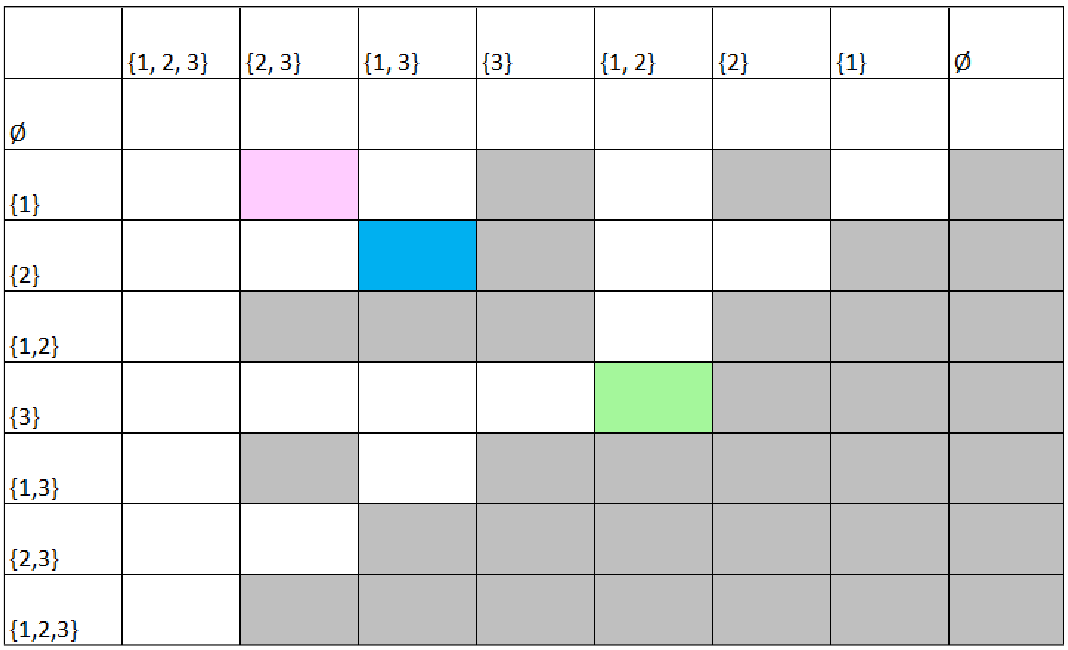


Figure 7.1: Young Diagram of $2^{[3]}$

For this section, our poset will be $2^{[n]}$ and Q_n will denote the split of $2^{[n]}$. Recall, also, that In(Q_n) denotes the incomparability graph of the split of $2^{[n]}$. The orientation of the Young Diagrams in this chapter have been selected for a reason. If we have YD(In(Q_n)), where Q_n denotes the split of the Boolean Lattice $2^{[n]}$, then each Young Diagram may be split into quarters rendering three Young Diagrams of In(Q_{n-1}). This organization will prove useful in a future proof.

Additionally, the colors in Figure 7.1 are called the *necessary colors* of the diagram. A *necessary color* is defined as a color in the diagram that absolutely cannot be changed.

Remark 7.0.1. *Note also that the Young diagram representation of In(Q_n) is called* generalized *since each difference graph from the difference graph cover of In(Q_n) will generate* *one Young diagram, and the* generalized *Young diagram is simply a culmination of these smaller Young diagrams.*

Each of these smaller Young diagrams is represented by a color in the generalized diagram. Hence, each color represents a Young diagram. For the rest of this thesis, we may assume that all Young diagrams of $\mathrm{In}(Q_n)$ are generalized.

Lemma 7.0.1. *The Young Diagram of* $\mathrm{In}(Q_3)$ *can be colored with at most* 2 *colors in each row and column. This implies that* $\mathrm{ldc}\,(\mathrm{In}(Q_3))) = 2$.

Proof. Notice the three necessary colors in Figure 7.1. If we index the diagram like a matrix, we get entry $(2, 2)$ is color 1, $(3, 3)$ is color 2, and $(5, 5)$ is color 3. First, we will show that $\mathrm{ldc}(2^{[3]}) \neq 1$. Assume to the contrary that we can color this diagram with at most one color in each row and column. Then all of row 2 ought to be Color 1. But then this implies that entry $(3, 4)$ must also be Color 1, which is a contradiction since entry $(3, 3)$ absolutely must be Color 2. So $\mathrm{ldc}(\mathrm{In}(Q_3)) \neq 1$.

Next, we need to show that $\mathrm{ldc}(\mathrm{In}(Q_3)) = 2$ exists. See the coloring below:

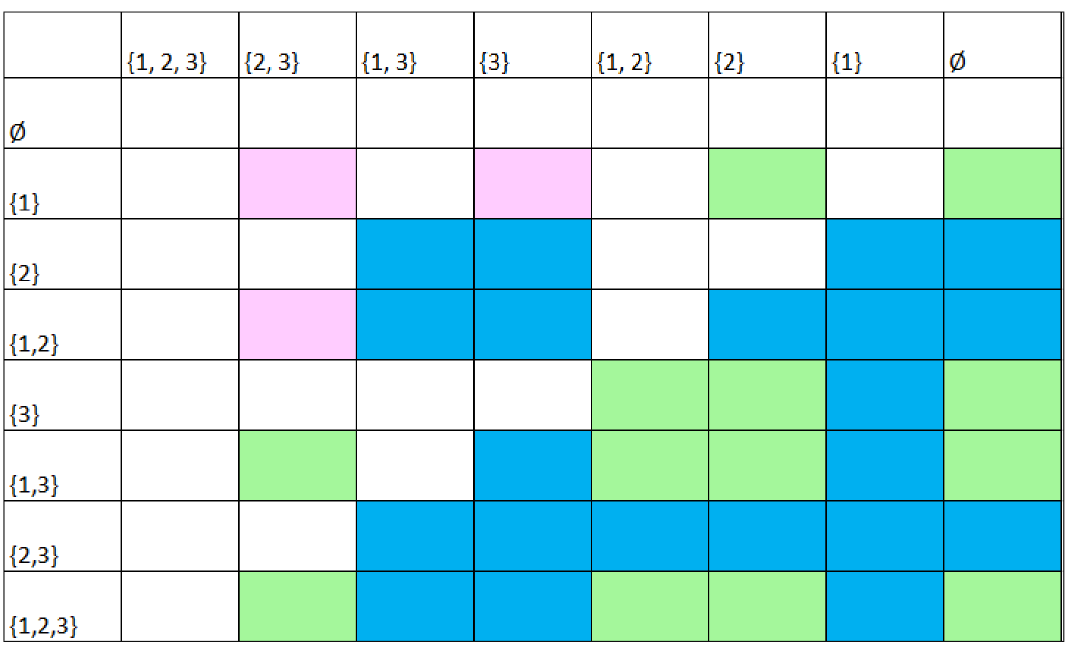


Figure 7.2: Colored Young Diagram of $\mathrm{In}(Q_3)$

This is one of many 2-colorings of YD(In(Q_3)), but it effectively shows that, since there exists a 2-coloring of the diagram, that ldc(In(Q_3)) = 2. □

Corollary 7.0.2.

$$\mathrm{ldc}\,(\mathrm{In}(Q_n)) \leq n - 1.$$

Proof. Each Young Diagram of In(Q_n) can be divided into quarters. Each quarter, except for the lower right hand, are identical to the Young Diagram of In(Q_{n-1}). If we color those three quarters identically to the coloring of the Young Diagram for In(Q_{n-1}), then we are adding no additional colors to each row or column. By simply coloring the square in the lower right hand entirely in the nth color for the diagram, we achieve an $n - 1$ coloring of the Young Diagram. This implies that we have, in the "worst" case scenario, an $n - 1$ coloring of the Young Diagram for In(Q_n). □

Conjecture 1.

$$\mathrm{ldc}\,(\mathrm{In}(Q_n)) = n - 1.$$

This conjecture follows from Corollary 7.0.2. Based on our work for this thesis, we have reason to believe that an $n - 1$ coloring is the best coloring for the local difference cover number of the Boolean Lattice. See the figure below:

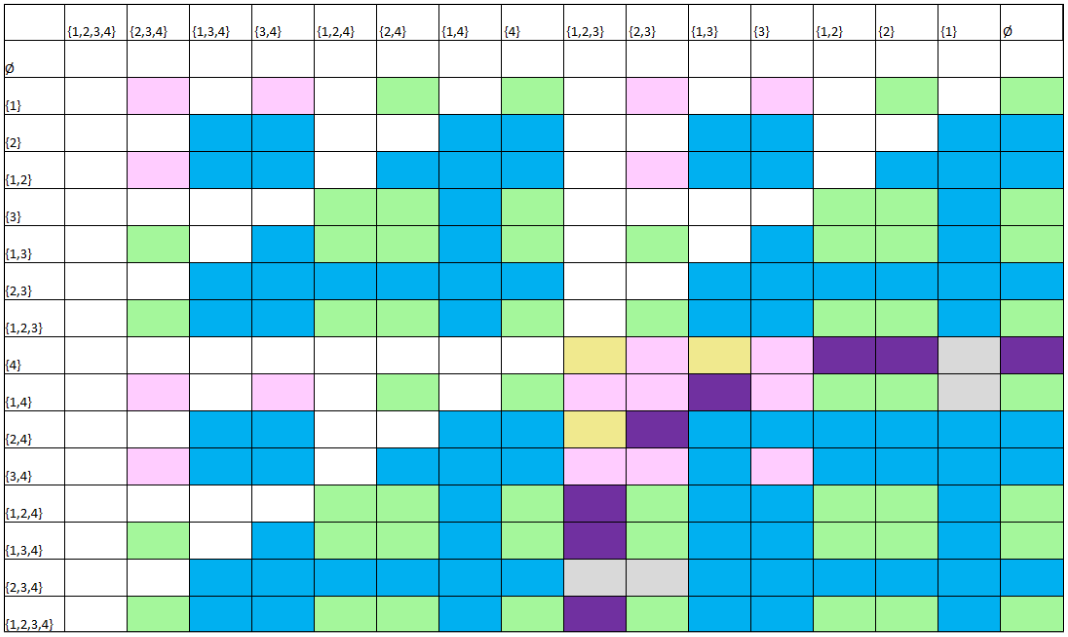


Figure 7.3: Partially Colored Young Diagram of $\mathrm{In}(Q_4)$

Since $n = 4$, we have four main colors: pink, blue, green, and yellow. The grey squares could contain any of the four colors. The purple squares, however, are the "problem" squares. Those squares are called problem squares because they exist at the intersection of a row and a column that already have distinct 2-colorings. Hence, regardless of the color chosen for problem squares, another color will be added to at least one row or column.

This diagram was colored with minimization in mind. It was colored attempting to use the minimum number of colors in each row and each column. Notice, however, that there remain eight problem squares in the coloring. Despite the choice of color for the purple squares, a third color will be added to a row and a column, giving a local difference graph cover number of three for this Young Diagram.

Finally, we state the following theorem, which motivates this thesis.

Theorem 7.0.3.

$$\mathrm{ldim}\left(2^{[n]}\right) = \Theta(n) \iff \mathrm{ldc}\left(\mathrm{In}(Q_n)\right) = \Theta(n).$$

Proof. The proof of this theorem is a corollary of a number of propositions and observations dicussed in this thesis. First of all, by Lemma 3.1.1, we have that

$$\mathrm{ldim}(Q_n) - 2 \leq \mathrm{ldim}(2^{[n]}) \leq 2\mathrm{ldim}(Q_n) - 1.$$

By Proposition 2.3.1, we have that

$$\mathrm{ldc}(\mathrm{In}(Q_n)) \leq \mathrm{ldim}(Q_n)) \leq \mathrm{ldc}(\mathrm{In}(Q_n)) + 2.$$

Therefore, by combining these two inequalities, we have that

$$\mathrm{ldc}(\mathrm{In}(Q_n)) - 2 \leq \mathrm{ldim}\left(2^{[n]}\right) \leq 2\mathrm{ldc}(\mathrm{In}(Q_n)) + 3.$$

$\square$

Remark 7.0.2. *If* $\mathrm{ldc}\,(\mathrm{In}(Q_n)) \geq n-1$, *then the local dimension of* Q_n *should also be greater than or equal to* $n - 1$. *This implies that* $\mathrm{ldim}\left(2^{[n]}\right) \geq n - 3$.

CHAPTER 8

FINAL REMARKS

8.1 Open Problems

Lewis presents several compelling problems in [2]. For instance, if we define $2^{[n]}_{1,k}$ as some function dependent on n, how does the function act as n goes to infinity? Dushnik shows in [12] that the dimension of $2^{[n]}_{1,k}$ is monotone for some specific values of k; however, the local dimension is not so.

Also presented in [4] is whether or not the bound for the local dimension of the Boolean lattice is tight. Due to the fact that this upper bound is trivial, it begs the question of whether or not that bound can be reduced.

And, of course, as mentioned in the previous chapter,

Conjecture 2.

$$\mathrm{ldc}\,(\mathrm{In}(Q_n)) = n - 1.$$

We have reasonable evidence to suggest the correctness of this statement; however, a proof is yet to be constructed. Based on the patterns with the the cases we have studied, it seems that there will always exist rows and columns containing problem squares; this implies that the local difference graph cover number is, indeed, $n - 1$. A reasonable way to prove this conjecture would be to write a program that generates the ideal colorings of the Young Diagrams. If the program could essentially "sort out" the good colorings from the bad, then it would give strong incentive that this answer is correct.

Over the course of writing this thesis, we have tried to color the Young Diagrams in a variety of ways. We tried splitting the diagram into smaller diagrams and coloring them alike, but this always generated problem squares. Additionally, we tried to strategically remove certain elements to get an $n-2$ coloring. We also looked for some sort of algorithm for coloring the diagram that would give a direct proof for the conjecture, similar to the proof for Lemma 7.0.1. But the sheer number of options for small cases like $2^{[4]}$ proved it a very difficult and tedious undertaking. There was also room for doubt since it is impossible to guarantee a square to be a certain color, unlike with $2^{[3]}$ where there are only three color choices to begin with.

Another method we considered is to somehow show that all colorings are, in a sense, isomorphic. Then, the individual colorings could be considered the same, providing a small simplification to this complicated problem.

The last and more crucial problem is that of the exact local dimension of $2^{[n]}$. Although we have various bounds for posets and the Boolean Lattice, there is nothing absolute, and certainly there are no proofs for anything exact. This is the question that began this whole thesis, and we believe that the most straight-forward method to finding the solution will come through the local difference graph cover number. The question remains: What exactly is the local dimension of the Boolean Lattice?

8.2 Conclusion

As for the concluding thoughts of this thesis, we have presented a variety of definitions from several different areas of mathematics. These key definitions provide the framework to understanding the main theorems presented in Chapter 3, Chapter 6, and Chapter 7.

In Chapter 3, we addressed the known bounds of the local dimension of a given poset, and although a definite formula for calculating $\mathrm{ldim}(P)$ for any given P is not known at this time, this theorem helps to prove other results.

In Chapter 6, we discuss the Boolean lattice. Most importantly, we present the result from [4] that the local dimension of the Boolean lattice is bounded above by n and below by $n/2e\log n$.

Finally, in Chapter 7, we discuss the importance of difference graphs and Young diagrams in proving the local difference graph cover number of the Boolean lattice. The goal, ultimately, is to use these facts and methods to prove the exact local dimension of the Boolean lattice, but, at this time, we have nothing definite except for Theorem 7.0.3, which states that the local dimension of the Boolean lattice is linear if and only if the local difference graph cover number of $\mathrm{In}(Q_n)$ is linear. One last remark is that the local difference graph cover number of $\mathrm{In}(Q_n)$ being greater than or equal to $n-1$ implies that the local dimension of the Boolean lattice will be greater than or equal to $n-3$.

REFERENCES

[1] I. V. Muñoz and U. Persson, "Interviews with three fields metalists," 2007.

[2] D. Lewis, *The local dimension of suborders of the boolean lattice*, 2020.

[3] T. Hiraguchi, *On the dimension of partially ordered sets.* 1955.

[4] J. Kim *et al.*, "On difference graphs and the local dimension of posets," *European Journal of Combinatorics*, vol. 86, p. 103 074, May 2020.

[5] F. Barrera-Cruz, T. Prag, H. C. Smith, L. Taylor, and W. T. Trotter, "Comparing dushnik-miller dimension, boolean dimension and local dimension," *Order*, vol. 37, no. 2, pp. 243–269, 2019.

[6] L. Csirmaz, P. Ligeti, and G. Tardos, "Erdős–pyber theorem for hypergraphs and secret sharing," *Graphs and Combinatorics*, vol. 31, no. 5, pp. 1335–1346, 2014.

[7] F. Chung, P. Erdős, and J. Spencer, "On the decomposition of graphs into complete bipartite subgraphs.," *Studies in Pure Mathematics*, pp. 95–101, 1983.

[8] B. Dushnik and E. W. Miller, "Partially ordered sets," *American Journal of Mathematics*, vol. 63, no. 3, pp. 600–610, 1941.

[9] N. Alon and J. H. Spencer, *The Probabilistic Method*, Third. New York: Wiley, 2004, ISBN: 0470170204.

[10] C. Crespelle, "In discussion at *algorithms seminar*, university of bergen, norway," February 2, 2018.

[11] G. Damásdi *et al.*, "A note on covering young diagrams with applications to local dimension of posets," 2020.

[12] B. Dushnik, "Concerning a certain set of arrangements," vol. 1, 1950.