NON-SEPARATING PATHS IN GRAPHS

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Yingjie Qian

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NON-SEPARATING PATHS IN GRAPHS

Thesis committee:

Dr. Anton Bernshteyn  
School of Mathematics  
*Georgia Institute of Technology*

Dr. Grigoriy Blekherman  
School of Mathematics  
*Georgia Institute of Technology*

Dr. Zi-Xia Song  
Department of Mathematics  
*University of Central Florida*

Dr. Zhiyu Wang  
School of Mathematics  
*Georgia Institute of Technology*

Dr. Xingxing Yu  
School of Mathematics  
*Georgia Institute of Technology*

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To my parents and grandparents.
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6.2 Subgraph $A_1$ of the 7-connected example ................................... 62
Motivated by Tutte’s result and Lovász’s conjecture, there is a series of work on non-separating paths in graphs and their applications. Let $G$ be a graph and $a_1, a_2, b_1, b_2$ be distinct vertices of $G$, we give a structural characterization for $G$ not containing a path $A$ from $a_1$ to $a_2$ and avoiding $b_1$ and $b_2$ such that removing $A$ from $G$ results in a 2-connected graph. Using this structure theorem, we construct a 7-connected such graph. We will also discuss potential applications to other problems, including the 3-linkage conjecture made by Thomassen in 1980. This is based on joint work with Shijie Xie and Xingxing Yu.
CHAPTER 1
INTRODUCTION

1.1 Notation and terminology

In this section, we give notation and terminology. For some (well-known) graph concepts that are omitted, we refer the readers to Graph Theory textbook by Bondy and Murty [2] and Diestel [5].

1.1.1 Graph operations

Let $G = (V(G), E(G))$ be a graph where $V(G)$ is its vertex set and $E(G)$ is its edge set. For all $x \in V(G)$, $d_G(x)$ (or $d(x)$ if $G$ is understood) denotes the degree of $x$ in $G$, i.e., $d_G(x) = |\{y \in V(G) : xy \in E(G)\}|$. For any $S \subseteq V(G)$, $N_G(S)$ is the neighborhood of $S$ in $G$, i.e., $N_G(S) = \{v \in V(G) \setminus S : \exists u \in S \text{ such that } uv \in E(G)\}$. We use $G[S]$ to denote the subgraph of $G$ induced by $S$, i.e., $V(G[S]) = S$ and $E(G[S]) = \{uv \in E(G) : \forall u, v \in S\}$. We also use $G - S$ to denote $G[V(G) \setminus S]$. When $S = \{s\}$, we write $G - s$ for $G - \{s\}$.

For two graphs $G$ and $H$, let $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$, $G \cap H = (V(G) \cap V(H), E(G) \cap E(H))$, and $G - H$ be the graph obtained from $G$ by deleting vertices of $H$ and all edges of $G$ incident with $H$. We call $H$ a subgraph of $G$, denoted as $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Let $G$ be a graph. For any subgraph $H \subseteq G$, and for any $S_1 \subseteq V(G)$ and $S_2 \in \binom{V(H) \cup S_1}{2}$ (i.e., $S_2$ is a set of 2-element subsets of $V(H) \cup S_1$), define $H + S_1 + S_2 = (V(H) \cup S_1, E(H) \cup S_2)$. For subgraphs $G_1, G_2 \subseteq G$, we say $(G_1, G_2)$ is a separation of $G$ if $E(G_1) \cap E(G_2) = \emptyset$, $G = G_1 \cup G_2$, and for $i = 1, 2$, $E(G_i) \setminus E(G_{3-i}) \neq \emptyset$ or $V(G_i) \setminus V(G_{3-i}) \neq \emptyset$. 

1
1.1.2 Paths

We call a path $P$ with ends $a, b$ an $a$-$b$ path. For $v_1, v_2 \in V(P)$, we define $P[v_1, v_2]$ to be
the subpath of $P$ with ends $v_1, v_2$. Let $P(v_1, v_2) = P[v_1, v_2] - v_1$, $P[v_1, v_2) = P[v_1, v_2] - v_2$
and $P(v_1, v_2) = P[v_1, v_2] - \{v_1, v_2\}$.

We call two paths $P_1, P_2$ disjoint if $V(P_1) \cap V(P_2) = \emptyset$. A collection of paths $P_1, \ldots, P_k$
are independent if no vertex of any path is an internal vertex of any other path in the collection. For any $a$-$b$ path $P$
in a graph $G$ and for any subgraph $H$ of $G$, $P$ is internally disjoint from $H$ if $(V(P) \setminus \{a, b\}) \cap V(H) = \emptyset$. For $A, B \subseteq V(G)$, $A$-$B$ paths in $G$ are
paths in $G$ from $A$ to $B$ and internally disjoint from $A \cup B$.

1.1.3 Connectivity

A graph is connected if there is a path from any vertex to any other vertex in the graph, and a graph that is not connected is disconnected.

We call a set $T \subseteq V(G)$ a cut of a graph $G$ if $G - T$ is disconnected; and if $|T| = k$, we call $T$ a $k$-cut. Note that for any separation $(G_1, G_2)$ of $G$, $V(G_1 \cap G_2)$ is a cut of $G$ if $V(G_i - G_{3-i}) \neq \emptyset$ for both $i \in [2]$.

For graph $G$ and its subgraph $H$, we call $C$ a component of $G - H$ if $C$ is a subgraph of $G - H$, $C$ is connected, and for any $C' \subseteq G - H$ such that $C'$ is connected and $C \subseteq C'$, $C = C'$.

Let $k$ be a positive integer. We call a graph $G$ $k$-connected if $|V(G)| \geq k + 1$ and for any $S \subseteq V(G)$ with $|S| < k$, $G - S$ is connected. For any set $A \subseteq V(G)$, we say $G$ is $(k, A)$-connected if for any cut $S \subseteq V(G)$ with $|S| < k$ and for every component $C$ of $G - S$, $|V(C) \cap A| \geq k - |S|$.

A subgraph $B$ of a graph $G$ is called a block if it is isomorphic to $K_2$ or 2-connected, and for any $B' \subseteq G$ such that $B'$ is isomorphic to $K_2$ or 2-connected, $B \subseteq B'$ implies $B = B'$. A block is non-trivial if $|V(B)| \geq 3$. 

1.1.4 Bridges

Let \( G \) be a graph and \( H \subseteq G \), we call \( X \subseteq G \) an \( H \)-bridge of \( G \), if either

1. \( X \) is induced by some edge \( e = uv \in E(G) \setminus E(H) \) with \( u, v \subseteq V(H) \), or
2. \( X = C + S \) where \( C \) is a component of \( G - H \) and \( S = \{ e, v : e = uv \in E(G), u \in V(C), v \in V(H) \} \).

When (1) holds, \( X \) is said to be trivial, and when (2) holds, \( X \) is non-trivial. The vertices in \( V(X \cap H) \) are called attachments of \( X \) on \( H \).

1.1.5 Plane graphs

A graph \( G \) is planar if it can be drawn in the plane with no edge crossing. Such a drawing is called a plane graph. Let \( G \) be a plane graph. The faces of \( G \) are the connected open regions of the complement of \( G \) in the plane. The boundary of a face \( F \) consists of vertices and edges incident with \( F \). The boundary of the unbounded (or infinite) face is called the outerwalk of \( G \). Two vertices of \( G \) are cofacial if they belong to the boundary of a common face. Note that if \( G \) is 2-connected, then all its faces are bounded by cycles. A triangular face in \( G \) is a face of \( G \) bounded by a triangle.

1.1.6 Lexicographic ordering

For any positive integer \( k \), we denote \([k] = \{1, 2, \ldots, k\}\).

Let \( \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \) be real numbers. We say that the sequence \((\alpha_1, \ldots, \alpha_n)\) is larger than the sequence \((\beta_1, \ldots, \beta_m)\) with respect to the lexicographic ordering, denoted by \((\alpha_1, \ldots, \alpha_n) > (\beta_1, \ldots, \beta_m)\), if either

(i) \( n > m \) and \( \alpha_i = \beta_i \) for \( i = 1, \ldots, m \), or

(ii) there exists \( j \in [\min(m, n)] \) with \( \alpha_j > \beta_j \) and \( \alpha_i = \beta_i \) for all \( i < j \).
1.2 Background on non-separating paths

When developing a theory of 3-connected graphs, Tutte [25] showed that

**Theorem 1.2.1** ([25]). For any 3-connected graph \( G \) and any distinct vertices \( a_1, a_2, b \) of \( G \), \( G - b \) has an \( a_1-a_2 \) path \( P \) such that \( G - P \) is connected.

We call such a path *non-separating*. The “3-connectedness” condition cannot be relaxed; for instance, when \( \{a_1, a_2\} \) is a 2-cut if \( G \) is allowed to be 2-connected. Lovász [15] made a conjecture which would generalize Tutte’s result.

**Conjecture 1.2.2** (Lovász, 1975). For each natural number \( k \), there exists a least natural number \( \beta(k) \) such that, for any two vertices \( a, b \) in any \( \beta(k) \)-connected graph \( G \), there exists a path \( P \) between \( a \) and \( b \) such that \( G - P \) is \( k \)-connected.

Thus, Tutte’s result showed that \( \beta(1) = 3 \). Chen, Gould and Yu [3], and, independently, Kriesell [13] showed \( \beta(2) = 5 \). Moreover, Kawarabayashi, Lee and Yu [11] showed that \( \beta(2) = 4 \) except for double wheels. Conjecture 1.2.2 for \( k \geq 3 \) is still open.

For \( m \geq 0 \) and \( k \geq 1 \), let \( \alpha(m, k) \) be the minimum connectivity such that for any \( \alpha(m, k) \)-connected graph \( G \) and distinct \( a_1, a_2, b_1, \ldots, b_m \in V(G) \), there exists an \( a_1-a_2 \) path \( P \), such that \( b_1, \ldots, b_m \notin V(P) \) and \( G - P \) is \( k \)-connected.

Note that \( \alpha(0, k) = \beta(k) \). See the first column of Table 1.1 for the discussion above on \( \alpha(0, k) = \beta(k) \) for \( k \in [2] \).

Now, let us look at the first row of Table 1.1. Theorem 1.2.1 also proved \( \alpha(1, 1) = 3 \). One can also deduce Theorem 1.2.1 from the following result of Tutte.

**Theorem 1.2.3** ([25]). For any 3-connected graph \( G \) and any distinct vertices \( a_1, a_2 \) of \( G \), \( G \) has independent \( a_1-a_2 \) paths \( P_1, P_2 \) such that \( G - P_i \) is connected for \( i \in [2] \).

Similarly, one can deduce from the following result of Chen, Gould and Yu [3] that finds a non-separating path avoiding arbitrarily \( m \) vertices in any \((22m + 24)\)-connected graph, and thus, \( \alpha(m, 1) \leq 22m + 24 \).
Table 1.1: Connectivity for non-separating paths avoiding $m$ vertices

<table>
<thead>
<tr>
<th>$\alpha(m, k)$-connected $G$</th>
<th>Avoiding $m$ vertices</th>
<th>$m = 0$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$\cdots m \cdots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G - P$ is $k$-connected</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k = 1$</td>
<td></td>
<td>3</td>
<td>3</td>
<td>6</td>
<td>6</td>
<td>$\leq 22m + 24$</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>5</td>
<td>5</td>
<td>$\geq 8$</td>
<td>$\alpha(m, 2)$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Lovász’ Conjecture</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\alpha(m, k)$</td>
</tr>
<tr>
<td>:</td>
<td>open for $k \geq 3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Theorem 1.2.4** ([3]). For any $(22m + 24)$-connected graph $G$ and any distinct vertices $a_1, a_2$ of $G$, $G$ has $m + 1$ independent $a_1$-$a_2$ paths $P_i$ such that $G - P_i$ is connected for all $i \in [m + 1]$.

It is worth mentioning that with higher connectivity, Wollan [26] showed that one can remove a subset of paths without disconnecting the graph.

**Theorem 1.2.5** ([26]). For any $83(m + 1)$-connected graph $G$ and any distinct $a_1, a_2$ of $G$, there exist independent $a_1$-$a_2$ paths $P_1, \ldots, P_m$ such that for any subset $I \subseteq [m]$, $G - (\bigcup_{i \in I} V(P_i))$ is connected.

Note that the above results (other than Theorem 1.2.1) involve graphs with high connectivity. In applications, one often needs to find a non-separating path that avoids specific vertices in graphs. For example, when proving the Kelmans-Seymour conjecture, He, Wang and Yu [6, 7, 8, 9] needed non-separating paths in $4$-connected graphs that avoids two vertices.

The result on 2-linked (defined later) graphs by Jung [10], Seymour [19], Shiloach [20], Thomassen [24], and Chakravarti and Robertson [17] showed that $\alpha(2, 1) = 6$. Thomas, Xie, and Yu [23] showed that $\alpha(3, 1) = 6$. One can easily deduce $\alpha(1, 2) = 5$ from a result
of Chen, Gould and Yu [3] and Kriesell [13], and we will present it as Corollary 2.1.4.

We are primarily interested in a structural characterization of graphs not containing non-separating paths between two given vertices and avoiding two other given vertices. Such a characterization should help determine $\alpha(2, 2)$, and we believe $\alpha(2, 2) = 8$.

1.3 Structure theorem

Given a graph $G$ and distinct vertices $a_1, a_2, b_1, b_2$ of $G$. We say that $(G, a_1, a_2, b_1, b_2)$ is feasible if $G - \{b_1, b_2\}$ contains an $a_1$-$a_2$ path $A$ such that $G - A$ is 2-connected. We say $(G, a_1, a_2, b_1, b_2)$ is infeasible if $G$ is not feasible.

Our aim is to provide structural information about $(G, a_1, a_2, b_1, b_2)$ when it is not feasible. We show that if $(G, a_1, a_2, b_1, b_2)$ is infeasible then $G$ is the edge disjoint union of three graphs $A_1, A_2$ and $H$, where $a_i \in V(A_i) \setminus V(A_3-i \cup H)$, $A_i$ is planar, and $H$ can be further decomposed into graphs of simple structures. See Figure 1.1 for an illustration.

![Diagram of edge disjoint subgraphs](image)

Figure 1.1: Decomposition into edge disjoint subgraphs $A_1, A_2$ and $H$.

**Theorem 1.3.1.** Let $G$ be an 8-connected graph and let $a_1, a_2, b_1, b_2 \in V(G)$ be distinct. Suppose $(G, a_1, a_2, b_1, b_2)$ is infeasible. Then, the following statements hold:

(i) $G - \{a_1, a_2\}$ contains three independent induced $b_1$-$b_2$ paths $B_1, B_2, B_3$ such that, for $i \in [2]$, the $(B_1 \cup B_2 \cup B_3)$-bridge of $G$ containing $a_i$, denoted as $A_i(B_1 \cup B_2 \cup B_3)$, satisfy the following properties (up to relabeling):
\(- A_1(B_1 \cup B_2 \cup B_3)\) has all its attachments on \(B_3\),
\(- A_1(B_1 \cup B_2 \cup B_3) \cup B_3\) has a plane representation in which \(B_3\) and \(a_1\) are on the boundary of the infinite face,
\(- A_2(B_1 \cup B_2 \cup B_3)\) has attachments on both \(B_1\) and \(B_2\).

(ii) There exists \(w \in V(\ A_1(B_1 \cup B_2 \cup B_3) \cap V(B_3))\) such that \(G - (A_1(B_1 \cup B_2 \cup B_3) - B_3) - w - a_2\) has three independent \(b_1\)-\(b_2\) paths \(P_1, P_2, P_3\), and the \((P_1 \cup P_2 \cup P_3)\)-bridge of \(G\) containing \(a_2\), denoted as \(A_2(P_1 \cup P_2 \cup P_3)\), satisfies the following properties:

\(- A_2(P_1, P_2, P_3)\) has all its attachments on \(P_3\),
\(- A_2(P_1, P_2, P_3) \cup P_3\) has a plane representation in which \(P_3\) and \(a_2\) are on the boundary of its infinite face.

(iii) \(H := G - (A_1(B_1 \cup B_2 \cup B_3) - (B_3 - w)) - (A_2(P_1 \cup P_2 \cup P_3) - P_3)\) is the edge disjoint union of subgraphs \(H_1, \ldots, H_{m+1}\), such that \(V(H_i \cap H_{i+1}) = \{u_i, v_i, w_i\}\) is a 3-cut of \(H\) separating \(b_1\) from \(b_2\), \(b_1, u_1, \ldots, u_m, b_2\) occur on \(P_3\) in order, \(b_1, v_1, \ldots, v_m, b_2\) occur on \(P_2\) in order, and \(b_1, w_1, \ldots, w_m, b_2\) occur on \(P_1\) in order.

(iv) For each vertex \(u \in V(\ A_2(P_1 \cup P_2 \cup P_3) \cap V(P_3))\), \(u = u_i\) for some \(i\).

(v) For each \(i \in [m] \setminus \{1\}\), \(H_i = (J_i, L_i)\), where \(J_i\) is a plane graph and \(L_i\) is a ladder consisting of rungs of simple structure.

See Figure 1.2 for an illustration of \(H\) in the above theorem. The concept of ladders and rungs will be described in Chapter 2.

Note that “8-connected” cannot be replaced by “7-connected”, as we have an example (see Chapter 6) on 7-connected infeasible graph.

We believe Theorem 1.3.1 will be enough to show that 8-connected graphs are feasible, i.e., \(\alpha(2, 2) = 8\), which is work in progress.
1.4 Related problems

1.4.1 Linkage problem

Theorem 1.3.1 should serve as a step towards the following conjecture of Thomassen [24].

**Conjecture 1.4.1** (Thomassen, 1980). Let $G$ be an 8-connected graph and let $a_1, a_2, b_1, b_2, c_1, c_2 \in V(G)$ be distinct. Then, $G$ contains disjoint paths from $a_1, b_1, c_1$ to $a_2, b_2, c_2$, respectively.

More generally, a graph $G$ is $k$-linked if, for any $k$ disjoint pairs of vertices $\{s_i, t_i\}, i \in [k]$, in $G$, $G$ has pairwise disjoint paths from $s_i$ to $t_i$ for $i \in [k]$. Note that if $(G, a_1, a_2, b_1, b_2)$ is infeasible then $G$ is not 3-linked as can be seen by taking $c_i \in N_G(b_i) \setminus \{a_1, a_2, b_1, b_2\}$ for both $i \in [2]$. Thomassen [24] initially conjectured that every $(2k + 2)$-connected graph is $k$-linked, but this is false for $k \geq 4$: the graph obtained from the complete graph $K_{3k-1}$ minus a matching of size $k$ is a counterexample. Robertson and Seymour [18] showed that there is a polynomial time algorithm for deciding whether a graph is $k$-linked (when $k$ is fixed). Bollobás and Thomason [1] showed that every $(22k)$-connected graph is $k$-linked. Thomas and Wollan [21] improved this further to that every $(2k)$-connected graph with average degree at least $10k$ is $k$-linked.

Figure 1.2: $H$ is a union of subgraphs $H_1, \ldots, H_{m+1}$.
Conjecture 1.4.1 states that 8-connected graphs are 3-linked, which is still open. The best result on this conjecture is due to Thomas and Wollan [22].

**Theorem 1.4.2** ([22]). Every 6-connected graph on \(n\) vertices with \(5n - 14\) edges is 3-linked.

As a consequence, every 10-connected graph is 3-linked. Theorem 1.4.2 combined with a result of Chen, Gould and Yu [3] (see Lemma 2.1.2) gives the following.

**Corollary 1.4.3.** For every 6-connected graph \(G\) on \(n\) vertices with \(5n - 14\) edges and distinct \(a_1, a_2, b_1, b_2 \in V(G)\), \((G, a_1, a_2, b_1, b_2)\) is feasible.

**Corollary 1.4.4.** For every 10-connected graph \(G\) and \(a_1, a_2, b_1, b_2 \in V(G)\), \((G, a_1, a_2, b_1, b_2)\) is feasible.

Note that the \(k\)-linked notion was further extended by Kostochka and G.Yu [12] to \(H\)-linked graphs for any fixed graph \(H\). Recent work of Liu, Rolek, Stephens, Ye and G.Yu [14] shows that every 7-connected graph is kite-linked, where a kite is a graph obtained from \(K_4\) by deleting two adjacent edges.

### 1.4.2 Signed graphs

A signed graph is a triple \((V(G), E(G), f)\) where \(f : E(G) \to \{1, -1\}\). The sign of a cycle is the product of the signs of its edges. We call a signed graph \(G\) balanced if every cycle is positive and imbalanced if \(G\) is not balanced.

Theorem 1.2.1 has a signed graph version by Tutte in [25], and we state it here.

**Theorem 1.4.5** ([25]). Let \(G\) be a 3-connected signed graph and \(b \in V(G)\). Suppose \(G - b\) is imbalanced, then \(G\) has a negative cycle \(C\) such that \(b \notin V(C)\) and \(G - C\) is connected.

Note that Theorem 1.4.5 implies Theorem 1.2.1: For any 3-connected graph \(G\) and distinct \(a_1, a_2, b \in V(G)\), let \(G' = G + a_1a_2\). We assign \(f : E(G') \to \{1, -1\}\) such that
\( f(a_1a_2) = -1 \) and \( f(e) = 1 \) for all \( e \in E(G') \setminus \{a_1a_2\} \). Then, Theorem 1.2.1 follows from Theorem 1.4.5.

Similarly, the following signed graph version of Corollary 2.1.4, by Devos, Nurse, Qian and Wollan [4], also implies Corollary 2.1.4.

**Theorem 1.4.6** ([4]). Let \( G \) be a 5-connected signed graph and \( b \in V(G) \). Suppose \( G - b \) is imbalanced, then \( G \) has a negative cycle \( C \) such that \( b \notin V(C) \) and \( G - C \) is 2-connected.

It is natural to ask the following:

**Question 1.4.7.** Can we extend other results in Table 1.1 to signed graphs?

The above known signed graph results, Theorem 1.4.5 and Theorem 1.4.6, imply Theorem 1.2.1 and Corollary 2.1.4.

**Question 1.4.8.** Can we find an example on other results in Table 1.1 whose signed graph version does not hold? A positive answer would imply that signed graph version could be strictly stronger than the graph version.

### 1.4.3 A general conjecture

Recall Table 1.1 and definition of \( \alpha(m, k) \). When \( m = 0 \), it centers around Lovász’s conjecture which is open for \( k \geq 3 \). For \( k = 1 \), \( \alpha(m, k) \) exists by Chen, Gould and Yu [3], and we have exact values when \( m \leq 3 \). Wollan \(^1\) conjectured that \( \alpha(m, 2) = 2m + C \) for some constant \( C \).

It is also natural to formulate a more general conjecture on non-separating paths avoiding more vertices.

**Conjecture 1.4.9** (Qian, Xie, Yu). For each natural number \( k \) and \( m \), there exists a least natural number \( \alpha(m, k) \) such that, for any two vertices \( a_1, a_2 \) in any \( \alpha(m, k) \)-connected graph \( G \), there exists an \( a_1 \)-\( a_2 \) path \( P \) avoiding a given set of \( m \) vertices such that \( G - P \) is \( k \)-connected.

\(^1\)Paul Wollan: Private communication
The rest of the thesis is organized as follows:

In Chapter 2, we state previous results on disjoint paths that we will use in the thesis. We first state and prove feasibility for 5-connected graphs with given conditions. The result also provides us with an equivalent condition for feasibility that is convenient to use. Then, we introduce Seymour’s characterization of 2-linked graphs and Yu’s characterization of graphs with special three paths.

In Chapter 3, for an infeasible 8-connected graph $G$, we use three special paths $B_1, B_2, B_3$ to give a decomposition of $G$ into three edge disjoint subgraphs $A_1, A_2$ and $H$. We will show $A_i$ is planar for both $i \in [2]$ and $H$ can be further decomposed into graphs with simple structures, called rungs.

Structure of $H$ is further explored in Chapter 4 and Chapter 5. In Chapter 4, we show that most rungs will avoid at least one of the special paths $B_i$ for all $i \in [3]$. In Chapter 5, we consider those rungs intersecting at most two $B_i$’s.

In Chapter 6, using the structure theorem we proved, we construct examples of $A_1$ and $H$, and we use them to form a 7-connected graph with special vertices $a_1, a_2, b_1, b_2$ such that $(G, a_1, a_2, b_1, b_2)$ is infeasible. Thus, $\alpha(2, 2) \geq 8$. 
CHAPTER 2
PREVIOUS RESULTS ON DISJOINT PATHS

In this chapter, we state and prove some known results on disjoint paths that we will use in the thesis.

First in section 2.1, we state and prove a result on feasibility for 5-connected graphs. That result gives Corollary 2.1.3, providing us with a convenient working condition on disjoint paths which is equivalent to feasibility. One other consequence is Corollary 2.1.4, which reproves $\alpha(1, 2) = \alpha(0, 2) = 5$.

In section 2.2, we introduce the concept of “3-planar” graphs and state Seymour’s characterization of 2-linked graphs.

In section 2.3, we introduce definitions of “rungs” and “ladders”, and state Yu’s characterization of graphs containing certain types of three disjoint paths.

2.1 Feasibility for 5-connected graphs

The following well-known result of Menger [16] is often used to find independent paths in graphs.

**Theorem 2.1.1** ([16]). For any positive integer $k$ and any $k$-connected graph $G$, and for any $A, B \subseteq V(G)$ with $|A| \geq k$ and $|B| \geq k$, there are at least $k$ disjoint $A$-$B$ paths.

Chen, Gould and Yu [3] proved a result that implies the following result. We give a proof for the sake of completeness.

**Lemma 2.1.2** ([3]). For any 5-connected graph $G$ and any distinct vertices $a_1, a_2, b_1, b_2$ of $G$, if there exist three independent paths $A, B_1, B_2$ such that $A$ is from $a_1$ to $a_2$ and $B_i$ is from $b_1$ to $b_2$ for both $i \in [2]$, then $(G, a_1, a_2, b_1, b_2)$ is feasible.
Proof. We may assume $A$ is induced. Let $C_1$ be the component of $G - A$ containing $\{b_1, b_2\}$ and $B$ be the block in $C_1$ containing $B_1 \cup B_2$. Let $B^1, B^2, \ldots, B^n$ denote the $B$-bridges of $C_1$, and let $C_2, \ldots, C_m$ be the other components of $G - A$. We may assume $|V(B^{i-1})| \geq |V(B^i)|$ for $2 \leq i \leq n$ and $|V(C_{i-1})| \geq |V(C_i)|$ for $2 \leq i \leq m$. Now, we further choose $A, B_1, B_2$ such that $(|V(B)|, |V(B^1)|, \ldots, |V(B^n)|, |V(C_1)|, \ldots, |V(C_m)|)$ is maximal with respect to the lexicographic ordering.

Suppose $m \geq 2$. Since $G$ is $5$-connected, by Theorem 2.1.1, there exist $5$ disjoint paths from $V(C_m)$ to $V(G - C_m)$. Since $V(C_i) \cap N_G(C_m) = \emptyset$ for all $i < m$, $|V(A) \cap N_G(C_m)| \geq 5$. Let $x, y \in V(A) \cap N_G(C_m)$ such that $A[a_1, x] \cap N_G(C_m) = \emptyset$ and $A(y, a_2) \cap N_G(C_m) = \emptyset$. Since $\{x, y\}$ is not a cut in $G$ separating $A(x, y)$ from $G - C_m$, there exists $z \in V(A(x, y))$ such that $N_G(z) \cap V(C_j) \neq \emptyset$ for some $j < m$. Choose minimum such $j$. Let $P$ be an induced $x$-$y$ path in $G[V(C_m) \cup \{x, y\}]$. Take $A' = A[a_1, x] \cup P \cup A[y, a_2]$. Note that $C_1, C_2, \ldots, C_{j-1}$ are components of $G - A'$, and if $j = 1$, the block in $G - A'$ containing $\{b_1, b_2\}$ still contains $B$. However, $|V(C_j')| > |V(C_j)|$, contradicting the choice of $A$ that $(|V(B)|, |V(B^1)|, \ldots, |V(B^n)|, |V(C_1)|, \ldots, |V(C_m)|)$ is maximal with respect to the lexicographic ordering.

So $m = 1$. If $n = 0$, we are done. So assume $n \geq 1$. Let $\{z\} = V(B) \cap V(B^n)$. Since $G$ is $5$-connected, $|N_G(B^n - z) \cap V(A)| \geq 2$. Let $x, y \in V(A) \cap N_G(B^n - z)$ such that $A[a_1, x] \cap N_G(B^n - z) = \emptyset$ and $A(y, a_2) \cap N_G(B^n - z) = \emptyset$, and let $P$ be an induced $x$-$y$ path in $G[V(B^n - z) \cup \{x, y\}]$. Take $A' = A[a_1, x] \cup P \cup A[y, a_2]$ and $B'$ be the block of $G - A'$ containing $\{b_1, b_2\}$.

Suppose $G$ has edges from distinct vertices of $B$ to $A(x, y)$. Then, $G - A'$ has block containing $B$ and a subpath of $A(x, y)$. So $A'$ contradicts the choice of $A$.

Hence, since $G$ is $5$-connected, $G$ has an edge from $A(x, y)$ to $B^i$ for some $i \in [n - 1]$. We choose minimum such $i$. Then, either (1) $G - A'$ has a block containing $B$ and part of $B^i \cup A(x, y)$, or (2) $B$ is a block of $G - A'$, $B^1, \ldots, B^{i-1}$ are $B$-bridges of $G - A'$, and $B^i$ is properly contained in a $B$-bridge of $G - A'$. Thus, $A'$ contradicts the choice of $A$. \qed
On the other hand, it is straightforward to see that feasibility implies the existence of such three paths in 5-connected graphs.

**Corollary 2.1.3.** For any 5-connected graph $G$ and any distinct vertices $a_1, a_2, b_1, b_2$ of $G$, the following statements are equivalent:

(i) There exist three pairwise independent paths $A, B_1, B_2$ such that $A$ is from $a_1$ to $a_2$ and $B_i$ is from $b_1$ to $b_2$ for both $i \in [2]$.

(ii) $(G, a_1, a_2, b_1, b_2)$ is feasible.

Hence, for the rest of the thesis, we also call $(G, a_1, a_2, b_1, b_2)$ feasible if $G$ is 5-connected and one can find three pairwise independent paths $A, B_1, B_2$ such that $A$ is from $a_1$ to $a_2$ and $B_i$ is from $b_1$ to $b_2$ for both $i \in [2]$.

Another consequence of Lemma 2.1.2 is the following result that $\alpha(1, 2) = 5$ (see Table 1.1).

**Corollary 2.1.4.** For any 5-connected graph $G$ and any distinct vertices $a_1, a_2, b$ of $G$, $G - b$ contains an $a_1$-$a_2$ path $P$ such that $G - P$ is 2-connected.

**Proof.** Since $G$ is 5 connected, by Menger’s Theorem, there exist two independent $a_1$-$a_2$ paths $P_1, P_2$ in $G - b$. By Menger’s Theorem again, there exist 5 paths from $b$ to $V(P_1 \cup P_2)$, with only $b$ in common. By Pigeonhole Principle, two of the paths, say $Q_1, Q_2$, are from $b$ to $P_i(a_1, a_2)$ for some $i \in [2]$. Let $B$ be the block of $G - P_{3-i}$ containing $Q_1 \cup Q_2$. By the same proof in Lemma 2.1.2, $G$ contains an $a_1$-$a_2$ path $P'$ such that $G - P'$ is 2-connected and $G - P'$ contains $Q_1 \cup Q_2$. Since $b \in V(Q_1 \cup Q_2)$, $P' \subseteq G - b$ and we are done. \qed

### 2.2 Characterization of 2-linked graphs

A result we use often is a characterization of 2-linked graphs, proved independently by Seymour [19], Shiloach [20], Thomassen [24], and Chakravarti and Robertson [17].
A more general result on finding $k$ disjoint paths can be found in [18] by Robertson and Seymour in their monumental project on graph minors over a series of papers.

To state Seymour’s version on 2-linked graphs, we introduce several concepts.

A $3$-planar graph $(G, \mathcal{A})$ consists of a graph $G$ and a set $\mathcal{A} = \{A_1, \ldots, A_k\}$ of pairwise disjoint subsets of $V(G)$ (let $\mathcal{A} = \emptyset$ when $k = 0$) such that

(i) for $i \neq j$, $N_G(A_i) \cap A_j = \emptyset$,

(ii) for $1 \leq i \leq k$, $|N_G(A_i)| \leq 3$, and

(iii) if $p(G, \mathcal{A})$ denotes the graph obtained from $G$ by (for each $i$) deleting $A_i$ and adding edges joining every pair of distinct vertices in $N_G(A_i)$, then $p(G, \mathcal{A})$ can be drawn in the plane without crossing edges.

If, in addition, $b_1, b_2, \ldots, b_n$ are vertices in $G$ such that $b_i \notin A$ for $i \in [n]$ and $A \in \mathcal{A}$, $p(G, \mathcal{A})$ can be drawn in a closed disk with no edge crossings, and $b_1, b_2, \ldots, b_n$ occur on the boundary of the disk in this cyclic order, then we say that $(G, \mathcal{A}, b_1, b_2, \ldots, b_n)$ is 3-planar. If there is no need to specify $\mathcal{A}$, we may simply say that $(G, b_1, b_2, \ldots, b_n)$ is 3-planar. If $\mathcal{A} = \emptyset$, we say that $(G, b_0, b_1, \ldots, b_n)$ is planar. If $G$ is planar and is drawn in a closed disk with no edge crossings, for any subgraph $H \subseteq G$, we say $(G, H)$ is planar if all vertices and edges of $H$ are contained in the boundary of the disk, in which case $H$ needs to be the union of disjoint paths.

Now, we can state Seymour’s characterization on 2-linked graphs.

**Lemma 2.2.1** (Seymour, 1980). Let $G$ be a graph with distinct vertices $x_1, x_2, x_3, x_4$. Then either $(G, x_1, x_2, x_3, x_4)$ is 3-planar, or $G$ has disjoint paths from $x_1, x_2$ to $x_3, x_4$, respectively.

### 2.3 Characterization of graphs with special three paths

While there is no known generalization of the above result to three paths with fixed ends (see Conjecture 1.4.1 of Thomassen), Yu [27, 28, 29] characterized graphs $G$ in which any
Let $G$ be a graph, $\{a, b, c\} \subseteq V(G)$, and $\{a', b', c'\} \subseteq V(G)$ must contain a path from $b$ to $b'$. To state this result, we need to describe **rungs** and **ladders**.

Let $G$ be a graph, $\{a, b, c\} \subseteq V(G)$, and $\{a', b', c'\} \subseteq V(G)$. (Here, $a, b, c$ are pairwise distinct, and $a', b', c'$ are pairwise distinct.) Suppose $\{a, b, c\} \neq \{a', b', c'\}$, and assume that $G$ has no separation $(G_1, G_2)$ such that $|V(G_1 \cap G_2)| \leq 3$, $\{a, b, c\} \subseteq V(G_1)$, and $\{a', b', c'\} \subseteq V(G_2)$. We say that $(G, (a, b, c), (a', b', c'))$ is a rung if one of the following holds up to symmetry between $\{a, b, c\}$ and $\{a', b', c'\}$, relabeling $a$ and $c$, and relabeling $a'$ and $c'$:

1. $b = b'$ or $\{a, c\} = \{a', c'\}$.
2. $a = a'$ and $(G - a, c, c', b', b)$ is 3-planar.
3. $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$ and $(G, a', b', c', c, b, a)$ is 3-planar.
4. $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, $G$ has a separation $(G_1, G_2)$ such that $V(G_1 \cap G_2) = \{x\}$, and $\{a, a', b, b'\} \subseteq V(G_1)$, $\{c, c'\} \subseteq V(G_2)$, and $(G_1, a, a', b', b)$ is 3-planar.
5. $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, and $G$ has a separation $(G_1, G_2)$ such that $V(G_1 \cap G_2) = \{z, b\}$, and $(G_1 + bz, a, a', b', b)$ is 3-planar, $\{a, a', b, b'\} \subseteq V(G_1)$, $\{c, c'\} \subseteq V(G_2)$, and $(G_2, c, c', z, b)$ is 3-planar.
6. $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, and there are pairwise edge disjoint subgraphs $G_a, G_c, M$ of $G$ such that $G = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{u, z\}$, $V(G_c \cap M) = \{p, q\}$, $V(G_a \cap G_c) = \emptyset$, and $\{a, a', b'\} \subseteq V(G_a)$, $\{c, c', b\} \subseteq V(G_c)$, and $(G_a, a, a', b', z, u)$ and $(G_c, c', c, b, p, q)$ are 3-planar.
7. $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, and there are pairwise edge disjoint subgraphs $G_a, G_c, M$ of $G$ such that $G = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{b, b', q\}$, $V(G_c \cap M) = \{b, b', p\}$, $V(G_a \cap G_c) = \{b, b'\}$, $\{a, a'\} \subseteq V(G_a)$, $\{c, c'\} \subseteq V(G_c)$, and $(G_a, a, a', b', q, b)$ and $(G_c, c', c, b, p, b')$ are 3-planar.

See Figure 2.1 for illustration of all types of rungs.

Let $L$ be a graph and let $R_1, \ldots, R_m$ be edge disjoint subgraphs of $L$ such that

(i) $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))$ is a rung for each $i \in [m]$. 

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Figure 2.1: All types of rungs
(ii) $V(R_i \cap R_j) = \{x_i, v_i, y_i\} \cap \{x_{j-1}, v_{j-1}, y_{j-1}\}$ for $i, j \in [m]$ with $i < j$.

(iii) for any $i, j \in [m] \cup \{0\}$, if $x_i = x_j$ then $x_k = x_i$ for all $i \leq k \leq j$, if $v_i = v_j$ then $v_k = v_i$ for all $i \leq k \leq j$, and if $y_i = y_j$ then $y_k = y_i$ for all $i \leq k \leq j$, and

(iv) $L = (\bigcup_{i=1}^{m} R_i) + S$, where $S$ consists of those edges of $L$ each of which has both ends in $\{x_i, v_i, y_i\}$ for some $i \in [m] \cup \{0\}$.

Then $(L, (x_0, v_0, y_0), (x_m, v_m, y_m))$ is a ladder with rungs $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i)), i \in [m]$, or simply, a ladder along $v_0 \ldots v_m$. See Figure 2.2 for an example of ladder $L$. Note that in this example, edge $x_jv_j$ and edge $x_jy_j$ are in $S$.

![Figure 2.2: Example of ladder L](image)

By definition, for any rung $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))$, $R_i$ has three disjoint paths from $\{x_{i-1}, v_{i-1}, y_{i-1}\}$ to $\{x_i, v_i, y_i\}$. So for any ladder $(L, (x_0, v_0, y_0), (x_m, v_m, y_m))$, $L$ has three disjoint paths from $\{x_0, v_0, y_0\}$ to $\{x_m, v_m, y_m\}$.

For a sequence $W$, the reduced sequence of $W$ is the sequence obtained from $W$ by removing all but one consecutive identical elements. For example, the reduced sequence of $aaabcca$ is $abca$. We can now state the main result in [27, 28, 29].

**Lemma 2.3.1** ([27, 28, 29]). Let $G$ be a graph, $\{a, b, c\} \subseteq V(G)$, and $\{a', b', c'\} \subseteq V(G)$ such that $\{a, b, c\} \neq \{a', b', c'\}$. Then any three disjoint paths in $G$ from $\{a, b, c\}$ to $\{a', b', c'\}$ must include a path from $b$ to $b'$ if, and only if, one of the following statements holds:
(i) $G$ has a separation $(G_1, G_2)$ of order at most 2 such that \( \{a, b, c\} \subseteq V(G_1) \) and 
\( \{a', b', c'\} \subseteq V(G_2) \).

(ii) \((G, (a, b, c), (a', b', c'))\) is a ladder.

(iii) $G$ has a separation $(J, L)$ such that $V(J \cap L) = \{w_0, \ldots, w_n\}$, $(J, w_0, \ldots, w_n)$ is
3-planar, \( \{a, b, c\} \cup \{a', b', c'\} \subseteq V(L) \), $(L, (a, b, c), (a', b', c'))$ is a ladder along a
sequence $v_0 \ldots v_m$, where $v_0 = b$, $v_m = b'$, and $w_0 \ldots w_n$ is the reduced sequence of
$v_0 \ldots v_m$.

![Figure 2.3: Structure (iii) of Yu’s characterization for graph $G$](image)

See Figure 2.3 for structure (iii) of Lemma 2.3.1, where $L$ is a ladder (see Figure 2.2).

Note that structure (ii) of the theorem is when $J = \emptyset$.

To help readers familiarize with the above concepts and for later applications, we prove
the following properties of rungs.

**Proposition 2.3.2.** For any rung $(G, (a, b, c), (a', b', c'))$, the following statements hold:

(i) \( \{a, b, c\} \) and \( \{a', b', c'\} \) are independent sets in $G$.

(ii) For any $x \in \{a, b, c\} \triangle \{a', b', c'\}$, \( N_G(x) \neq \emptyset \). When \( \{a, b, c\} \cap \{a', b', c'\} = \emptyset \),
\( |N_G(x)| \geq 2 \).
(iii) Suppose \( \{a, b, c\} \cap \{a', b', c'\} = \emptyset \) or \(|\{a, b, c\} \cup \{a', b', c'\}| = 5 \) and \( b \neq b' \). Then, for any \( x \in \{b, b'\} \), \( N_G(x) \cap \{a, c, a', c'\} = \emptyset \). Moreover, any three disjoint paths in \( G \) from \( \{a, b, c\} \) to \( \{a', b', c'\} \) must be from \( a, b, c \) to \( a', b', c' \), respectively.

**Proof.** Suppose (i) fails and without loss of generality, let \( e \in E(G[\{a, b, c\}]) \). Let \( G_1 = (\{a, b, c\}, \{e\}) \) and \( G_2 = G - e \). Then \( (G_1, G_2) \) is a separation in \( G \) contradicting the definition of a rung. Hence, (i) holds.

To prove (ii), let \( x \in \{a, b, c\} \Delta \{a', b', c'\} \) and, without loss of generality, assume \( x \in \{a, b, c\} \setminus \{a', b', c'\} \). Then, \( N_G(x) \neq \emptyset \); otherwise \( \{a, b, c\} \setminus \{x\} \) is a 2-cut in \( G \) separating \( \{a, b, c\} \) from \( \{a', b', c'\} \), contradicting the definition of a rung. Now suppose \( \{a, b, c\} \cap \{a', b', c'\} = \emptyset \). If \( |N_G(x)| = 1 \) then \( (\{a, b, c\} \setminus \{x\}) \cup N_G(x) \) is a 3-cut in \( G \) separating \( \{a, b, c\} \) from \( \{a', b', c'\} \), contradicting the definition of a rung. So \(|N_G(x)| \geq 2\).

We now prove (iii). First, suppose \(|\{a, b, c\} \cup \{a', b', c'\}| = 5 \) and \( b \neq b' \). By symmetry, we may assume \( a = a' \) and \( (G - a, b, b', c', c) \) is 3-planar. By applying Lemma 2.2.1, we see that any three disjoint paths in \( G \) from \( \{a, b, c\} \) to \( \{a', b', c'\} \) must be from \( a, b, c \) to \( a', b', c' \), respectively. Now, \( ba' \notin E(G) \) by (i) as \( a = a' \), and \( bc' \notin E(G) \) as \( \{a, b, c'\} \) cannot be a cut in \( G \) separating \( \{a, b, c\} \) from \( \{a', b', c'\} \). Similarly, \( b'c, b'a \notin E(G) \).

It remains to consider the case when \( \{a, b, c\} \cap \{a', b', c'\} = \emptyset \). Then \( (G, (a, b, c), (a', b', c')) \) is a rung of type (3)-(7).

First, assume that \( (G, (a, b, c), (a', b', c')) \) is of Type (3). Then \( (G, a, b, c, c', b', a') \) is 3-planar. By applying Lemma 2.2.1, we see that any three disjoint paths in \( G \) from \( \{a, b, c\} \) to \( \{a', b', c'\} \) must be from \( a, b, c \) to \( a', b', c' \), respectively. Now, \( ba', bc', b'a, b'c \notin E(G) \). For, otherwise, by symmetry, assume \( bc' \in E(G) \). Then, \( \{a, b, c\} \) is a 3-cut in \( G \) separating \( \{a, b, c\} \) from \( \{a', b', c'\} \), a contradiction.

Next, assume \( (G, (a, b, c), (a', b', c')) \) is of Type (4). Then \( G \) has a separation \( (G_1, G_2) \) such that \( V(G_1 \cap G_2) = \{x\} \), \( \{a, a', b, b'\} \subseteq V(G_1) \), \( \{c, c'\} \subseteq V(G_2) \), and \( (G_1, a, a', b', b) \) is 3-planar. By applying Lemma 2.2.1, we see that any three disjoint paths in \( G \) from
\{a, b, c\} to \{a', b', c'\} must be from \(a, b, c\) to \(a', b', c'\), respectively. Now, we prove \(ba', bc', b'a, b'c \notin E(G)\). By structure of \(G\), \(bc', b'c \notin E(G)\). So, by symmetry, suppose \(ba' \in E(G)\). Then, \{a', b, c\} is a 3-cut in \(G\) separating \{a, b, c\} from \{a', b', c'\}, a contradiction.

Suppose \((G, (a, b, c), (a', b', c'))\) is of Type (5). Then \(G\) has a 2-separation \((G_1, G_2)\) such that \(V(G_1 \cap G_2) = \{x, b\}, \{a, a', b, b'\} \subseteq V(G_1), \{c, c'\} \subseteq V(G_2)\), and \((G_1 + xb, a, a', b, b)\) and \((G_2, c, c', x, b)\) are 3-planar. By applying Lemma 2.2.1, we see that any three disjoint paths in \(G\) from \{a, b, c\} to \{a', b', c'\} must be from \(a, b, c\) to \(a', b', c'\), respectively. Now, we prove \(ba', bc', b'a, b'c \notin E(G)\). By structure of \(G\), \(b'c \notin E(G)\). If \(bc' \in E(G)\), then \{a, b, c\} is a 3-cut in \(G\) separating \{a, b, c\} from \{a', b', c'\}, a contradiction. So, by symmetry, assume \(ba' \in E(G)\). Then, \{a', b, c\} is a 3-cut in \(G\) separating \{a, b, c\} from \{a', b', c'\}, a contradiction.

Now assume \((G, (a, b, c), (a', b', c'))\) is of Type (6). Then there are pairwise edge disjoint subgraphs \(G_a, G_c, M\) of \(G\) such that \(G = G_a \cup G_c \cup M, V(G_a \cap M) = \{u, z\}, V(G_c \cap M) = \{p, q\}, V(G_a \cap G_c) = \emptyset, \{a, a', b'\} \subseteq V(G_a), \{c, c', b\} \subseteq V(G_c)\), and \((G_a, a, a', b', z, u)\) and \((G_c, c', c, b, p, q)\) are 3-planar. By applying Lemma 2.2.1, we see that any three disjoint paths in \(G\) from \{a, b, c\} to \{a', b', c'\} must be from \(a, b, c\) to \(a', b', c'\), respectively. Now, we prove \(ba', bc', b'a, b'c \notin E(G)\). By structure of \(G\), \(ba', b'c \notin E(G)\). So, by symmetry, suppose \(bc' \in E(G)\). Then, \{a, b, c\} is a 3-cut in \(G\) separating \{a, b, c\} from \{a', b', c'\}, a contradiction.

Finally, assume \((G, (a, b, c), (a', b', c'))\) is of Type (7). Then there are pairwise edge disjoint subgraphs \(G_a, G_c, M\) of \(R\) such that \(G = G_a \cup G_c \cup M, V(G_a \cap M) = \{b, b', q\}, V(G_c \cap M) = \{b, b', p\}, V(G_a \cap G_c) = \{b, b'\}, \{a, a'\} \subseteq V(G_a), \{c, c'\} \subseteq V(G_c)\), and \((G_a, a, a', b', q, b)\) and \((G_c, c', c, b, p, b')\) are 3-planar. By applying Lemma 2.2.1, we see that any three disjoint paths in \(G\) from \{a, b, c\} to \{a', b', c'\} must be from \(a, b, c\) to \(a', b', c'\), respectively. Now, we prove \(ba', bc', b'a, b'c \notin E(G)\). For, otherwise, by symmetry, assume \(bc' \in E(G)\). Then, \{a, b, c\} is a 3-cut in \(G\) separating \{a, b, c\} from \{a', b', c'\}, a
contradiction.
CHAPTER 3
FRAMES AND CONSTRAINTS

Let $G$ be a graph and $a_1, a_2, b_1, b_2 \in V(G)$ be distinct. Recall that by Corollary 2.1.3 in Chapter 2, $(G, a_1, a_2, b_1, b_2)$ is feasible if $G$ contains three pairwise independent paths $A, B_1, B_2$, such that $A$ is from $a_1$ to $a_2$, and $B_i$ is from $b_1$ to $b_2$ for $i \in [2]$.

Our main Theorem 1.3.1 gives a structural result on infeasible 8-connected graphs. In this chapter, we give the decomposition of $G$ into edge disjoint subgraphs $A_1, A_2$ and $H$.

Suppose $(G, a_1, a_2, b_1, b_2)$ is infeasible.

In section 3.1, we find the subgraphs $A_1, A_2, H$ in $G$, and prove that $A_1$ and $A_2$ are both planar by applying Lemma 2.2.1 on 2-linked graphs.

In section 3.2, by choosing favorite $A_1$ and $A_2$ and applying Lemma 2.3.1 on three special paths, we show that there exists $w \in V(H)$ such that $H - w$ is a ladder of rungs.

We give an illustration of the structure of $G$ in Figure 3.1.

3.1 Frame and its properties

For any three independent $b_1$-$b_2$ paths $B_1, B_2, B_3$ in $G - \{a_1, a_2\}$, we use $A_i(B_1 \cup B_2 \cup B_3)$, for $i \in [2]$, to denote the $(B_1 \cup B_2 \cup B_3)$-bridge of $G$ containing $a_i$.

We say that $B_1, B_2, B_3$ form a frame in $(G, a_1, a_2, b_1, b_2)$, if they satisfy (C1)-(C4), up to relabeling $a_1$ and $a_2$ and relabeling $b_1$ and $b_2$.

(C1) $B_1, B_2, B_3$ are independent induced $b_1$-$b_2$ paths in $G - \{a_1, a_2\}$,

(C2) $A_1(B_1 \cup B_2 \cup B_3)$ has all its attachments on $B_3$,

(C3) $A_2(B_1 \cup B_2 \cup B_3)$ has attachments on both $B_1(b_1, b_2)$ and $B_2(b_1, b_2)$, and

(C4) subject to (C1)-(C3), $A_1(B_1 \cup B_2 \cup B_3)$ is maximal.
In this section, we prove the existence of such a frame in 8-connected infeasible graphs, as well as some related properties. Since $G$ is 8-connected, by Theorem 2.1.1, there exist three independent $b_1$-$b_2$ paths in $G - \{a_1, a_2\}$. Take such three paths $B_1, B_2, B_3$ to be induced; so (C1) holds.

Now, we show that (C2) holds for any three independent $b_1$-$b_2$ paths $B_1, B_2, B_3$ in $G - \{a_1, a_2\}$.

**Lemma 3.1.1.** Suppose $(G, a_1, a_2, b_1, b_2)$ is infeasible and $B_1, B_2, B_3$ are three independent $b_1$-$b_2$ paths in $G - \{a_1, a_2\}$. Then there exist $i \in [2]$ and $j \in [3]$ such that $A_i(B_1 \cup B_2 \cup B_3)$ has all its attachments contained in $B_j$.

**Proof.** For, suppose such $i, j$ do not exist. Then there exists some $k \in [2]$ such that, for $s \in [2]$, $A_s(B_1 \cup B_2 \cup B_3)$ has an attachment $a_s' \in V(B_k(b_1, b_2))$. Let $Q_s$ denote an $a_s$-$a_s'$ path in $A_s(B_1 \cup B_2 \cup B_3)$ internally disjoint from $B_1 \cup B_2 \cup B_3$. Without loss of generality, let $k = 1$. Then $B_2, B_3, Q_1 \cup B_1[a_1', a_2'] \cup Q_2$ show that $(G, a_1, a_2, b_1, b_2)$ is feasible, a contradiction. \qed

Next, we show that if $B_1, B_2, B_3$ satisfy (C1)-(C3), then $(A_1(B_1 \cup B_2 \cup B_3)) \cup B_3, B_3 + a_2)$ is planar.

**Lemma 3.1.2.** Suppose $(G, a_1, a_2, b_1, b_2)$ is infeasible and $G$ is 4-connected, and suppose $B_1, B_2, B_3$ are independent $b_1$-$b_2$ paths in $G - \{a_1, a_2\}$. For any $i \in [2]$ and $j \in [3]$, if $A_i(B_1 \cup B_2 \cup B_3) \cap (B_1 \cup B_2 \cup B_3) \subseteq B_j$ and $A_{3-i}(B_1 \cup B_2 \cup B_3)$ intersects $B_k(b_1, b_2)$ for both $k \in [3] \setminus \{j\}$, then $(A_i(B_1 \cup B_2 \cup B_3) \cup B_j, B_j + a_i)$ is planar.

**Proof.** Without loss of generality, we may assume $i = 1$ and $j = 3$. Let $H$ be the graph obtained from $G$ by contracting $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3)$ to a single vertex $w$. Suppose there exist disjoint paths $P_1, P_2$ in $H$ from $b_1, a_1$ to $b_2, w$, respectively. Let $w' \in N(w) \cap V(P_2) \subseteq V(B_3)$. By symmetry between $B_1$ and $B_2$, we may assume that $G$ has a path $Q$ from $w'$ to $B_1$ and internally disjoint from $A_1(B_1 \cup B_2 \cup B_3) \cup B_1 \cup B_2 \cup B_3$. Since $A_2(B_1 \cup B_2 \cup B_3)$ has attachments on $B_1(b_1, b_2)$, it contains an $a_2$-$w'$ path, say $P$, internally
disjoint from $B_1 \cup B_2 \cup B_3$. Now $(P_2 - w) \cup Q \cup B_1(b_1, b_2) \cup P$ contains an $a_1$-$a_2$ path independent of $P_1$ and $B_2$, which, together with $P_1$ and $B_2$, shows that $(G, a_1, a_2, b_1, b_2)$ is feasible, a contradiction.

So such paths $P_1, P_2$ do not exist in $H$. By Lemma 2.2.1, $(H, \mathcal{A}, \{w, b_1, a_1, b_2\})$ is 3-planar, where $\mathcal{A}$ is a collection of disjoint subsets of $V(H) \setminus \{w, b_1, a_1, b_2\}$. If $\mathcal{A} = \emptyset$, we are done. Hence we may assume there exists $A \in \mathcal{A}$. Since $|N_H(A)| \leq 3$ and $G$ is 4-connected, $V(B_3) \cap A \neq \emptyset$. Therefore, $w \in N_H(A)$ and, thus, $|N_H(A) \cap V(B_3)| = 2$. Hence, $H[A] \subseteq B_3$ by definition of $A_1(B_1 \cup B_2 \cup B_3)$ and $B_3$. This implies $(H[A \cup N_H(A)], N_H(A))$ is planar for all $A \in \mathcal{A}$. Hence, $(A_i(B_1 \cup B_2 \cup B_3) \cup B_j, B_j + a_i)$ is planar. \hfill \square

Before we prove the existence of a frame, we need the following lemma for 8-connected graphs when $(A_1(B_1 \cup B_2 \cup B_3), B_3 + a_1)$ is planar.

**Lemma 3.1.3.** Suppose $(G, a_1, a_2, b_1, b_2)$ is infeasible and $G$ is 8-connected. Let $B_1, B_2, B_3$ be three independent induced $b_1$-$b_2$ paths in $G - \{a_1, a_2\}$ such that $(A_1(B_1 \cup B_2 \cup B_3), B_3 + a_1)$ is planar. Then there exists $w \in V(A_1(B_1 \cup B_2 \cup B_3)) \cap V(B_3(b_1, b_2))$ such that $w$ is not contained in any 3-cut of $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3) - \{a_2\}$ separating $b_1$ from $b_2$.

**Proof.** For convenience, let $A_1 := A_1(B_1 \cup B_2 \cup B_3)$ and $H = G - (A_1 - B_3) - \{a_2\}$. Suppose such $w$ does not exist. Then every vertex in $V(A_1) \cap V(B_3(b_1, b_2))$ is contained in a 3-cut of $H$ separating $b_1$ from $b_2$. Let $V(A_1) \cap V(B_3(b_1, b_2)) = \{w_1, \ldots, w_m\}$ such that $b_1, w_1, \ldots, w_m, b_2$ occur on $B_3$ in order. For $i \in [m]$, let $u_i \in V(B_1(b_1, b_2))$, $v_i \in V(B_2(b_1, b_2))$ such that $T_i := \{u_i, v_i, w_i\}$ is a 3-cut of $H$ separating $b_1$ from $b_2$. We may assume that

- (1) for all $i \in [m - 1]$, $b_1, u_i, u_{i+1}, b_2$ occur on $B_1$ in order and $b_1, v_i, v_{i+1}, b_2$ occur on $B_2$ in order.

To see this, we choose $T_i$ such that the $T_i$-bridge of $H$ containing $b_1$, denoted by $H_i$, is minimal. Suppose (1) fails. Then by symmetry between $B_1$ and $B_2$, we may assume that for some $i$, $b_1, u_{i+1}, u_i, b_2$ are in order on $B_1$. 

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First, suppose \( b_1, v_i, v_{i+1}, b_2 \) occur on \( B_2 \) in order. By the choice of \( \{u_i, v_i, w_i\} \), \( \{u_{i+1}, v_i, w_i\} \) is not a cut in \( H \) separating \( b_1 \) from \( b_2 \). Hence, there exists a \( b_1-u_i \) path \( P \) in \( H_i - \{u_{i+1}, v_i, w_i\} \). But then \( P \cup B_1[u_i, b_2] \) is a \( b_1-b_2 \) path in \( H - T_{i+1} \), a contradiction.

Now assume that \( b_1, v_{i+1}, v_i, b_2 \) are in order on \( B_2 \). By the choice of \( \{u_i, v_i, w_i\} \), \( \{u_{i+1}, v_i, w_i\} \) is not a cut in \( H \) separating \( b_1 \) from \( b_2 \). So there exists a \( b_1-w_{i+1} \) path \( Q \) in \( H_{i+1} - \{u_{i+1}, v_{i+1}, w_i\} \). But again, \( Q \cup B_1[w_{i+1}, b_2] \) is a \( b_1-b_2 \) path in \( H - T_i \), a contradiction.

\[
(2) \quad V(H) = \{b_1, b_2\} \cup \left( \bigcup_{i \in [m]} T_i \right).
\]

Otherwise suppose there exists \( x \in V(H) \) such that \( x \notin \{b_1, b_2\} \cup \left( \bigcup_{i \in [m]} T_i \right) \). Then, \( x \) is not contained in the \( T_i \)-bridge of \( H \) containing \( b_1 \); as otherwise, \( T_1 \cup \{b_1, a_2\} \) is a \( 5 \)-cut in \( G \) separating \( x \) from \( b_2 \), a contradiction. Similarly, \( x \) is not contained in the \( T_m \)-bridge of \( H \) containing \( b_2 \). Hence, there exists \( i \in [m] \) such that \( x \) is contained in both the \( T_{i+1} \)-bridge of \( H \) containing \( b_1 \) and the \( T_i \)-bridge of \( H \) containing \( b_2 \). Now \( T_i \cup T_{i+1} \cup \{a_2\} \) is a cut in \( G \) of order at most 7 and separates \( x \) from \( \{a_1, a_2\} \), a contradiction.

Since \( d_G(b_i) \geq 8 \) for both \( i \in [2] \), it follows from (2) that

\[
(3) \quad d_{A_1}(b_i) \geq 5 \text{ for } i \in [2].
\]

\[
(4) \quad \text{There exists } i \in [m] \text{ such that } d_H(w_i) \geq 7.
\]

Suppose for a contradiction, \( d_H(w_i) < 7 \) for all \( i \in [m] \). Then, since \( G \) is 8-connected, \( d_{A_1}(w_i) \geq 2 \) for all \( i \in [m] \).

Let \( H' \) be the graph obtained from \( A_1 \cup B_2 \) by adding a new vertex \( a \) and an edge from \( a \) to each vertex in \( B_3 \). Then, \( (H', a_1, b_1, a, b_2) \) is planar. We take an embedding of \( H' \) in the plane such that \( a_1, b_1, a, b_2 \) occur on the outer cycle of \( H' \) in clockwise order. Let \( F(H') \) denote the set of faces of \( H' \). For convenience, for the rest proof of the lemma, we write \( d(x) := d_{H'}(x) \) for \( x \in V(H') \cup F(H') \). When \( x \in F(H') \), \( d(x) \) is the number of edges incident to \( x \).
Note that \( d(a) = |V(B_3)|, d(w) \geq 5 \) for all \( w \in V(B_3) \), and \( d(v) \geq 8 \) for all \( v \in V(A_1 - B_3) \). Moreover, \( d(a) \geq 8 \); otherwise \( V(B_3) \) is a cut of size \( \leq 7 \) in \( G \) separating \( a_1 \) from \( a_2 \), a contradiction.

We now apply the discharging method to \( H' \). First, define \( \sigma(x) := d(x) - 4 \) as the charge of \( x \) for all \( x \in V(H') \cup F(H') \). Then, \( \sigma(x) \geq -1 \) for all \( x \in F(H') \), \( \sigma(x) \geq 1 \) for all \( x \in V(B_3) \), and \( \sigma(x) \geq 4 \) for all \( x \in V(A_1 - B_3) \cup \{a\} \). So \( \sigma(x) < 0 \) only if \( x \in F(H') \) is a triangular face of \( H' \). By Euler’s formula,

\[
\sum_{x \in V(H') \cup F(H')} \sigma(x) = -8.
\]

Next, we move charges from vertices to faces as follows: For every \( v \in V(H' - B_3) \), we discharge \( \frac{d(v)-4}{d(v)} \geq \frac{1}{2} \) (since \( d(v) \geq 8 \)) from \( v \) to each of the triangular faces of \( H' \) incident to \( v \). So the new charge \( \tau(v) \) for each vertex \( v \) satisfies

\[
\tau(v) \geq \sigma(v) - (d(v) - 4) \geq 0,
\]

and the new charge \( \tau(f) \) for each triangular face \( f \) with at most one vertex on \( B_3 \) satisfies

\[
\tau(f) \geq \sigma(f) + 2 \cdot \frac{1}{2} \geq 0.
\]

For each \( w \in V(B_3) \), we perform the discharging as follows. If \( d(w) \geq 6 \), we discharge \( \frac{d(w)-4}{d(w)} \geq \frac{1}{3} \) from \( w \) to each of the triangular faces incident to \( w \); the new charge of \( w \) is

\[
\tau(w) \geq \sigma(w) - (d(w) - 4) \geq 0.
\]

If \( d(w) = 5 \), we discharge \( \frac{1}{4} \) from \( w \) to each triangular face \( f \) incident to \( w \) and having two
vertices from $B_3$ (there are at most four such faces); so the new charge of $w$ is

$$\tau(w) \geq \sigma(w) - 4 \cdot \frac{1}{4} = 1 - 1 = 0.$$ 

Now, consider any triangular face $f$ with two vertices on $B_3$. $f$ gets at least $\frac{1}{2}$ from its vertex in $V(A_1 - B_3)$ and $\frac{1}{4}$ from each of its vertices in $V(B_3)$. So the new charge of $f$ is

$$\tau(f) \geq \sigma(f) + \frac{1}{2} + 2 \cdot \frac{1}{4} = 0.$$ 

Note that the infinity face of $H'$, say $f_0$, is incident to at least 4 vertices, so $\tau(f_0) \geq 0$. Thus, $\sum_{x \in V(H') \cup F(H')} \tau(x) \geq 0$. Since the total charge is preserved, we have

$$0 \leq \sum_{x \in V(H') \cup F(H')} \tau(x) = \sum_{x \in V(H') \cup F(H')} \sigma(x) = -8,$$

a contradiction. So we have (4).

By (4), let $j \in [m]$ be such that $d_H(w_j) \geq 7$. By (2) and the pigeonhole principal, $|V(B_i(b_1,b_2)) \cap N_G(w_j)| \geq 3$ for some $i \in [2]$. By symmetry, assume $|V(B_1(b_1,b_2)) \cap N_G(w_j)| \geq 3$.

(5) $N_G(w_j) \cap V(B_1) = \{u_{j-1}, u_j, u_{j+1}\}$ is disjoint from $\{b_1, b_2\}$ and $u_{j-1}, u_j, u_{j+1}$ are pairwise distinct, $N_G(u_j) \cap V(B_2) \subseteq \{v_{j-1}, v_j, v_{j+1}\}$, $N_G(u_j) \cap V(B_3) \subseteq \{w_{j-1}, w_j, w_{j+1}\}$, and if $w_{j-1}, w_j, w_{j+1}$ are pairwise distinct then $N_G(v_j) \cap V(B_1) \subseteq \{u_{j-1}, u_j, u_{j+1}\}$.

Let $x \in N_G(w_j) \cap V(B_1)$. If $x \in V(B(u_{j+1}, b_2))$ then $w_j x$ contradicts the existence of the 3-cut $T_{j+1}$ of $H$; and if $x \in V(B_1[b_1, u_{j-1}])$ then $w_j x$ contradicts the existence of the 3-cut $T_{i-1}$ of $H$. So by (2), $N_G(w_j) \cap V(B_1) = \{u_{j-1}, u_j, u_{j+1}\}$, and $u_{j-1}, u_j, u_{j+1}$ are pairwise distinct.

Now, consider $N_G(u_j)$. Clearly, $b_1, b_2 \not\in N_G(u_j)$ as $B_1$ is induced path in $G$. 

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Since \( u_{j-1}, u_j, u_{j+1} \) are pairwise distinct, similar arguments in last paragraph shows
\( N_G(u_j) \cap V(B_2) \subseteq \{v_{j-1}, v_j, v_{j+1}\} \) and \( N_G(u_j) \cap V(B_3) \subseteq \{w_{j-1}, w_j, w_{j+1}\} \). Similarly, if \( w_{j-1}, w_j, w_{j+1} \) are pairwise distinct then \( N_G(v_j) \cap V(B_1) \subseteq \{u_{j-1}, u_j, u_{j+1}\} \).

(6) \( a_2 \notin N_G(u_j) \).

Suppose \( u_ja_2 \in E(G) \). Then, \( N_G(u_j) \cap V(B_3) = \{w_j\} \). Otherwise there exists \( w_l \in V(B_3(b_1, b_2)) - \{w_j\} \) such that \( u_jw_l \in E(G) \). By symmetry, we may assume \( l < j \).

Let \( P \) be a \( w_l-a_1 \) path in \( A_1 \) independent of \( B_3 \). Then, \( P' = P \cup w_l u_j a_2 \) is an \( a_1-a_2 \) path, and, \( B_1[b_1, u_{j-1}] \cup u_{j-1}w_j \cup B_3[w_j, b_2] \) and \( B_2 \) are two disjoint \( b_1-b_2 \) paths in \( G - P' \), showing that \( (G, a_1, a_2, b_1, b_2) \) is feasible, a contradiction. But then, by (2), \( d_G(u_j) \leq |\{u_{j-1}, u_{j+1}, v_{j-1}, v_j, v_{j+1}, w_j, a_2\}| \leq 7 \), a contradiction.

By (5) and (6), \( N_G(u_j) = \{u_{j-1}, u_{j+1}, w_{j-1}, w_j, w_{j+1}, v_{j-1}, v_j, v_{j+1}\} \). Note that \( a_2 \in N_G(v_j) \), to avoid 7-cut \( \{u_{j-1}, u_{j+1}, w_{j-1}, w_j, w_{j+1}, v_{j-1}, v_j, v_{j+1}\} \) in \( G \) separating \( \{u_j, v_j\} \) from \( \{b_1, b_2\} \). Since \( d_G(v_j) \geq 8 \), there exists \( w_l \in V(B_3(b_1, b_2)) - \{w_j\} \) such that \( v_jw_l \in E(G) \). By symmetry, we may assume \( l < j \). Let \( P \) be a \( w_l-a_1 \) path in \( A_1 \) independent of \( B_3 \). Then, \( P' = P \cup w_l v_j a_2 \) is an \( a_1-a_2 \) path, and \( B_1[b_1, u_{j-1}] \cup u_{j-1}w_j \cup B_3[w_j, b_2] \) and \( B_2[b_1, v_{j-1}] \cup v_{j-1}u_j \cup B_1[u_j, b_2] \) are two independent \( b_1-b_2 \) paths in \( G - P' \). This shows that \( (G, a_1, a_2, b_1, b_2) \) is feasible, a contradiction.  

\[ \square \]

**Corollary 3.1.4.** Suppose \( (G, a_1, a_2, b_1, b_2) \) is infeasible and \( G \) is 8-connected. Then \( G - \{a_1, a_2\} \) contains three independent induced \( b_1-b_2 \) paths \( B_1, B_2, B_3 \) such that for some \( i \in [2], A_i(B_1 \cup B_2 \cup B_3) \) has all its attachments contained in \( B_3 \) and \( A_{3-i}(B_1 \cup B_2 \cup B_3) \) has attachments on both \( B_1(b_1, b_2) \) and \( B_2(b_1, b_2) \).

**Proof.** Let \( B_1, B_2, B_3 \) be three independent induced \( b_1-b_2 \) paths in \( G - \{a_1, a_2\} \). Choose \( B_1, B_2, B_3 \) so that \( A_1(B_1 \cup B_2 \cup B_3) \) is maximal.

We may assume that \( A_1(B_1 \cup B_2 \cup B_3) \) has all its attachments on \( B_3 \). For, otherwise, since \( (G, a_1, a_2, b_1, b_2) \) is infeasible, \( A_2(B_1 \cup B_2 \cup B_3) \) has all its attachments on \( B_j \) for exactly one \( j \in [2] \). Then by relabeling, we see that \( B_1, B_2, B_3 \) are desired paths.
Let $H := G - (A_1(B_1 \cup B_2 \cup B_3) - B_3) - a_2$. By the maximality of $A_1(B_1 \cup B_2 \cup B_3)$, we see that each $w \in V(A_1(B_1 \cup B_2 \cup B_3)) \cap V(B_3(b_1, b_2))$ is contained in a 3-cut in $H$ separating $b_1$ from $b_2$.

Let $G'$ be obtained from $G - a_2$ by contracting $G - (A_1(B_1 \cup B_2 \cup B_3) \cup B_3)$ to a single vertex $a'_2$. Suppose $G'$ contains disjoint paths $Q_a, Q_b$ from $a_1, b_1$ to $a'_2, b_2$, respectively. Then the independent $b_1$-$b_2$ paths $B_1, B_2, Q_b$ give the desired paths, as $(G, a_1, a_2, b_1, b_2)$ is infeasible and $A_1(B_1 \cup B_2 \cup Q_b)$ has attachments on both $Q_b$ and $B_1(b_1, b_2) \cup B_2(b_1, b_2)$.

So, such paths do not exist in $G'$. Hence, by Lemma 2.2.1, $(G', A, a_1, b_1, a'_2, b_2)$ is 3-planar, where $A$ is a collection of disjoint subsets of $V(G') \setminus \{a_1, b_1, a'_2, b_2\}$.

We claim that $(A_1(B_1 \cup B_2 \cup B_3) \cup B_3, B_3 + a_1)$ is planar. If $A = \emptyset$, we are done. Hence we may assume there exists $A \in A$. Since $|N_{G'}(A)| \leq 3$ and $G$ is 8-connected, $V(B_3) \cap A \neq \emptyset$. Therefore, $a'_2 \in N_{G'}(A)$ and, thus, $|N_{G'}(A) \cap V(B_3)| = 2$. Hence, $G'[A] \subseteq B_3$ by definition of $A_1(B_1 \cup B_2 \cup B_3)$ and $B_3$. This implies $(G'[A \cup N_{G'}(A)], N_{G'}(A))$ is planar for all $A \in A$. So $(A_1(B_1 \cup B_2 \cup B_3) \cup B_3, B_3 + a_1)$ is planar.

This is a contradiction to Lemma 3.1.3.

Hence, by Corollary 3.1.4, we may choose three independent $b_1$-$b_2$ paths $B_1, B_2, B_3$ in $G - \{a_1, a_2\}$ which satisfy (C1)-(C4). Moreover, by Lemma 3.1.2, $(A_1(B_1 \cup B_2 \cup B_3), B_3 + a_1)$ is planar. Let

$$S := V(A_1(B_1 \cup B_2 \cup B_3)) \cap V(B_3).$$

By Lemma 3.1.3, there exists $w \in S \setminus \{b_1, b_2\}$ such that $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3) - \{a_2, w\}$ has three independent $b_1$-$b_2$ paths $P_1, P_2, P_3$.

### 3.2 Ladders and rungs

In this section, we show that $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3) - \{a_2, w\}$ can be obtained from a plane graph and a ladder (which consists of rungs as defined in section 2.3) by gluing them along a path.
Let \( A_2(P_1 \cup P_2 \cup P_3) \) be the \((P_1 \cup P_2 \cup P_3)\)-bridge of \( G \) containing \( a_2 \). We choose \( B_1, B_2, B_3, w, P_1, P_2, P_3 \), such that

\[(C5) \text{ subject to } (C1)-(C4), \quad A_2(P_1 \cup P_2 \cup P_3) \text{ is maximal.} \]

By the maximality of \( A_1(B_1 \cup B_2 \cup B_3) \) (see (C4)), all attachments of \( A_2(P_1 \cup P_2 \cup P_3) \) are contained in exactly one of \( P_1, P_2, P_3 \), as otherwise, if \( A_1(P_1 \cup P_2 \cup P_3) \) has attachments on at least two of \( P_i \)'s for \( i \in [3] \), \((G, a_1, a_2, b_1, b_2)\) is feasible; and if \( A_1(P_1 \cup P_2 \cup P_3) \) has attachments only on one \( P_j \) for some \( j \in [3] \), \( P_1, P_2, P_3, w \) would contradict the choice of \( B_1, B_2, B_3, w \). So we may assume that

\[(C6) \text{ subject to } (C1)-(C5), \quad A_2(P_1 \cup P_2 \cup P_3) \text{ on } P_1 \cup P_2 \cup P_3 \text{ are contained in } P_3. \]

Let

\[ H = G - (A_1(B_1 \cup B_2 \cup B_3) - (B_3 - w)) - (A_2(P_1 \cup P_2 \cup P_3) - P_3). \]

Label the vertices in \( V(A_2(P_1 \cup P_2 \cup P_3)) \cap V(P_3) \) as \( u_1, \ldots, u_m \) in order from \( b_1 \) to \( b_2 \). Then by the maximality of \( A_2(P_1 \cup P_2 \cup P_3) \) (see (C5)), each \( u_i \) is in a 3-cut of \( H \) separating \( b_1 \) from \( b_2 \).

**Lemma 3.2.1.** For \( i \in [m] \), there are 3-cuts \( T_i = \{u_i, v_i, w_i\} \) in \( H \) separating \( b_1 \) from \( b_2 \) such that \( b_1, u_1, \ldots, u_m, b_2 \) occur on \( P_3 \) in order, \( b_1, v_1, \ldots, v_m, b_2 \) occur on \( P_2 \) in order, and \( b_1, w_1, \ldots, w_m, b_2 \) occur on \( P_1 \) in order.

**Proof.** The proof is the same as (1) in the proof of Lemma 3.1.3. \qed

Let \( H_1 \) denote the \( T_1 \)-bridge of \( H \) containing \( b_1 \), and \( H_{m+1} \) denote the \( T_m \)-bridge of \( H \) containing \( b_2 \). Let \( \text{Int}(H_1) = V(H_1) \setminus (T_1 \cup \{b_1\}) \) and \( \text{Int}(H_{m+1}) = V(H_{m+1}) \setminus (T_m \cup \{b_2\}) \). For \( i \in [m] \setminus \{1\} \), let \( H_i \) denote the union of those \((T_{i-1} \cup T_i)\)-bridges of \( H \) containing the subpaths of \( P_j \) between \( T_{i-1} \) and \( T_i \) for \( j \in [3] \).
Lemma 3.2.2. For \(i \in [m] \setminus \{1\}\), any three disjoint paths in \(H_i\) from \(T_{i-1}\) to \(T_i\) contains a \(u_{i-1} - u_i\) path.

Proof. Suppose for some \(i \in [m] \setminus \{1\}\), \(H_i\) has three disjoint paths \(Q_u, Q_v, Q_w\) from 
\(u_{i-1}, v_{i-1}, w_{i-1}\), respectively, to \(T_i\), with no \(u_{i-1} - u_i\) path. Then, \(u_i \in V(Q_v \cup Q_w)\). Let 
\(P'_1, P'_2, P'_3\) be formed by taking the union of \(Q_u, Q_v, Q_w\), respectively, with the subpaths of \(P_1, P_2, P_3\) outside of \(H_i\). We may assume that \(P'_1 \supseteq Q_u\) and \(P'_2\) contains \(Q_v\) (if \(u_i \in V(Q_v)\)) or \(Q_w\) (if \(u_i \in V(Q_w)\)). Then, \(P'_1, P'_2, P'_3\) are three independent \(b_1 - b_2\) paths in 
\(G - (A_1(B_1 \cup B_2 \cup B_3) - B_3) - \{a_2, w\}\) such that \(A_2(P'_1 \cup P'_2 \cup P'_3)\) has attachments on \(P'_1\) and \(P'_2\). Hence, \(P'_1, P'_2, P'_3, w\) contradict the choice of \(B_1, B_2, B_3, w\), or \((G, a_1, a_2, b_1, b_2)\) is feasible, a contradiction. \(
\)

Thus, by Lemma 2.3.1, \(H_i = J_i \cup L_i\), where \(J_i\) is planar and \(L_i\) is a ladder from 
\((v_{i-1}, u_{i-1}, w_{i-1})\) to \((v_i, u_i, w_i)\). Let 
\[
L^* = H_1 \cup H_{m+1} \cup \bigcup_{i=2}^{m} L_i.
\]

We further choose \(P_1, P_2, P_3\) such that 

(C7) subject to (C1)-(C6), \((P_1 \cup P_2 \cup P_3) \cap H_i \subseteq L_i\) for \(i \in [m] \setminus \{1\}\) (and hence, 
\(P_1 \cup P_2 \cup P_3 \subseteq L^*\)), and \(A'_2(P_1 \cup P_2 \cup P_3) := A_2(P_1 \cup P_2 \cup P_3) \cup J_2 \cup \ldots \cup J_m\) is maximal.

See the following Figure 3.1 for an illustration for all the above results.

The following observation will be convenient.

Observation 3.2.3. There exists no path in \(H\) from \(S \setminus \{b_1, b_2\}\) to \(P_3(b_1, b_2)\) disjoint from 
\(P_1 \cup P_2\).

Proof. For, suppose \(Q\) is a path between \(s \in S \setminus \{b_1, b_2\}\) and \(t \in V(P_3(b_1, b_2))\) internally disjoint from \(P_1 \cup P_2\). We may further assume \(Q\) is independent of \(A_1(B_1 \cup B_2 \cup B_3) \cup \ldots \cup J_m\). 

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Figure 3.1: Structure of infeasible \((G, a_1, a_2, b_1, b_2)\)

\(A_2(P_1 \cup P_2 \cup P_3)\). Note that \(A_1(B_1 \cup B_2 \cup B_3)\) contains an \(a_1\)-s path independent of \(H\).

Let \(b \in V(P_3(b_1, b_2)) \cap V(A_2(P_1 \cup P_2 \cup P_3))\). Then \(A_2(P_1 \cup P_2 \cup P_3)\) contains an \(a_2\)-b path \(Q_2\) independent of \(P_3\). Now \(Q_1 \cup Q \cup P_3(t, b) \cup Q_2\) is an \(a_1\)-\(a_2\) path disjoint from \(P_1 \cup P_2\).

This shows that \((G, a_1, a_2, b_1, b_2)\) is feasible, a contradiction.

We conclude this section with a useful lemma concerning the rungs \((R, (a, b, c), (a', b', c'))\) in \(L^*\) with \(|\partial R| = 6\) or \(|\partial R| = 5\) and \(b \neq b'\).

**Lemma 3.2.4.** \((R, (a, b, c), (a', b', c'))\) in \(L^*\) with \(|\partial R| = 6\) or \(|\partial R| = 5\) and \(b \neq b'\). Then

(a) any three disjoint paths in \(R\) from \(\{a, b, c\}\) to \(\{a', b', c'\}\) must be from \(a, b, c\) to \(a', b', c'\), respectively, and

(b) there are disjoint induced paths \(P_a, P_c\) in \(R - \{b, b'\}\) from \(a, c\) to \(a', c'\), respectively, such that \(R - (P_a \cup P_c)\) is connected and \(S \cap V(R) \subseteq V(P_a \cup P_c)\).

**Proof.** By (iii) of Proposition 2.3.2, we have (a). So there are disjoint induced paths \(P_a, P_c\) in \(R - \{b, b'\}\) from \(a, c\) to \(a', c'\), respectively, such that \(\{b, b'\}\) is contained in a \((P_a \cup P_c)\)-bridge of \(R\), say \(R_b\). Note that \((S \cup N(w)) \cap V(R_b) = \emptyset\) by Observation 3.2.3.
To prove (b), let us assume by symmetry that if \( |\partial R| = 5 \) then \( a = a' \). If \( R_b \) is the only component of \( R - (P_a \cup P_c) \) then (b) holds; for otherwise \((G, a_1, a_2, b_1, b_2)\) would be feasible as \( A_2(P_1 \cup P_2 \cup P_3) \) has attachments on \( P_2(b_1, b_2) \). For any component \( X \) of \( R - (P_a \cup P_c) \) with \( X \neq R_b \), it follows from 3-planarity of \( R \) or \( R - a' \) (when \( a = a' \)) that we may assume \( X \) has neighbors only on \( P_c \) unless \( a = a' \). Moreover, the two neighbors of \( X \) on \( P_c \) that are furtherest apart form a cut (with \( a \) if \( a = a' \)) in \( R \), and these two neighbors might be the same.

Hence, let \( \{y_i, z_i\} \) be the cut of size at most 2 in \( R \) (or \( R - a \) when \( a = a' \)) separating \( R_b \) from \( P_c[y_i, z_i] \) and at least one vertex of \( R - (P_a \cup P_c) \), such that \( P_c[y_i, z_i] \) are maximal. Then by planarity, we may assume that \( c, y_1, z_1, \ldots, y_t, z_t, c' \) occur on \( P_c \) in order. Let \( X_i \) denote the union of \( P[y_i, z_i] \) and all \( (P_a \cup P_c) \)-bridges of \( R \) with all attachments contained in \( P_c[y_i, z_i] \) (or \( P_c[y_i, z_i] + a \) if \( a = a' \)). Let \( X_i^* = R[X_i + w] \) and \( \text{Int}(X_i^*) = V(X_i^*) \setminus \{a, w, y_i, z_i\} \). Note that \( X_i^* - (P_a \cup P_c) \neq \emptyset \); so \( V(B_3) \cap \text{Int}(X_i^*) \neq \emptyset \) (to avoid the cut \( \{a, w, y_i, z_i\} \)). Let \( r_1, r_2 \in V(B_3) \cap \{a, w, y_i, z_i\} \) with \( N(r_i) \cap \text{Int}(X_i^*) \neq \emptyset \) for \( i \in [2] \), such that \( B_3[r_1, r_2] \) is maximal.

First, we claim that \( \{r_1, r_2\} \neq \{y_i, z_i\} \) for \( i \in [t] \). For, suppose \( \{r_1, r_2\} = \{y_i, z_i\} \) for some \( i \in [t] \). Then \( B_i \cap \text{Int}(X_i^*) = \emptyset \) for \( i \in [2] \). If there exists \( s \in (S \cap \text{Int}(X_i^*)) \setminus V(P_c[y_i, z_i]) \) then letting \( B_3' := (B_3 - B_3(y_i, z_i)) \cup P_c[y_i, z_i] \) we see that \( A_1(B_1 \cup B_2 \cup B_3') \) contains \( A_1(B_1 \cup B_2 \cup B_3) + s \), contradicting (C4). So \( S \cap \text{Int}(X_i^*) \subseteq V(P_c[y_i, z_i]) \).

Now let \( Y \) be a \( (P_a \cup P_c) \)-bridge of \( R \) contained in \( X_i^* \) and \( y, z \in V(Y) \cap V(P_a \cup P_c) \) with \( P_c[y, z] \) maximal, such that no other \( (P_a \cup P_c) \)-bridges of \( R \) has attachments in \( P_c(y, z) \). Note \( Y \) is well defined because of planarity. Now there exists \( s \in S \cap V(P_c(y, z)) \) to avoid the cut \( \{a, w, y, z\} \). Let \( B_3' \) denote the union of \( (B_3 - B_3(y, z)) \) and an induced \( y-z \) path in \( Y - V(P_c(y, z)) \). Then \( A_1(B_1 \cup B_2 \cup B_3') \) contains \( A_1(B_1 \cup B_2 \cup B_3) + s \), contradicting (C4).

Thus, for any \( i \in [t] \), we have \( w' \in \text{Int}(X_i^*) \), or \( a = a' \) and \( B_3 \) enters \( \text{Int}(X_i^*) \) at \( a \); for otherwise, \( B_3 \cap X_i^* \) would be a \( y_i-z_i \) path. This, in particular, implies \( t \leq 2 \).
Case 1. $a \neq a'$.

Then $t = 1$. First, suppose $S \cap \text{Int}(R) \subseteq V(X^*_1)$. Let $R'$ be obtained from $R^* - \text{Int}(X^*_1)$ by adding edges $\{ab, bc, y_1z_1, c'b', b'a'\}$ (or $\{ab, bc, c'b', b'a'\}$ when $y_1 = z_1$), as well as edges from $w$ to $K := \{a, b, c, y_1, z_1, c', b', a'\}$. Then $R'$ is a planar graph. Let $k = |K|$ and $m = |V(R') \setminus (K \cup \{w\})|$. By the Hand-shaking lemma and Euler’s formula, we see that $k \times 4 + k + 8(|V(R')| - k - 1) \leq 6|V(R')| - 12$, which implies $|V(R')| \leq 3k/2 - 2$. So $m \leq 3k/2 - 2 - (k + 1) = k/2 - 3 \leq 1$. This implies that there exists $u \in \text{Int}(R)$ such that $N_R(u) = K$. (Note $N(w) \cap V(R_b) = \emptyset$ as $(G, a_1, a_2, b_1, b_2)$ is infeasible.) By planarity of $R$, $\{a, u, c\}$ is a cut in $R$ separating $\{a, b, c\}$ from $\{a', b', c'\}$, a contradiction.

Now, suppose there exists $s \in S \cap \text{Int}(R)$ and $s \notin V(X^*_1)$. By symmetry, assume $s \in V(P_c(c, y_1))$. We choose such $s$ with $P_c[c, s]$ minimal. We consider the paths $B_i \cap R$ for $i \in [3]$. If we can find disjoint paths in $R^* - s$ linking the same ends of $B_i \cap R^*$ for $i \in [3]$, then by replacing $B_i \cap \text{Int}(R)$ with such paths in $R^* - s$, we obtain independent $b_1b_2$ paths $B'_1, B'_2, B'_3$ such that $A_1(B'_1 \cup B'_2 \cup B'_3)$ contains $A_1(B_1 \cup B_2 \cup B_3) + s$, contradicting (C4). So such paths do not exist. Hence by 3-planarity of $(R, a, b, c, c', b'a')$ we see that $R$ has a 4-cut $\{s, v_1, v_2, a'\}$ separating $\{a, b, c\}$ from $\{y_1, a', b', c'\} \cup V(X^*_1)$. Let $R'$ denote the $\{w, s, v_1, v_2, a'\}$-bridge of $R^*$ containing $\{a, b, c\}$ and assume notation is chosen so that $(R' - w, a, b, c, s, v_1, v_2, a')$ is planar. Let $R''$ be obtained from $R'$ by adding edges in $\{ab, bc, sv_1, v_1v_2, v_2a'\}$, as well as edges from $w$ to all vertices in $K := \{a, b, c, s, v_1, v_2, a'\}$. Let $k := |K|$ and $m := |V(R'') \setminus (K \cup \{w\})|$. By Hand-shaking lemma and Euler’s formula, we see that $k \times 4 + k + 8(|V(R'')| - k - 1) \leq 6|V(R'')| - 12$, which implies $|V(R'')| \leq 3k/2 - 3$. So $m \leq 3k/2 - 3 - (k + 1) = k/2 - 3 \leq 1/2$. This leads to a contradiction to (ii) of Proposition 2.3.2.

Case 2. $a = a'$ and $t = 1$.

Suppose $S \cap (\text{Int}(R) \setminus V(X^*_1)) = \emptyset$. Let $R'$ be obtained from $R^* - a - (X^*_1 - \{y_1, z_1\})$ by adding edges $bc, b'c'$ and $y_1z_1$ if $y_1 \neq z_1$, as well as edges from $w$ to all vertices in $K := \{b, b', c, c', y_1, z_1\}$. Let $k := |K|$ and $m := |V(R') \setminus (K \cup \{w\})|$. By Hand-shaking
lemma and Euler’s formula, we see that
\[ 4 \times k + k + 7(|V(R')| - k - 1) \leq 6|V(R')| - 12, \]
which implies \(|V(R')| \leq 2k - 5\). Thus, \(m \leq k - 6\). Since \(k \leq 6\), \(V(R') = K\). This leads to a contradiction to (ii) of Proposition 2.3.2.

Now assume there exists \(s \in S \cap Int(R)\) and \(s \notin inV(X^*_1)\). By symmetry, assume \(s \in V(P_c(c, y_1))\). We choose such \(s\) with \(P_c[c, s]\) minimal. We consider the paths \(B_i \cap R^*\) for \(i \in [3]\). If we can find disjoint paths in \(R^* - s\) linking the same ends of \(B_i \cap R^*\) then by replacing \(B_i \cap Int(R)\) with such paths in \(R^* - s\), we obtain independent \(b_1-b_2\) paths \(B'_1, B'_2, B'_3\) such that \(A_1(B'_1 \cup B'_2 \cup B'_3)\) contains \(A_1(B_1 \cup B_2 \cup B_3) + s\), contradicting (C4). So such paths do not exist. Hence by 3-planarity of \((R - a, b, c, c', b')\) we see that \(R - a\) has a 3-cut \(\{s, v_1, v_2\}\) separating \(\{b, c\}\) from \(\{y_1, b', c'\}\). Let \(R'\) denote the \(\{w, s, v_1, v_2\}\)-bridge of \(R - a\) containing \(\{b, c\}\) and assume notation is chosen so that \((R' - w, b, c, s, v_1, v_2)\) is planar.

Let \(R''\) be obtained from \(R'\) by adding edges \(\{bc, sv_1, v_1v_2\}\), as well as edges from \(w\) to all vertices in \(K := \{b, c, s, v_1, v_2\}\). Let \(k := |K|\) and \(m := |V(R'') \setminus (K \cup \{w\})|\). By Hand-shaking lemma and Euler’s formula, we see that
\[ k \times 4 + k + 7(|V(R'')| - k - 1) \leq 6|V(R'')| - 12, \]
which implies \(|V(R'')| \leq 2k - 5\). So \(m \leq k - 6 < 0\), a contradiction.

Case 3. \(a = a'\) and \(t = 2\).

Then since \(B_3 \cap Int(X^*_1) \neq \emptyset\) cannot be a \(y_i - z_i\) path for \(i \in [2]\), we see that \(B_3\) enters \(Int(X^*_1)\) at \(a\) and leaves \(Int(X^*_2)\) at \(w\). Thus \(S \cap Int(R) \subseteq V(X^*_1) \cup V(P_c[z_1, y_2]) \cup V(X^*_2)\).

Suppose \(S \cap Int(R) \subseteq V(X^*_1 \cup X^*_2)\). Let \(R'\) be obtained from \(R^* - a - (X^*_1 - \{y_1, z_1\} - (X^*_2 - \{y_2, z_2\}))\) by adding edges \(bc, b'c'\) and \(y_iz_i\) for \(i \in [2]\) with \(y_i \neq z_i\), as well as edges from \(w\) to all vertices in \(K := \{b, b', c, c', y_1, z_1, y_2, z_2\}\). Let \(k = |K|\) and \(m := |V(R') \setminus (K \cup \{w\})|\). By Hand-shaking lemma and Euler’s formula, we see that
\[ 4 \times k + k + 7(|V(R')| - k - 1) \leq 6|V(R')| - 12, \]
which implies \(|V(R')| \leq 2k - 5\). So \(m \leq k - 6 \leq 2\). Using planarity of \(R' - w\) and every vertex inside \(R' - (K \cup \{w\})\) has degree at least 7, we see that \(m = 1\) and the only vertex in \(V(R') \setminus (K \cup \{w\})\), say \(u\), is adjacent to both \(b\) and \(b'\) (and \(bb' \in E(G)\)) by (ii) of Proposition 2.3.2. Hence, by letting
\( P'_3 = (P_3 - bb') \cup bb' \), we see that \( A'_2(P_1 \cup P_2 \cup P'_3) \) contains \( A'_2(P_1 \cup P_2 \cup P_3) + bb' \).

Hence, \( B_1, B_2, B_3, w, P_1, P_2, P'_3 \) contradict (C7).

Now assume there exists \( s \in S \cap V(P_c(z_1, y_2)) \). We choose such \( s \) with \( P_c[z_1, s] \) minimal. We consider the paths \( B_i \cap R^* \) for \( i \in [3] \). If we can find disjoint paths in \( R^* - s \) linking the same ends of \( B_i \cap R^* \), then by replacing \( B_i \cap \text{Int}(R) \) with such disjoint paths in \( R^* - s \), we obtain independent \( b_1-b_2 \) paths \( B'_1, B'_2, B'_3 \) such that \( A_1(B'_1 \cup B'_2 \cup B'_3) \) contains \( A_1(B_1 \cup B_2 \cup B_3) + s \), contradicting (C4). So such paths do not exist. Hence by 3-planarity of \( (R - a, b, c, c', b') \) we see that \( R - a \) has a 3-cut \( \{s, v_1, v_2\} \) separating \( \{b, c\} \cup V(X^*_1) \) from \( \{b', c'\} \cup V(X^*_2) \). Let \( R' \) denote the graph obtained from the \( \{w, s, v_1, v_2\} \)-bridge of \( R^* - a \) containing \( \{b, c\} \) by deleting \( \text{Int}(X^*_1) \), and assume notation is chosen so that \( (R' - w, b, c, y_1, z_1, s, v_1, v_2) \) is planar.

Let \( R'' \) be obtained from \( R' \) by adding edges in \( \{bc, sv_1, v_1v_2\} \) and \( y_1z_1 \) (if \( y_1 \neq z_1 \)), as well as edges from \( u \) to all vertices in \( K := \{b, c, y_1, z_1, s, v_1, v_2\} \). Let \( k = |K| \) and \( m := |V(R'') \setminus (K \cup \{w\})| \). By Hand-shaking lemma and Euler’s formula, we see that \( k \times 4 + k + 7(|V(R'')| - k - 1) \leq 6|V(R'')| - 12 \), which implies \( |V(R'')| \leq 2k - 5 \). Hence \( m \leq k - 6 \leq 1 \). By planarity and (ii) of Proposition 2.3.2, we have \( m = 1 \). So the unique vertex in \( V(R') \setminus \{b, c, v_1, v_2, u\} \cup \{y_1, z_1\} \), say \( u \), must be adjacent to \( w \). However, this means \( N(w) \cap V(R_0) \neq \emptyset \), a contradiction. \( \square \)
CHAPTER 4
RUNGS INTERSECTING THREE SPECIAL PATHS

For any rung \((R, (a, b, c), (a', b', c'))\), let \(\partial R = \{a, b, c, a', b', c'\}\) and \(\text{Int}(R) = V(R) \setminus \partial R\). In this chapter, we consider the rungs \(R\) in \(L^*\) such that \(\text{Int}(R) \cap V(B_i) \neq \emptyset\) for all \(i \in [3]\), including \(H_1\) and \(H_{m+1}\), and prove that only \(H_1\) or \(H_{m+1}\) could intersect all three paths.

First, in section 4.1, we prove a technical lemma that will be used to deal with such rungs. In section 4.2, we use Lemma 4.1.1 to obtain structure results of \(H_1\) and \(H_{m+1}\) in subsection 4.2.1, and that of the other rungs \(R\) in subsection 4.2.2. Last in Lemma 4.2.3, we show that such rungs do not exist except when they are contained in \(H_1 \cup H_{m+1}\) or when \(|\partial R| = 5\) and \(b = b'\).

Let \(w', w'' \in N(w) \cap V(B_3)\) such that \(b_1, w', w, w'', b_2\) occur on \(B_3\) in order.

4.1 Technical lemma

In this section, we prove a technical lemma to deal with rungs intersecting \(B_i\) for all \(i \in [3]\).

**Lemma 4.1.1.** Let \(\{a', b', c'\}\) be a 3-cut of \(L^*\) with \(b' \in V(P_3)\) and separating \(\{b_1, w'\}\) from \(\{b_2, w''\}\), let \(R\) denote the \(\{a', b', c'\}\)-bridge of \(L^*\) containing \(\{b_1, w'\}\), and let \(R^* = R + \{w, wx : x \in N(w) \cap V(R)\}\). Suppose there exists \(w^* \in S \cap V(B_3(b_1, w'))\) such that \(R^* - w^*\) contains three independent paths \(Q_a, Q_c, Q_w\) from \(b_1\) to \(a', c', w\), respectively, such that \(b' \in V(Q_a)\), or \(Q_a \cap P_3\) is a subpath of \(P_3[b_1, b']\) and the \((Q_a \cup Q_c \cup Q_w)\)-bridge of \(R^*\) containing \(b'\) has an attachment on \(Q_a\). Suppose \(A_2(P_1 \cup P_2 \cup P_3)\) has attachments on both \(P_3(b_1, b')\) and \(P_3(b', b_2)\).

Then \(L^*\) has a 3-cut \(\{a'', b'', c''\}\) with \(b'' \in V(P_3)\) separating \(\{a', b', c'\} \cup (N(w) \cap \)
$V(L^*)$ from $b_2$, and $A_2(P_1 \cup P_2 \cup P_3)$ has no attachment in $P_3(b', b'')$. Moreover, if $R''$ denotes the graph obtained from $H$ by deleting the components of $L^* - (\{a'', b'', c''\} \cup \{a', b', c'\})$ containing $b_1$ or $b_2$ then $R'' = J'' \cup L''$ with $b' \in V(J'' - L'')$, $(J'', J'' \cap L'')$ planar, $J'' \cap L''$ an $a'$-$b''$ path, and $L''$ a ladder from $\{a', c', w\}$ to $\{a'', b'', c''\}$ along $J'' \cap L''$.

**Proof.** Let $a'' = b'' = c'' = b_2$ if $L^*$ has no 3-cut separating $\{a', b', c'\} \cup (N(w) \cap V(L^*))$ from $b_2$, and otherwise let $\{a'', b'', c''\}$ be a 3-cut of $L^*$ separating $\{a', b', c'\} \cup (N(w) \cap V(L^*))$ from $b_2$ and let $b'' \in V(P_3)$. Moreover, let $R'$ denote the graph obtained from $L^*$ by deleting the components of $L^* - (\{a'', b'', c''\} \cup \{a', b', c'\})$ containing $b_1$ or $b_2$, and choose $\{a'', b'', c''\}$ to minimize $R'$. By the choice of $R'$, $R'$ has no cut of size at most 3 separating $\{a', b', c'\} \cup (N(w) \cap V(L^*))$ from $\{a'', b'', c''\}$. Let $R_v = R' + \{v, w, va', vb', wx : x \in N(w) \cap V(R')\}$, where $v$ is a new vertex.

Note that $R_v$ contains three independent paths from $v, c', w$, respectively, to $\{a'', b'', c''\}$. For, otherwise, $R_v$ has a cut $T$ of size at most 2 separating $\{v, c', w\}$ from $\{a'', b'', c''\}$. Then $v \in T$ as, otherwise, $T$ would separate $\{a', b', c'\} \cup (N(w) \cap V(L^*))$ from $\{a'', b'', c''\}$, a contradiction. Moreover, $w \not\in T$ because of the existence of three independent paths $P_i \cap R'$, $i \in [3]$, in $R'$. Now $\{b', a'\} \cup (T \setminus \{v\})$ is a 3-cut in $L^*$ contradicting the choice of $\{a'', b'', c''\}$ (i.e., the minimality of $R'$).

We claim that $A_2(P_1 \cup P_2 \cup P_3)$ has no attachment on $P_3(b', b'')$ (and, hence, $\{a'', b'', c''\}$ is a cut in $L^*$). For, otherwise, there exists $b^* \in V(A_2(P_1 \cup P_2 \cup P_3) \cap V(P_3(b', b'')))$, and we choose $b^*$ so that $P_3[b^*, b'']$ is minimal. Note that $b^*$ is contained in a 3-cut $\{a^*, b^*, c^*\}$ of $L^*$ separating $\{a', b', c'\}$ from $\{a'', b'', c''\}$. Let $M$ denote the graph obtained from $L^*$ by deleting the components of $L^* - (\{a^*, b^*, c^*\} \cup \{a'', b'', c''\})$ containing $b_1$ or $b_2$, and let $M^* = M + \{w, wx : x \in N(w) \cap V(M)\}$. By the choice of $\{a^*, b^*, c^*\}$ (minimality of $R'$), $w$ has a neighbor in $V(M^*) \setminus \{a^*, b^*, c^*\}$. By the choice of $\{a^*, b^*, c^*\}$ again, $M^* - b^*$ contains independent paths $P_a, P_c, P_w$ from $a^*, c^*, w$, respectively, to $\{a'', b'', c''\}$. Now we obtain three independent $b_1$-$b_2$ paths $P'_1, P'_2, P'_3$ from $Q_a \cup Q_c \cup Q_w \cup P_a \cup P_c \cup P_w, (P_1 \cup$
paths $P_2 \cap (R' - (M - \{a', c'\}))$, and three independent paths from $b_2$ to $a'', b'', c''$, respectively, in the $\{a'', b'', c''\}$-bridge of $L^*$ containing $b_2$. Then $B_1, B_2, B_3, w^*, P'_1, P'_2, P'_3$ contradict (C5), as $A_2(P'_1 \cup P'_2 \cup P'_3)$ contains $A_2(P_1 \cup P_2 \cup P_3) + b^*$.

We further claim that any three disjoint paths in $R_v$ from $\{v, c', w\}$ to $\{a'', b'', c''\}$ must contain a $v-b''$ path. For, suppose $P_v, P_c, P_w$ are disjoint paths in $R_v$ from $v, c', w$, respectively, to $\{a'', b'', c''\}$ with no $v-b''$ path. Then $b'' \in V(P_c)$ or $b'' \in V(P_w)$. If $b'' \in V(P_w)$, let $v' \in \{b', a'\}$ such that $v' \in V(P_v)$. Then, there is an $a_1-a_2$ path in union of $A_1(B_1 \cup B_2 \cup B_3) - (B_3 - w)$, $P_w$ and $A_2(P_1 \cup P_2 \cup P_3) - P_3[b_1, b']$ (as $A_2(P_1 \cup P_2 \cup P_3)$ has attachment on $P_3(b', b_2)$), which is independent from the two $b_1-b_2$ paths obtained from two independent paths from $b_1$ to $\{v', c'\}$ (subpaths of $P_1 \cup P_2 \cup P_3$), $P_v - v, P_c$, and the two independent paths from $b_2$ to $\{a'', c''\}$ (subpaths of $P_1 \cup P_2 \cup P_3$). So $(G, a_1, a_2, b_1, b_2)$ is feasible. Thus, $b'' \in V(P_c)$. By symmetry, assume $c'' \in V(P_v)$ and $a'' \in V(P_w)$. If $a' \in V(P_v)$, let $Q_a' = Q_a$; otherwise if $b' \in V(P_v)$, let $Q_a'$ be the $b_1-b'$ path in union of $Q_a$ and the $(Q_a \cup Q_c \cup Q_w)$-bridge of $R^*$ containing $b'$. Then we obtain three independent $b_1-b_2$ paths $B_1', B_2', B_3'$ in $H - w^*$ from $Q_a' \cup Q_c \cup Q_w \cup (P_v - v) \cup P_c \cup P_w$ and the three independent paths from $b_2$ to $\{a'', b'', c''\}$ (subpaths of $B_1, B_2, B_3$), such that, $A_1(B_1' \cup B_2' \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + w^*$ and $A_2(B_1' \cup B_2' \cup B_3')$ has attachments on both $B_1'$ and $B_2'$ (by assumption on $Q_a$). So $B_1', B_2', B_3', w^*$ contradict (C4).

Hence, by applying Lemma 2.3.1 to $(R_v, (w, v, c'), (a'', b'', c''))$, we see that $R_v = J_v \cup L_v$, where $L_v$ is a ladder from $(w, v, c')$ to $(a'', b'', c'')$ and $(J_v, J_v \cap L_v)$ is planar.

Case 1. $J_v \subseteq L_v$.

Then by the choice of $R'$, $L_v$ is a single rung. By relabeling $a''$ and $c''$ if necessary, we may assume $c' = a''$ when $c' \in \{a'', c''\}$. Then, since $v, w \notin \{a'', b'', c''\}$, it follows from definition of rungs that either $c' \neq a''$ and $(L_v, w, v, c', a'', b'', c'')$ is 3-planar, or $c' = a''$ and $(L_v - c', w, v, b'', c'')$ is 3-planar.

Hence, because of $P_1, P_2, P_3$ and the choice of $R'$, $R' - b'$ contains three disjoint paths $P_a, P_c, P_w$ from $a', c', w$ to $b'', a'', c''$, respectively. Now these three paths, $Q_a \cup Q_c \cup Q_w$, ...
and \((P_1 \cup P_2 \cup P_3) - ((R^* - a'') + \text{Int}(R'))\) form three independent \(b_1-b_2\) paths \(X_1, X_2, X_3\) in \(H-w^*\) such that \(a' \in V(X_1), c' \in V(X_2),\) and \(w \in V(X_3).\) Note that \(A_1(X_1 \cup X_2 \cup X_3)\) contains \(A_1(B_1 \cup B_2 \cup B_3) + w^*\).

If \(A_2(X_1 \cup X_2 \cup X_3)\) has attachments on both \(X_1\) and \(X_2\) then \(X_1, X_2, X_3, w^*\) contradict (C4), or all attachment of \(A_1(X_1 \cup X_2 \cup X_2)\) are on \(X_3\), then \((G, a_1, a_2, b_1, b_2)\) is feasible with \(X_1, X_2\) and an \(a_1-a_2\) path in union of \(X_3\) and \(A_1(X_1 \cup X_2 \cup X_3)\). So assume \(A_2(X_1, X_2, X_3)\) has all its attachments on \(X_1\). Then the \((P_a \cup P_c \cup P_w)-\text{bridge of } R_v - v\) containing \(b'\), say \(J''\), has all its attachments in \(P_a\). By choosing \(P_a, P_c, P_w\) to maximize \(J''\) and by the planarity of \(L_v\) (when \(|\partial R| = 6\) or \(L_v - c'\) when \(|\partial R| = 5\), we see that \(J''\) and \(L'' := (R_v - v) - (J'' - P_a)\) satisfies the conclusion of the lemma.

**Case 2.** \(J_v - L_v \neq \emptyset\).

By the minimality of \(R'\), we see that the boundary of \(J_v\) has a path from \(v\) to \(b''\) and avoiding \(L_v - \{v, b''\}\), which we denote by \(Q\). Note \(b' \in V(Q)\) or \(a' \in V(Q)\). If \(b' \in V(Q)\) then \(R'' = R_v - v, J'' = J_v - v\) and \(L'' = L_v - v\) satisfy the conclusion. So assume \(a \in V(Q)\).

We claim that \(R_v - Q - w\) contains disjoint paths \(B_b, B_c\) from \(b', c'\), respectively, to \(\{a'', c''\}\); for otherwise, there is a cut vertex \(t\) in \(R_v - Q - w\) separating \(\{b, c\}\) from \(\{a'', c''\}\). However, this contradicts the existence of the disjoint paths \(P_i \cap (R_v - w), i \in [3]\).

Now \((P_1 \cup P_2 \cup P_3) - \text{Int}(R'), \) and \((Q - v) \cup B_b \cup B_c\) give three independent \(b_1-b_2\) paths \(B'_{1}, B'_{2}, B'_{3}\) in \(L^*\), such that \(A_1(B'_{1} \cup B'_{2} \cup B'_{3})\) contains \(A_1(B_1 \cup B_2 \cup B_3) + w\) and \(A_2(B'_{1} \cup B'_{2} \cup B'_{3})\) attaches to two of \(B'_{1}, B'_{2}, B'_{3}\) (as \(A_2(P_1 \cup P_2 \cup P_3)\) has attachments on both \(P_5(b_1, b')\) and \(P_5(b', b_2)\)). Hence, either \((G, a_1, a_2, b_1, b_2)\) is feasible, or \(B'_{1}, B'_{2}, B'_{3}, w\) contradict (C4).

**4.2 Structures**

In this section, we apply Lemma 4.1.1 to obtain structures for \(H_1\) and \(H_{m+1}\) and rungs \(R\) in \(L^*\) not contained in \(H_1 \cup H_{m+1}\). Then, in Lemma 4.2.4, we conclude that only \(H_1\) or
and $H_{m+1}$ could intersect all three paths.

4.2.1 $H_1$ and $H_{m+1}$

First, consider $H_1$ and $H_{m+1}$ in $L^*$.

Lemma 4.2.1. If $B_i \cap \text{Int}(H_1) \neq \emptyset$ for $i \in [3]$ and if $w' \in V(H_1) \setminus T_1$ and $w'' \in V(H_1)$, then, there exists $w^* \in S \cap (V(H_1) \setminus (T_1 \cup \{b_1\}))$ such that,

(a) for each $s \in S \cap V(B_3(b_1, w*))$, $s$ is contained in a 3-cut of $H_1^*: = H_1 + \{w, wv : v \in N(w) \cap \text{Int}(H_1)\}$ separating $b_1$ from $T_1 \cup \{w\}$, and

(b) for each $s \in S \cap V(B_3(w^*, w))$, $s$ is contained in a 3-cut of $H_1^*$ separating $\{b_1, x_3\}$ from $\{w, x_1, x_2\}$, where for $i \in [2]$, $x_i$ denotes the end of $B_i \cap H_1$ other than $b_1$, and $x_3 \in T_1 \setminus \{x_1, x_2\}$.

The same holds for $H_{m+1}$ and $b_2$.

Proof. By symmetry, we prove the assertion for $H_1$. By definition, $B_i \cap H_1^*$, $i \in [3]$, are paths in $H_1$ from $b_1$ to $\{u_1, v_1, w_1, w\}$ with only $b_1$ in common. Let $w^* \in S \cap (V(H_1) \setminus (T_1 \cup \{b_1\}))$ such that $w^*$ is contained in some 3-cut $T$ of $H_1^*$ separating $b_1$ from $T_1 \cup \{w\}$; and if such $w^*$ does not exist we set $w^* = b_1$. We choose $w^*$ such that $B_3[b_1, w^*]$ is maximal.

Let $H_1'$ denote the $T$-bridge of $H_1^*$ containing $b_1$ (with $V(H_1') = \{b_1\}$ if $w^* = b_1$).

We claim that for any $s \in S \cap V(B_3(b_1, w*))$, $s$ is contained in some 3-cut of $H_1'$ separating $b_1$ from $T$. For, otherwise, $H_1' - s$ contains independent paths from $b_1$ to $T$ with only $b_1$ in common. Now these three paths and $B_i - (H_1' - T)$ for $i \in [3]$ form three independent $b_1$-$b_2$ paths $B_1', B_2', B_3'$ in $H - s$ such that $A_1(B_1' \cup B_2' \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + s$ and $A_2(B_1' \cup B_2' \cup B_3')$ has attachments on both $B_1'$ and $B_2'$. Hence, $B_1', B_2', B_3'$ contradict (C4).

Now let $s \in S \cap V(B_3(w^*, w))$ be arbitrary. By the choice of $w^*$, $s$ is not contained in any 3-cut of $H_1^*$ separating $b_1$ from $T_1 \cup \{w\}$. For $i \in [2]$, let $x_i$ be the end of $B_i \cap H_1$ other than $b_1$. Thus, $x_1, x_2 \in T_1$, and let $x_3 \in T_1 \setminus \{x_1, x_2\}$.
If $H_1^* - s$ contains no independent paths from $b_1$ to $x_1, x_2, w$, respectively, then $s$ is contained in a 3-cut $T'$ of $H_1^*$ separating $b_1$ from $\{x_1, x_2, w\}$. Since $T'$ cannot separate $b_1$ from $T_1 \cup \{w\}$, $T'$ must separate $\{b_1, x_3\}$ from $\{w, x_1, x_2\}$.

So assume that $H_1^* - s$ contains independent paths $Q_1, Q_2, Q_3$ from $b_1$ to $x_1, x_2, w$, respectively. These paths, $B_3[w, b_2]$, and the parts of $B_1, B_2$ outside $H_1$ form three independent $b_1$-$b_2$ paths $B'_1, B'_2, B'_3$ in $H - s$. Since $A_1(B'_1 \cup B'_2 \cup B'_3)$ contains $A_1(B_1 \cup B_2 \cup B_3) + s$ and $A_2(B_1 \cup B_2 \cup B_3)$ has attachments on both $B_1$ and $B_2$, it follows from (C4) that $A_2(B'_1 \cup B'_2 \cup B'_3)$ has all its attachments on $Q_i + b_2$ for exactly one $i \in [2]$, and that $V(A_2(B_1 \cup B_2 \cup B_3)) \cap V(B_{3-i}) \subseteq V(H_1)$. So $u_1 \notin V(B_1 \cup B_2 \cup B_3), u_1 \in A_2(B'_1 \cup B'_2 \cup B'_3)$, and $\{x_1, x_2\} = \{v_1, w_1\}$.

Thus, we may apply Lemma 4.1.1 with the cut $T_1$ as $\{a', b', c'\}$ and $u_1$ as $b'$. So $L^*$ has a 3-cut $\{a'', b'', c''\}$ with $b'' \in V(P_3)$ separating $\{a', b', c'\} \cup (N(w) \cap V(L^*))$ from $b_2$, and $A_2(P_1 \cup P_2 \cup P_3)$ has no attachment in $P_3(b', b'')$. Moreover, if $R''$ denotes the graph obtained from $H$ by deleting the components of $L^* - (\{a'', b'', c''\} \cup \{a', b', c'\})$ containing $b_1$ or $b_2$, then $R'' = J'' \cup L''$ with $b' \in V(J'' - L'')$, where $(J'', J'' \cap L'')$ is planar, $J'' \cap L''$ is an $a'$-$b''$ path, and $L''$ is a ladder from $\{a', c', w\}$ to $\{a'', b'', c''\}$ along $J'' \cap L''$. Let $P'_1, P'_2, P'_3$ be three independent $b_1$-$b_2$ paths in $H - w^*$ obtained from $Q_1 \cup Q_2 \cup Q_3$, three disjoint paths in $L''$ from $\{v_1, w_1, w\}$ to $\{a'', b'', c''\}$, and the subpaths of $P_i, i \in [3]$, from $\{a'', b'', c''\}$ to $b_2$. Since $b' = u_1 \in V(A_2(B_1, B_2, B_3))$, we see that $A_2(P'_1 \cup P'_2 \cup P'_3)$ contains $A_2(P_1 \cup P_2 \cup P_3) \cup J''$. Thus, $B_1, B_2, B_3, w^*, P'_1, P'_2, P'_3$ contradict (C7).

4.2.2 Rungs not in $H_1 \cup H_{m+1}$

Next, consider rungs $(R, (a, b, c), (a', b', c'))$ not contained in $H_1 \cup H_{m+1}$. First, we show results of such rungs $R$ with $w' \in Int(R)$ and $w'' \notin V(R)$. We discuss them in two cases: $|\partial R| = 5$ and $b = b'$ in Lemma 4.2.2, and $|\partial R| = 6$ or $|\partial R| = 5$ and $b \neq b'$ in Lemma 4.2.3.

**Lemma 4.2.2.** Suppose $(R, (a, b, c), (a', b', c'))$ is a rung in $L^*$ such that $R \not\subset H_1 \cup H_{m+1}$.
and \(|\{w', w''\} \cap \text{Int}(R)| = 1 = |\{w', w''\} \cap V(R)|\). Moreover, assume that \(b = b'\) and 
\(V(B_i) \cap \text{Int}(R) \neq \emptyset\) for \(i \in [3]\). Then, for all \(s \in S \cap \text{Int}(R)\), \(s\) is contained in a 3-cut of \(R^* = R + \{w, wv : v \in N(w) \cap \text{Int}(R)\}\) separating \(\{a, b, c\}\) from \(\{a', c', w\}\), or for all \(s \in S \cap \text{Int}(R)\), \(s\) is contained in a 3-cut of \(R^*\) separating \(\{a', b', c'\}\) from \(\{a, c, w\}\).

**Proof.** By symmetry, let \(w' \in \text{Int}(R)\) and \(w'' \notin V(R)\), and we may assume that \(b_1, w', w'', b_2\) occur on \(B_3\) in order. Note that \(b \in V(P_3)\), and we may assume that \(a \in V(P_2)\), and \(c \in V(P_1)\). Suppose for a contradiction that there exists some \(w^1 \in S \cap \text{Int}(R)\) such that \(R^* - w^1\) contains disjoint paths \(Q_a, Q_b, Q_c\) from \(a, b, c\), respectively, to \(\{a', c', w\}\).

Observe that \(w \notin V(Q_b)\). For otherwise, by replacing \((P_1 \cup P_2) \cap R\) with \(Q_a \cup Q_c\), we obtain from \(P_1, P_2\) independent \(b_1-b_2\) paths \(P_1'\) and \(P_2'\) such that \(G - (P_1' \cup P_2')\) contains an \(a_1-a_2\) path. This shows that \((G, a_1, a_2, b_1, b_2)\) is feasible, a contradiction.

Hence, by symmetry, we may assume that \(a' \in V(Q_b), c' \in V(Q_a),\) and \(w \in V(Q_c)\). Let \(K^*\) denote the \(\{a', b', c'\}\)-bridge of \(L^*\) containing \(\{b_1, w'\}\), and let \(K^* = K + \{w, wx : x \in N(w) \cap V(K)\}\). Then \(Q_a, Q_b, Q_c, P_1[b_1, c], P_2[b_1, a],\) and \(P_3[b_1, b]\) form three independent paths \(Q_a^1, Q_c^1, Q_w^1\) in \(K^* - w^1\) from \(b_1\) to \(a', c', w\), respectively, with \(b \in V(Q_a')\). Hence, \(Q_a^1 \cap P_3 = P_3[b_1, b]\).

Since \(R \not\subseteq H_1 \cup H_{m+1}\), \(A_2(P_1 \cup P_2 \cup P_3)\) has an attachment on both \(P_3[b_1, b]\) and \(P_3(b, b_2)\). Since \(b \in V(Q_a')\), we may apply Lemma 4.1.1 with the paths \(Q_a^1, Q_c^1, Q_w^1\). So \(L^*\) has a 3-cut \(\{a^2, b^2, c^2\}\) with \(b^2 \in V(P_3)\) separating \(\{a', b', c'\} \cup (N(w) \cap V(L^*))\) from \(b_2\), and \(A_2(P_1 \cup P_2 \cup P_3)\) has no attachment in \(P_3(b', b^2)\). Moreover, if \(R^2\) denotes the graph obtained from \(H\) by deleting the components of \(L^* - (\{a^2, b^2, c^2\} \cup \{a', b', c'\})\) containing \(b_1\) or \(b_2\), then \(R^2 = J^2 \cup L^2\) with \(b' \in V(J^2 - L^2)\), where \((J^2, J^2 \cap L^2)\) is planar, \(J^2 \cap L^2\) is an \(a'-b^2\) path, and \(L^2\) is a ladder from \((c', a', w)\) to \((a^2, b^2, c^2)\) along the path \(J^2 \cap L^2\). Note that \(L^2\) contains three disjoint paths \(P_a^2, P_c^2, P_w^2\) from \(a', c', w\), respectively, to \(\{a^2, b^2, c^2\}\), with \(P_a^2 = J^2 \cap L^2\).

If \(N(w) \cap V(R^* \setminus \{a, b, c\}) = \emptyset\) then let \(P_i^2, P_i^2, P_i^2\) be three independent \(b_1-b_2\) paths in \(H - w^*\) obtained from \(Q_a^1 \cup Q_c^1 \cup Q_w^1, P_a^2 \cup P_c^2 \cup P_w^2\), and the subpaths of \(P_i, i \in [3]\),
from \(\{a^2, b^2, c^2\}\) to \(b_2\). We see that 
\(A'_2(P_1^2 \cup P_2^2 \cup P_3^2)\) contains 
\(A'_2(P_1 \cup P_2 \cup P_3) \cup J^2\); so 
\(B_1, B_2, B_3, w^1, P_2^2, P_3^2\) contradict (C7).

So assume \(N(w) \cap V(R^* \setminus \{a, b, c\}) \neq \emptyset\).

We may assume that there exists \(w^2 \in S \cap \text{Int}(R)\) such that \(R^* - w^2\) contains disjoint paths \(Q_a^2, Q_b^2, Q_c^2\) from \(a', b', c'\), respectively, to \(\{a, c, w\}\); otherwise the assertion of the lemma holds. Hence, we may apply the same argument as above with respect to \(R\) and \(b_1\), and conclude that \(L^*\) has a 3-cut \(\{a^1, b^1, c^1\}\) with \(b^1 \in V(P_3)\) separating \(\{a, b, c\} \cup (N(w) \cap V(L^*))\) from \(b_1\), and \(A_2(P_1 \cup P_2 \cup P_3)\) has no attachment in \(P_3(b, b^1)\). Moreover, if \(R^1\) denotes the graph obtained from \(H\) by deleting the components of \(L^* - (\{a^1, b^1, c^1\} \cup \{a, b, c\})\) containing \(b_1\) or \(b_2\) then \(R^1 = J^1 \cup L^1\) with \(b \in V(J^1 - L^1)\), where \(J^1, J^1 \cap L^1\) is planar, \(J^1 \cap L^1\) is an \(a-b^1\) path, and \(L^1\) is a ladder from \((c, a, w)\) to \((a^1, b^1, c^1)\) along \(J^1 \cap L^1\). Note that \(L^1\) contains three disjoint paths \(P_a^1, P_c^1, P_w^1\) from \(a, c, w\), respectively, to 
\(\{a^1, b^1, c^1\}\), with \(P_a^1 = J^1 \cap L^1\).

If \(R \setminus b\) has disjoint paths from \(a, c\) to \(c', a'\), respectively, then, by definition of rung, these paths can be chosen to avoid some \(s \in S \cap \text{Int}(R)\). So these two paths, \(P_a^i \cup P_c^i \cup P_w^i\), \(i \in [2]\), and subpaths of \(P_j\), \(j \in [3]\), from \(b_i\) to \(\{a^i, b^i, c^i\}\), form three independent \(b_1\)-\(b_2\) paths \(B_1', B_2', B_3'\). We can show that \((G, a_1, a_2, b_1, b_2)\) is feasible or there exists \(s \in S \cap \text{Int}(R)\) such that \(B_1', B_2', B_3', s\) contradict (C4).

Thus, \((R \setminus b, a, a', c', c)\) is planar. Let \(X_a, X_c\) denote the disjoint paths in \(R \setminus b\) from 
\(a, c\) to \(a', c'\), respectively, such that \(X_a \cup X_c\) is contained in the outer walk of \(R \setminus a\). Then 
\(S \cap \text{Int}(R) \subseteq V(X_c)\) by (C4). Moreover, \((R, a, b, a', c', c)\) is 3-planar. For, otherwise, there exists \(s \in S \cap V(P_c(c, c'))\) such that \(R \setminus s\) has disjoint paths from \(c, s\) to \(a', b\), respectively, or disjoint paths from \(c', s\) to \(a, b\), respectively. The \(b-s\) path can be used to find an \(a_1-a_2\) path that is disjoint from two \(b_1-b_2\) paths using the other paths. So \((G, a_1, a_2, b_1, b_2)\) is feasible, a contradiction.

Let \(P_1', P_2', P_3'\) be three independent \(b_1-b_2\) paths in \(H\) obtained from \(X_a \cup X_c\), \(P_a^i \cup P_c^i \cup P_w^i\) (for \(i \in [2]\)), and the subpaths of \(P_j\), \(j \in [3]\), from \(\{a^i, b^i, c^i\}\) to \(b_i\) (for \(i \in [2]\)). We see that
Lemma 4.2.3. Suppose \((R, (a, b, c), (a', b', c'))\) is a rung in \(L^*\) such that \(|\partial R| = 6\) or \(|\partial R| = 5\) and \(b \neq b'\), \(R \not\subset H_1 \cup H_{m+1}\), and \(|\{w', w''\} \cap \text{Int}(R)| = 1 = |\{w', w''\} \cap V(R)|.
Then there exists \(i \in [2]\) such that \(V(B_i) \cap \text{Int}(R) = \emptyset\).

Proof. By symmetry, let \(w' \in \text{Int}(R)\) and \(w'' \notin V(R)\), and assume that \(b_1, w', w'', b_2\) occur on \(B_3\) in order. Note that \(b, b' \in V(P_3)\) and, since \(R \not\subset H_1 \cup H_{m+1}\), \(A_2(P_1 \cup P_2 \cup P_3)\) has attachments on both \(P(b_1, b)\) and \(P(b', b_2)\). Since \(|\partial R| = 6\) or \(|\partial R| = 5\) and \(b \neq b'\), it follows from (b) of Lemma 3.2.4 (with appropriate relabeling) that \(R\) contains induced paths \(P_a, P_c\) from \(a, c\) to \(a', c'\), respectively, such that \(R - (P_a \cup P_c)\) is connected and contains \(\{b, b'\}\), and \(S \cap \text{Int}(R) \subseteq V(P_a \cup P_c)\). Let \(R^* = H[R + w]\). We claim that

1. \(N(w) \cap V(R) \subseteq V(P_a \cup P_c)\).

For otherwise, \(R^* - (P_a \cup P_c)\) contains a path \(P_w\) from \(w\) to \(\{b, b'\}\). Let \(P'_1, P'_2\) be the \(b_1\)-\(b_2\) paths in \(L^*\) obtained from \(P_1, P_2\) by replacing \((P_1 \cup P_2) \cap R\) with \(P_a \cup P_c\). Since \(A_2(P_1 \cup P_2 \cup P_3)\) has attachments on both \(P_3(b_1, b)\) and \(P_3(b', b_2)\), \((R^* - (P_a \cup P_c)) \cup (A_1(B_1 \cup B_2 \cup B_3) \cup (B_3 - w)) \cup (A_2(P_1 \cup P_2 \cup P_3) \cup (P_3(b_1, b) - R) \cup P_w\) contains an \(a_1\)-\(a_2\) path independent of \(P'_1, P'_2\). This shows that \((G, a_1, a_2, b_1, b_2)\) is feasible, a contradiction.

By symmetry, let \(w' \in V(P_c)\). Then \(c \neq c'\). Suppose the assertion of the lemma fails, i.e., \(V(B_i) \cap \text{Int}(R) \neq \emptyset\) for \(i \in [3]\). Then by planarity of \((R, a, b, c, c', b', a')\) or \((R - a, b, c, c', b')\), \(S \cap \text{Int}(R) \subseteq V(P_c(c, c'))\). Let \(s \in S \cap V(P_c(c, w'))\) with \(P_c[c, s]\) minimal.

2. \(s\) is not contained in any cut of \(R^*\) of order at most 3 separating \(\{a, a', b, c\}\) from \(\{b', c', w\}\).

For, suppose \(R^*\) has a 3-cut containing \(s\), say \(\{s, v_1, v_2\}\), separating \(\{a, a', b, c\}\) from \(\{b', c', w\}\).
First, assume \( a = a' \). Let \( K \) denote the \( \{ s, v_1, v_2 \} \)-bridge of \( R^* \) containing \( \{ a, b, c \} \). By choosing notation of \( v_1 \) and \( v_2 \), we may assume that \( (K, b, c, s, v_1, v_2) \) is planar. Let \( K' \) be obtained from \( K + \{ bc, sv_1, v_1 v_2 \} \) by adding a new vertex \( v \) and edges from \( v \) to all of \( \{ b, c, s, v_1, v_2 \} \). Then by Hand-shaking lemma and Euler’s formula, \( 5 \times 4 + 5 + 7(|V(K')| - 6) \leq 6|V(K')| - 12 \). This implies \( |V(K')| \leq 5 \), a contradiction.

Now consider the case when \( a \neq a' \). Let \( K \) denote the \( \{ s, v_1, v_2 \} \)-bridge of \( R^* \) containing \( \{ a, a', b, c \} \). By choosing notation of \( v_1 \) and \( v_2 \), we may assume that \( (K, a, b, c, s, v_1, v_2, c') \) is planar. Let \( K' \) be obtained from \( K + \{ ab, bc, sv_1, v_1 v_2, v_2 a' \} \) by adding a new vertex \( v \) and edges from \( v \) to all of \( \{ a, b, c, s, v_1, v_2, a' \} \). Then by Hand-shaking lemma and Euler’s formula, \( 7 \times 4 + 7 + 8(|V(K')| - 8) \leq 6|V(K')| - 12 \). This implies \( |V(K')| \leq 10 \). Hence, \( U := V(K) \setminus \{ a, b, c, s, v_1, v_2, a' \} \) contains at most two vertices. Since each vertex in \( U \) must have degree at least 8 and \( U \cap N(w) = \emptyset \) by (1), we have \( |U| = 2 \). However, this contradicts the planarity of \( (R, a, b, c, c', b', a') \).

We claim that

\[ (3) \quad a = a'. \]

For, suppose \( a \neq a' \). Then \( (R, a, b, c, c', b', a') \) is planar. Thus, \( B_3 \cap R^* \) must be a \( c-w \) path, and \( B_1 \cap R^* \) and \( B_2 \cap R^* \) must be an \( a-\{ a', b' \} \) path and a \( b-\{ b', c' \} \) path. By (2) and planarity of \( R \), we see that \( R^* - s \) contains disjoint induced paths \( B'_a, B'_b, B'_c \) connecting the ends of \( B_1 \cap R^*, B_2 \cap R^*, B_3 \cap R^* \), respectively. Thus, by replacing \( B_1 \cap R^*, B_2 \cap R^*, B_3 \cap R^* \) with \( B'_a, B'_b, B'_c \), we obtain from \( B_1, B_2, B_3 \) three independent induced \( b_1-b_2 \) paths \( B'_1, B'_2, B'_3 \). Since \( A_2(B_1 \cup B_2 \cup B_3) \) has attachments on both \( B_1 \) and \( B_2 \) and has no attachment in \( Int(R) \), \( A_2(B'_1 \cup B'_2 \cup B'_3) \) has attachments on both \( B'_1 \) and \( B'_2 \). Clearly, \( A_1(B'_1 \cup B'_2 \cup B'_3) \) contains \( A_1(B_1 \cup B_2 \cup B_3) + s \). So \( B'_1, B'_2B'_3 \) contradicts (C4).

By (3), \( (R - a, b, c, c', c') \) is planar. Hence, by (2), \( (R^* - a) - s \) contains disjoint paths \( B^2_b, B^2_c \) from \( b, c \) to \( c', w \), respectively. Note that \( B^2_b, B^2_c \) and the subpaths of \( P_i, \ i \in [3], \) between \( b_1 \) and \( \{ a, b, c \} \) form three independent induced paths \( Q^2_a, Q^2_c, Q^2_w \) from \( b_1 \)
to $a, c', w$, respectively. Moreover, we see that $Q_c^2 \cap P_3$ contains $P_3[b_1, b]$ and has an attachment of $A_2(P_1 \cup P_2 \cup P_3) - b_1$. Note that $b' \in V(Q_c^2)$ or the $(Q_a^2 \cup Q_c^2 \cup Q_w^2) \cap R^*$-bridge of $R^*$ containing $b'$ has an attachment in $Q_c^2$. We can now apply Lemma 4.1.1 to obtain a 3-cut $\{a^2, b^2, c^2\}$ in $L^*$ separating $\{a', b', c'\} \cup (N(w) \cap V(L^*))$ from $b_2$. Moreover, if $R^2$ denotes the graph obtained from $H$ by deleting the components of $L^* - (\{a', b', c'\} \cup \{a^2, b^2, c^2\})$ containing $b_1$ or $b_2$, then $R^2 = J^2 \cup L^2$ with $b' \in V(J^2 - L^2)$, where $(J^2, J^2 \cap L^2)$ is planar, $J^2 \cap L^2$ is an $c'-b^2$ path, and $L^2$ is a ladder from $(a', c', w)$ to $(a^2, b^2, c^2)$ along $J^2 \cap L^2$. Moreover, $L^2 - J^2$ has disjoint paths from $\{a, w\}$ to $\{a^2, c^2\}$ which, we may assume, are $P_a^2, P_w^2$ from $a, w$ to $a^2, c^2$, respectively.

Let $L_1$ denote the $\{a, b, c\}$-bridge of $L^*$ containing $H_1$.

**Case 1.** $N(w) \cap V(L_1 - \{a, b, c\}) = \emptyset$.

Let $P_b$ be the $b-c'$ path in the boundary of $R - a$ containing $b'$ but not $c$. Suppose $P_b$ is an induced path. Then let $P'_3 : = P_3[b_1, b] \cup P_b \cup (J^2 \cap L^2) \cup P_3[b^2, b_2]$, and let $P', P''$ be obtained from $P_1, P_2$ by replacing $(P_1 \cup P_2) \cap (R \cap R^2)$ with $P_a^2, B_c \cup P_w^2$. We see that $A'_2(P'_1 \cup P'_2 \cup P'_3)$ contains $A'_2(P_1 \cup P_2 \cup P_3) \cup J^2$; so $B_1, B_2, B_3, s, P'_1, P'_2, P'_3$ contradict (C7).

Hence, $P_b$ is not an induced path. Thus, let $xy \in E(G) \setminus E(P_b)$ with $x, y \in V(P_b)$. Choose $x, y$ with $P_b[x, y]$ maximal. To avoid the cut set $\{x, y, w, a, b\}$ in $G$, we may assume that $xy, b'y \in E(P_b)$ and $x, b', y$ occur on $P_b$ in this order. Let $P'_3 : = P_3[b_1, b] \cup (P_b[b, x] \cup xy \cup P_b[y, c']) \cup (J^2 \cap L^2) \cup P_3[b^2, b_2]$. Let $P'_1, P'_2$ be defined as above.

If $b'a \notin E(G)$, then we see that $A'_2(P'_1 \cup P'_2 \cup P'_3)$ contains $A'_2(P_1 \cup P_2 \cup P_3) \cup J^2$; so $B_1, B_2, B_3, s, P'_1, P'_2, P'_3$ contradict (C7). Thus, $b'a \in E(G)$. By symmetry between $a^2$ and $c^2$, let $P'_c, P'_w$ be disjoint induced paths in $L^2 - \{a, b^2\}$ from $c, w$ to $a^2, c^2$, respectively. Let $B'_1, B'_2, B'_3$ be obtained from $(P_1 \cup P_2 \cup P_3) - ((R \cup R^2) - \{a, b, c, a^2, b^2, c^2\})$ by adding $(P_b[b, x] \cup xy \cup P_b[y, c']) \cup P'_c, ab' \cup (J^2 - (L^2 - b^2)), B_c \cup P'_w$. By choosing notation, we may assume $w \in V(B'_3), P_3[b_1, b] \subseteq B'_1$, and $P_3[b^2, b_2] + a \subseteq B'_2$. Now $A_2(B'_1 \cup B'_2 \cup B'_3)$ has attachments on both $B'_1$ and $B'_2$. Clearly, $A_1(B'_1 \cup B'_2 \cup B'_3)$ contains $A_1(B_1 \cup B_2 \cup B_3) + s$;
so $B'_1, B'_2, B'_3$ contradict (C4).

**Case 2.** $N(w) \cap V(L_1 - \{a, b, c\}) = \emptyset$.

Then consider disjoint paths $B^1_b, B^1_c$ in $(R^* - a) - s$ from $b', c'$ to $c, w$, respectively, which exists by planarity of $(R - a, b, c, c', b')$. Now $B^1_b, B^1_c$ and the subpaths of $P_i, i \in [3]$, between $b_2$ and $\{a', b', c'\}$ form three independent induced paths $Q^1_a, Q^1_c, Q^1_w$ from $b_2$ to $a, c, w$, respectively. Moreover, we see that $Q^1_c \cap P_3$ contains $P_3[b_2, b']$ and has an attachment of $A_2(P_1 \cup P_2 \cup P_3) - b_2$. We can now apply Lemma 4.1.1 to obtain a 3-cut $\{a^1, b^1, c^1\}$ in $L^*$ separating $\{a, b, c\} \cup (N(w) \cap V(L^*))$ from $b_1$. Moreover, if $R^1$ denotes the graph obtained from $H$ by deleting the components of $L^* - (\{a, b, c\} \cup \{a^1, b^1, c^1\})$ containing $b_1$ or $b_2$, then $R^1 = J^1 \cup L^1$ with $b \in V(J^1 \cup L^1)$, where $(J^1, J^1 \cap L^1)$ is planar, $J^1 \cap L^1$ is an $c-b^1$ path, and $L^1$ is a ladder from $(a, c, w)$ to $(a^1, b^1, c^1)$ along $J^1 \cap L^1$. Note, $L^1 - J^1$ has disjoint paths from $\{a, w\}$ to $\{a^1, c^1\}$ which, we may assume, are $P^1_a, P^1_w$ from $a, w$ to $a^1, c^1$, respectively.

Let $Q$ denote an induced $c-c'$ path with $V(Q)$ contained in the boundary of $R - a$ disjoint from $P_c(c, c')$. Let $P'_3 = P_3[b_1, b^1] \cup (J^1 \cap L^1) \cup Q \cup (J^2 \cap L^2) \cup P_3[b^2, b_2]$, and let $P'_1, P'_2$ be the $b_1-b_2$ paths obtained from $P_1 \cup P_2$ by replacing $(P_1 \cup P_2) \cap (R^1 \cup R^2)$ with $P^1_a \cup P^2_a$ and $P^1_w \cup P^2_w$. We see that $A'_2(P'_1 \cup P'_2 \cup P'_3)$ contains $A'_2(P_1 \cup P_2 \cup P_3) \cup J^1 \cup J^2$; so $B_1, B_2, B_3, s, P'_1, P'_2, P'_3$ contradict (C7).

We conclude this section with the following result.

**Lemma 4.2.4.** Let $(R, (a, b, c), (a', b', c'))$ be a rung in $L^*$ with $R \not\subset H_1 \cup H_{m+1}$ and $b \neq b'$. Then there exists $i \in [2]$ such that $Int(R) \cap V(B_i) = \emptyset$.

**Proof.** Suppose $Int(R) \cap V(B_i) \neq \emptyset$ for $i \in [2]$. Then, since $G$ is 8-connected, $S \cap Int(R) \neq \emptyset$ and, hence, $V(B_3) \cap Int(R) \neq \emptyset$. Since $R \not\subset H_1 \cup H_{m+1}$, it follows from Lemma 4.2.3 that $w', w'' \in V(R)$ or $\{w', w''\} \cap Int(R) = \emptyset$. Hence, $|\partial R| = 6$. By Lemma 3.2.4, let $P_a, P_c$ be the induced paths in $R$ from $a, c$ to $a', c'$, respectively, such that $R - (P_a \cap P_c)$ is connected and contains $\{b, b'\}$. Note that $N(w) \cap Int(R) \subset V(P_a \cup P_c)$;
as otherwise \((G, a_1, a_2, b_1, b_2)\) would be feasible.

Suppose \(\{w', w''\} \cap \text{Int}(R) = \emptyset\) or \(B_3 \cap R \subseteq P_a\) or \(B_3 \cap R \subseteq P_c\). Then \(B_i \cap R, i \in [3]\), are \(\{a, b, c\} - \{a', b', c'\}\) paths. By Lemma 3.2.4, we may assume that \(B_1 \cap R = P_a\) and \(B_3 \cap R = P_c\). So there exists \(s \in S \cap V(P_c(c, c'))\). By definition of rung, \(R\) has no 3-cut separating \(\{a, b, c\}\) from \(\{a', b', c'\}\). Hence, \(R - s\) has three disjoint paths \(Q_a, Q_b, Q_c\) from \(a, b, c\), respectively, to \(\{a', b', c'\}\). By Lemma 3.2.4 again, \(a' \in V(Q_a), b' \in V(Q_b), c' \in V(Q_c)\). For each \(i \in [3]\), let \(B'_i\) be obtained from \(B_i\) by replacing \(B_i \cap R\) with one of \(Q_a, Q_b, Q_c\). Now \(A_1(B'_1 \cup B'_2 \cup B'_3)\) contains \(A_1(B_1 \cup B_2 \cup B_3) + s\), and \(A_2(B'_1 \cup B'_2 \cup B'_3)\) has attachments on both \(B'_1\) and \(B'_2\) (as \(R \not\subseteq H_1 \cup H_{m+1}\)). So \(B'_1, B'_2, B'_3, w\) contradict (C4).

Hence, we may assume \(w' \in V(P_a)\), and \(w'' \in V(P_c)\). Choose \(w_a \in N(w) \cap V(P_a)\) with \(P_a[a, w_a]\) minimal, and choose \(w_c \in N(w) \cap V(P_c)\) with \(P_c[w_c, c']\) minimal. Note that \(S \cap \text{Int}(R) \subseteq V(P_a[a, w_a]) \cup V(P_c[w_c, c'])\). For otherwise, we could modify \(B_3\) by replacing \(B_3(w_a, w_c)\) with \(w_a w w_c\) to obtain a new \(b_1\)-\(b_2\) path \(B'_3\). Now \(A_1(B_1 \cup B_2 \cup B'_3)\) contains \(A_1(B_1 \cup B_2 \cup B_3)\) and some vertex \(s \in S \cap \text{Int}(R)\). Moreover, \(A'_2(P_1 \cup P_2 \cup P_3)\) has attachments on both \(B'_1\) and \(B'_2\). So \(B_1, B_2, B'_3, s\) contradict (C4).

Suppose there exists \(s \in S\) with \(s \in V(P_a(a, w_a)) \cup V(P_c(w_c, c'))\). By symmetry, assume \(s \in V(P_c(w_c, c'))\). Since \(R\) has no 3-cut separating \(\{a, b, c\}\) from \(\{a', b', c'\}\), \(R - (P_a[a, a'] \cup P_c[c, w]) - s\) contains two disjoint paths \(Q_a, Q_b\) from \(a, b\) to \(b', c'\), respectively. Without loss of generality, we may assume \(a, a' \in V(P_1)\) and \(c, c' \in V(P_2)\). Let \(B'_1 = P_1[b_1, a] \cup Q_a \cup P_3[b', b_2], B'_2 = P_3[b_1, b] \cup Q_b \cup P_1[c', b_2], B'_3 = P_2[b_1, c] \cup P_c[c, w_c] \cup w_c w a \cup P_a[w_a, a'] \cup P_1[a', b_2]\). Now \(B'_1, B'_2, B'_3\) are independent \(b_1\)-\(b_2\) paths. Moreover, \(A_1(B'_1 \cup B'_2 \cup B'_3)\) contains \(A_1(B_1 \cup B_2 \cup B_3) + s\), and \(A'_2(P_1 \cup P_2 \cup P_3)\) has attachments on both \(B_1\) and \(B_2\) (as \(R \not\subseteq H_1 \cup H_{m+1}\)), contradicting (C4).

Thus, \(S \cap (V(P_a(a, w_a)) \cup V(P_c(w_c, c'))) = \emptyset\). This implies that \(S \cap \text{Int}(R) \subseteq \{w_a, w_c\}\). Let \(R^*\) be the plane graph obtained from \(G[R + w]\) by adding \(ba, bc, b'a', b'c'\) and all edges from \(w\) to \(V(P_a \cup P_c) \cup \{b, b'\}\). Now \(|E(R^*)| \geq 8(|R^*| - 8) + 6 \times 4 + 2 \times 5 = 8|R^*| - 30\). So \(8|R^*| - 30 \leq 6|R^*| - 12\). This implies \(|R^*| \leq 9\), a contradiction as
$|N(b) \cap \text{Int}(R)| \geq 2$ by (ii) of Proposition 2.3.2.
CHAPTER 5
STRUCTURE OF OTHER RUNGS

In this chapter, we consider rungs $R$ in $L^*$ such that $\text{Int}(R) \cap B_i = \emptyset$ for some $i \in [2]$.

First, in section 5.1, we prove technical lemmas for separation $(G', G'')$ of $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3)$ in which $B_i \cap (G' - G'') = \emptyset$ for some $i \in [2]$. Then, we deal with $H_1$, $H_{m+1}$ in subsection 5.2.1 and all other rungs in subsection 5.2.2.

For $x, y \in V(B_j)$ for some $j \in [3]$, we denote $x \preceq y$ if $B_j[b_1, x] \subseteq B_j[b_1, y]$; and $x < y$ if $x \preceq y$ and $x \neq y$.

5.1 Technical lemmas

We begin by showing that for any rung $R$ in $L^*$ or for $H_1, H_{m+1}$, if neither $B_1$ nor $B_2$ intersects $\text{Int}(R)$ or $\text{Int}(H_1)$, or $\text{Int}(H_{m+1})$, then $\text{Int}(R), \text{Int}(H_1), \text{Int}(H_{m+1}) \subseteq S$. For convenience, we prove a more general statement in terms of separations in $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3)$.

**Lemma 5.1.1.** Suppose $(G', G'')$ is a separation of $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3)$ such that $|V(G' \cap G'')| \leq 7$, $V(G' - G'') \neq \emptyset$, and $V(G'' - G') \neq \emptyset$. Suppose $V(G' - G'') \cap V(B_1 \cup B_2) = \emptyset$. Then $V(G' - G'') \subseteq S$.

**Proof.** Note $S \cap V(G' - G'') \neq \emptyset$; otherwise $V(G' \cap G'')$ is a cut of $G$ contradicting the $(8, S)$-connectivity. Let $r_1, r_2 \in V(B_3) \cap V(G' \cap G'')$ be such that $B_3[r_1, r_2]$ is maximal. Note that it is possible $B_3[r_1, r_2] \not\subseteq G'$.

Suppose $V(G' - G'') \not\subseteq S$ and let $X$ be an $S$-bridge of $G' - (V(G' \cap G'') \setminus \{r_1, r_2\})$ with $X - S \neq \emptyset$. Let $x_1, x_2 \in V(X) \cap (S \cup \{r_1, r_2\})$ such that $B_3[x_1, x_2]$ is maximal. Then $|V(X) \cap (S \cup \{r_1, r_2\})| \geq 3$; otherwise, $(V(X) \cap (S \cup \{r_1, r_2\})) \cup (V(G' \cap G'') \setminus \{r_1, r_2\})$
is a cut of $G$ separating $V(X) \setminus (S \cup \{r_1, r_2\})$ from $V(G'' - G')$, a contradiction to the $(8, S)$-connectivity of $G$.

Hence, there exists $s \in V(B_3(x_1, x_2)) \cap S$. Let $A$ be any induced $x_1$-$x_2$ path in $X - s$, and $B'_3 = (B_3 - B_3(x_1, x_2)) \cup Q$. Then $B_1, B_2, B'_3$ are independent $b_1$-$b_2$ paths in $G$ such that $A_1(B_1 \cup B_2 \cup B'_3) \cap \{r, t\}$ contains $A_1(B_1 \cup B_2 \cup B_3) + w$ and $A_2(B_1 \cup B_2 \cup B'_3)$ attaches to both $B_1$ and $B_2$, a contradiction. \hfill \Box

Next, we consider rungs $R$ when $\text{Int}(R) \cap B_i = \emptyset$ for exactly one $i \in [2]$. Again we prove statements in terms of separations in $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3)$. We first show that all internal vertices are in $V(B_i) \cup S$ in Lemma 5.1.2. Then, we give structural results of such rungs in Lemma 5.1.3.

**Lemma 5.1.2.** Suppose $(G', G'')$ is a separation of $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3)$ such that $|V(G' \cap G'')| \leq 7$, $V(G' - G'') \neq \emptyset$, and $V(G'' - G') \neq \emptyset$. Let $i \in [2]$ such that $V(G' - G'') \cap V(B_i) \neq \emptyset$, and $V(G'' - G') \cap V(B_{3-i}) = \emptyset$. Then $V(G' - G'') \subseteq V(B_i) \cup S$.

**Proof.** Note $S \cap V(G' - G'') \neq \emptyset$; otherwise $V(G' \cap G'')$ is a cut of $G$ contradicting the $(8, S)$-connectivity of $G$. Let $r_1, r_2 \in V(B_3) \cap V(G' \cap G'')$ and $t_1, t_2 \in V(B_i) \cap V(G' \cap G'')$ be such that $B_3[r_1, r_2]$ and $B_i[t_1, t_2]$ are maximal. For convenience, let $G^* = G' - \{V(G' \cap G'') \setminus \{r_1, r_2, t_1, t_2\}\}$. We may assume $r_1 < r_2$ and $t_1 < t_2$.

Suppose for a contradiction, $V(G' - G'') \setminus (V(B_i) \cup S) \neq \emptyset$. Then $G^*$ has a $((B_i \cup S) \cap G^*)$-bridge $X$ such that $V(X) \setminus (V(B_i) \cup S) \neq \emptyset$. Choose $X$ and modify $B_i \cap G^*$ (if necessary) so that

1. $|V(X) \cap (S \cup \{r_1, r_2\})|$ is maximal, and
2. subject to (1), $X$ is maximal.

Let $x_1, x_2 \in V(X) \cap (S \cup \{r_1, r_2\})$ with $B_3[x_1, x_2]$ maximal, and let $x_1 < x_2$. We claim that

3. $|V(X) \cap (S \cup \{r_1, r_2\})| \leq 2$. 

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For, otherwise, there exists \( s \in V(X) \cap V(B_3(x_1, x_2)) \cap S \). Let \( Q \) be any induced \( x_1-x_2 \) path in \( X - (B_i + (S \setminus \{x_1, x_2\})) \), and let \( B'_3 = (B_3 - B_3(x_1, x_2)) \cup Q \). Then \( B_1, B_2, B'_3 \) are independent \( b_1-b_2 \) paths in \( G - \{a_1, a_2\} \) such that \( A_1(B_1 \cup B_2 \cup B'_3) \) contains \( A_1(B_1 \cup B_2 \cup B_3) + s \) and \( A_2(B_1 \cup B_2 \cup B'_3) \) attaches to both \( B_1 \) and \( B_2 \), a contradiction.

By (3), \( V(B_3(x_1, x_2)) \cap S = \emptyset \); so we may choose \( B_3 \) such that

\[
(4) \quad B_3[\{x_1, x_2\}] \subseteq X.
\]

Then, \( |V(X \cap B_i)| \geq 2 \); otherwise by (3), \( V(X \cap B_i) \cup (V(X) \cap (S \cup \{r_1, r_2\})) \cup (V(G' \cap G'') \setminus \{r_1, r_2, t_1, t_2\}) \) is a cut in \( G \) of size \( \leq 7 \) separating \( V(X - (B_i \setminus S)) \) from \( G'' - G' \), contradicting the \( (8, S) \) connectivity of \( G \). Let \( y_1, y_2 \in V(X \cap B_i) \) with \( y_1 < y_2 \) such that \( B_i[y_1, y_2] \) is maximal. Then

\[
(5) \quad G^* \text{ has no path from } B_i(y_1, y_2) \text{ to } B_i B_i[y_1, y_2] \text{ and internally disjoint from } B_i \cup B_3.
\]

For otherwise, let \( Q \) be an induced path in \( G^* \) from \( z_1 \in V(B_i(y_1, y_2)) \) to \( z_2 \in V(B_i - B_i[y_1, y_2]) \), and let \( B'_i \) be an induced \( b_1-b_2 \) path in \( (B_i - B_i(z_1, z_2)) \cup Q \). Then, the \( ((B_i \cup S) \cap G^*) \)-bridge of \( G^* \) containing \( X \) also contains \( z_2 \), contradicting (2).

\[
(6) \quad |V(X) \cap (S \cup \{r_1, r_2\})| \geq 1.
\]

For, suppose \( V(X) \cap (S \cup \{r_1, r_2\}) = \emptyset \). Then by (1), no \( ((B_i \cup S) \cap G^*) \)-bridge of \( G^* \) has attachment in \( S \cup \{r_1, r_2\} \). Hence by (5) and since \( S \cap V(G^*) \subseteq V(B_3) \), there exists an induced path \( Q' \) in \( G^* \) from some vertex \( y \in V(B_i(y_1, y_2)) \) to some vertex \( s \in (S \cup \{r_1, r_2\}) \cap V(G^*) \), internally disjoint from \( X \cup B_i + S \). Let \( Q'' \) be an induced \( y_1-y_2 \) path in \( X - B_i(y_1, y_2) \) and \( B'_i := (B_i - B_i(y_1, y_2)) \cup Q'' \). Then, the \( ((B'_i \cup S) \cap G^*) \)-bridge of \( G^* \) containing \( Q' \) also contains \( s \), contradicting (1).

By (3) and (6), we have two cases.

**Case 1.** \( |V(X) \cap (S \cup \{r_1, r_2\})| = 2 \).
Since \( \{x_1, x_2, y_1, y_2\} \cup (V(G' \cap G'') \setminus \{r_1, r_2, t_1, t_2\}) \) is not a cut in \( G \) separating \( X - \{x_1, x_2, y_1, y_2\} \) from \( G'' - G' \), it follows from (5) that there is a \( y_3 \)-\( x_3 \) path \( Q \) in \( G^* \) internally disjoint from \( X \cup B_i \cup S \), with \( y_3 \in V(B_i(y_1, y_2)) \) and \( x_3 \in V(B_3) \). Since \( B_3 \cap B_i \subseteq \{b_1, b_2\} \), if \( B_3 \cap Q \neq \emptyset \) then \( x_3 \) may be chosen so that \( x_3 \in (S \cup \{r_1, r_2\}) \setminus \{x_1, x_2\} \).

Note that \( x_j \in V(B_3(x_{3-j}, x_3)) \) for some \( j \in [2] \), and thus, \( x_j \in S \). By symmetry, we may assume \( j = 2 \) and \( x_1 \prec x_2 \prec x_3 \), and that there exists \( z \in V(A_2(B_1 \cup B_2 \cup B_3) \cap B_i[b_1, y_2]) \). Let \( Q' \) be any \( x_1 \)-\( y_2 \) path in \( X - y_3 \) internally disjoint from \( B_i \cup S \).

Then, the following paths show that \((G, a_1, a_2, b_1, b_2)\) is feasible: \( B_3 - i, B_3[b_1, x_1] \cup Q' \cup B_i[y_2, b_2], \) and an \( a_1 \)-\( a_2 \) path in the union of \( A_1(B_1 \cup B_2 \cup B_3) - (B_3 - x_3), B_i[z, y_3] \cup Q \cup B_3[x_3, x_2], \) and \( A_2(B_1 \cup B_2 \cup B_3) - (B_1 \cup B_2 - z) \).

**Case 2.** \( |V(X) \cap (S \cup \{r_1, r_2\})| = 1 \).

So \( x_1 = x_2 \). Since \( \{x_1, y_1, y_2\} \cup (V(G' \cap G'') \setminus \{r_1, r_2, t_1, t_2\}) \) is not a cut in \( G \) separating \( X - \{x_1, y_1, y_2\} \) from \( G'' - G' \), it follows from (5) that there exist disjoint paths \( Q_1, Q_2 \) from \( z_1, z_2 \in V(B_i(y_1, y_2)) \) to \( x_2, x_3 \in V(B_3 - x_1) \), respectively, internally disjoint from \( X \cup B_i \cup S \). We may choose \( x_2, x_3 \in (S \cup \{r_1, r_2\}) \setminus \{x_1\} \). (If \( Q_1, Q_2 \) intersect \( B_3 - S \) then we obtain a new bridge contradicting (1).) Since the order of \( z_1, z_2 \) will not matter in the rest of our argument, we may assume \( x_1 \prec x_2 \prec x_3 \) or \( x_2 \prec x_1 \prec x_3 \).

First, suppose \( x_1 \prec x_2 \prec x_3 \). Let \( Q \) be an induced \( x_1 \)-\( y_2 \) path in \( X \) independent of \( B_i \), and let \( B'_i = B_i[1, z_2] \cup Q_2 \cup B_3[x_3, b_2] \) and \( B'_3 = B'[1, x_1] \cup Q \cup B_i[y_2, b_2] \). Note that \( A_2(B_3 - i \cup B'_i \cup B'_3) \) attaches to \( B_3 - i \) as well as \( B'_i \) or \( B'_3 \). If \( A_2(B_3 - i \cup B'_i \cup B'_3) \) attaches to \( B'_3 \) then, since \( x_1 \in S \), we see that \((G, a_1, a_2, b_1, b_2)\) is feasible. If \( A_2(B_3 - i \cup B'_i \cup B'_3) \) attaches to \( B'_i \) then \( B_3 - i, B'_i, B'_3, x_1 \) contradict (C4) as \( A_1(B_3 - i \cup B'_i \cup B'_3) \) contains \( A_1(B_1 \cup B_2 \cup B_3) + x_1 \).

Now suppose \( x_2 \prec x_1 \prec x_3 \). Let \( Y \) be an induced \( y_1 \)-\( y_2 \) path in \( X - x_1 \) independent of \( B_i \), let \( B'_3 = B_3[b_1, x_2] \cup Q_1 \cup B_i[z_1, z_2] \cup Q_2 \cup B_3[x_3, b_2] \) and let \( B'_i = B_i[1, y_1] \cup Y \cup B_i[y_2, b_2] \). Then \( A_1(B_3 - i \cup B'_i \cup B'_3) \) contains \( A_1(B_1 \cup B_2 \cup B_3) + x_1 \). So \( B_3 - i, B'_i, B'_3, x_1 \) contradict (C4). \( \square \)
Lemma 5.1.3. Suppose \((G', G'')\) is a separation of \(G - (A_1(B_1 \cup B_2 \cup B_3) - B_3)\) such that \(|V(G' \cap G'')| \leq 7\), \(V(G' - G'') \neq \emptyset\) and \(V(G'' - G') \neq \emptyset\). Suppose for some \(i \in [2]\), \(V(G' - G'') \cap V(B_i) \neq \emptyset\), and \(V(G' - G'') \cap V(B_{3-i}) = \emptyset\). Let \(r_1, r_2 \in V(G' \cap G'' \cap B_3)\) and \(t_1, t_2 \in V(G' \cap G'' \cap B_i)\) such that \(B_3[r_1, r_2] \) and \(B_i[t_1, t_2] \) are maximal, and \(N_{G' - G''}(r_j) \cap S \neq \emptyset\) for both \(j \in [2]\). Let \(V' = V(G' \cap G'') \setminus (\{r_1, r_2\} \cup V(B_i))\).

Then, for some \(e\) with \(e = \emptyset\) or \(e \in E(G' - V')\) incident to either \(r_1\) or \(r_2\), if \(x_jy_j \in E(G' - V') \setminus (E(B_i \cup B_3) \cup \{e\})\) with \(x_j \in V(B_i)\) and \(y_j \in V(B_3)\) for \(j \in [2]\), then \(x_1 \preceq x_2\) implies \(y_1 \preceq y_2\).

Proof. By Lemma 5.1.2, \(V(G' - G'') \subseteq V(B_i) \cup S\). Thus, \(G^* := G' - V'\) is obtained from \(G^* \cap (B_i[t_1, t_2] \cup B_3[r_1, r_2])\) by adding edges with one end in \(B_i\) and the other end in \(B_3\). For any distinct \(x_1, x_2 \in V(B_i \cap G^*)\) and distinct \(y_1, y_2 \in V(B_3 \cap G^*)\), we say \((x_1, x_2, y_1, y_2)\) is a cross if \(x_1 \prec x_2, y_1 \prec y_2\) and \(x_1y_2, x_2y_1 \in E(G^*)\). If there is no cross, the lemma holds with \(e = \emptyset\). So assume there is a cross.

For any cross \((x_1, x_2, y_1, y_2)\), we have \(S \cap V(B_3[b_k, y_k]) = \emptyset\) for some \(k \in [2]\). For otherwise, both \(b_1\)-\(b_2\) paths \(B'_i := B_i[b_1, x_1] \cup \{x_1y_2\} \cup B_3[y_2, b_2], B'_3 := B_3[b_1, y_1] \cup \{y_1x_2\} \cup B_i[x_2, b_2]\) have an internal vertex in \(A_2(B_1 \cup B_2 \cup B_3)\). Since \(A_2(B_1 \cup B_2 \cup B_3)\) attaches to \(B_{3-i}\) and one of \(B'_i\) or \(B'_3\), we see that \((G, a_1, a_2, b_1, b_2)\) is feasible.

Thus, since \(V(G' - G'') \subseteq V(B_i) \cup S\), we have, for any cross \((x_1, x_2, y_1, y_2)\), \(y_j = r_j \notin S\) for some \(j \in [2]\). For convenience, let \(t_1 < t_2\) and \(r_1 < r_2\).

Next, we show that, for any cross \((x_1, x_2, y_1, y_2)\), if \(y_1 = r_1\) then \(B_i(t_1, x_2) = \emptyset\), and if \(y_2 = r_2\) then \(B_i(t_2, x_1) = \emptyset\). For, otherwise, suppose \(y_1 = r_1\) and there exists \(x \in V(B_i(t_1, x_2))\). Since \(B_i\) is induced and \(G\) is 8-connected, \(|N_{G^*}(x) \cap (S \cup \{r_1, r_2\})| \geq 3\); so let \(s_1, s_2, s_3 \in N_{G^*}(x) \cap (S \cup \{r_1, r_2\})\) with \(s_1 < s_2 < s_3\). Then \(s_2 \in S \setminus \{r_1, r_2\}\). Let \(B'_{3-i} := B_{3-i}, B'_i := B_i[b_1, x] \cup \{x_1s_3\} \cup B_3[s_3, b_2], B'_3 := B_3[b_1, y_1] \cup \{y_1x_2\} \cup B_i[x_2, b_2]\).

Then we see that \(A_2(B'_1 \cup B'_2 \cup B'_3)\) has attachments on \(B'_{3-i}\) and one of \(B'_i\) and \(B'_3\), and \(A_1(B'_1 \cup B'_2 \cup B'_3)\) contains \(A_1(B_1 \cup B_2 \cup B_3) + s_2\). It is easy to see that \((G, a_1, a_2, b_1, b_2)\) is feasible or \(B'_1, B'_2, B'_3, s_2\) contradict (C4).
Now let \((x_1, x_2, y_1, y_2)\) be a cross with \(y_1 = r_1 \notin S\), and we further choose this cross to maximize \(B_i[t_1, x_2]\). By above, we see that \(V(B[t_1, x_2]) = \{x_1, x_2\}\). If all crosses use the edge \(y_1, x_2\), then the assertion of the lemma holds with \(e = y_1, x_2\). So assume there is a cross \((x_1', x_2', y_1', y_2')\) with \(y_1' x_2' \neq y_1, x_2\). Then \(y_1' \neq y_1\). Hence, \(y_1' \in S\) and \(y_2' = r_2 \notin S\). This implies that \(V(B_i[x_1', t_2]) = \{x_1', x_2'\}\). Note that \(x_1' \neq x_1\) and \(x_2' \neq x_2\) (as \(V(B_i) \cap V(G' - G'') = \emptyset\)). Thus, \(x_2 < x_1'\). By the maximality of \(B_i[t_1, x_2]\), we see that \(y_2 \neq y_1'\).

Let \(B'_i := B_i[b_1, x_1] \cup \{x_1, y_2\} \cup B_3[y_2, y_1'] \cup \{y_1', x_2\} \cup B_i[x_2, b_2]\) and \(B'_3 := B_3[b_1, r_1] \cup \{r_1, x_2\} \cup B_i[x_2, x_1'] \cup \{x_1', r_2\} \cup B_3[r_2, b_2]\). Then, both \(B'_i(t_1, t_2)\) and \(A_2(B_1 \cup B_2 \cup B_3)\) contains \(y_1'\) and \(y_2\); so \(G - (B'_3 \cup B_{3-i})\) has an \(a_1\)-\(a_2\) path, showing that \((G, a_1, a_2, b_1, b_2)\) is feasible.

\[\square\]

5.2 Structures

In this section, we use technical lemmas from previous section to give structural results of \(H_1, H_{m+1}\) and rungs \(R \in L^* \) not contained in \(H_1 \cup H_{m+1}\).

5.2.1 \(H_1\) and \(H_{m+1}\)

First, we consider \(H_1, H_{m+1}\) when \(Int(H_1), Int(H_{m+1})\) intersects \(B_i\) for at most one \(i \in [2]\).

**Lemma 5.2.1.** If \(B_i \cap Int(H_1) \neq \emptyset\) for at most one \(i \in [2]\), then one of the following holds:

(a) \(Int(H_1) \subseteq S\).

(b) \(Int(H_1) \subseteq V(B_i) \cup S\) for some \(i \in [2]\) and the following holds:

- Let \(G' := G[V(H_1) \cup \{w\}]\); let \(r_1, r_2 \in V(B_3) \cap (T_1 \cup \{w, b_1\})\) and \(t_1, t_i \in V(B_i) \cap (T_1 \cup \{w, b_1\})\) such that \(N(r_j) \cap S \cap Int(H_1) \neq \emptyset\) for both \(j \in [2]\) and subject to this, \(B_3[r_1, r_2]\) and \(B_i[t_1, t_2]\) are maximal; let \(V' = (T_1 \cup \{w, b_1\}) \setminus \)
Then, there exists e with e = ∅ or e incident to either r₁ or r₂, such that, if xⱼyⱼ ∈ E(G' − V') \ E(Bᵢ ∪ B₃) with xⱼ ∈ V(Bᵢ) and yⱼ ∈ V(B₃) for j ∈ [2], then x₁ ≤ x₂ implies y₁ ≤ y₂.

The same holds for Hₘ₊₁ and b₂.

Proof. If Int(H₁) = ∅ then (a) holds. So assume Int(H₁) ≠ ∅. Then |S ∩ Int(H₁)| ≥ 3; otherwise T₁ ∪ {w, b₁} ∪ (S ∩ Int(H₁)) is a cut in G of size ≤ 7 separating Int(H₁) from b₂, contradicting the (8, S)-connectivity of G.

Suppose V(B₁ ∪ B₂) ∩ Int(H₁) = ∅. Let G' := G[V(H₁) ∪ {w}] and G'' := G − Int(H₁) − E(G[T₁ ∪ {w, b₁}]). Then, by Lemma 5.1.1, Int(H₁) = V(G' − G'') ⊆ S, and thus, (a) holds.

So V(Bᵢ) ∩ Int(H₁) ≠ ∅ for some i ∈ [2] and V(B₃₋ᵢ) ∩ Int(H₁) = ∅. By Lemma 5.1.2 with G' := G[V(H₁) ∪ {w}] and G'' := G − Int(H₁) − E(G[T₁ ∪ {w, b₁}]), Int(H₁) ⊆ V(Bᵢ) ∪ S.

Let r₁, r₂ ∈ V(B₃) ∩ (T₁ ∪ {w, b₁}) and t₁, t₂ ∈ V(Bᵢ) ∩ (T₁ ∪ {w, b₁}) such that N(rⱼ) ∩ S ∩ Int(H₁) ≠ ∅ for both j ∈ [2] and subject to this, B₃[r₁, r₂] and Bᵢ[t₁, t₂] are maximal. Let V' = (T₁ ∪ {w, b₁}) \ {r₁, r₂, t₁, t₂}. Then, G' − V' = G[V(B₃[r₁, r₂] ∪ Bᵢ[t₁, t₂])], and (b) follows from Lemma 5.1.3. □

5.2.2 Rungs not in H₁ ∪ Hₘ₊₁

We now consider rungs R in L* such that R ⊆ H₁ ∪ Hₘ₊₁. First, we show that if a rung R is 3-planar then R is planar, except in a very special situation which can occur in at most twice in all rungs of L*.

Corollary 5.2.2. Suppose (R', R'') is a separation of rung R in L* such that |V(R' ∩ R'')| ≤ 3 and ∂R ⊆ V(R'). Then,

(a) (R'', V(R' ∩ R'')) is planar, or
(b) \( \{w', w''\} \cap V(R'' - R') \neq \emptyset \) and \( \{w', w''\} \not\subseteq V(R''), |V(R' \cap R'')| = 3, R'' - R' \neq \emptyset, V(R' - R') \subseteq B_i \cup S \) for some \( i \in [2] \) and there exists \( e = \emptyset \) or \( e \) has one end in \( V(R' \cap R'') \) such that \( (R'' - e, V(R' \cap R'')) \) is planar.

**Proof.** Note that \( V(R'' - R') \cap V(B_3) \neq \emptyset \) as \( G \) os 8-connected. Suppose \( |V(R' \cap R'')| \cap V(B_3) | \geq 2 \). Since \( |V(R' \cap R'')| \leq 3, V(R'' - R') \cap V(B_1 \cup B_2) = \emptyset \). Let \( G' := R'' \) and \( G'' := G[V(G) \setminus V(R'' - R')] \). Then, \( |V(G' \cap G'')| \leq 3 \) and \( V(G' - G'') \cap V(B_1 \cup B_2) = \emptyset \).

By Lemma 5.1.1, \( V(R'' - R') = V(G' - G'') \subseteq S \) and thus, \( R'' - R' \) is a subpath of \( B_3 \).

Now, \( R'' - B_3 = \emptyset \) or is a single vertex, and hence, \( (R'', V(R' \cap R'')) \) is planar and (a) holds.

Now, assume \( |V(R' \cap R'' \cap B_3)| = 1 \). Then, \( \{w', w''\} \cap V(R'' - R') \neq \emptyset \) and \( \{w', w''\} \not\subseteq V(R'') \). Let \( G' := G[V(R'') \cup \{w\}] \) and \( G'' := G - (R'' - R') - E\{G[V(R' \cap R'') \cup \{w\}]\} \).

If \( V(R'' - R') \cap V(B_1 \cup B_2) = \emptyset \), then by Lemma 5.1.1, \( V(R'' - R') = V(G' - G'') \subseteq S \), and thus, \( R'' - R' \) is a subpath of \( B_3 \). Now, \( V(R'' - B_3) \) is a set of two vertices (in \( V(R' \cap R'') \)) and, hence, \( (R'', V(R' \cap R'')) \) is planar.

So \( V(R'' - R') \cap V(B_1 \cup B_2) \neq \emptyset \). Indeed, there exists unique \( i \in [2] \) such that \( V(R'' - R') \cap V(B_i) \neq \emptyset \). Then, \( |V(G' \cap G'')| = |V(R' \cap R'') \cup \{w\}| = 4, V(G' - G'') \cap V(B_i) \neq \emptyset \), and \( V(G' - G'') \cap V(B_{3-i}) = \emptyset \). By Lemma 5.1.2, \( V(G' - G'') = V(R'' - R') \subseteq (V(B_i) \cup S) \setminus \{w\} \). Hence, (b) follows from Lemma 5.1.3.

Next, we make the following observation to be used.

**Observation 5.2.3.** \( (N(w) \cup S) \cap V(P_3(b_1, b_2)) = \emptyset \).

**Proof.** For, suppose there exists \( v \in (N(w) \cup S) \cap V(P_3(b_1, b_2)) \). If \( v \in S \), then let \( Q_1 \) be an \( a_1 - a_2 \) path in the union of \( A_1(B_1 \cup B_2 \cup B_3) - (B_3 - v) \) and \( A_2(P_1 \cup P_2 \cup P_3) \cup P_3(b_1, b_2) \); and if \( v \in N(w) \) then let \( Q_2 \) be an \( a_1 - a_2 \) path in the union of \( A_1(B_1 \cup B_2 \cup B_3) - (B_3 - w), \{wv\} \) and \( A_2(P_1 \cup P_2 \cup P_3) \cup P_3(b_1, b_2) \). Now, \( P_1, P_2 \) and \( Q_1 \) or \( Q_2 \) show that \( (G, a_1, a_2, b_1, b_2) \) is feasible.
Now, we show structures for all rungs \( R \) in \( L^* \) with \( \text{Int}(R) \cap B_j = \emptyset \) for some \( j \in [2] \) in Lemma 5.2.4.

**Lemma 5.2.4.** For any rung \( (R, (a, b, c), (a', b', c')) \) in \( L^* \) with \( R \not\subseteq H_1 \cup H_{m+1} \) and \( \text{Int}(R) \cap B_j \neq \emptyset \) for at most one \( j \in [2] \), \(|\partial R| \leq 5\) and one of the following holds:

(a) \( \text{Int}(R) \subseteq S \), and if \(|\partial R| = 5\) then \( b = b' \), or

(b) \( b = b' \) and, for some \( i \in [2] \), \( V(B_i) \cap \text{Int}(R) \neq \emptyset \) and \( \text{Int}(R) \subseteq V(B_i) \cup S \).

Moreover, let \( r_1, r_2 \in (V(B_i) \cap \partial R) \cup \{w\} \) with \( N_{\text{Int}(R)}(r_j) \cap S \neq \emptyset \) for \( j \in [2] \), and let \( t_1, t_2 \in V(B_i) \cap \partial R \) such that \( B_3[r_1, r_2] \) and \( B_3[t_1, t_2] \) are maximal. Let \( R^* = R + \{w, wv : v \in V(R)\} \) and \( V' = \partial R \setminus (\{r_1, r_2\} \cup V(B_i)) \). Then, there exists \( e \) with \( e = \emptyset \) or \( e \in E(R^*) \) incident to either \( r_1 \) or \( r_2 \), such that, if \( x_jy_j \in E(R^* - V') \setminus (E(B_i \cup B_3) \cup \{e\}) \) with \( x_j \in V(B_i) \) and \( y_j \in V(B_3) \), for \( j \in [2] \), then \( x_1 \preceq x_2 \) implies \( y_1 \preceq y_2 \).

**Proof.** Suppose \( S \cap \text{Int}(R) = \emptyset \). Then \( \text{Int}(R) = \emptyset \) to avoid the cut \( \partial R \cup \{w\} \) in \( G \) (of size \( \leq 7 \)). By (ii) and (iii) of Proposition 2.3.2, if \(|\partial R| = 6 \) or \(|\partial R| = 5 \) and \( b \neq b' \), \( N_{\text{Int}(R)}(b) \neq \emptyset \). So for \( \text{Int}(R) = \emptyset, |\partial R| \leq 5 \) and if \(|\partial R| = 5 \) then \( b = b' \).

Now, assume \( \text{Int}(R) \neq \emptyset \). First, suppose \( V(B_1 \cup B_2) \cap \text{Int}(R) = \emptyset \). Let \( G' := G[V(R) \cup \{w\}] \) and \( G'' := G - \text{Int}(R) - E(G[\partial R]) \). Then, by Lemma 5.1.1, \( \text{Int}(R) = V(G' - G'') \subseteq S \). Assume (a) fails. Then, \(|\partial R| = 5 \) and \( b \neq b' \) or \(|\partial R| = 6 \). By (b) of Lemma 3.2.4, let \( P_a, P_c \) be disjoint paths in \( R - \{b, b'\} \) from \( a, c \) to \( a', c' \), respectively, such that \( R - (P_a \cup P_c) \) is connected and contains \( \{b, b'\} \). Then, \( \text{Int}(R) \subseteq P_a \cup P_c \); otherwise by replacing \( (P_1 \cup P_2) \cap R \) with \( P_a \cup P_c \), we obtain from \( P_1, P_2 \) independent \( b_1-b_2 \) paths \( P_1' \) and \( P_2' \) such that \( G - (P_1' \cup P_2') \) contains an \( a_1-a_2 \) paths, which shows that \( (G, a_1, a_2, b_1, b_2) \) is feasible, a contradiction. By (ii) and (iii) of Proposition 2.3.2, \( N(b) \cap \text{Int}(R) \neq \emptyset \). So by symmetry, assume there exists \( s \in N_{\text{Int}(R)}(b) \cap V(P_c) \). Then, \( \{a, b, s\} \) is a 3-cut in \( R \) separating \( \{a, b, c\} \) from \( \{a', b', c'\} \), contradicting the definition of rung.
So $V(B_i) \cap \text{Int}(R) \neq \emptyset$ for some $i \in [2]$. Hence $V(B_{3-i}) \cap \text{Int}(R) = \emptyset$. By Lemma 5.1.2, with $G' := G[V(R) \cup \{w\}]$ and $G'' := G - \text{Int}(R) - E(G[\partial R])$, we obtain $\text{Int}(R) \subseteq V(B_i) \cup S$. By Observation 5.2.3, we see that $\{b, b'\}$ has no neighbors in $S \cap \text{Int}(R)$. Thus, $\{b, b'\} \cap \{r_1, r_2\} = \emptyset$ by definition of $r_1$ and $r_2$. By Lemma 5.1.3, to prove (b), we need to show $b = b'$. Suppose for a contradiction $b \neq b'$.

First, consider that case when $|\partial R| = 4$. Then, $\{b, b'\} \cap \{t_1, t_2\} \neq \emptyset$ and at least one of the vertices in $\{b, b'\} \cap \{t_1, t_2\}$, say $b = t_1$, has a neighbor $v$ such that $v \in V(B_i) \cap \text{Int}(R)$ and $vb \in E(B_i)$. Let $s \in N(v) \cap \text{Int}(R) \cap S$, which exists since $B_i$ is induced and $G$ is 8-connected. Now, there is an $a_1$-$a_2$ path in the union of $A_1(B_1 \cup B_2 \cup B_3)$, $bvs$, $P_3(b_1, b_2)$ and $A_2(P_1 \cup P_2 \cup P_3)$, which is disjoint from $b_1$-$b_2$ paths $P_1, P_2$. So $(G, a_1, a_2, b_1, b_2)$ is feasible, a contradiction.

Now assume $|\partial R| \geq 5$. Then $|\partial R| = 5$ and $b \neq b'$ or $|\partial R| = 6$. When $|\partial R| = 5$, we may assume $(R - a, b, b'c, c')$ is planar. By (b) of Lemma 3.2.4, let $P_a, P_c$ be disjoint paths in $R - \{b, b'\}$ from $a, c$ to $a', c'$, respectively, such that $R - (P_a \cup P_c)$ is connected and contains $\{b, b'\}$. As before, $\text{Int}(R) \cap S \subseteq V(P_a \cup P_c)$ (as otherwise, $(G, a_1, a_2, b_1, b_2)$ would be feasible) and $N_{\text{Int}(R)}(b) \cap V(P_a \cup P_c) = \emptyset$. Then, by (iii) of Proposition 2.3.2 and by Observation 5.2.3, $N_{\text{Int}(R)}(b) \cap V(B_i) \neq \emptyset$. Let $v \in N_{\text{Int}(R)}(b) \cap V(B_i)$. Since $B_i$ is induced, $|N(v) \cap V(B_i)| = 2$. Since $G$ is 8-connected and $N(v) \subseteq V' \cup V(B_i) \cup \{r_1, r_2\} \cup (\text{Int}(R) \cap S)$, there exists $s \in N(v) \cap \text{Int}(R) \cap S$. By 3-planarity and since $\text{Int}(R) \subseteq V(B_i) \cup S$, $\{a, v, s\}$ or $\{a', v, s\}$ is a 3-cut in $R$ separating $\{a, b, c\}$ from $\{a', b', c'\}$, contradicting the definition of rung. 

\[\square\]
In this chapter, we give a 7-connected graph $G$ with distinct vertices $a_1, a_2, b_1, b_2 \in V(G)$ such that $(G, a_1, a_2, b_1, b_2)$ is infeasible. As shown below, $G$ is obtained by gluing $H$ in Figure 6.1 and $A_1$ in Figure 6.2 together along the $b_1$-$b_2$ path $B_3$.

As shown in Figure 6.1, $B_1, B_2, B_3$ are 3 independent $b_1$-$b_2$ paths. $H$ is the graph with $V(H) = \{a_2, b_1, b_2, w_1, w_0, w_{m+1}, u_i, v_i, w_i : i \in [m]\}$ and $E(H) = \bigcup_{i \in [3]} E(B_i) \cup \{a_3 x : x \in V(B_1 \cup B_2)\} \cup \{u_jv_j, u_jw_j, v_jw_j, u_jv_{j+1}, v_ju_{j+1}, w_ju_{j+1}, w_jv_{j+1} : j \in [m-1]\} \cup \{w_{-1}u_1, w_{-1}v_1, w_0u_1, w_0v_1, w_{m+1}u_m, w_{m+1}v_m\}$, where $m = 7^5 - 11$. 

![Figure 6.1: H with $m = 7(7^4 - 2) + 3 = 7^5 - 11$](image)

![Figure 6.2: A1](image)
We construct $A_1$ as in Figure 6.2, $S_i$’s are the horizontal paths from $b_1, w_0, w_1, w_2$ to $b_2, w_{m+1}, w_m, w_{m-1}, w_{m-2}$, respectively for $i \in [5]$. So $V(A_1) = \{a_1\} \cup \bigcup_{i \in [5]} V(S_i)$. For each $i \in [5]$, let $x^i_j$ be the $j$-th vertex from left to right on $S_i$. For any vertex $x \in V(S_i) \setminus \{w_1, w_{m-1}\}$ where $i \in \{0\} \cup \{4\}$, $|N_{S_{i+1}}(x)| = 7$.

In the following sections, we show that $G$ is infeasible and 7-connected.

6.1 Infeasibility

Suppose $G$ is feasible and let $P$ be the $a_1$-$a_2$ path such that there exist two independent $b_1$-$b_2$ paths $Q_1, Q_2$ in $G - P$. Denote $T_i = \{u_i, v_i, w_i\}$ for $i \in [m]$.

Now, let $w_j \in V(P)$ be such that $V(P[a_2, w_j]) \cap V(B_3) = \emptyset$. Since $P \cap S_i \neq \emptyset$ for all $i \in [5]$, $|(V(Q_1) \cup V(Q_2)) \cap T_k| = 2$ for $k \in \{j, j + 1\}$. Let $x \in N_{P[a_2, w_j]}(w_j)$. Then, $x \in \{u_i, v_i\} \cap \{j, j + 1\}$. Suppose $x \in \{u_j, v_j\}$. Then, $|V(P) \cap T_j| = 2$, and thus, $|(V(Q_1) \cup V(Q_2)) \cap T_j| = 1$, a contradiction.

So $x \in \{u_{j+1}, v_{j+1}\}$. Since $w_{j+1}u_j, w_{j+1}v_j \notin E(G)$, $(V(Q_1) \cup V(Q_2)) \cap T_{j+1} = \{u_{j+1}, v_{j+1}\} \setminus \{x\}$, a contradiction.

Hence, $(G, a_1, a_2, b_1, b_2)$ is indeed infeasible.

6.2 7-connectivity

Suppose not, let $T$ be a minimum cut of $G$. Then, $|T| \leq 6$. Note that with our construction, $V(H) \cap V(A_1) = V(B_3)$. For simplicity, paths will be represented as sequences of vertices with consecutive vertices adjacent. For path $P$ and $u, v \in V(P)$, we denote $uPv$ be the subpath of $P$ from $u$ to $v$. For vertices $u, v, w$ such that $uv, vw$ are edges, we use $uvw$ to denote the $v$-$w$ path of length 2.

Claim 6.2.1. All components of $G - T$ intersect $V(B_3)$.

Proof. Suppose for a contradiction, there exists a component $C$ of $G - T$ such that $V(C) \cap V(B_3) = \emptyset$. Then, $V(C) \subseteq V(H - B_3)$ or $V(C) \subseteq V(A_1 - B_3)$.
First, suppose \( V(C) \subseteq V(A_1 - B_3) \). Then, there exists \( x_j \in V(C) \cap V(S_j) \) for some \( 0 \leq j \leq 4 \). For any \( j \leq i \leq 4 \) and \( x_i \in V(C) \cap V(S_i) \), since \( |N_{S_i+1}(x_i)| = 7 > |T| \), there exists \( x_{i+1} \in V(C) \cap N_{S_i+1}(x_i) \). Hence, there exists \( x_5 \in V(C) \cap V(S_5) \), a contradiction.

So \( C \subseteq V(H - B_3) \). Clearly, \( C \neq H - B_3 \); otherwise \( V(B_3) \subseteq V(T) \), a contradiction.

We claim that \( a_2 \notin C \). Suppose \( a_2 \in C \). Since \( a_2 \in N_H(x) \) for all \( x \in V(B_1 \cup B_2) \), \( |N_C(a_2)| \geq \deg(a_2) - |T| = 2m + 2 - 6 = 2m - 4 \). Since \( N_C(a_2) \setminus T \subseteq V(B_1 \cup B_2) \), \( |N_{B_3}(C)| \geq \frac{|N_C(a_2) \setminus T|}{2} \geq m - 2 > |T| \), a contradiction.

Hence there exists \( x \in V(C) \) and \( y \in V(H - B_3) \setminus V(C) \). Since \( u_iv_i, u_ju_{j+1} \in E(H) \) for all \( i \in [m] \) and \( j \in [m - 1] \), \( \{x,y\} \neq \{u_i,v_i\} \) and \( \{x,y\} \neq \{u_j,u_{j+1}\} \). By symmetry and without loss of generality, \( \{x,y\} = \{u_i,u_j\} \) for some \( 1 \leq i < i + 1 < j \leq m \). But, there exist the following 7 independent \( u_iu_j \) paths in \( G \): \( u_ia_2u_j, u_1B_1u_j, u_iv_i+1B_2v_j-1u_j, u_iw_1B_3w_j-1u_j, u_iB_1b_1s_1B_2B_1u_j, u_iB_2v_1w_1S_2w_{m+1}v_mB_2v_ju_j, u_iw_{i-1}B_3w_0S_3w_mB_3w_ju_j \).

By Claim 6.2.1, there exist \( x, y \in V(B_3) \) such that \( x, y \) belongs to different components of \( G - T \). Note that \( xy \notin E(B_3) \). But we can find 7 independent \( x-y \) paths in all cases as the following, which leads to a contradiction:

**Case 1.** \( x = b_1, y = w_0 \).

The 7 independent \( x-y \) paths are: \( b_1a_2v_2w_1w_0, b_1u_1w_0, b_1v_1w_0, b_1B_3w_0, b_1x_2^2x_3^3S_3w_0, b_1x_3^2x_{3,7}^3x_{3,7_2}^4S_4x_3^4w_0, b_1S_1B_2B_3x_7^5x_2^4w_0 \).

**Case 2.** \( x = b_1, y = w_i \) for \( i \in [m - 1] \).

The 7 independent \( x-y \) paths are: \( b_1a_2u_{i+1}w_i, b_1B_1u_iw_i, b_2B_2v_iw_i, b_1B_3w_i, b_1x_2^2x_3^3S_3w_mB_3w_i, b_1S_1B_2B_3v_i+1w_i \).

**Case 3.** \( x = b_1, y = w_i \) for \( i \in \{m, m + 1\} \).

The 7 independent \( x-y \) paths are: \( b_1a_2w_mw_i, b_1B_1u_{m-1}w_{m-1}B_3w_i, b_1B_2v_mw_i, b_1a_1b_2B_3w_i, b_1S_1x_6^1x_{6,7}^2x_{6,7_2}^3S_3w_m \) or \( b_1S_1x_6^1x_{6,7}^2S_2w_{m+1} \).
\[ b_1 x_3^2 x_3^3 x_7^4 \cdot x_4^4 \cdot x_{7_3-3} w_m \quad \text{or} \quad b_1 x_3^2 x_3^3 x_7^4 \cdot x_4^4 \cdot x_{7_3-3} w_{m+1} \quad \text{or} \quad b_1 x_3^2 x_3^3 x_7^4 \cdot x_4^4 \cdot x_{7(7_3-1)} \cdot x_{7_3-1} w_{m+1}. \]

**Case 4.** \( x = b_1, y = b_2. \)

The 7 independent \( x-y \) paths are: \( b_1 a_2 b_2, b_1 B_1 b_2, b_1 B_2 b_2, b_1 B_3 b_2, b_1 a_0 b_2, b_1 S_1 b_2, b_1 x_2^2 S_2 x_{7_2-1} b_2. \)

**Case 5.** \( x = w_i, y = w_j \) where \( i \in \{-1, 0\} \) and \( j \in [m - 1]. \)

The 7 independent \( x-y \) paths are: \( w_i u_1 a_2 u_{j+1} w_j, w_i v_1 B_2 v_j w_j, w_i B_3 w_1 u_2 B_1 u_j w_j, w_i B_2 b_1 S_1 b_2 v_{j+1} w_j, w_i S_2 w_1 m + 1 B_2 w_j \) or \( w_0 S_3 w_m B_3 w_j, w_i x_3 x_7^4 S_4 w_{m+1} \) or \( w_0 x_4 S_4 w_{m+1} \) where \( \{ s \} = S_4(w_j), w_{-1} x_2 x_4 x_7 S_4 x_{7_2+2} B_2 w_j \) or \( w_0 x_2 w_2 B_3 w_j. \)

**Case 6.** \( x = w_i, y = w_j \) where \( i \in \{-1, 0\} \) and \( j \in \{ m, m + 1 \}. \)

The 7 independent \( x-y \) paths are: \( w_i u_1 a_2 u_{m+w}, w_i B_3 w_1 u_2 B_1 u_{m-1} w_{m+1} B_2 w_j, w_i v_1 B_2 v_{m+w}, w_i B_2 b_1 S_1 b_2 B_3 w_j, \) and the other three paths are \( X_1, X_2, X_3, \) where \( \{ X_1, X_2, X_3 \} \) is one of the following:

\( \{ w_{-1} x_3^3 S_3 w_m, w_{-1} x_3^3 x_7^4 S_4 x_{7_4-2} w_m, w_{-1} x_3 x_7^4 x_7^5 S_5 w_{m-2} x_7^4 w_m \}, \)

\( \{ w_{-1} S_2 w_{m+1}, w_{-1} x_3^3 x_7^3 - 3 w_{m+1}, w_{-1} x_3 x_7^4 S_4 x_{7(7_3-1)} w_{m+1} \}, \)

\( \{ w_0 S_3 w_m, w_0 x_3 S_4 x_{7_4-2} w_m, w_0 x_3 x_7^5 S_5 w_{m-2} x_7^4 w_m \}, \)

\( \{ w_0 x_4 S_4 x_{7(7_3-2)}^3 x_{7_3-2} w_{m+1}, w_0 x_3 x_7^5 S_5 x_{7(7_3-1)-1} x_{7(7_3-1)} x_{7_3-1} w_{m+1} \}. \)

**Case 7.** \( x = w_i, y = b_2 \) where \( i \in \{-1, 0\}. \)

The 7 independent \( x-y \) paths are: \( w_i u_1 a_2 b_2, w_i v_1 B_2 b_2, w_i B_3 w_1 u_2 B_1 b_2, w_i B_3 b_1 a_1 b_2, \) and the other three paths are \( X_1, X_2, X_3, \) where \( \{ X_1, X_2, X_3 \} \) is one of the following:

\( \{ w_{-1} S_2 x_2^2 x_{7_2-2} b_2, w_{-1} x_3 S_3 x_{7(7_2-1)} x_{7_2-1} x_{7_2-1} b_2, w_{-1} x_3 x_7^4 x_{7(7_2-1)} B_3 b_2 \} \) or \( w_0 S_3 x_{7(7_2-2)} x_{7_2-2} b_2, w_0 x_3 S_4 x_{7(7_2-1)} x_{7(7_2-1)} x_{7_2-1} b_2, w_0 x_3 x_7^5 B_3 b_2. \)

**Case 8.** \( x = w_i, y = w_j \) for \( 1 \leq i < i + 1 < j \leq m - 1. \)

The 7 independent \( x-y \) paths are: \( w_i u_i a_2 u_{j+1} w_j, w_i u_{i+1} B_1 u_j w_j, w_i v_{i+1} B_2 v_j w_j, w_i B_3 w_j, w_i v_i B_2 b_1 S_1 b_2 B_2 v_{j+1} w_j, w_i B_3 w_0 S_3 w_m B_3 w_j, w_i S_4 t w_j \) where \( \{ s \} = S_4(w_i) \) and \( \{ t \} = S_4(w_j). \)

**Case 9.** \( x = w_i, y = w_j \) where \( i \in [m - 1] \) and \( j \in \{ m, m + 1 \}. \)
Note that \( \{x, y\} \neq \{w_{m-1}, w_m\} \). The 7 independent \( x-y \) paths are: 
\[ w_iu_ia_2u_mw_j, \quad w_iu_{i+1}B_1u_{m-1}w_{m-1}B_3w_j, \quad w_iv_{i+1}B_2v_mw_j, \quad w_iviB_2b_1b_2B_3w_j, \quad w_iviB_3w_{m-1}S_2w_{m+1}(w_m), \]
\[ w_iB_3w_{m-2}x_{74-1}w_m(w_{m+1}), \quad wiS_4x_{74-2}w_m \text{ or } w_isS_4x_{74}(7^2-2)x_{73-2}w_{m+1} \text{ where } \{s\} = N_S_4(w_i). \]

**Case 10.** \( x = w_i, y = b_2 \) for \( i \in [m - 1] \).

The 7 independent \( x-y \) paths are:
\[ w_iu_ia_2b_2, \quad w_iu_{i+1}B_1b_2, \quad w_iv_{i+1}B_2b_2, \quad w_iB_3b_2, \]
\[ wiB_2b_1S_1b_2, \quad wiB_3w_{m-1}S_2x_{72-3}b_2, \quad wiS_4x_{72(7^2-2)}x_{72-2}^3x_{72-2}^2b_2 \text{ where } \{s\} = N_S_4(w_i). \]

**Case 11.** \( x = w_m, y = b_2 \).

The 7 independent \( x-y \) paths are:
\[ w_mu_mb_2, \quad w_mv_mb_2, \quad w_mB_3b_2, \quad w_mw_{m-1}u_{m-1}a_2b_2, \]
\[ w_mx_{74-1}w_{m-2}B_3b_1S_1b_2, \quad w_mx_{74-2}S_4x_{72(7^2-2)}x_{72-2}^3x_{72-2}^2b_2, \quad w_mS_3x_{7(7^2-1)}x_{72-1}^2b_2. \]

Hence, \( G \) is a 7-connected infeasible example as desired.
REFERENCES


