MATHEMATICAL MODELING OF THE OVERALL FLOTATION DEINKING PROCESS: PROBABILITY OF ATTACHMENT BY SLIDING IMPROVEMENTS

Project F00903

Report 6

to the

MEMBER COMPANIES OF THE INSTITUTE OF PAPER SCIENCE AND TECHNOLOGY

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1 Executive Summary

This report develops a closed-form approximation for the probability of attachment by sliding, $P_{asl}$, which is considered to be one of the most important microprocesses in flotation deinking. The expression presented here is a function of fluid properties, bubble and particle physical properties, and the ratio of initial-to-critical film thickness separating the bubble and particle, $h_0/h_{crit}$, and is believed to be the first closed-form solution of this microprocess. Using the expression developed here, we have shown that, in general, $P_{asl}$ decreases with increasing $h_0/h_{crit}$ and increases with increasing bubble and particle radii and particle density. However, deviations from the general trends are observed where local minima and discontinuities exist.

In the derivation of the expression for $P_{asl}$, we completed a force balance in the radial direction on the particle as it slides over the bubble surface. In previous literature, investigators applied various forces outside the range of applicability and/or neglected certain forces in their derivation of $P_{asl}$. In our derivation, we include the resistive force due to film drainage, the gravitational force, and the flow force between the bubble and particle, and account for both Stokes and non-Stokes flow conditions. To obtain a solution, we assumed the bubble and particle were spherical with $R_p \leq R_B$, which are common assumptions applied in flotation modeling.

This result for $P_{asl}$ is an improvement over what is previously available because it allows for $P_{asl}$ predictions when the fluid properties, bubble and particle physical properties (i.e., bubble and particle size, particle density), and $h_0/h_{crit}$ are known. This microprocess probability will be incorporated into our overall flotation model and future modifications to this microprocess could include the addition of London-Van der Waals dispersion forces, electrostatic forces, and/or long-range hydrophobic attraction forces.
2 Introduction

In the flotation process, swarms of small bubbles rise through agitated liquid tanks with suspended pulp and contaminant particles. The bubbles preferentially attach to naturally or chemically hydrophobized contaminant particles, carrying them to a froth layer at the surface of the agitated tank where they are removed. Flotation cells were originally designed for mineral flotation and now have been specifically designed for flotation deinking with cell designs varying with respect to their geometry, operating parameters, and flow configurations. Despite the many design differences, however, flotation deinking cells all operate on similar principles, and in all modern flotation systems, three separate processes take place in tandem:

1. aeration, whereby air bubbles are introduced into the grey pulp;

2. mixing, where bubbles and stock are intimately mixed to maximize bubble/particle interaction; and

3. separation, where bubbles and bubble/particle aggregates are allowed to separate from the bulk mixture and are skimmed away.

One consistent theme in the modeling of flotation has been to treat the overall process as a multistage probability process; such an approach is directly tied to the idea of treating the overall flotation mechanism as a sequence of microprocesses. A survey of attempts to model the overall flotation process may be found in the authors’ recent reports and papers (1 - 6). As for the sequence of microprocesses themselves, these can generally be ordered as follows:
(a) the approach of a particle to an air bubble with the subsequent collision with, or interception of the particle by the bubble (for particles the size of a typical ink particle, the main focus here is on the zone of possible interaction which is created when the particle approaches to within a sufficiently small distance of the bubble);

(b) the formation of a three-phase contact angle after sliding of the particle along the thin liquid film which separates the particle from the bubble and the subsequent thinning and rupture of this film; and

(c) the stabilization of the bubble/particle aggregate and its transport to the froth layer for removal.

Referring to Fig. 1 we note that for bubble/particle collisions, with respect to particles which are of a size that will be considered in this paper, interception of a particle by a bubble can only take place if the trajectory of the particle is within a streaming tube of radius $R_c$, (the so-called capture radius). The interception probability is then given by $(R_c/R_B)^2$, and the determination of an expression for $R_c$ then depends on whether one models the flow in the flotation cell around a particular bubble as a Stokes flow, a potential flow, or a flow intermediate to these. A survey of earlier results for $P_c$, the probability of capture (or ‘interception’), along with some new exact and approximate expressions for this important microprocess probability, may be found in Report 4 of this project (3); precise meanings for all of the quantities indicated in Fig. 1 will be given below.

Once interception has taken place, a process ensues by which the particle slides along the surface of the thin liquid (disjoining) film surrounding the bubble with a resultant weak surface deformation. In this paper we will develop some new expressions for $P_{ast}$, the probability of adhesion by sliding. In order to predict $P_{ast}$, one must model the particle motion
in the flow field of the bubble as it moves (in an assumed circular path) over the surface of the disjoining film and must also model the drainage and subsequent rupture of that film. Indeed, during the sliding process, the disjoining film (assumed to have some initial thickness $h_0$, as indicated in Fig. 1) may thin down to that critical thickness $h_{crit}$ at which point rupture of the film occurs with the subsequent development of a three-phase contact between the particle, liquid film, and bubble. As the particle slides over the surface of the disjoining film surrounding the bubble, a minimal time $\tau_i$, the so-called induction time, is required in order for the film to thin out to the point where rupture can occur. If $\tau_{sl}$ is the 'sliding time' associated with the motion of the particle over the film's surface, then for attachment to occur we must have $\tau_{sl} \geq \tau_i$.

The motion of the particle over the surface of the disjoining film is, of course, governed by a force balance which will be discussed, in detail, below; the various possible forces which can act on the particle during the sliding process are depicted in Fig. 2. The key to the modeling of the force balance governing $P_{asl}$ is the determination of an appropriate expression for $F_T$, the resistive force which is generated during the drainage of the liquid film surrounding the bubble surface. Expressions for $F_T$ are determined by using the theory of capillary hydrodynamics for thin films; a comprehensive discussion of the situation may be found in (4) along with the derivation of the expression for $F_T$ which will be employed in the present work. We note that the form of $F_T$ used in the model presented below takes into account only the (assumed constant) surface (interfacial) tension of the disjoining film and does not reflect the influence surfactant concentration may have on London-Van der Waals dispersion (as gauged by the Hamaker constant), electrostatic interactions, or long-range hydrophobic attraction forces.
Theories of thin-film capillary hydrodynamics have been widely discussed in the fluid dynamics literature. The computation of the expression for $F_T$, which is used in the present work is based on the analysis presented in Derjaguin et al. (7) and related work by these authors, e.g., Rulev and Dukhin (8), which has been referenced in (4) and summarized by Schulze in (9). Analyses similar to that presented in (7) appear in the work of Ruckenstein and Jain (10), in which variations in the surface concentration of surfactant are discussed, Scheludko et al. (11), and Jain and Ivanov (12). A careful discussion of thin film dynamics which incorporates London-Van der Waals dispersion and examines nonlinear effects on film rupture may be found in Williams and Davis (13). Other discussions related to modeling the thinning out of the disjoining film surrounding a bubble during the sliding process may be found in the recent work of Paulsen et al. (14) which also considers the effect of variable surface tension on film rupture. A discussion of the possible role of long-range hydrophobic attraction forces in the thinning and rupture of disjoining films has been presented in some detail by Paulsen et al. (15), as well as in the recent work by Yoon and Mao (16).

Systems of equations that can be used to model the sliding of a particle over the surface of a disjoining film surrounding a bubble have been presented by Schulze (17, 18). Because of the inclusion of all the forces depicted in Fig. 2 in the analyses presented in (17) and (18), it is not possible to generate a closed form approximate expression for $P_{ast}$ from the equations for $h_p(t)$ and $\varphi_p(t)$ which are presented in these papers; here, $\varphi = \varphi_p(t)$ (see Fig. 3) describes the angular position of the particle at time $t$, where $\varphi_p(0) = \varphi_0 \equiv \varphi_T$ is the touching angle, and $h_p(t)$ is the height (or thickness) of the disjoining film between the particle and bubble at the current position of the particle at time $t$. In the present work, we will argue against the inclusion of some of those forces in the force balance equations that have been employed, e.g., in (18), to derive a system of equations for $(\varphi_p(t), h_p(t))$;
some of these arguments are predicated on the relative magnitudes of the various forces involved, while others involve considerations related to the physical relevance of particular forces, and the expressions employed for them within the context of the actual problem under consideration.

Discussions of the computation (numerically) of $P_{asl}$, which are based on computing $\tau_{sl}$ and $\tau_i$ directly from the equations governing the motion of a particle (over the surface of the disjoining film), and the equations governing the thinning of the disjoining film, respectively, may be found in Dobby and Finch (19) and Schulze (9), as well as in Yoon and Luttrell (20). In (20), what appear to be closed form analytical expressions for $P_{asl}$ are presented for Stokes flow, intermediate flow, and potential flow conditions; these expressions, however, all turn out to depend implicitly on the angle $\varphi_{crit}$, where $\varphi_{crit}$ is the largest value of the touching angle $\varphi_T$, for a given value of $h_0$, such that film rupture will occur at an angle $\varphi = \varphi_{crit} \leq \pi/2$. However, $P_{asl} = \sin^2 \varphi_{crit}$; therefore, knowledge of $\varphi_{crit}$ allows for a direct computation of $P_{asl}$, thus, negating the potential utility of the referenced expressions in (20). Indeed, it is believed by the present authors that the expression for $P_{asl}$ which is derived in this report may represent the first, analytical, closed-form (albeit, approximate) formula for this key flotation microprocess probability that has appeared in the literature.

To begin the analysis (and with reference to Fig. 3), we let $h(x,t)$ be the height of the disjoining film at the position $x = R_B \varphi$ along the bubble surface, where $0 \leq \varphi \leq \pi/2$. Thus,

$$h_p(t) = h(R_B \varphi_p(t), t)$$  \[2.1\]

As already indicated, $\varphi = \varphi_p(t)$ describes the angular position of the particle at time $t \geq 0$ where $\varphi_p(0) = \varphi_0 \equiv \varphi_T$ is the touching angle. The radial position of the particle at time $t$ is given by

$$r_p(t) = R_B + R_p + h_p(t)$$  \[2.2\]
In Section 3 it will be demonstrated that balance of forces in the radial \((r)\) and angular \((\varphi)\) directions leads to a coupled system of nonlinear ordinary differential equations of the form

\[
\begin{align*}
\frac{d\varphi_p}{dt} &= f(\varphi_p(t), h_p(t)) \\
\frac{dh_p}{dt} &= g(\varphi_p(t), h_p(t))
\end{align*}
\]  

[2.3]

with associated initial data

\[
\varphi_p(0) = \varphi_0(\equiv \varphi_T), \quad h_p(0) = h_0
\]  

[2.4]

Systems of the form [2.3], [2.4] have appeared previously in the literature, e.g., Schulze (18), in connection with the computation of \(P_{ast}\).

From [2.3], [2.4] one may, in principle, obtain either of the initial-value problems

\[
\begin{align*}
\frac{dh_p}{d\varphi_p} &= \mathcal{F}(\varphi_p, h_p) \\
\varphi_p(0) &= \varphi_0
\end{align*}
\]  

[2.5]

or

\[
\begin{align*}
\frac{d\varphi_p}{dh_p} &= \mathcal{G}(h_p, \varphi_p) \\
\varphi_p(h_0) &= \varphi_0
\end{align*}
\]  

[2.6]

where we assume that, as a consequence of [2.3], [2.4], \(\varphi_p = \varphi_p(t), \quad h_p = h_p(t)\) so that \(t = \varphi_p^{-1}(\varphi_p(t))\); then \(h_p(\varphi_p^{-1}(\varphi_p(t)))\), i.e., \(h_p\) is some function \(\tilde{h}_p(\varphi_p)\), with \(\tilde{h}_p\) the composition of \(h_p\) and \(\varphi_p^{-1}\). By virtue of [2.3], we have \(\mathcal{F} = g/f\) and \(\mathcal{G} = 1/\mathcal{F}\).

At some thickness \(h_{crit}\) the film can be expected to spontaneously rupture. It has been common in the literature to set

\[
\varphi_{crit} = \varphi(h_{crit})
\]  

[2.7]

and to define

\[
\varphi^*_{crit} = \max\{\varphi_0| \text{ for a given } h_0, \varphi_{crit} \leq \frac{\pi}{2}\}
\]  

[2.8]
Because of the assumed symmetry of the flow around the bubble it can be argued that the inequality in [2.8] may be replaced by equality. If, in [2.6], we set \( \bar{h}_p = h_p - h_0 \) then

\[
\frac{d\varphi_p}{dh_p} = \frac{d\varphi_p}{dh_p} = G(\bar{h}_p + h_0, \varphi_p)
\]

or

\[
\frac{d\varphi_p}{dh_p} = G_0(\bar{h}_p, \varphi_p)
\]

with associated initial data

\[
\varphi_p \bigg|_{h_p=h_0} = \varphi_p \bigg|_{\bar{h}_p=0} = \varphi_0
\]

Thus, without loss of generality, the relevant initial-value problem for \( \varphi_p \), as a function of \( h_p \), may be recast in the form

\[
\begin{cases}
\frac{d\varphi_p}{dh_p} = G_0(h_p, \varphi_p) \\
\varphi_p(0) = \varphi_0
\end{cases}
\]

[2.9]

with \( h_p \) the difference between the actual film thickness, just below the particle at time \( t \), and the initial film thickness \( h_0 \).

Returning to the remarks, above, relative to the definition [2.8] of \( \varphi^{*\text{crit}} \) we note that the solution of the initial-value problem [2.6] will have the form \( \varphi_p = \varphi_p(h_p; h_0, \varphi_0) \) so that

\[
\varphi_{\text{crit}} = \varphi_p(h_{\text{crit}}; h_0, \varphi_0)
\]

[2.10]

Equation [2.10] defines, for fixed \( h_0 \) and \( h_{\text{crit}} \), a mapping

\[
\Phi_p : \varphi_0 \to \varphi_{\text{crit}}
\]

[2.11]

If \( h_{\text{crit}} \) is fixed, but \( h_0 \) varies, then the mapping in [2.11] can be considered to be parametrized by \( h_0 \), i.e.,

\[
\Phi_{p,h_0} : \varphi_0 \to \varphi_{\text{crit}}
\]

[2.12]
Thus, corresponding to [2.8] we would have

$$\varphi_{\text{crit}}^* = \max \left\{ \varphi_0 \left| \Phi_{p,h_0}(\varphi_0) \leq \frac{\pi}{2} \right. \right\}$$  \hspace{1cm} [2.13]

However, for an assumed symmetrical flow around the bubble it may be argued that for a fixed $h_0$, $\Phi_{p,h_0}$ is both continuous and monotone so that, in fact,

$$\varphi_{\text{crit}}^* = \max \left\{ \varphi_0 \left| \text{for a given } h_0, \varphi_{\text{crit}} = \frac{\pi}{2} \right. \right\}$$  \hspace{1cm} [2.14]

By virtue of the continuity and monotonicity of $\Phi_{p,h_0}$, for fixed $h_0$, it follows that $\Phi_{p,h_0}$ is invertible, that $\Phi_{p,h_0}^{-1}$ is continuous, and

$$\varphi_{\text{crit}}^* = \Phi_{p,h_0}^{-1} \left( \frac{\pi}{2} \right)$$  \hspace{1cm} [2.15]

It may, of course, be the case that for a given $h_0$, a $\varphi_{\text{crit}}^*$ satisfying [2.15] does not exist with $0 < \varphi_{\text{crit}}^* < \pi/2$.

Once $\varphi_{\text{crit}}^*$ has been determined, standard arguments (e.g., Heindel (6), Schulze (18), Yoon and Luttrell (20)) lead to the conclusion that

$$P_{\text{asl}} = \sin^2 \varphi_{\text{crit}}^*$$  \hspace{1cm} [2.16]

Alternatively, for some function $F$, such that $F(1) = 1$,

$$P_{\text{asl}} = F(\tau_i/\tau_{sl})$$  \hspace{1cm} [2.17]

We note that the sliding time $\tau_{sl}$ results from modeling a particle's motion, in the flow field of the bubble, as it slides over the bubble surface, while the induction time $\tau_i$ results, in essence, from modeling the drainage/rupture of the disjoining film separating a particle from the bubble.

With this background in place, we now focus on the development of a closed-form approximate solution for the probability of attachment by sliding.
3 The Probability of Attachment by Sliding ($P_{asl}$)

3.1 Derivation of the Differential Equation Governing $P_{asl}$

The probability of adhesion by sliding, $P_{asl}$, depends on (i) $h_{crit}$, (ii) the flow field around the bubble, (iii) the mobility of the bubble surface, (iv) the (relative) particle and bubble sizes $R_p$ and $R_B$, and (v) the bubble rise velocity $v_B$. In the present work, our assumptions will be similar to those made, e.g., in Schulze (18):

A1. The particle executes a quasi-stationary motion and moves in an almost circular path across the bubble surface.

A2. $L >> h_p$ and $dL/dt >> d\bar{h}_p/dt$, where $L$ is the length of the sliding path, while $\bar{h}_p(t)$ is the average film thickness during the sliding process.

A3. Boundary-layer effects around the bubble surface are ignored.

A4. The tangential fluid velocity $u_\varphi$ is given by potential flow for the case of an unretarded bubble surface and by the intermediate flow of Yoon and Luttrell (20) in the case of a completely retarded bubble surface.

In addition, we shall make the assumption that

A5. The direction of a rising bubble is the (+) direction while the direction of a settling particle is the (−) direction; this sign convention will be respected with reference to all vectorial quantities (forces, velocities, and accelerations) which enter the discussion in this section. In particular, $\text{sgn } v_{ps} = -\text{sgn } v_B$, where $v_{ps}$ is the particle settling velocity and, by convention, $v_B > 0$. 
In accordance with assumption A1, we ignore inertial effects in modeling the sliding motion of a particle. The tangential particle velocity $v_{p\phi}^{rel}$ relative to the bubble is, therefore, given by

$$v_{p\phi}^{rel} = \frac{dL}{dt} \simeq r \frac{d\varphi}{dt} = u_\varphi - v_{ps} \sin \varphi$$ \[3.1\]

where $v_{ps}$, the particle settling velocity, is given by

$$v_{ps} = \lambda \tilde{v}_{ps}$$ \[3.2\]

and

$$\tilde{v}_{ps} = -\frac{2R_p^2 \Delta \rho g}{9\mu_\ell}$$ \[3.3a\]

$$\lambda \equiv 6\pi \mu_\ell R_p/f \equiv 18Re_p/\text{Ar}$$ \[3.3b\]

**Remarks:** In actuality, as $r_p(t) = R_B + R_p + h_p(t)$, $\frac{dL}{dt} = r \frac{d\varphi}{dt} + \varphi \frac{dh_p}{dt}$ in [3.1]; however, the second term on the right-hand side of this equation has been dropped in view of assumption A2, above.

We now define the dimensionless particle settling velocity $G$ by

$$G = \frac{v_{ps}}{v_B}$$ \[3.4\]

By virtue of the sign convention laid down in assumption A5, $G < 0$. In [3.3a], $\tilde{v}_{ps}$ is the particle settling velocity which corresponds to the case of Stokesian flow while $f$ is the fluid flow friction factor, $\Delta \rho = \rho_p - \rho_\ell$, is the difference of the particle and fluid densities, $g$ is the acceleration due to gravity, and $\mu_\ell$ is the fluid viscosity; also, $Re_p$ is the particle Reynolds number, $Ar = \frac{\Delta \rho d_p^3 g}{\rho_\ell \nu_\ell^2}$ is the Archimedes number where $d_p = 2R_p$ is the particle diameter and $\nu_\ell$ is the fluid kinematic viscosity, $\nu_\ell = \mu_\ell/\rho_\ell$. For the Stokesian case $\lambda = 1$. In fact, for Stokesian particles it is well known that $f = 6\pi \mu_\ell R_p$ as the drag force is given
by \( \mathbf{F}_d = 6\pi \mu \ell R_p \mathbf{v}_p \) with \( \mathbf{v}_p \) the particle velocity. For non-Stokesian particles we have, in general, \( \mathbf{F}_d = f \mathbf{v}_p \) while the coefficient of drag, \( C_D \), is defined to be

\[
C_D \equiv \frac{|\mathbf{F}_d|}{\frac{1}{2} \rho \ell |\mathbf{v}_p|^2 \pi R_p^2}
\]  

[3.5]

In view of the definition of \( \mathbf{F}_d \) in terms of \( f \),

\[
C_D = \frac{f}{\frac{1}{2} \rho \ell |\mathbf{v}_p| \pi R_p^2}
\]  

[3.6]

In the Stokesian case, with \( f = 6\pi \mu \ell R_p \) and \( C_D = C_D^{st} \), [3.6] yields

\[
C_D^{st} = \frac{12 \nu \ell / R_p |\mathbf{v}_p|}{4}
\]  

[3.7]

If we define, in the usual manner, the Reynolds number for the particle to be

\[
Re_p = \frac{2 R_p |\mathbf{v}_p|}{\nu \ell}
\]  

[3.8]

then [3.7], [3.8] yield the widely known result (e.g., Cheremisinoff (21)) that \( C_D^{st} = 24/Re_p \).

In the general case, however, it is easily seen that [3.6], [3.8] combine so as to yield

\[
C_D = \frac{4 f}{(\pi \mu \ell R_p) Re_p}
\]  

[3.9]

It is generally accepted (e.g. (21)) that \( C_D = C_D^{st} = 24/Re_p \) holds for \( Re_p < 2 \). For the situation in which inertial forces acting on the particle are ignored, the particle velocity corresponds to the particle settling velocity (\( \mathbf{v}_p = \mathbf{v}_{ps} \)). In this case it can be demonstrated (i.e., (21)) that

\[
C_D Re_p^2 = \frac{4}{3} Ar
\]  

[3.10]

For the Stokes' law range (\( Re_p < 2 \)), the use of \( C_D = C_D^{st} = \frac{24}{Re_p} \) in [3.10] leads to \( Re_p = \frac{Ar}{18} \).

In the intermediate or transitional range for which \( 2 < Re_p < 500 \), empirical results must be used; from the results reported in (21) one may infer that

\[
C_D = \frac{18.5}{Re_p^{0.6}}, \quad 2 < Re_p < 500
\]  

[3.11]
the use of which in [3.10] yields

\[ Re_p = 0.152 A_r^{0.715}, \quad 2 < Re_p < 500 \]  

[3.12]

By combining [3.9] with [3.11], solving for \( f \), and then using [3.12] to eliminate \( Re_p \), one may express, (for the case in which inertial effects are ignored) the friction factor \( f \) in terms of the Archimedes number as is implicit in [3.3b].

The sensitivity of [3.3b] to particle radius and density is shown in Figs. 4 and 5. Figure 4 reveals \( \lambda \) as a function of particle radius for selected particle densities, which are in the range of toner particles typically found in flotation deinking. Particles following Stokes flow conditions correspond to \( \lambda = 1 \). As the particle size increases, drag on the particle surface causes \( \lambda \) to decrease monotonically. The discontinuity in \( \lambda \) coincides to the transition point of \( Re_p = 2 \) (i.e., [3.12]). This transition occurs at smaller particle radii for heavier particles. Knowledge of this discontinuity will be important in Section 3.3 when analyzing \( \rho_{ast} \) predictions.

Figure 5 shows \( \lambda \) as a function of particle density for selected particle radii. Again, the deviation from \( \lambda = 1 \) is due to the particle not satisfying Stokes flow conditions. For densities common to particles encountered during flotation deinking (\( \rho_p \approx 1.3 \text{ g/cm}^3 \)), particles follow Stokes flow only when \( R_p < 100 \mu m \). However, for larger particles, typical of toner particles in flotation deinking, the deviation from Stokes flow can have a significant influence on \( \lambda \).

In analyzing particle motion during sliding in (17, 18), Schulze begins by taking as the form of the equation representing balance of forces in the tangential direction the relation

\[ |F_{g\varphi}| - |F_{w\varphi}| = 0 \]  

[3.13]

where \( F_{g\varphi} \) is the tangential component of the weight of the particle while \( F_{w\varphi} \) is the resistive (or drag) force acting on the particle in the vicinity of the bubble surface; this latter force
depends on the nature of the flow field and on the degree of covering of the bubble surface with surfactant molecules. In all that follows we shall denote, by the corresponding scalar, the magnitude of an indicated force, i.e., $|\mathbf{F}_{g\phi}| = F_{g\phi}$, etc. For the force component $F_{w\phi}$ near a completely retarded bubble ($\hat{h}_p/R_p > 10^{-3}$) Goldman et al. (22) have shown that for the case of a Stokes flow about the bubble

$$\tilde{F}_{w\phi} \simeq \frac{16}{5} \pi \mu \dot{V}_p R_p \ln \left( \frac{h_p}{R_p} \right)$$

By modifying the analysis in (22) to cover those cases in which the dimensionless friction factor $\lambda \equiv 6\pi \mu R_p/f \neq 1$, it is easy to deduce that the analysis in (22) leads to

$$F_{w\phi} \simeq \frac{16}{5\lambda} \pi \mu \dot{V}_p R_p \ln \left( \frac{h_p}{R_p} \right)$$

[3.14]

For $F_{g\phi}$ one has

$$F_{g\phi} = \frac{4}{3} \pi R_p^3 \Delta \rho g \sin \phi_p$$

[3.15]

Substitution of [3.14] and [3.15] into [3.13], and subsequent simplification, yields upon solving for $v_{p\phi}^{rel}$

$$v_{p\phi}^{rel} = \frac{v_{p\phi} \sin \phi_p}{\left( \frac{8}{15} \right) \ln(h_p/R_p)}$$

[3.16]

However, for a neutrally buoyant particle, it follows from [3.2] and [3.3a] that $v_{p\phi} = 0$; in this case [3.16], which is a direct consequence of the assumed form of the tangential balance law in Schulze (17, 18), i.e., [3.13] with [3.14], [3.15], yields $v_{p\phi}^{rel} = 0$ which is, of course, nonsense. In fact, if $v_{p\phi} = 0$ then, by virtue of [3.1], $v_{p\phi}^{rel} = u_\phi$ where, for an assumed intermediate flow over the bubble surface, it follows from the work of Yoon and Luttrell (20) that

$$u_\phi = v_B \left( 1 - \frac{3R_B}{4r} - \frac{R_B^3}{4r^3} \right) \sin \phi_p(t)$$

$$+ v_B R_T \left( \frac{R_B}{r} + \frac{R_B^3}{r^3} - \frac{2R_B^4}{r^4} \right) \sin \phi_p(t)$$

[3.17]

$$\equiv v_B g(r) \sin \phi_p(t)$$
with $Re_B^* = \frac{1}{15} Re_B^{0.72}$, and

$$g(r) = \left(1 - \frac{3R_B}{4r} - \frac{R_B^3}{4r^3}\right) + Re_B^* \left(\frac{R_B}{r} + \frac{R_B^3}{r^3} - \frac{2R_B^4}{r^4}\right) [3.18]$$

The contradiction we have arrived at above, in the case where $v_{ps} = 0$, has resulted, of course, from the specious form [3.13] of the tangential force balance employed in (17, 18); the correct form of the force balance in the tangential direction must include the angular component $F_{up}$ of the fluid flow force no matter how small in magnitude this force is in comparison with the other force magnitudes in the balance equation. In deriving an expression for $P_{asi}$ in this section we will not need to make use of a force balance equation for the sliding particle in the tangential direction; as will be seen in the analysis to follow, a judicious use of [3.1], in combination with the appropriate form of the force balance equation in the radial direction, suffices to produce the desired approximate analytical expression for $P_{asi}$. The use of both a radial and a tangential force balance equation would be needed only if we were actually interested in monitoring the evolution in time of both the film thickness and the angular position of the particle.

We now consider the form assumed by the quasi-static force balance in the radial direction; the most general structure for such an equation, under the present set of assumptions relative to the motion of the particle, is

$$-F_{gr} + F_c + F_T - F_{ur} + F_L = 0 [3.19]$$

where $F_{gr}$ is the magnitude of the component of the particle weight in the radial direction, $F_c$ is the magnitude of the centrifugal force exerted on the particle, $F_T$ is the magnitude of the resistive force generated during the drainage of the disjoining film, $F_{ur}$ is the magnitude of the radial component of the flow force acting on the particle in the vicinity of the bubble surface, and $F_L$ is the magnitude of the lift force.
Remarks: Not all of the nomenclature for the various forces displayed in [3.19] is standard, e.g., in Luttrell and Yoon (23), $F_T$ is called the 'resistance force' and is denoted by $F_r$; for Stokes flow in the case of a sphere approaching an infinitely flat plate (23)

$$\begin{align*}
F_r &= 6\pi \mu \ell R_p \nu \frac{\beta}{p} \\
\beta &= R_p/h_p \text{ (the Stokes correction factor)}
\end{align*}$$

[3.20]

It is argued, in (23), that [3.20] remains a good approximation for the case of a particle sliding over the thin film around a bubble if (approximately) $h_p < 0.1R_p$; even so, [3.20] holds only for the special case in which one assumes an immobilized bubble surface. A more general result for $F_r$ (or $F_T$, as we shall label it), which includes [3.20] as a special case, will be given below. Also, in (23), the magnitude of the radial component of the flow force $F_{ur}$, or force pressing a particle against a bubble, is labeled $F_p$; as $F_p = F_d$, the hydrodynamic drag force, in (23)

$$F_p = 6\pi \mu \ell R_p |u_r|$$

[3.21]

Clearly, [3.21] holds only for the case of Stokes flow around the bubble and must be modified for non-Stokesian flow.

The magnitude of the component, in the radial direction, of the particle weight, $F_{gr}$, is easily computed as

$$F_{gr} = \frac{4}{3} \pi R_p^3 \Delta \rho g \cos \varphi_p(t)$$

[3.22]

while the magnitude of the centrifugal force, $F_c$, acting on the particle has the form

$$F_c = \frac{4}{3r} \pi R_p^3 \Delta \rho (v_{r, p}^r)^2$$

[3.23]

with $r = R_p + R_B + h_p(t)$. 
In, e.g., (17) and (18), Schulze has used a classical result of Saffman (24) to express the magnitude of the lift force experienced by a particle as it slides over the disjoining film which separates the particle from a bubble; the result in question has the form

\[
F_L = 3.24 \mu \ell R_p \nu_{rel}^2 \sqrt{\text{Re}_S}
\]  

where \( \nu_{rel}^2 \) is given by [3.1] and \( \text{Re}_S \) is the Reynolds number of shear which is defined to be

\[
\text{Re}_S = \frac{4R_p^2 \partial u_\varphi}{\nu_\ell \partial r}
\]

The result given by [3.24] was derived in (24) for flows at small but nonzero Reynolds numbers \( \text{Re} \) (i.e., spheres moving through a very viscous liquid). Most theoretical attempts to explain lift (see Clift et al. (25) for a survey of such efforts) have focused on flows at small but nonzero \( \text{Re} \) and have used the technique of matched asymptotic expansions in order to obtain approximate results such as [3.24] (as indicated in (25), theoretical or numerical analyses of the lift problem are lacking beyond the near-Stokesian range although they are available at higher Reynolds numbers). The approximate result [3.24] was also derived in (24) by using the technique of matched asymptotic expansions and it is specifically noted there that the derived expression for the magnitude of \( F_L \) becomes invalid for large values of \( \nu_{rel}^2 \) because a key sequence of steps in the analysis requires that

\[
\nu_{rel}^2 << \sqrt{\nu_\ell \frac{\partial u_\varphi}{\partial r}}
\]

By virtue of [3.17], \( u_\varphi = v_B g(r) \sin \varphi \), where \( g(r) \) is defined by [3.18], so that

\[
\frac{\partial u_\varphi}{\partial r} = v_B g'(r) \sin \varphi
\]

with

\[
g'(r) = \frac{3}{4} \left( \frac{R_B}{r^2} + \frac{R_B^3}{r^4} \right) + \text{Re}_B \left( \frac{R_B}{r^2} - \frac{3R_B^3}{r^4} + \frac{8R_B^4}{r^5} \right)
\]
Using [3.1], [3.27], and the fact that \( u_\varphi = v_B g(r) \sin \varphi \) in [3.26], we see that this latter requirement is equivalent to

\[
(v_B g(r) - v_p) \sin \varphi \ll \sqrt{\nu_i v_B g'(r)} \sin \varphi \tag{3.29}
\]

with \( g(r) \) given by [3.18] and \( g'(r) \) by [3.28]. By employing physical and geometrical parameters in ranges that are typical for spherical particles and spherical air bubbles in a flotation deinking system, and estimating \( r \approx R_B + R_p \), Fig. 6 shows that [3.29] is violated except when \( \varphi_p(t) \approx 0 \), which corresponds to a particle approaching a bubble on the stagnation streamline. Therefore, the application of the expression [3.24] for \( F_L \) is invalid under the present circumstances. It should be noted that [3.29] is valid if \( R_p \) is small \((\sim 1 \mu m)\) and \( \rho_p \) is large \((\sim 7 g/cm^3)\), conditions common in mineral flotation.

**Remarks:** In (25) the authors note that it follows from the work of Poe and Acrivos (26) that [3.24] represents an applicable result for \( F_L \) only if \( Re_s << 0.1 \), i.e., only if

\[
\sin \varphi_p \ll \frac{0.025 \nu_t}{v_B R_p} \frac{1}{g'(r)} \tag{3.30}
\]

where we have used [3.25] and [3.27] and where \( g'(r) \) is given by [3.28] for the intermediate flow of Yoon and Luttrell (20). As shown in Fig. 7, [3.30] also places severe restrictions on allowable angular positions for a sliding particle when \( R_p > 5 \mu m \), thus, reinforcing the conclusion that [3.24] is invalid for flotation deinking conditions.

In an interesting paper, Mileva (27) has studied the feasibility of including the result for \( F_L \), which was given by Saffman (24), in the radial force balance which governs the motion of a solid particle in the boundary layer of a rising bubble. In (27) the flow around the bubble is not modeled by the intermediate flow of Yoon and Luttrell (20) but, rather, by the boundary layer part of Moore’s solution (28) for spherical gas bubbles rising steadily through a liquid of low viscosity. It is concluded by the author in (27) that “the major forces...
carrying the particles towards the bubble’s surface are the gravity and the hydrodynamic driving forces · · · if the flow field is modeled by potential (flow), or by Stokes’ equations, the hydrodynamic driving force plays a decisive role and gravity is only a correction factor. The migration force of Saffman’s type is a first-order correction to the other two forces pressing the particle towards the bubble’s surface · · · beside the drag force, the centrifugal force also hampers the mutual approach; it is, however, two orders of magnitude smaller than the first-order correction $F_L$ and can be neglected in the force balance.”

**Remarks:** The work of Mileva (27) supports the philosophy of retaining only $F_T$ and $F_{ur}$ in the quasi-static radial force balance equation [3.19] and further evidence to that effect appears in the work of Luttrell and Yoon (23), and Dobby and Finch (19), which we will cite, below; however, the conclusions in (27) may be limited by the same considerations that have been raised earlier, i.e., the essential applicability, in a specific situation, of the Saffman result [3.24] for $F_L$. As a consequence of the analysis in (28), the flow field employed in (27) satisfies

$$\frac{\partial u_\varphi}{\partial r} \sim \frac{3v_B \sin \varphi}{R_B} \left[ \frac{\sqrt{\pi}}{2} + \mathcal{O} \left( \frac{Y}{2X^{1/2}(\varphi)} \right) \right]$$

[3.31]

where

$$\left\{ \begin{array}{l}
Y = \frac{r - R_B}{R_B} \sqrt{Re_B} \\
X(\varphi) = \frac{2}{3} \csc 4\varphi \left( \frac{2}{3} - \cos \varphi + \frac{1}{3} \cos^3 \varphi \right) \end{array} \right. \quad [3.32]$$

and it is assumed that $\frac{Y}{2X^{1/2}(\varphi)} < 1$. The criteria of Poe and Acrivos (26), namely $Re_S << 0.1$, implies that $\frac{4R_p^2}{\nu_t} \left( \frac{\partial u_\varphi}{\partial r} \right) << 0.1$ where, for the specific flow under consideration, $(\frac{\partial u_\varphi}{\partial r})$ denotes the right-hand side of [3.31] with the term $\mathcal{O} \left( \frac{Y}{2X^{1/2}(\varphi)} \right)$ deleted. Thus, the criteria of (26) implies in this case that we must satisfy

$$\sin \varphi_p \ll \frac{1}{60\sqrt{\pi}} \left( \frac{\nu_t R_B}{v_B R_p^2} \right)$$

[3.33]
If, for example, we were to take

\[
\begin{align*}
\mu_t &= 1.0 \times 10^{-2} \text{ gm/cm} \cdot \text{sec} \\
\rho_t &= 1 \text{ gm/cm}^3 \\
v_B &= 10 \text{ cm/sec} \\
R_p &= 5 \times 10^{-3} \text{ cm} \\
R_B &= 5 \times 10^{-2} \text{ cm}
\end{align*}
\]

then the requirement expressed by [3.33] becomes \( \sin \varphi_p << 1/30\sqrt{\pi} \) (i.e., \( \varphi_p << 1^\circ \)).

In Dobby and Finch (19), the computation of \( P_{asl} \) is accomplished by estimating the values of \( \tau_i \) and \( \tau_{sl} \). The values of \( \tau_{sl} \) and \( \tau_i \) in (19) are computed by ignoring any possible lift force \( F_L \) as well as inertial effects; as the authors of (19) note, “the particle actually experiences a velocity gradient across its dimension. This gradient will impart spin to the particle the consequences of which are ignored here. Also ignored are possible particle bounce and inertia effects; detailed analysis through trajectory calculations indicate that these factors are not very important and this model fitted the experimental results of Schulze and Gottschalk (29).” Finally, in (30), McLaughlin has considered the motion of a small, rigid sphere in a linear shear flow, extending Saffman’s analysis to those asymptotic cases in which the particle Reynolds number based on its slip velocity is comparable with or larger than the square root of the particle Reynolds number based on the velocity gradient. In all the cases considered in (30), the particle Reynolds numbers are assumed to be small compared to unity. It was shown, in (30), that as the Reynolds number based on particle slip velocity becomes larger than the square root of the Reynolds number based on particle shear rate, the magnitude of the inertial migration velocity rapidly decreases to very small values thus suggesting, again, that the lift force in such a situation plays only a minor role in the force balance [3.19].

To determine the hydrodynamic drag force, \( F_{ur} (\equiv F_p \text{ in (23)}) \) must be modified from the form displayed in [3.21], which holds in Stokes flow, to \( F_{ur} = f|u_r| \), where \( f \) is the friction
factor for the flow and $u_r$ is the radial component of the fluid velocity for an intermediate flow in the vicinity of the bubble surface. As we have indicated, in general

$$f = \frac{\pi \mu \ell R_p}{3} \left( \frac{Ar}{Re_p} \right)$$

where $Ar$ is the Archimedes number and $Re_p$ is the particle Reynolds number so that for Stokesian particles $Re_p = \frac{Ar}{18}$ and we recover the fact that $f = 6\pi \mu \ell R_p$. Alternatively, we may introduce the dimensionless friction factor $\lambda \equiv 18Re_p/Ar$ and write that

$$f = \frac{6\pi \mu \ell R_p}{\lambda}$$ \hspace{1cm} [3.34]

Therefore, $F_{ur}$ assumes the form

$$F_{ur} = \frac{6\pi \mu \ell R_p}{\lambda} |u_r|$$ \hspace{1cm} [3.35]

with $u_r$ (as given by Yoon and Luttrell (20) for intermediate flow in the vicinity of the bubble surface) having the form

$$u_r = v_B k(r) \cos \varphi_p$$ \hspace{1cm} [3.36]

with

$$k(r) = -\left\{ \left( 1 - \frac{3R_B}{2r} + \frac{R^3_B}{2r^3} \right) + 2Re_B \left( \frac{R^4_B}{r^4} - \frac{R^3_B}{r^3} - \frac{R^2_B}{r^2} + \frac{R_B}{r} \right) \right\}$$ \hspace{1cm} [3.37]

Combining [3.36] with [3.35] we have

$$F_{ur} = \frac{6\pi \mu \ell R_p v_B}{\lambda} |k(r)| \cos \varphi_p$$ \hspace{1cm} [3.38]

where $k(r)$ is given by [3.37] with $r = R_B + R_p + h_p$.

Comparing the magnitudes of the hydrodynamic drag force, $F_{ur}$, the gravitational force, $F_{gr}$, and the centrifugal force, $F_c$, for conditions common to flotation disinking (Figs. 8-11),
one clearly sees that $F_c$ is several orders of magnitude less than $F_{ur}$ and $F_{gr}$ and can be neglected in the force balance [3.19]. Luttrell and Yoon (23) and Dobby and Finch (19) have assumed that $|F_{gr} - F_c| << F_{ur}$, but as shown in Figs. 8-11, this assumption is not applicable for flotation deinking. It is true that $F_{gr} < F_{ur}$ for most conditions, but the magnitudes of these forces differ by a factor of five or less, unless $p_p \lesssim 1.2$ g/cm$^3$ (Fig. 10). We will include $F_{gr}$ in our analysis. Therefore, [3.19] can be rewritten as

$$F_T = F_{ur} + F_{gr}$$ [3.39]

In this report, the following expression will be employed for the magnitude of the resistive force $F_T$ which is generated during drainage of the disjoining film:

$$F_T = \frac{6\pi \mu_k R^2 v_{pr}}{h_p C_B}$$ [3.40]

where $C_B$ (Schulze’s notation, e.g., (17)) varies between one (for a completely immobilized or rigid bubble surface) and four (for an unrestrained bubble surface) and, thus, characterizes the degree of immobilization of the bubble surface due to the influence of the adsorption layer of surfactant on the surface of the bubble. The expression [3.40] for $F_T$ is a consequence of the theory of capillary hydrodynamics; it has been derived, for the case $C_B = 4$, in Bloom and Heindel (4), as well as, e.g., in Schulze (9), by computing the integral of the disjoining pressure $P_\sigma$ over the bubble surface. In particular, [3.40] is a consequence of working with that part of the capillary pressure which depends on surface tension $\sigma$ only and does not take into consideration either London-Van der Waals dispersion or electrostatic interactions; it also does not include the effects of variable interfacial tension due to possible variations in surfactant levels.

The original derivation of the expression for $F_T$ which is delineated in [3.40] may, as we have already indicated, be found in the paper of Derjaguin et al. (7); in this paper, in fact,
the authors derive for $F_T$ an expression of the form

$$F_T = 6\pi \mu \rho p v_{pr} \gamma_A \left( \frac{h_p}{R_p} \right)$$  \[3.41\]

with the function $\gamma_A \left( \frac{h_p}{R_p} \right)$ having the structure

$$\gamma_A \left( \frac{h_p}{R_p} \right) = \frac{R_p}{4h_p} f(A)$$  \[3.42\]

In [3.42] the parameter $A$ is given by

$$A = \frac{\alpha^2 R \Gamma C}{D \mu \ell}$$  \[3.43\]

where $\alpha = \Gamma_{\infty}/C_{\infty}$ is the ratio of the adsorption and concentration of surfactant at the periphery of the gap between the bubble and particle surfaces, $D$ is the coefficient of bulk diffusion of the surfactant, $R$ is the universal gas constant, and $T$ is the absolute temperature.

The function $f(A)$ has the form

$$f(A) = 1 - \frac{3Ab}{(A+4)(d-b)} \left\{ 1 - \frac{d}{d-b} \ln \frac{d}{b} \right\}$$  \[3.44\]

where $b = 2R_p \rho p$ and

$$d = \left( 8R_p \alpha \frac{D_s}{D} + (A+4)b \right)/(A+4)$$

with $D_s$ the coefficient of surface diffusion of the surfactant. As noted in (7), the quantity $A$ may take on values over the entire range $(0, \infty)$ and characterizes the degree of immobilization of the bubble surface; it is also noted in (7) that

$$\gamma_A \left( \frac{h_p}{R_p} \right) \sim \left( 1 - \frac{3}{A} \right) \frac{R_p}{h_p}, \quad \text{as } A \to \infty$$  \[3.45\]

and

$$\gamma_A \left( \frac{h_p}{R_p} \right) \sim \left\{ 1 + \frac{3}{4} \frac{ADh_p}{D_s} \ln \frac{D_s \alpha}{Dh_p} \right\} \frac{R_p}{4h_p}, \quad \text{as } A \to 0$$  \[3.46\]
For $A \to \infty$ a situation is created in which the bubble surface may be considered to be rigid or completely immobilized; thus, $\gamma_A \left( \frac{h_p}{R_p} \right) \to \frac{R_p}{h_p}$, as $A \to \infty$, and, by virtue of [3.41],

$$F_T \to \frac{6\pi \mu \ell R_p^2 v_{pr}}{h_p}.$$  

On the other hand, for the case in which $A \to 0$, it is noted in (7) that we are dealing with the case of an unrestrained bubble surface; thus, $\gamma_A \left( \frac{h_p}{R_p} \right) \to \frac{R_p}{4h_p}$ and, again, by [3.41],

$$F_T \to \frac{6\pi \mu \ell R_p^2 v_{pr}}{4h_p}.$$  

By combining [3.41] with [3.42] we easily obtain

$$F_T = \frac{6\pi \mu \ell R_p^2 v_{pr}}{4h_p} f(A)$$

which we write in the form

$$F_T = \frac{6\pi \mu \ell R_p^2 v_{pr}}{4h_p} \left\{ \frac{4}{f(A)} \right\} = \frac{6\pi \mu \ell R_p^2 v_{pr}}{4h_p} \left( \frac{4}{f(A)} \right)$$

Comparing [3.47] with [3.40] we see that $C_B = 4/f(A)$ and, indeed, as $A \to \infty$, $f(A) \to 4$ and $C_B \to 1$, while as $A \to 0$, $f(A) \to 1$ and $C_B \to 4$.

Returning to [3.40] and noting that

$$v_{pr} = -\frac{dr}{dt} = -\frac{dh_p}{dt}$$

we have

$$F_T = -\frac{6\pi \mu \ell R_p^2}{C_B} \frac{1}{h_p} \frac{dh_p}{dt} = \frac{6\pi \mu \ell R_p^2}{C_B h_p} \frac{dh_p}{dt}$$

[3.48]

With regard to [3.48] we note that during the course of film thinning, prior to rupture, $\frac{dh_p}{dt} < 0$ so that [3.48] indeed represents the magnitude of $F_T$.

Therefore, as a consequence of [3.39], [3.38], [3.22], and [3.48], quasi-static force balance in the radial direction assumes the form

$$\frac{6\pi \mu \ell R_p^2}{C_B h_p} \frac{dh_p}{dt} = -\left( \frac{6\pi \mu \ell R_p}{\lambda} v_B k(r) + \frac{4}{3} \pi R_p^3 \rho g \right) \cos \phi_p$$

[3.49]
where $k(r)$ is given by [3.37] with $r = R_B + R_p + h_p$. Simplifying [3.49] we obtain the equation

$$\frac{1}{h_p} \frac{dh_p}{dt} = -\frac{C_B}{R_p} \left( \frac{v_B}{\lambda} |k(r)| - \bar{v}_{ps} \right) \cos \varphi_p \tag{3.50}$$

with $\bar{v}_{ps}$ the particle settling velocity for Stokes flow as given by [3.3a]. By virtue of [3.1] and [3.17]

$$r \frac{d\varphi_p}{dt} = (v_B g(r) - v_{ps}) \sin \varphi_p \tag{3.51}$$

where $g(r)$ is given by [3.18]. Therefore,

$$r \frac{d\varphi_p}{dh_p} \frac{dh_p}{dt} = (v_B g(r) - v_{ps}) \sin \varphi_p \tag{3.52}$$

If we now eliminate $\frac{dh_p}{dt}$ between [3.50] and [3.52] we are easily led to the relation

$$\frac{d\varphi_p}{dh_p} = \frac{-(v_B g(r) - v_{ps}) \tan \varphi_p}{\left( \frac{C_B}{R_p} \right) r(r - (R_B + R_p)) \left[ \frac{v_B}{\lambda} |k(r)| - \bar{v}_{ps} \right]} \tag{3.53}$$

where we have used the fact that $h_p = r - (R_B + R_p)$. As a direct consequence of [3.53] it follows that

$$\frac{dh_p}{d\varphi_p} = \frac{1}{(d\varphi_p/dh_p)} < 0 \tag{3.54}$$

indicating that film thinning is proceeding as the particle executes it's sliding motion.

**Remarks:** An approximate equation of the form

$$\frac{dh_p}{d\varphi_p} \approx -R_B \left( G \sqrt{Re_B \sin 2\varphi_p} + \frac{1}{\sqrt{Re_B \sin^2 \varphi_p}} \right) \tag{3.55}$$

was derived by Mileva in (27) (based on the solution in Moore's paper (28) for the steady motion of spherical gas bubbles rising in a small viscosity liquid, and assuming that the particles in question are actually entrained in the boundary layer around the bubble). Analytical consequences of [3.55] were not explored by Mileva in (27).
3.2 The Analytical Expression for $P_{asl}$

In this section we will integrate the differential equation [3.53] so as to obtain a closed-form expression for $P_{asl}$. We begin by making the change of independent variable

$$h_p \rightarrow r = R_B + R_p + h_p$$

in [3.53], in which case $\frac{d\varphi_p}{dh_p} = \frac{d\varphi_p}{dr} \frac{dr}{dh_p} \equiv \frac{d\varphi_p}{dr}$ and [3.53] becomes

$$\cot \varphi_p \frac{d\varphi_p}{dr} = A(r) \quad [3.56]$$

with

$$A(r) = \frac{v_{ps} - v_{Bg}(r)}{(C_B/R_p) r(r - (R_B + R_p)) \left[ \frac{v_B}{\lambda} |k(r)| - \bar{v}_{ps} \right]} \quad [3.57]$$

Integrating [3.56] from $\varphi_0$ to $\varphi_{crit}$, on the left-hand side of the equation, and from $R_B + R_p + h_0$ to $R_B + R_p + h_{crit}$, on the right-hand side of the equation, and using the fact that

$$\int \cot \varphi d\varphi = \ln \sin \varphi + \text{const.}$$

we obtain

$$\ln(\sin \varphi_{crit}) - \ln(\sin \varphi_0) = \int_{R_B + R_p + h_0}^{R_B + R_p + h_{crit}} A(r)dr$$

or

$$\ln \left( \frac{\sin \varphi_{crit}}{\sin \varphi_0} \right) = Q(h_0, h_{crit}, R_B + R_p) \quad [3.58]$$

with

$$Q(h_0, h_{crit}, R_B + R_p) = \int_{R_B + R_p + h_0}^{R_B + R_p + h_{crit}} \frac{v_{ps} - v_{Bg}(r)}{(C_B/R_p) r(r - (R_B + R_p)) \left[ \frac{v_B}{\lambda} |k(r)| - \bar{v}_{ps} \right]} \quad [3.59]$$
Alternatively,
\[ Q(h_0, h_{\text{crit}}, R_B + R_p) = \int_{R_B + R_p + h_{\text{crit}}}^{R_B + R_p + h_0} \frac{(v_B g(r) - \bar{v}_{ps})}{(C_B R_p) (r - (R_B + R_p)) \left( \frac{v_B}{\lambda} |k(r)| - \bar{v}_{ps} \right)} \, dr \quad [3.60] \]

From [3.58] we readily obtain
\[ \varphi_{\text{crit}} = \sin^{-1} \left( (\sin \varphi_0) e^{Q} \right) \quad [3.61] \]
which, for given \( h_0 \) and \( h_{\text{crit}} \), is precisely of the form [2.10]. Therefore, the mapping [2.12] is given explicitly by
\[ \Phi_{p,h_0} (\varphi_0) = \sin^{-1} \left( (\sin \varphi_0) e^{Q} \right) \quad [3.62] \]
and, by virtue of the definition [2.14] of \( \varphi_{\text{crit}}^* \), and the assumed monotonicity and continuity of \( \Phi_{p,h_0} \), it follows as a consequence of [3.61] that
\[ \varphi_{\text{crit}} = \max \left\{ \varphi_0 | \sin^{-1} (e^Q \sin \varphi_0) = \frac{\pi}{2} \right\} \quad [3.63] \]

Remarks: In view of the structure of [3.60], the fact that both \( v_{ps} \leq 0 \) and \( \bar{v}_{ps} \leq 0 \), as well as the fact that \( h_{\text{crit}} \leq h_0 \), we see that \( Q(h_0, h_{\text{crit}}, R_B + R_p) \geq 0 \) with
\[ Q(h_0, h_{\text{crit}}, R_B + R_p) = 0 \iff h_0 = h_{\text{crit}} \quad [3.64] \]

Also, in view of the continuity of \( g(r) \) and \( |k(r)| \) in [3.60], if \( h_{\text{crit}} \approx h_0 \) then \( Q \approx 0 \) in which case, by [3.63], \( \varphi_{\text{crit}}^* \approx \frac{\pi}{2} \) and \( P_{\text{asl}} = \sin^2 \varphi_{\text{crit}}^* \approx 1 \). As \( P_{\text{asl}} = \sin^2 \varphi_{\text{crit}}^* \) and, \( \varphi_{\text{crit}} = \sin^{-1} (e^{-Q}) \), it follows that
\[ P_{\text{asl}} = e^{-2Q} \quad [3.65] \]

Returning to [3.60] and noting that, for the application at hand, the difference \( h_0 - h_{\text{crit}} \) is very small, we may approximate \( Q \) by
\[ Q \approx \frac{v_B g(R_B + R_p + h_{\text{crit}}) - \bar{v}_{ps}}{\left( \frac{C_B}{R_p} \right) (R_B + R_p + h_{\text{crit}}) \left[ \frac{v_B}{\lambda} |k(R_B + R_p + h_{\text{crit}})| - \bar{v}_{ps} \right]} \left( \frac{h_0}{h_{\text{crit}}} - 1 \right) \quad [3.66] \]
Making the further approximation in [3.66] that

$$R_B + R_p + h_{\text{crit}} \simeq R_B + R_p$$  \[3.67\]

and noting that

$$\frac{v_{ps}}{v_B} = G, \quad \frac{\dot{v}_{ps}}{v_B} = \frac{1}{\lambda} G$$

where $G < 0$, we are led to the approximate relation

$$Q \simeq \left( \frac{\lambda}{C_B} \right) \frac{R_p}{R_B + R_p} \left\{ \frac{g(R_B + R_p) - G}{|k(R_B + R_p)| - G} \right\} \left( \frac{h_0}{h_{\text{crit}}} - 1 \right)$$  \[3.68\]

As $P_{ast} \simeq e^{-2Q}$ we may make the following observations based on the structure of $Q$ in [3.68]:

(i) If we fix $h_0$ then as $h_{\text{crit}}$ decreases, $\frac{h_0}{h_{\text{crit}}}$ increases, as does $\left( \frac{h_0}{h_{\text{crit}}} - 1 \right)$; thus, $Q$ also increases in which case $P_{ast}$ decreases.

(ii) As $C_B$ increases (from one to four), $Q$ decreases and, thus, $P_{ast}$ increases. However, this result must be viewed cautiously as $h_{\text{crit}}$ and $h_0$ will both vary with $C_B$.

(iii) As $\lambda$ increases, $Q$ increases and $P_{ast}$ decreases.

Combining [3.65] and [3.68] the final approximate expression for $P_{ast}$, which follows from the analysis described above, is:

$$P_{ast} = \exp \left[ -2 \left( \frac{\lambda}{C_B} \right) \left( \frac{R_p}{R_B + R_p} \right) \left\{ \frac{g(R_B + R_p) - G}{|k(R_B + R_p)| - G} \right\} \left( \frac{h_0}{h_{\text{crit}}} - 1 \right) \right]$$  \[3.69\]

where $g(r)$ is given by [3.18], $k(r)$ by [3.37], $C_B$, $1 \leq C_B \leq 4$, gauges the mobility of the bubble surface, while $\lambda \equiv \frac{18Re_p}{Ar}$ gauges, by virtue of [3.34], the deviation of the friction factor $f$ from the usual friction factor for Stokes flow; for $2 < Re_p < 500$ the empirical relation $Re_p = 0.152Ar^{0.715}$ may be used to compute $\lambda$, as has been explained in Section 3.1.
As a special case of [3.69] we may obtain an approximate expression for $P_{asl}$ which holds for Stokes flow around, say, a bubble which is idealized to have a rigid (or completely immobilized) surface; in this situation $\lambda = 1$, $C_B = 1$, and we set $Re_B = 0$. Then $G = \tilde{G}$ where

$$\tilde{G} = \frac{\bar{v}_{ps}}{v_B}$$

with

$$\bar{v}_{ps} = -\frac{2R_p^2 \Delta \rho g}{9\mu_l}$$

the particle settling velocity for Stokes flow, while $g(r)$, $k(r)$ reduce, respectively, to

$$\bar{g}(r) = 1 - \frac{3R_B}{4r} - \frac{R_B^3}{4r^3}$$

and

$$\bar{k}(r) = -\left(1 - \frac{3R_B}{2r} + \frac{R_B^3}{2r^3}\right)$$

Thus, $P_{asl}$ may, in this case, be approximated by

$$\tilde{P}_{asl} = \exp \left[-2 \left(\frac{R_p}{R_B + R_p}\right) \left\{\frac{\bar{g}(R_B + R_p)}{|\bar{k}(R_B + R_p)| - \bar{G}}\right\} \left(\frac{h_o}{h_{crit}} - 1\right)\right]$$

We now have an approximate expression for $P_{asl}$ for intermediate flow or Stokes flow conditions. Since most flotation deinking systems typically operate in intermediate flow conditions, selected predictions will now be presented for this case.

### 3.3 $P_{asl}$ Predictions

In performing the calculations to predict $P_{asl}$, certain parameters must be known. In our calculations, we assumed that all fluid properties correspond to those of water. Particle density must also be specified, and for most calculations, we assumed $\rho_p = 1.3 \, g/cm^3$, which approximates that of toner particles (31).
The bubble surface mobility coefficient, $C_B$, has been shown to vary between 1 and 4, depending on the concentration of surface active agents in the system. For pure water, $C_B = 4$. However, in deinking operations, the system is contaminated with surface active agents. In this case, the bubble surface is more likely to be rigid, which corresponds to $C_B = 1$. This value was used in our calculations and is a good approximation of a deinking system.

The bubble rise velocity must also be specified for our $P_{asl}$ predictions, and is known to be a function of bubble radius (25). In our calculations, we assumed the system to be water contaminated with surface active agents. The data presented in Clift et al. (i.e., see Fig. 7.3 in (25)) was then curve-fitted to yield the following relationships for bubble rise velocity:

\[ v_B = \begin{cases} 
230d_B^{1.11} & \text{0.0002 m} \leq d_B < 0.001 \text{ m} \\
-9.11 \times 10^7d_B^4 + 2.20 \times 10^6d_B^3 - 1.84 \times 10^4d_B^2 \\
+7.03 \times 10^4d_B + 5.12 \times 10^{-2} & \text{0.001 m} \leq d_B \leq 0.01 \text{ m} 
\end{cases} \]  

[3.73a]

[3.73b]

where $d_B$ is the bubble diameter $(= 2R_B)$ and measured in meters. The transition from one correlation to the other at $d_B = 0.001 \text{ m} (R_B = 0.5 \text{ mm})$ corresponds to a change in bubble shape from spherical to ellipsoidal. Although we assume the bubble in our model to be spherical for all conditions, and recent flash x-ray work reveals the bubble remains spherical at larger equivalent diameters in a fiber suspension (32, 33), we do not know what the bubble rise velocity is in a fiber suspension. Therefore, the correlations determined above from the Clift et al. data will be used as a first approximation.

The bubble and particle radius must also be designated in the $P_{asl}$ calculations. These values were varied between $0.1 \text{ mm} \leq R_B \leq 5 \text{ mm}$ and $1 \text{ µm} \leq R_p \leq 500 \text{ µm}$, which
encompass expected ranges in flotation deinking operations.

Finally, the ratio of initial to critical film thickness, $h_0/h_{\text{crit}}$, must also be defined to determine $P_{\text{asl}}$. Schulze has specified two different equations for $h_{\text{crit}}$ (18, 34), which are functions of the surface tension and contact angle. These equations appear to also be system dependent. Schulze (17) has also indicated that $h_0$ is a function of particle diameter, fluid viscosity, particle settling velocity, surface tension, and surface mobility, and this function depends on the specific system of interest. Rulev and Dukhin (8) concluded that both $h_0$ and $h_{\text{crit}}$ are functions of the surface tension and collision process. They determined that for quasi-elastic collisions ($St = \frac{\rho d^2 v_B}{9 \mu l d_B} > 1$, where $St$ is the Stokes number), they indicate that $h_0/h_{\text{crit}} \approx 3$. For inelastic collisions ($0.1 < St \leq 1$), $h_0/h_{\text{crit}} \approx 4$. Therefore, although we do not know the specific values of $h_0$ or $h_{\text{crit}}$, the ratio $h_0/h_{\text{crit}}$ is typically on the order of 3 to 4. We have completed our calculations for $h_0/h_{\text{crit}} = 2, 3, 4,$ and 5. The sensitivity of $P_{\text{asl}}$ to $h_0/h_{\text{crit}}$ has also been determined.

### 3.3.1 $h_0/h_{\text{crit}}$ Variations

Figure 12 reveals $P_{\text{asl}}$ predictions for a range of $h_0/h_{\text{crit}}$ values for selected particle radii. The bubble radius and particle density were fixed at 0.5 mm and 1.3 g/cm³, respectively. The range $2 \leq h_0/h_{\text{crit}} \leq 5$ is marked in the figure. There is not much difference between the results when $R_p \leq 100 \mu m$, but at $R_p = 200, 300,$ and $500 \mu m$, $P_{\text{asl}}$ increases considerably for a fixed $h_0/h_{\text{crit}}$. This is due to the particle no longer following Stokes flow and deviating from the fluid streamlines. This will be further discussed in Section 3.3.3. As $h_0/h_{\text{crit}}$ increases, there is a sharp decrease in $P_{\text{asl}}$ (note the figure has a log-log scale). If $h_0/h_{\text{crit}} > 5$, $P_{\text{asl}} \leq 0.001$ when $R_p < 100 \mu m$ and particle attachment is unlikely (a chance of less than 1 in 1000). The range specified by Rulev and Dukhin (8), $3 \leq h_0/h_{\text{crit}} \leq 4$, does provide a
reasonable estimate for $P_{asl}$.

Figures 13 and 14 are similar plots with $R_B = 1$ and 5 mm, respectively. The general trends remain the same. The most significant difference is the improvement in $P_{asl}$ as $R_B$ increases, which is clearly apparent for $R_p \leq 100 \mu m$. This is reasonable because with a larger bubble, a fixed particle radius would allow more sliding time for film rupture as it slides over the bubble surface.

Figure 15 results when $h_0/h_{crit}$ is varied with selected bubble radii and a fixed particle radius and density of 50 $\mu m$ and 1.3 g/cm$^3$, respectively. There is little difference in the $P_{asl}$ predictions when $0.1 mm \leq R_B \leq 1 mm$ for a fixed $h_0/h_{crit}$. Very large bubbles result in an increase in $P_{asl}$ because more time is allowed for particle sliding over the bubble surface.

### 3.3.2 Bubble Radius Variations

Calculations were performed over a range of bubble diameters (0.1 mm $\leq R_B \leq 5$ mm) for selected particle radii. Figure 16 shows the resulting $P_{asl}$ predictions for $h_0/h_{crit} = 2$ and $\rho_p = 1.3$ g/cm$^3$. The large filled circles on the $R_p = 200$, 300, and 500 $\mu m$ curves represent conditions when $R_p = R_B$, values where the calculations are terminated (i.e., the model is valid for $R_p \leq R_B$). The results are very similar for $1 \mu m \leq R_p \leq 100 \mu m$, which will be elaborated upon in Section 3.3.3. For these particle radii, there is a local minimum at $R_B \approx 0.5 mm$, then $P_{asl}$ increases with increasing $R_B$. Increasing the particle radius to $R_p = 200$ mm increases $P_{asl}$ and reveals similar trends, but the local minimum is not very pronounced. Further increases in the particle radius to $R_p = 300$ and 500 $\mu m$ reduces the sensitivity of $P_{asl}$ to the bubble radius, with $R_p = 500 \mu m$ revealing $P_{asl}$ is nearly independent of $R_B$ at these particle radii for 0.5 mm $\leq R_B \leq 5$ mm.

Figures 17-19 reveal analogous trends for $h_0/h_{crit} = 3$, 4, and 5, respectively. As pre-
viously shown, increasing \( h_{0}/h_{\text{crit}} \) reduces \( P_{\text{asl}} \), but the effect of \( R_{B} \) remains similar. Even though the scales differ on Figs. 16-19, the most significant difference is that at small bubble radii, larger differences in \( P_{\text{asl}} \) are apparent for \( 1 \ \mu m \leq R_{p} \leq 100 \ \mu m \) as \( h_{0}/h_{\text{crit}} \) increases. The local minimum at \( R_{B} \approx 0.5 \ mm \) is still very apparent. Increasing the bubble radius shows all curves for \( 1 \ \mu m \leq R_{p} \leq 100 \ \mu m \) become insensitive to \( R_{p} \). However, this trend is not observed when \( R_{p} = 200, 300, \) or \( 500 \ \mu m \).

Figure 20, obtained for \( R_{p} = 50 \ \mu m \) and \( \rho_{p} = 1.3 \ \text{g/cm}^{3} \), emphasizes that increasing \( h_{0}/h_{\text{crit}} \) reduces \( P_{\text{asl}} \) but the general trends remain the same. The local minimum at \( R_{B} \approx 0.5 \ mm \) is also more distinct as \( h_{0}/h_{\text{crit}} \) increases. Similar conclusions are obtained at other particle radii.

3.3.3 Particle Radius Variations

The particle radius was varied between \( 1 \ \mu m \leq R_{p} \leq 500 \ \mu m \) to determine its influence on \( P_{\text{asl}} \) at selected bubble radii. All calculations were completed with \( \rho_{p} = 1.3 \ \text{g/cm}^{3} \). Figure 21 shows the predicted \( P_{\text{asl}} \) values for \( h_{0}/h_{\text{crit}} = 2 \). The two large filled circles on the \( R_{B} = 0.1 \) and \( 0.3 \ mm \) curves correspond to \( R_{p} \) values (100 and \( 300 \ \mu m \)) where the calculations were terminated to satisfy \( R_{p} \leq R_{B} \). In all figures with \( R_{p} \) as the abscissa, there is a sharp transition when \( \rho_{p} = 1.3 \ \text{g/cm}^{3} \) at \( R_{p} \approx 112 \ \mu m \). This discontinuity corresponds to the particle radius at which the particle settling velocity transitions from Stokes flow to non-Stokes flow, and is a function of particle density (e.g., see Figs. 4 and 5). This translates into a sharp rise in \( P_{\text{asl}} \) because a particle settling under non-Stokes flow conditions will cross fluid streamlines and increase the rate of film thinning between a bubble and particle, thereby increasing \( P_{\text{asl}} \).

When \( R_{p} \leq 112 \ \mu m \), \( R_{B} = 0.5 \ mm \) results in a minimum \( P_{\text{asl}} \) for all considered bubble
radii. Also, there is only a small effect on $P_{asl}$ when the bubble radius varies from 0.1 mm to 1 mm. For these conditions, increasing $R_p$ increases $P_{asl}$ slightly. When $R_p \geq 112 \mu m$, increasing $R_p$ increases $P_{asl}$ considerably and $P_{asl}$ is nearly independent of $R_B$ for $0.1 \text{ mm} \leq R_B \leq 1 \text{ mm}$. When $R_B = 3$ or 5 mm, $P_{asl}$ is much larger, but the transition at $R_p \approx 112 \mu m$ is still observed. For these bubble radii (and particle density), $P_{asl}$ is approximately independent of $R_p$ for $R_p \leq 112 \mu m$ and increases with increasing $R_p$ for $R_p \geq 112 \mu m$. For all bubble radii considered, $P_{asl}$ asymptotes to the same value at $R_p = 500 \mu m$. Figures 22-24 show similar trends when $h_0/h_{crit} = 3, 4,$ and 5, respectively.

Figure 25, generated for $R_B = 0.5 \text{ mm}$ and $\rho_p = 1.3 \text{ g/cm}^3$, reveals that increasing $h_0/h_{crit}$ decreases $P_{asl}$ but results in the same general trends. Analogous results are obtained for other values of $R_B$. Note that the exact particle radius at which the transition occurs in these figures ($R_p \approx 112 \mu m$) is a function of particle density (i.e., see Figs. 4 and 5). If the particle density is decreased, the transition would be delayed to a larger particle radius. In contrast, increasing the particle density would cause the transition to occur at a smaller particle radius.

### 3.3.4 Particle Density Variations

Particle density affects $P_{asl}$ through the dimensionless particle settling velocity, $G$, and through the dimensionless friction factor, $\lambda$. To determine the sensitivity of $P_{asl}$ to particle density, calculations were completed for $1 \text{ g/cm}^3 \leq \rho_p \leq 3 \text{ g/cm}^3$ with $h_0/h_{crit} = 4$. This particle density range encompasses particles expected in flotation deinking applications, where toner particles typically have a density in the range of $1.1 - 1.6 \text{ g/cm}^3$ (31).

Figure 26 reveals $P_{asl}$ as a functions of particle density for selected particle radii at $R_B = 0.5 \text{ mm}$ and $h_0/h_{crit} = 4$. For all particle radii, increasing the particle density increases
$P_{asl}$, which is reasonable because a heavier particle will cross fluid streamlines and increase the thinning rate of the liquid film separating the particle from the bubble. For $R_p = 1$, 10, and 50 $\mu m$, this increase is smooth and continuous because the particle is settling under Stokes flow conditions for all particle densities considered. When $R_p = 100$ $\mu m$, there is a discontinuity in the $P_{asl}$ predictions at $\rho_p \approx 1.4$ g/cm$^3$, which corresponds to the particle deviation from Stokes flow (i.e., see Fig. 5 - note the scales are different). This deviation causes $P_{asl}$ to increase more than if the particle was settling under Stokes flow conditions. At $R_p = 200, 300, \text{and } 500$ $\mu m$, the deviation from Stokes flow occurs at low particle densities ($\rho_p < 1.1$ g/cm$^3$), which causes $P_{asl}$ to increase substantially as the particle density increases, and then asymptote to a constant value depending on $R_p$. This corresponds to $\lambda$ asymptoting to a constant as a function of $R_p$ in Fig. 5. Figures 27 and 28 show similar trends for $R_B = 1$ and 5 mm, respectively. The most significant difference is that at $R_B = 5$ mm (Fig. 28), $P_{asl}$ is approximately independent of $R_p$ for $1 \mu m \leq R_p \leq 100 \mu m$ for all considered values of $\rho_p$ because the flow conditions are Stokssian except when $R_p = 100 \mu m$ and $\rho_p > 1.4$ g/cm$^3$; in this case, the deviation from Stokes flow is not too large for these considered particle densities.

Figures 29-31 show $P_{asl}$ as a function of particle density for selected bubble radii at $R_p = 50, 100, \text{and } 200 \mu m$, respectively, and $h_0/h_{crit} = 4$. All figures show that $P_{asl}$ is minimized when $R_B \approx 0.5$ mm for all but very small particle densities. Additionally, as $\rho_p \rightarrow 1$ when $\rho_p > 1$ g/cm$^3$, $0.002 < P_{asl} < 0.008$ and $P_{asl}$ is not that sensitive to $R_B$. This implies that neutrally buoyant particles will not effectively attach to bubbles by the sliding process and will not be successfully removed by flotation.
4 Conclusions

A closed-form approximate expression for the probability of attachment by sliding has been developed in this report. This expression is the first of its kind, accounts for the effect of interfacial tension on the disjoining film for both Stokes and non-Stokes flow conditions, and assumes that the bubble and particle are spherical with \( R_p \leq R_B \). Future extensions of this approximation could include London-Van der Waals dispersion forces, electrostatic interactions, or long-range hydrophobic attraction forces.

The model can be used to determine \( P_{asl} \) as a function of fluid properties, bubble and particle physical properties, and the ratio \( h_0/h_{crit} \) (which has been shown in the literature to vary between 3 and 4). Therefore, the probability of this important microprocess can be determined from a rather simple expression when these flotation characteristics are known. This is a large improvement over what was previously available in the literature for \( P_{asl} \). This expression will be incorporated into our overall flotation model.

Selected \( P_{asl} \) predictions have also been presented in this report using our new expression and encompassed the ranges \( 2 \leq h_0/h_{crit} \leq 5 \), \( 0.1 \text{ mm} \leq R_B \leq 5 \text{ mm} \), \( 1 \mu m \leq R_p \leq 500 \mu m \), and \( 1 \text{ g/cm}^3 < \rho_p \leq 3 \text{ g/cm}^3 \). In general, \( P_{asl} \) decreases with increasing \( h_0/h_{crit} \) and increases with increasing \( R_B \), \( R_p \), and \( \rho_p \). Deviations in these general trends produce local minima in the predictions. The particle settling velocity was shown to be an important parameter and, when deviations from Stokes flow exist, a discontinuity results, with \( P_{asl} \) increasing considerably.
5 References


6 Nomenclature

$A = \frac{\alpha^2 RTC_\infty}{D \mu_c}$

$Ar$ - Archimedes number

$A(r) = \frac{v_{ps} - v_B \theta(r)}{\left(\frac{C_B}{R_p}\right) \tau(r - (R_D + R_p)) \left[\frac{v_B}{A} |k(r)| - v_{ps}\right]}$

$b = (2R_p h_p)$

$C_B$ - measure of bubble surface mobility

$C_{\infty}$ - concentration of surfactant at the periphery of the gap between the bubble and particle surface

$C_D$ - coefficient of drag (for a particle)

$C_D^S$ - coefficient of drag in Stokes flow

$D$ - coefficient of bulk diffusion of the surfactant

$D_s$ - coefficient of surface diffusion of the surfactant

$d = \left(\left[8R_p \alpha \frac{D_s}{D} + (A + 4)b\right] / [A + 4]\right)$

$d_B$ - bubble diameter

$d_p$ - particle diameter

$F_c$ - magnitude of the centrifugal force exerted on a particle

$F_d$ - magnitude of the hydrodynamic drag force acting on a particle

$F_g$ - magnitude of the gravitational force acting on a particle

$F_{gr}$ - magnitude of the radial component of the particle weight

$F_{gt}$ - magnitude of the tangential component of the particle weight

$F_p$ - notation for $F_{ur}$ in (23)

$F_r$ - notation for $F_T$ in (23)

$F_{ur}$ - magnitude of the flow force acting on a particle in the vicinity of the bubble surface
$F_{w\varphi}$ - magnitude of the drag force acting on the particle in the vicinity of the bubble surface

$F_L$ - magnitude of the lift force acting on a particle

$F_T$ - magnitude of the resistive force generated during the drainage of the disjoining film

$\tilde{F}_{w\varphi}$ - magnitude of the force $F_{w\varphi}$ for the case of a Stokes flow about the bubble

$F_d$ - drag force acting on a particle

$f$ - friction factor

$G$ - dimensionless particle settling velocity $\left(= \lambda \frac{\bar{v}_{ps}}{v_B} \equiv \frac{v_{ps}}{v_B}\right)$

$g$ - acceleration due to gravity

$g(r) = \left(\left[1 - \frac{3R_B}{4r} - \frac{R_B^3}{4r^3}\right] + Re_B \left[\frac{R_B}{r} + \frac{R_B^3}{r^3} - 2R_B^2\right]\right)$

$g'(r) = \left(\frac{3}{4} \left[\frac{R_B}{r^2} + \frac{R_B^3}{r^4}\right] + Re_B^* \left[-\frac{R_B}{r^2} - \frac{3R_B^3}{r^4} + 8R_B^4\right]\right)$

$h(x,t)$ - height of the disjoining film at the position $x = R_B\varphi$ along the bubble surface

$h_{crit}$ - critical film thickness at rupture

$h_0$ - ($h_p(0)$), disjoining film thickness at the instant of contact with the particle

$h_p(t)$ - thickness of the disjoining film below the current position of the particle at time $t$

$\tilde{h}_p$ - ($h_p(t) - h_0$)

$\tilde{h}_p$ - composition of $h_p$ and $\varphi_p^{-1}$

$k(r) = \left(- \left[\left[1 - \frac{3R_B}{2r} + \frac{R_B^3}{2r^3}\right] + 2Re_B^* \left[\frac{R_B^4}{r^4} - \frac{R_B^3}{r^3} - \frac{R_B^2}{r^2} + \frac{R_B}{r}\right]\right]\right)$

$L$ - length of the particle sliding path

$O$ - order symbol

$P_{asl}$ - microprocess probability of adhesion by sliding

$P_c$ - microprocess probability of capture of a particle by a bubble
\( P_\omega \) - disjoining pressure due to surface tension
\( R \) - universal gas constant
\( R_c \) - limiting capture radius of a streaming tube
\( Re_p \) - particle Reynolds number
\( Re_S \) - Reynolds number of shear
\( Re_B \) - bubble Reynolds number
\( Re_B^* \) - \( \left( \frac{1}{15} Re_B^{0.72} \right) \)
\( R_p \) - particle radius
\( R_B \) - bubble radius
\( R_T \) - touching radius from stagnation streamline (Figs. 1 and 3)
\( r \) - radial distance of a particle from a bubble
\( r_p(t) \) - \( (R_B + R_p + h_p(t)) \), the radial position of the particle at time \( t \)
\( T \) - absolute temperature
\( t \) - time
\( u_r \) - radial component of the fluid velocity
\( u_\phi \) - angular component of the fluid velocity
\( v_p \) - particle velocity
\( v_{p\text{rel}} \) - \( u_\phi - v_{ps} \sin \varphi_p \)
\( v_{pr} \) - radial component of the particle velocity
\( v_{p\varphi} \) - tangential component of the particle velocity
\( v_{ps} \) - particle settling velocity
\( v_{ps} \) - magnitude of the particle settling velocity
\( \bar{v}_{ps} \) - magnitude of the particle settling velocity in Stokes flow
\( v_B \) - bubble rise velocity
\[ X(\varphi) = \left( \frac{2}{3} \csc 4\varphi \left[ \frac{2}{3} - \cos \varphi + \frac{1}{3} \cos^3 \varphi \right] \right) \]

\[ Y = \left( \frac{r - R_B}{R_B} \sqrt{R_B} \right) \]

\[ \alpha = \frac{\Gamma_{\infty}}{C_{\infty}} \]

\[ \beta = \frac{R_p}{h_p}, \text{ the Stokes correction factor} \]

\[ \Delta \rho = (\rho_p - \rho_t) \]

\[ f(A) = \left( 1 - \frac{3Ab}{(A + 4)(d - b)} \left\{ 1 - \frac{d}{d - b} \ln \frac{d}{b} \right\} \right) \]

\[ \gamma_A = \text{function of } \frac{h_p}{R_p}, \text{ parametrized by } A, \text{ which serves to determine } F_T \]

\[ \lambda = \text{dimensionless friction factor (} 6\pi \mu_t R_p / f \) \]

\[ \mu_t = \text{fluid viscosity} \]

\[ \nu_t = \text{kinematic fluid viscosity} \]

\[ \rho_t = \text{fluid density} \]

\[ \rho_p = \text{particle density} \]

\[ \sigma = \text{surface tension} \]

\[ \tau_i = \text{fluid film induction time} \]

\[ \tau_{sl} = \text{particle sliding time} \]

\[ \varphi_p(t) = \text{angular position of the particle at time } t \]

\[ \varphi_T = \text{particle touching angle (} = \varphi_p(0) = \varphi_0 \) \]

\[ \varphi_{\text{crit}} = \text{angular position at which film rupture occurs} \]

\[ \varphi^{*}_{\text{crit}} = \text{largest value of } \varphi_T, \text{ for a given } h_0, \text{ such that film rupture will occur at an angle } \varphi = \varphi_{\text{crit}} \leq \pi/2 \]

\[ \Gamma_{\infty} = \text{adsorption rate of surfactant at the periphery of the gap between the bubble and particle surface} \]

\[ \Phi_p = \text{mapping of } \varphi_0 \rightarrow \varphi_{\text{crit}} \text{ at an arbitrary value of } h_0 \]
\( \Phi_{p,h_0} \) - \( \Phi_p \) at a fixed value of \( h_0 \)

\( \Phi_{p,h_0}^{-1} \) - inverse map to \( \Phi_{p,h_0} \)
7 Figures

Figure 1: Schematic representation of a bubble intercepting a particle.
gravitational force (apparent particle weight)

centrifugal force

lift force

resistive force during film drainage

flow force

resistive or drag force

Figure 2: The forces acting on a particle as it slides over a bubble surface.
Figure 3: Physical interpretation of $\varphi_p(t)$ and $h_p(t)$.
Figure 4: Effect of particle radius, $R_p$, on the dimensionless particle friction factor, $\lambda$, for selected particle densities, $\rho_p$. 

\( \rho_p = 1.1 \text{ g/cm}^3 \)
\( \rho_p = 1.3 \text{ g/cm}^3 \)
\( \rho_p = 1.5 \text{ g/cm}^3 \)
Figure 5: Effect of particle density, $\rho_p$, on the dimensionless particle friction factor, $\lambda$, for selected particle radii, $R_p$. 
Figure 6: Comparison of the terms in [3.29].
Figure 7: Comparison of the terms in [3.30].
Figure 8: The magnitudes of various forces as a function of particle radius, $R_p$. 

$R_B = 0.5 \text{ mm}$

$\rho_p = 1.3 \text{ g/cm}^3$

$\varphi_p (t) = 45^\circ$
Figure 9: The magnitudes of various forces as a function of bubble radius, $R_B$. 

$R_p = 50 \mu m$

$\rho_p = 1.3 \text{ g/cm}^3$

$\varphi_p(t) = 45^\circ$
Figure 10: The magnitudes of various forces as a function of particle density, $\rho_p$. 

- $|F_{ur}|$ 
- $|F_{gr}|$ 
- $|F_c|$ 

$R_p = 50 \, \mu m$  
$R_B = 0.5 \, mm$  
$\varphi_p(t) = 45^\circ$
Figure 11: The magnitudes of various forces as a function of angular position of the particle on the bubble surface, \( \varphi_p(t) \).
Figure 12: \( P_{asl} \) as a function of \( h_0/h_{crit} \) for selected particle radii, \( R_p \), with \( R_B = 0.5 \text{ mm} \) and \( \rho_p = 1.3 \text{ g/cm}^3 \).
Figure 13: $P_{asl}$ as a function of $h_0/h_{crit}$ for selected particle radii, $R_p$, with $R_B = 1 \text{ mm}$ and $\rho_p = 1.3 \text{ g/cm}^3$. 
Figure 14: $P_{asl}$ as a function of $h_0/h_{crit}$ for selected particle radii, $R_p$, with $R_B = 5 \text{ mm}$ and $\rho_p = 1.3 \text{ g/cm}^3$. 
Figure 15: $P_{asl}$ as a function of $h_0/h_{crit}$ for selected bubble radii, $R_B$, with $R_p = 50 \ \mu m$ and $\rho_p = 1.3 \ \text{g/cm}^3$. 
Figure 16: $P_{asl}$ as a function of bubble radius, $R_B$, for selected particle radii, $R_p$, with $h_0/h_{crit} = 2$ and $\rho_p = 1.3 \text{ g/cm}^3$. 

$h_0/h_{crit} = 2$

$\rho_p = 1.3 \text{ g/cm}^3$
Figure 17: $P_{asl}$ as a function of bubble radius, $R_B$, for selected particle radii, $R_p$, with $h_0/h_{crit} = 3$ and $\rho_p = 1.3 \text{ g/cm}^3$. 
Figure 18: $P_{asl}$ as a function of bubble radius, $R_B$, for selected particle radii, $R_p$, with $h_0/h_{crit} = 4$ and $\rho_p = 1.3 \text{ g/cm}^3$. 
Figure 19: $P_{asl}$ as a function of bubble radius, $R_B$, for selected particle radii, $R_p$, with $h_0/h_{crit} = 5$ and $\rho_p = 1.3 \text{ g/cm}^3$. 
Figure 20: $P_{asl}$ as a function of bubble radius, $R_B$, for selected $h_0/h_{crit}$, with $R_p = 50 \mu m$ and $\rho_p = 1.3 \text{ g/cm}^3$.
Figure 21: $P_{asl}$ as a function of particle radius, $R_p$, for selected bubble radii, $R_B$, with $h_0/h_{crit} = 2$ and $\rho_p = 1.3 \text{ g/cm}^3$. 
Figure 22: $P_{as}l$ as a function of particle radius, $R_p$, for selected bubble radii, $R_B$, with $h_0/h_{crit} = 3$ and $\rho_p = 1.3 \text{ g/cm}^3$. 

$\rho_p = 1.3 \text{ g/cm}^3$
Figure 23: $P_{ast}$ as a function of particle radius, $R_p$, for selected bubble radii, $R_B$, with $h_0/h_{crit} = 4$ and $\rho_p = 1.3 \text{ g/cm}^3$. 
Figure 24: $P_{asl}$ as a function of particle radius, $R_p$, for selected bubble radii, $R_B$, with $h_0/h_{crit} = 5$ and $\rho_p = 1.3 \text{ g/cm}^3$. 

$R_B = 0.5 \text{ mm}$

$R_B = 0.7 \text{ mm}$

$R_B = 0.1 \text{ mm}$

$R_B = 3 \text{ mm}$

$R_B = 5 \text{ mm}$
Figure 25: $P_{asl}$ as a function of particle radius, $R_p$, for selected $h_0/h_{crit}$, with $R_B = 0.5 \text{ mm}$ and $\rho_p = 1.3 \text{ g/cm}^3$. 
Figure 26: $P_{asl}$ as a function of particle density, $\rho_p$, for selected particle radii, $R_p$, with $R_B = 0.5 \text{ mm}$ and $h_0/h_{crit} = 4$. 
Figure 27: $P_{asl}$ as a function of particle density, $\rho_p$, for selected particle radii, $R_p$, with $R_B = 1 \text{ mm}$ and $h_0/h_{crit} = 4$. 
Figure 28: $P_{asl}$ as a function density, $\rho_p$, for selected particle radii, $R_p$, with $R_B = 5$ mm and $h_0/h_{crit} = 4$. 

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Figure 29: $P_{asl}$ as a function of particle density, $\rho_p$, for selected bubble radii, $R_B$, with $R_p = 50 \mu m$ and $h_0/h_{crit} = 4$. 
Figure 30: $P_{asl}$ as a function of particle density, $\rho_p$, for selected bubble radii, $R_B$, with $R_p = 100 \mu m$ and $h_0/h_{crit} = 4$. 
Figure 31: $P_{asl}$ as a function of particle density, $\rho_p$, for selected bubble radii, $R_B$, with $R_p = 200 \, \mu m$ and $h_0/h_{crit} = 4$. 