Bifurcations, Normal Forms and their Applications

A Thesis
Presented to
The Academic Faculty

by

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In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

School of Mathematics
Georgia Institute of Technology
August 2005

Bifurcations, Normal Forms and their Applications

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DEDICATION

For my family and friends

ACKNOWLEDGEMENTS

There are a number of people I would like to acknowledge in connection with the successful completion of this work. I would like to take this opportunity to express my appreciation to all of them who have influenced, stimulated, expedited, and warmly supported my work in various ways. Specifically, I would like to extend special thanks to the follwoing individuals:

First and foremost, it is an immense pleasure to express my deep and sincere gratitude to my advisor, Dr. Yingfei Yi, for his guidance, assistance, encouragement, and hearty support in all the phases of my doctoral program at Georgia Tech.

I am greatly indebted to Dr. Zhilan Feng at Purdue and Dr. Xaing Zhang at Shanghai Jiaotong University for their stimulating discussions.

I am thankful to Dr. Shui-Nee Chow, Dr. Luca Dieci, Dr. Christopher Klausmeier and Dr. Yang Wang for taking time to serve on my dissertation reading committee and for their useful comments and suggestions.

I thank Dr. Alfred Andrew, Ms. Rena Brakebill, Dr. William Green, Ms. Cathy Jacobson, Dr. Evans Harrell, Dr. Konstantin Mischaikow, Dr. Liang Peng and all the staffs in School of Math for their support and assistance during all the years at Georgia Tech.

I am grateful to Marcio Gameiro, Luis Hernandez-Urena, Wen Jiang, Hwa Kil Kim, Rafal Komendarczyk, Yongfen Li, , Jose Sanchez, Zixia Song, Suleyman Ulusoy, Jorge Viveros Rogel, Ying Wang and Hua Xu for their friendship at Georgia Tech.

While I was writing my thesis, I was hired as an intern at Best Software/Sage group. Here especially I want to show my gratitude to my supervisor, Mr. Jim Needle and VP, Mr. Ian Oxman, for their always understanding and support when I took days off or put their assignments aside to work on my thesis.

At last, but by no means the least, my heart-felt appreciations go to my parents and my wife for their ”time-invariant” support.

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SUMMARY

The thesis consists of two loosely connected parts.

The first part is a study of an ecological model with one herbivore and N plants. The system has a new type of functional response due to the speculation that the plants compete with each other and have different levels of toxin which inhibit the herbivore's ability to eat up to a certain amount. We first derive the model mathematically and then investigate, both analytically and numerically, the possible dynamics for this model, including the bifurcation and chaos. We also discuss the conditions under which all the species can coexist.

The second part is a study in the normal form theory. In particular, we study the relations between the normal forms and the first integrals in analytic vector fields. We are able to generalize one of Poincaré's classical results on the nonexistence of first integrals in an autonomous system. To be precise, we find a formula which can determine the maximum number of the first integrals in an analytic quasi-periodic vector field. Then in the space of analytic autonomous systems in $\mathbb{C}^{2n}$ with exactly n resonances and n functionally independent first integrals, we obtain some results related to the convergence and generic divergence of the normalizations. Lastly we give a new proof of the necessary and sufficient conditions for a planar Hamiltonian system to have an isochronous center.

CHAPTER 1

OUTLINE OF THE THESIS

The thesis consists of two loosely connected parts.

The first part concerns an ecological model with one herbivore and N plants. The system has a new type of functional response due to the speculation that the plants compete with each other and have different levels of toxin which inhibit the herbivore's ability to eat up to a certain amount. The functional response has a new feature that for some regions in the parameter space, it is of Holling type II, i.e., hyperbolic and monotonically increasing; however, for some other regions, it is of Holling type IV, i.e., nonmonotonic. We will explain why a system can exhibit two types of functional response in an ecological sense. After constructing the model, we study, both analytically and numerically, possible dynamics for this model, including bifurcation and chaos. We will mainly focus on the study of the 3D model. Mathematically, planar systems are better understood due to the availability of the Poincaré-Bendixson Theorem, while the study of 3D dynamics is much more challenging. Ecologically, it is rare that a system is described only by two species, while a 3D model is the first step to implement the objective for which we try to model the interaction of more species. A fundamental problem for such ecological model is whether the species can coexist, because if they can't, then they are not observable and therefore the model has less ecological meaning. In this thesis, we will give a co-existence result by using the ergodic decomposition theorem in smooth dynamical systems.

The second part is a study in normal form theory. In particular, we study the relations between the normal forms and first integrals for analytic vector fields. Poincaré had a well-known result about the nonexistence of first integrals in an analytic autonomous system, see [37] and [38]. We extend his result to the analytic quasi-periodic vector field by giving a formula on the maximal number of first integrals. There are other questions one can ask in this regard. For example, suppose that we have n functionally independent first integrals in an analytic vector field, then what is the connection to its normalization? If it has a normalization, is it formal or convergent? What is the condition to determine the convergence of the normalization? Is the normal form unique? These questions have been well studied for analytic Hamiltonian systems. In this thesis, we consider some of these questions for general analytic vector fields. In particular, we give some results concerning the convergence and generic divergence of the normalizations for an analytic autonomous systems in $\mathbb{C}^{2n}$ with exactly n resonances and n functionally independent first integrals. As an application of our results, we give a new proof of the necessary and sufficient conditions for a planar Hamiltonian system to have an isochronous center.

Bifurcations, Normal Forms and their Applications

PART I
Global Analysis in a Predator-prey System with a New Type of Functional Response

by

Jian Chen

CHAPTER 2

INTRODUCTION

2.1 Motivation

In population dynamics, limit cycles and chaos are intrinsically fascinating. There has been a long lasting interest in finding mechanisms behind such complexities. Once an ecologist said (see [46]), if physicists were able to formulate general laws of physical motions by studying periodic orbits of planets, perhaps ecologists will be able to formulate general laws of population dynamics by studying periodic oscillations in population density. We are moving towards this direction. Although we are still not able to make any claim on such general law of population ecology, more and more newly found models have helped a great deal in describing the population phenomena we observe. We would like to quote from Turchin [46] by saying that " population ecology may be on the brink of maturity, rapidly becoming a quantitative and predictive science."

For mathematical ecology, we are interested in modeling three types of pattern changes. The first, which is the core of population ecology, is the temporal population dynamics. The second is the population structure change, e.g., age distribution. The third, which is in most recent progress, is the spatiotemporal population dynamics. In this thesis, we mainly study the temporal population dynamics of a model which is described in Section 2.3. And at the end of part I, a brief discussion will be given for a possible extension of the model to its spatiotemporal counterpart.

Mathematically, the complexity of population dynamics comes from two sources:

one is the structural assumption of the model, and the other is the parameter adjustment. In this thesis, we will first derive the model and then focus on exploring how and what parameter changes can affect the dynamics of the model.

2.2 Overview of the Functional Response

The very first model on population dynamics can be traced back to Malthus (1798), [33]. But he made a false assumption that a single species should increase at an exponential rate. The second major contribution was due to Verhulst (1838) [47], who introduced the first population self-regulation model — the logistic equation. For interacting species, the first model may be credited to Lotka(1925) [31] and Volterra (1931) [48]. The interaction terms in Lotka-Volterra model was speculated by the mass-action law, that is, in the form of $b_{ij}P_iP_j$ if species i interacts with species j. The problem with the mass-action formulation is that it implies the rate of the prey consumption by each predator will become arbitrarily large if the prey density is sufficiently high. In practice, the rate at which the predator can consume prey is limited by many factors, for example, the time to handle each prey item. This observation leads to the notion of the functional response.

In population dynamics, a predator *functional response* to prey density refers to the change in the density of prey attacked per unit of time per predator as the prey density changes. See Freedman [16]. There has been a rich literature on this topic. Categorically, it can be divided into the following types, (see Turchin [46]):

Linear Response This is the simplest and the component of the Lotka-Volterra predation model . The function form is $f(N) = aN$. The derivation is based on the mass-action principle.

Hyperbolic Response The hyperbolic response is solidly based on mechanisms at the individual level, see [20], which is also called Holling's type II functional response.

The function form is:

$$f(N) = \frac{aN}{1 + ahN} = \frac{cN}{d + N}.$$

Here a is the predator searching rate, h is the handling time , $c = h^{-1}$ is the maximum killing rate, and $d = (ah)^{-1}$ is the half-saturation constant (prey density at which the killing rate is half of the maximum). It can be easily seen that the hyperbolic response has two limits: as $N \to 0$, it becomes a linear response; as $N \to \infty$, it becomes a constant response.

Sigmoid Response This is another effort to model the nonlinearity of the transition between low predation rate at N near 0 and the saturated level of predation at $N \to \infty$. The general function form is

$$f(N) = \frac{cN^\theta}{d^\theta + N^\theta}.$$

It has been argued by the ecologists that specialist predators should be characterized by the hyperbolic response, while generalists are expected to exhibit a sigmoid response. The sigmoid response can model the phenomenon by which the predator switches to kill two different types of prey. That is, assume that one generalist predator usually lives for killing one type of prey, however, if the density of this prey is low, the predator will turn to kill the other type of prey. In the literature, a common type of the sigmoid response is

$$f(N) = \frac{cN^2}{d^2 + N^2}.$$

This is obtained by assuming that the searching rate for the predator in the hyperbolic response function is a linear increasing function $a(N) = bN$.

Predator Interference Response The above three types of functional responses only involve the prey density, i.e., $f(\cdot) = f(N)$. The derivation is at the individual predator level. However, it is common that the predator, as a species, may cooperate or compete with each other to hunt the prey. So at the species level, the functional response may also involve the density of the predator. One function form of this type of functional response is

$$f(N,P) = \frac{aN}{1+awP}$$

where a is the searching rate and w is the wasted time when predators encounter each other. The derivation follows the same logic as that of the hyperbolic response. By considering the time spent for the predator to handle the prey, the functional response becomes

$$f(N,P) = \frac{aN}{1+awP+ahN}.$$

Ratio-Dependent Response This is the one which drops the constant "1" in the above function forms. The advantage is that it has fewer parameters so to make mathematical analysis easier. Assuming $aw \gg 1$, one has the function form:

$$f(N,P) = \frac{aN}{1+awP+ahN} \cong \frac{aN}{awP+ahN} = \frac{(1/h)(N/P)}{(w/h)+(N/P)}.$$

Nonmonotonic Response This type of functional response is used to model the situation where the prey can better protect themselves by using group defense when the population density becomes large enough. So as the density of prey is small, the functional response just behaves like the hyperbolic one; however, once the density passes through a threshold, the functional response will drop due to the group defense, see [1]. One example of such a form is:

$$f(N) = \frac{cN}{d^2+N^2}.$$

It differs from the sigmoid response by changing the exponent of the numerator part to 1.

A list of different types of the functional response is summarized in Table 2.1.

2.3 Assumptions of the Model and Outline of the Results

In this section, we will describe the model we are studying and then outline the results we obtain.

Table 2.1: Some Functional Responses. Variable: N, prey density; P, predator density. Parameters: c, maximum killing rate; a, predator searching rate; h, handling time; w, waste time; d, half-saturation constant; θ, an exponent

Type	Function Form
Constant	c
Linear (Holling type-I)	aN
Hyperbolic (Holling type-II)	$\frac{aN}{1+ahN}$
Sigmoid (Holling type-III)	$\frac{cN^2}{d^2+N^2}$
θ-Sigmoid (Holling type-III)	$\frac{cN^\theta}{d^\theta+N^\theta}$
Predator Inference	$\frac{aN}{1+awP}$
Ratio Dependent	$\frac{cN}{dP+N}$
Nonmonotonic (Holling type-IV)	$\frac{cN}{d^2+N^2}$

Ecologically, we consider a landscape of n plant species $\mathbf{N} = (N_1, N_2, \cdots, N_n)$ and one herbivore P . Here $P = P(t)$ and $N_i = N_i(t)$ denote the densities of the herbivore and plant species i. We assume that each plant species may have a different level of toxicity and competition ability for resources, and the herbivore's functional responses to plant abundance may be dependent upon their toxicity. The toxin-dependent intake of plant i per herbivore has been modeled to take the form

$$f_i(\mathbf{N}) \left(1 - \frac{a_i T_i f_i(\mathbf{N})}{M_i}\right) \tag{2.3.1}$$

where $f_i(\mathbf{N})$ is the consumption rate of plant i in the absence of a toxin, which is defined by

$$f_i(\mathbf{N}) = \frac{C_i N_i}{1 + \sum_{j=1}^{n} h_j C_j N_j}. \tag{2.3.2}$$

The following is a description of the parameters. C_i is the consumption rate of plant N_i, T_i the amount of toxin contained per unit plant N_i content that is toxic to the herbivore, M_i the maximal amount of toxin of plant N_i that can be eaten before a herbivore dies. a_i and h_i are constants determining the asymptote of $f_i(1-a_iT_if_i/M_i)$ (maximum daily intake of plant N_i); specifically, a_i is the searching rate for the herbivore to find plant N_i, and h_i is the handling time needed before the herbivore can eat the plant N_i.

Then the mathematical model can be described as follows:

$$
\begin{aligned}
\frac{dN_i}{dt} &= r_i N_i \left(1 - \frac{N_i + \sum\limits_{j=1, j\neq i}^{n} \beta_{ij} N_j}{K_i}\right) - P f_i(\mathbf{N}) \left(1 - \frac{a_i T_i f_i(\mathbf{N})}{M_i}\right), \\
\frac{dP}{dt} &= P\left(\sum_{j=1}^{n} B_j f_j(\mathbf{N}) \left(1 - \frac{a_j T_j f_j(\mathbf{N})}{M_j}\right) - D\right),
\end{aligned}
\tag{2.3.3}
$$

for $i = 1, 2, \ldots, n$. We now explain what the remaining parameters not described above stand for. B_i is the conversion rate of consumed plant N_i into new herbivores, D the death rate of herbivore due to causes unrelated to plant toxicity, r_i the growth rate of plant N_i under the best circumstances in the local environment, i.e., no competition for resources by plants, β_{ji} the competition parameter which measures the competition intensity of plant N_j against plant N_i, and K_i the carrying capacity of plant N_i.

For simplicity, we denote $\mu_j = \frac{\alpha_j T_j}{M_j}$ and

$$
F_j(\mathbf{N}) = \mathbf{f_j}(\mathbf{N})\left(\mathbf{1} - \mu_\mathbf{j} \mathbf{f_j}(\mathbf{N})\right) \tag{2.3.4}
$$

for $j = 1, \ldots, n$. Then the system (2.3.3) becomes

$$
\begin{aligned}
\frac{dN_i}{dt} &= r_i N_i \left(1 - \frac{N_i + \sum\limits_{j=1, j\neq i}^{n} \beta_{ij} N_j}{K_i}\right) - P F_i(N_i), \\
\frac{dP}{dt} &= P\left(\sum_{j=1}^{n} B_j F_j(N_j) - D\right),
\end{aligned}
\tag{2.3.5}
$$

for $i = 1, 2, \ldots, n$.

The functional response in the system (2.3.5) is complicated. It is the combination of the Holling type II and Holling type IV. That is, for some region of the parameters, the functional response of this system is hyperbolic, i.e., monotonic; but for the other, it is nonmonotonic. Recently Zhu and co-authors [50] have given a very detailed bifurcation analysis on the Holling type-IV functional response for one predator-one prey system, (see also [40]). It has been shown that the Holling type-IV system is very complicated and there are rich bifurcation phenomena. Especially, there exist, in some parameter region, limit cycles and homoclinic loops which exhibit the so-called paradox of enrichment phenomena. However, comparing to the richness and completeness of results on planar systems, the high-dimensional systems have fewer theoretic results. For a two predator-one prey system, Smith [44] obtained a limit cycle in the interior of the first octant via the interaction of a saddle-node bifurcation and a Hopf bifurcation on the subsystem plane, (see also [21] and [6]). And Liu and the coauthors [30] showed that there exists a relaxation cycle in the interior of the first octant by using singular perturbation analysis. For the multi-level food web, we refer to the work by Deng and the coauthors [11], [12], [13], and [14] on higher order bifurcations and chaos. There are also extensive numerical studies on high-dimensional predator-prey systems. See, for example, [26] and the references therein. Finally, we mention that recently Schreiber [41], [42] obtained some nice co-existence results on a general predator-prey system in which the prey species share one predator by using the ergodic decomposition theorem.

In the next two chapters, we will discuss the possible dynamics which can come from the system (2.3.5). Section 3.1 deals with the planar case, i.e., there is only one predator and one prey. We conduct a linear analysis on the equilibria and detect Hopf bifurcation which generates the stable limit cycle. In Section 3.2, we explore the case $n = 2$ by showing that a limit cycle in the $N_i - P$ plane $i = 1, 2$ can be bifurcated into an limit cycle in the interior of the first octant of the $N_1 - N_2 - P$ space via a saddle-node bifurcation. Then by using smooth ergodic theory, we prove that for certain parameters, the solution of the system is contained in a compact region in the interior of the first octant of the $N_1 - N_2 - P$ space. Ecologically, this means that the three species coexist. In Chapter 4, some numerical experiments are carried out. We especially show that the three-dimensional case can be very complicated, even chaotic. We end with Chapter 5 by giving some discussions on possible future work.

CHAPTER 3

THEORETIC STUDIES OF THE MODEL

3.1 The 2-D Model

In this section, we discuss the case when there are only one prey and one predator. The system (2.3.5) is changed to

$$\begin{aligned} \frac{dN_1}{dt} &= r_1 N_1 (1 - \frac{N_1}{K_1}) - P F_1(N_1), \\ \frac{dP}{dt} &= P\left(B_1 F_1(N_1) - D\right), \end{aligned} \qquad (3.1.1)$$

where

$$F_1(N_1) = f_1(N_1)\left(1 - \mu_1 f_1(N_1)\right) \qquad (3.1.2)$$

and $f_1(N_1)$ is given by (2.3.2).

We first study properties of the functional response $F_1(N_1)$. Differentiating F_1 with respect to N_1, we have that

$$F_{1N_1}(N_1) := \frac{dF_1}{dN_1} = \frac{C_1(1 + (h_1 - 2\mu_1)C_1 N_1)}{(1 + h_1 C_1 N_1)^3}. \qquad (3.1.3)$$

The following is easily seen from (3.1.3),

(a) If $h_1 \geq 2\mu_1$, then F_1 is monotonically increasing, and has the limit $\frac{h_1 - \mu_1}{h_1^2}$ as N_1 approaches positive infinity.

(b) If $h_1 < 2\mu_1$, then F_1 is monotonically increasing when $N_1 \in [0, \frac{1}{C_1(2\mu_1 - h_1)}]$ and is monotonically decreasing when $N_1 \in [\frac{1}{C_1(2\mu_1 - h_1)}\ , \ +\infty)$. It attains maximum at $N_1 = \frac{1}{C_1(2\mu_1 - h_1)}$. As N_1 approaches positive infinity, F_1 approaches $\frac{h_1 - \mu_1}{h_1^2}$. We note that the limit can be negative if $0 < h_1 < \mu_1$.

The monotonicity of the functional response therefore depends on the ratio of $\frac{h_1}{\mu_1}$. See Fig. 3.1. In Section 3.1.2 we will focus on the case $\frac{h_1}{\mu_1} \geq \frac{1}{2}$. The case $\frac{h_1}{\mu_1} < \frac{1}{2}$ is much more complicated. We give some discussion of it in Chapter 5. We refer to the works in [39], [40] and [50] for some analysis on nonmonotonic functional response.

Ecologically, we notice that μ_1 is related to the level of toxin in the plant. If it is low, toxins do little harm to the herbivore, then the relationship between the herbivore and the plant is similar to the Holling type II, i.e., the change of the density of the plant is proportional to the efficiency of the herbivore's consumption up to the saturation level. However, if the toxin level is high, then there is a limit for the herbivore's daily intake. What's more, if the herbivore's consumption exceeds this limit, the herbivore will die and the density of plant will increase. So the functional response behaves like a hump.

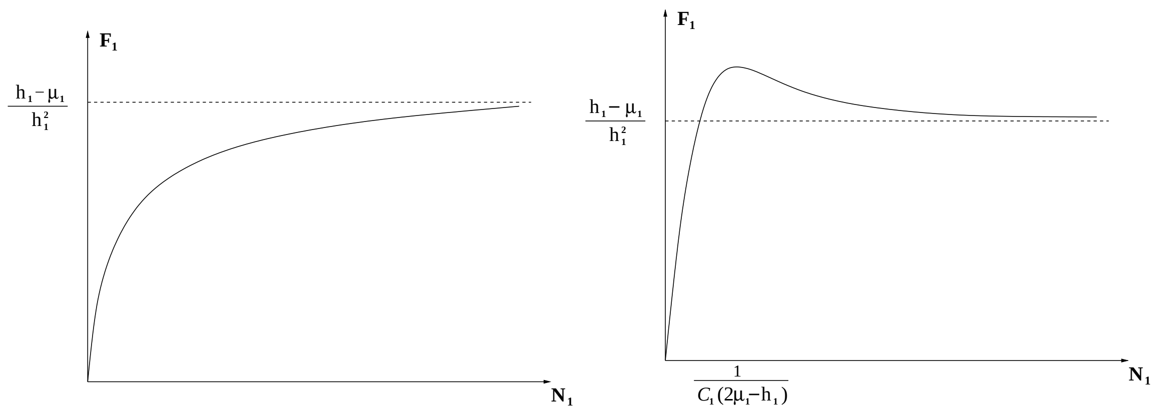

Figure 3.1: the predator response function $F_1(N_1)$

3.1.1 The Equilibria and Linear Stability Analysis

In this section, we are only interested in the equilibrium solutions that lie in the nonnegative cones including the positive N_1- axis and the positive $P-$ axis. we first note that the N_1- axis and the $P-$ axis are invariant, and $E_0 = (0, 0)$ and $E_{K_1} = (K_1, 0)$ are two equilibria of the system (3.1.1).

Here and after in this chapter, we refer to an interior equilibrium for (3.1.1) as an equilibrium in the interior of the positive $(N_1, P)-$ plane. From the second equation of the system (3.1.1), we can see that if there is an interior equilibrium (N_1, P), then

$$B_1 F_1(N_1) - D = 0. \tag{3.1.4}$$

Lemma 3.1. *If $F_1(K_1) \leq \frac{D}{B_1}$, then there is no interior equilibrium.*

Proof. We require $P > 0$ and $N_1 > 0$. By (3.1.4), we have $F_1(N_1) = D/B_1$. Substituting it into (3.1.1) and solving for P, we have

$$\begin{aligned} P &= \tfrac{r_1 N_1 (1 - N_1/K_1)}{F_1(N_1)} \\ &= \tfrac{r_1 N_1 (1 - N_1/K_1)}{D/B_1} \\ &= \tfrac{B_1 r_1}{D} N_1 (1 - N_1/K1) \\ &> 0 \end{aligned}$$

which implies $N_1 < K_1$. By the monotonicity of $F_1(N_1)$, there exists at most one interior equilibrium denoted by $(\bar{N}_1, \bar{P})$. It follows that $F_1(K_1) > F_1(\bar{N}_1) = \frac{D}{B_1}$. □

Lemma 3.2. *If $F_1(K_1) > \frac{D}{B_1}$, then the system (3.1.1) has a unique interior equilibrium $E_1^* = (N_1^*, P^*)$ satisfying*

$$B_1 F_1(N_1^*) = D \tag{3.1.5}$$

and

$$P^* = \frac{B_1 r_1 N_1^* (1 - \frac{N_1^*}{K_1})}{D}. \tag{3.1.6}$$

Proof. The uniqueness follows from the monotonicity of the functional response. The formulas (3.1.5) and (3.1.6) follow from a simple algebraic calculation. □

We next investigate the linear stability for each equilibrium. The variational matrix at any equilibrium $(\bar{N}_1, \bar{P})$ of (3.1.1) reads

$$J(\bar{N}_1, \bar{P}) = \begin{bmatrix} r_1(1 - \frac{2N_1}{K_1}) - P\frac{dF_1}{dN_1} & -F_1(N_1) \\ P(B_1 \frac{dF_1}{dN_1}) & B_1 F_1(N_1) - D \end{bmatrix}_{(\bar{N}_1, \bar{P})}. \tag{3.1.7}$$

Lemma 3.3. *E_0 is always a saddle point. E_{K_1} is a stable node if $F_1(K_1) \le D/B_1$, and a saddle point if $F_1(K_1) > D/B_1$.*

Proof. E_0 is a saddle point because $\det J(0,0) = -r_1 D < 0$.

For E_{K_1},

$$\mathrm{tr}J(K_1, 0) = -r_1 + B_1 F_1(K_1) - D$$

and

$$\det J(K_1, 0) = r_1(D - B_1 F_1(K_1)).$$

So if $F_1(K_1) \le D/B_1$, then there are two negative eigenvalues for the variational matrix at $(K_1, 0)$ and hence it is a stable node; and if $F_1(K_1) > D/B_1$, then there are one negative and one positive eigenvalues for the variational matrix at $(K_1, 0)$ and hence it is a saddle point. ☐

We denote

$$p_1(K_1) = \mathrm{tr}J(N_1^*, P^*) = \left(r_1(1 - \frac{2N_1}{K_1}) - P(\frac{dF_1}{dN_1}) \right)_{(N_1^*, P^*)} \tag{3.1.8}$$

and

$$q_1(K_1) = \det J(N_1^*, P^*) = \left(DP\frac{dF_1}{dN_1} \right)_{(N_1^*, P^*)} \tag{3.1.9}$$

where (N_1^*, P^*) is defined in (3.1.5) and (3.1.6).

Lemma 3.4. *$p_1(K_1)$ is strictly increasing and there exists a unique $K_1^0 > 0$ such that $p_1(K_1^0) = 0$.*

Proof. Substituting (3.1.3) and (3.1.6) into (3.1.8), we have

$$\begin{aligned} p_1(K_1) &= r_1(1 - \frac{2N_1^*}{K_1}) - \frac{B_1 r_1 N_1^*(1 - \frac{N_1^*}{K_1})}{D} \frac{C_1(1 + (h_1 - 2\mu_1)C_1 N_1^*}{(1 + h_1 C_1 N_1^*)^3} \\ &= r_1(1 - \frac{2N_1^*}{K_1}) - \frac{r_1 N_1^*(1 - \frac{N_1^*}{K_1})}{\frac{C_1 N_1^*(1+(h_1-\mu_1)C_1 N_1^*)}{(1+h_1 C_1 N_1^*)^2}} \frac{C_1(1 + (h_1 - 2\mu_1)C_1 N_1^*}{(1 + h_1 C_1 N_1^*)^3} \\ &= r_1(1 - \frac{2N_1^*}{K_1}) - r_1(1 - \frac{N_1^*}{K_1}) \frac{1 + (h_1 - 2\mu_1)C_1 N_1^*}{(1 + h_1 C_1 N_1^*)(1 + (h_1 - \mu_1)C_1 N_1^*)}. \end{aligned}$$

Then taking the derivative with respect to K_1, we have that

$$\frac{dp_1(K_1)}{K_1} = \frac{r_1 N_1^*}{K_1^2}\left(2 - \frac{1+(h_1-2\mu_1)C_1N_1^*}{(1+h_1C_1N_1^*)(1+(h_1-\mu_1)C_1N_1^*)}\right) > 0.$$

So $p_1(K_1)$ is strictly increasing.

K_1^0 is obtained by an easy algebraic calculation:

$$K_1^0 = \frac{2 - \frac{1+(h_1-2\mu_1)C_1N_1^*}{(1+h_1C_1N_1^*)(1+(h_1-\mu_1)C_1N_1^*)}}{1 - \frac{1+(h_1-2\mu_1)C_1N_1^*}{(1+h_1C_1N_1^*)(1+(h_1-\mu_1)C_1N_1^*)}} N_1^*. \tag{3.1.10}$$

□

Remark 3.5. $K_1^0 > 2N_1^*$.

Lemma 3.6. $q_1(K_1) > 0$ *for any* $K_1 > 0$.

Proof. This follows from the monotonicity of $F_1(N_1)$. □

Lemma 3.7. E_1^* *is a node or focus. It is stable if* $K_1 < K_1^0$ *and unstable if* $K_1 > K_1^0$.

Proof. The eigenvalues of J at E_1^* are

$$\lambda_{1,2} = \frac{p_1(K_1) \pm \sqrt{p_1(K_1)^2 - 4q_1(K_1)}}{2}.$$

If $K < K_1^0$, then $p_1(K_1) < 0$ and $q_1(K_1) > 0$. So both of the eigenvalues are negative and E_1^* is a stable node or focus. If $K > K_1^0$, then $p_1(K_1) > 0$ and $q_1(K_1) > 0$. So both of the eigenvalues are positive and E_1^* is an unstable node or focus. □

3.1.2 Hopf Bifurcations

Lemma 3.8. *A supercritical Hopf bifurcation occurs at* $K_1 = K_1^0$.

Proof. As $K_1 < K_1^0$, E_1^* is a stable node or focus. It changes stability and becomes an unstable node or focus when K_1 passes through K_1^0 from left to the right. By the Hopf bifurcation theorem, see [34], there is a supercritical Hopf bifurcation occurring at $K_1 = K_1^0$. □

In the following theorem we summarize and classify the dynamics of the system (3.1.1) in the first quadrant.

Theorem 3.9. *If $F_1(K_1) \le \frac{D}{B_1}$, then the system (3.1.1) has two equilibria E_0 and E_{K_1} and no interior equilibrium in the $(N_1, P)-$ plane. In this case, E_0 is a saddle and E_{K_1} is a globally stable node. All the orbits in the first quadrant are attracted to E_{K_1}.*

If $F_1(K_1) > \frac{D}{B_1}$, then the system (3.1.1) has three equilibria E_0, E_{K_1} and a unique interior equilibrium E_1^* defined by (3.1.5) and (3.1.6). In this case, both E_0 and E_{K_1} are saddles. Let K_1^0 be defined by (3.1.10).

1. *If $K_1 < K_1^0$, then E_1^* is a stable node or focus;*
2. *If $K_1 > K_1^0$, then E_1^* is an unstable node or focus. There exists at least one stable limit cycle in the interior of the first quadrant;*
3. *There is a supercritical Hopf bifurcation occurring at $K_1 = K_1^0$.*

Proof. The appearance of the limit cycle is via the supercritical Hopf bifurcation as K_1 passes from the left to the right of K_1^0. □

3.2 The 3-D Model

In this section, we mainly focus on the system (2.3.5) for $n = 2$. That is,

$$\begin{aligned} \frac{dN_1}{dt} &= r_1 N_1 \left(1 - \frac{N_1 + \beta_{12} N_2}{K_1}\right) - P\bar{F}_1(N_1, N_2), \\ \frac{dN_2}{dt} &= r_2 N_2 \left(1 - \frac{N_2 + \beta_{21} N_1}{K_2}\right) - P\bar{F}_2(N_1, N_2), \\ \frac{dP}{dt} &= P(B_1 \bar{F}_1(N_1, N_2) + B_2 \bar{F}_2(N_1, N_2) - D). \end{aligned} \tag{3.2.1}$$

where

$$\bar{F}_i(N_1, N_2) = \frac{C_i N_i}{1 + h_1 C_1 N_1 + h_2 C_2 N_2} \left(1 - \frac{C_i N_i}{1 + h_1 C_1 N_1 + h_2 C_2 N_2}\right)$$

for $i = 1, 2$.

3.2.1 An Interior Limit Cycle

We have showed that in the 2D case in a large region of the parameter space, there exists at least one stable limit cycle in the interior of the first quadrant of the (N_1, P)–plane. Now we show that this limit cycle can be bifurcated into the interior of the positive octant of the (N_1, N_2, P) space.

First, we denote

$$a_1 = \frac{1}{T}\int_0^T \frac{P(t)}{1 + h_1 C_1 N_1(t)}\, dt, \tag{3.2.2}$$

and

$$b_1 = \frac{1}{T}\int_0^T N_1(t)\, dt, \tag{3.2.3}$$

for some positive T.

Theorem 3.10. *Let r_1, K_1, C_1, h_1 and μ_1 be given such that $(N_1(t), P(t))$ is a locally unique periodic orbit with period T in the interior of the first quadrant of the N_1-P plane. If*

$$\left| \frac{K_2}{\beta_{21}} - b_1 \right| \geq \epsilon_0 > 0, \tag{3.2.4}$$

then there exists an r_2^* denoted by

$$r_2^* = \frac{C_2 a_1}{1 - \frac{\beta_{21}}{K_2} b_1} \tag{3.2.5}$$

such that as $|r_2 - r_2^*| \ll 1$, there is a periodic orbit in the interior of the first octant arbitrarily near the N_1-P plane.

The proof originates from Butler and Waltman [6]. The principal idea is to apply Lemma 3.11 (see below) to the Poincaré map on a section of the locally unique periodic orbit in the plane. All the parameters about N_1, N_2 and P except r_2 will be fixed and β_{21} and K_2 satisfy the condition (3.2.4). Then as r_2 passes through r_2^* from one side to the other, a saddle-node bifurcation of a fixed point of the Poincaré map will occur.

Before we prove our main theorem, we require the following three technical lemmas. The proofs can be found in [34].

Lemma 3.11. *Let W be an open neighborhood of $0 \in R^n$ and I be an open interval about $0 \in R$. Let $\Phi_\nu : W \to R^n$ be such that the map $(\nu, x) \to \Phi_\nu(x)$ is C^k $(k \geq 1)$ from $I \times W$ to R^n, and $\Phi_\nu(0) = 0$ for all $\nu \in I$. Define L_ν to be the differential map $d\Phi_\nu(0)$ and assume that all eigenvalues of L_ν lie inside the unit circle of the complex plane for $\nu < 0$. Also assume that there is a real, simple eigenvalue $l(\nu)$ of L_ν such that $l(0) = 1$ and $(dl/d\nu)(0) > 0$. Let ν_0 be the eigenvector corresponding to $l(0)$. Then there is a C^{k-1} curve $\mathcal{C}$ of fixed points of $\Phi : (\nu, x) \to (\nu, \Phi_\nu(x))$ near $(0, 0)$ in $I \times R^n$ which, together with the points $(\nu, 0)$, are the only fixed points near $(0, 0)$. The curve $\mathcal{C}$ is tangent to ν_0 at $(0, 0)$ in $I \times R^n$.*

Lemma 3.12. *If X is a C^k vector field on a manifold M and γ is a closed orbit of X, then there exists a Poincaré map of γ. Moreover, the spectrum of the linearization of the Poincaré map union $\{1\}$ is equal to the spectrum of the linearization of the solution map of X.*

Lemma 3.13. *Let $A(t)$, $t \in \mathcal{R}$, be a periodic matrix-valued function of period τ and suppose that the linear system*

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = A(t) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

has Floquet exponents 0 and $-\mu < 0$. Let $b(t)$, $c_1(t)$ and $c_2(t)$ be functions of period τ such that the mean value of b is equal to ν. Then the linear system

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = B(t) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

where

$$B(t) = \begin{pmatrix} & A(t) & & c_1(t) \\ & & & c_2(t) \\ 0 & 0 & & b(t) \end{pmatrix}$$

has Floquet exponents $0, -\mu, \nu$.

We now prove Theorem 3.10.

Proof. Let Γ denote the orbit corresponding to $(N_1(t), 0, P(t))$ where $(N_1(t), P(t))$ is the locally unique periodic orbit in the $(N_1, P)-$ plane. Let Ω be a two dimensional, local, transverse section of Γ. For each value of r_2 the Poincaré map $Q : W_0 \to W_1$ exists where W_0 and W_1 are open subsets of Ω. Given a periodic orbit, the relationship between the linearization about the orbit and the linearization of the Poincaré map about the corresponding fixed point according to Lemma 3.12. That is, there is an eigenvalue 1, an eigenvalue determined by the stability of the periodic orbit and an eigenvalue determined by the linearization of system (3.2.1) at the periodic orbit, followed from Lemma 3.13. Without loss of generality, we assume that the periodic orbit in the $(N_1, P)-$ plane is a stable limit cycle. Then there is one Floquet multiplier inside the unit circle. In order to apply the bifurcation theorem, it is necessary to show that the remaining eigenvalue crosses the unit circle transversally. This can be accomplished by showing that one of the Floquet exponents passes through zero transversally.

Let $(N_1(t), P(t))$ be the locally unique periodic solution of (3.1.1). As assumed, the Floquet exponents are 0 and $-\mu < 0$. The linearization about the periodic orbit of a solution of (3.1.1) has coefficient matrix of the form

$$V = \begin{pmatrix} r_1(1 - \frac{2N_1}{K_1}) - F_{1N_1}(N_1) & -F_1(N_1) \\ P(B_1 F_{1N_1}(N_1)) & B_1 F_1(N_1) - D \end{pmatrix} \qquad (3.2.6)$$

where $F_1(N_1)$ is defined in (3.1.2) and $F_{1N_1}(N_1)$ defined in (3.1.3). For a solution of

(3.2.1), with $N_2 = 0$, the linearised system takes the form

$$\begin{pmatrix} & & -r_1\beta_{12}\frac{N_1}{K_1} - P\frac{\partial \bar{F}_1}{\partial N_2} \\ V & & \\ & & P(B_1\frac{\partial \bar{F}_1}{\partial N_2} + B_2\frac{\partial \bar{F}_2}{\partial N_2}) \\ 0 & 0 & r_2(1-\beta_{21}\frac{N_1}{K_2}) - P\frac{\partial \bar{F}_2}{N_2} \end{pmatrix} \tag{3.2.7}$$

where

$$\bar{F}_1(N_1, N_2) = \frac{C_1 N_1}{1 + h_1 C_1 N_1 + h_2 C_2 N_2}\left(1 - \mu_1 \frac{C_1 N_1}{1 + h_1 C_1 N_1 + h_2 C_2 N_2}\right)$$

and

$$\bar{F}_2(N_1, N_2) = \frac{C_2 N_2}{1 + h_1 C_1 N_1 + h_2 C_2 N_2}\left(1 - \mu_2 \frac{C_2 N_2}{1 + h_1 C_1 N_1 + h_2 C_2 N_2}\right).$$

By Lemma 3.13, the Floquet exponents for the linearization of (3.2.1) about $(N_1(t), 0, P(t))$ are 0, $-\mu$ and ν where

$$\begin{aligned} \nu &= \frac{1}{T}\int_0^T r_2(1-\beta_{21}\frac{N_1}{K_2}) - P\frac{\partial \bar{F}_2}{N_2}\, dt \\ &= \frac{1}{T}\int_0^T r_2(1-\beta_{21}\frac{N_1(t)}{K_2}) - \frac{C_2 P(t)}{1 + h_1 C_1 N_1(t)}\, dt \\ &= r_2(1 - \frac{\beta_{21}}{K_2} b_1) - C_2 a_1. \end{aligned} \tag{3.2.8}$$

As r_2 approaches r_2^* from one side to the other, by condition (3.2.4), ν crosses zero transversally. Hence the Poincaré map has one eigenvalue $e^{-\mu}$ inside the unit circle and one eigenvalue e^{ν} crossing the unit circle transversally as ν passes through zero. Let Γ be the orbit corresponding to $(N_1(t), 0, P(t))$. Consider a $q_0 \in \Gamma$, we identify the transverse section Ω of Γ through q_0 with $\mathcal{R}^2$, identifying q_0 with $0 \in \mathcal{R}^2$. Let Φ_ν denote the Poincar e map associated with Γ, q_0, and the section Ω, for (3.2.1) with $\nu = (r_2 - r_2^*)(1 - \frac{\beta_{21}}{K_2} b_1)$. From the analytic dependence of the vector field defined by (3.2.1) on its parameters it follows from [10] p. 36 that solutions are analytic in parameters and initial conditions, and so is the Poincar e map. It follows that there is a neighborhood W of q_0 in Ω such that for all ν sufficiently close to 0, say

$\nu \in I$, the map Φ_ν is defined on W. Making our identification with $\mathcal{R}^2$, we see that $(\nu, x) \to \Phi_\nu(x)$ is analytic from $I \times W$ to $\mathcal{R}^2$, $\Phi_\nu(0) = 0$ for all $\nu \in I$, and $d\Phi_\nu(0)$ has eigenvalues $e^{-\mu}$ and e^{ν}. Applying Lemma 3.11, we obtain an analytic curve $\mathcal{C}$ of fixed points of $\Phi : (\nu, x) \to (\nu, \Phi_\nu(x))$ bifurcating from $(\nu, 0)$ at $(0, 0)$.

For such (ν, x), we have $x = \Phi_\nu(x)$, so x is a fixed point of the Poincaré map Φ_ν. $\mathcal{C}$ therefore corresponds to a $1-$parameter family of periodic solutions of (3.2.1). In addition, $\mathcal{C}$ is tangent to the eigenvector ν_0 associated with eigenvalue 1 of $d\Phi_0$. The direction of ν_0 is transverse to the $(N_1, P)-$ plane for the system (3.2.1) (the other eigenvector of $d\Phi_0$ lies in this plane). It follows that there is a branch of periodic solutions in the first octant for $|r_2 - r_2^*| = |\nu|$ small. □

Remark 3.14. There will also be a branch of periodic solutions not in the positive octant, but these have no ecological interest.

3.2.2 Coexistence of the Three Species

Already the 2-D system (3.1.1) is complicated enough to analyze the stability of the interior equilibria or possible limit cycles. In this section, we use another way to think about "stability". From the ecological point of view, a stable system is a system where all the species can coexist. In this section, we show that in some region of the parameter space, the system (3.2.1) is C^∞ robustly permanent (see, for example, [41] and the references therein).

Let $x = (x_1, x_2, x_3) := (N_1, N_2, P)$ and denote

$$\begin{aligned}
g_1(x) &:= r_1\left(1 - \frac{x_1 + \beta_{12}x_2}{K_1}\right) - \frac{C_1x_3}{A}\left(1 - \mu_1\frac{C_1x_1}{A}\right) \\
g_2(x) &:= r_2\left(1 - \frac{x_2 + \beta_{21}x_1}{K_2}\right) - \frac{C_2x_3}{A}\left(1 - \mu_2\frac{C_2x_2}{A}\right) \\
g_3(x) &:= \frac{B_1C_1x_1}{A}\left(1 - \mu_1\frac{B_1C_1x_1}{A}\right) + \frac{B_2C_2x_2}{A}\left(1 - \mu_2\frac{B_2C_2x_2}{A}\right) - D
\end{aligned} \tag{3.2.9}$$

where $A = 1 + h_1x_1 + h_2x_2$. Then the system (3.2.1) becomes

$$\dot{x}_i = x_i g_i(x), \qquad i = 1, 2, 3, \tag{3.2.10}$$

defined on the nonnegative cone:

$$\mathbf{C} = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}.$$

Let $\phi : U \to \mathbf{C}$, where U is some open subset of $\mathbf{R} \times \mathbf{C}$, be the flow defined by the solutions of Eq. (3.2.10). Denote $\phi_t(x) = \phi(t, x)$. Given sets $I \subseteq \mathbf{R}$ and $K \subseteq \mathbf{C}$, we let $\phi_I K = \{\phi_t x : t \in I, x \in K\}$. A set $K \subseteq \mathbf{C}$ is called *invariant* if $\phi_t K = K$ for all $t \in \mathbf{R}$. The ω – limit set of a set $K \subseteq \mathbf{C}$ equals $\omega(K) = \bigcap_{t\geq 0} \overline{\phi_{[t,\infty)}K}$. The α – limit set of a set $K \subseteq \mathbf{C}$ equals $\alpha(K) = \bigcap_{t\leq 0} \overline{\phi_{(-\infty,t]}K}$. Given an invariant set K, $A \subset K$ is called an *attractor* for $\phi|K$ provided there exists an open neighborhood $U \subseteq K$ of A such that $\omega(U) = A$. The *basin of attraction* of A for $\phi|K$ is the set of points $x \in K$ such that $\omega(x) \subseteq A$. The flow ϕ is *dissipative* if there exists a compact attractor $A \subset \mathbf{C}$ for ϕ whose basin of attraction is $\mathbf{C}$. A compact invariant set K is called *isolated* if there exists a neighborhood V of K such that K is the maximal compact invariant set in V. A collection of sets $\{M_1, \ldots, M_k\}$ is a *Morse decomposition* for a compact invariant set K if $M_1, \ldots, M_k$ are pairwise disjoint, compact isolated invariant sets for $\phi|K$ with the property that for each $x \in K$ there are integers $l = l(x) \leq m = m(x)$ such that $\alpha(x) \subseteq M_m$ and $\omega(x) \subseteq M_l$ and if $l = m$ then $x \in M_l = M_m$.

Let $\mathcal{P}_n^r$ be the space of C^r vector fields $F = (F_1, F_2, \ldots, F_n) : \mathbf{R}_+^n \to \mathbf{R}^n$ that satisfy $F_i(x) = 0$ whenever $x_i = 0$. F is *permanent* provided that $\dot{x} = F(x)$ generates a dissipative flow ϕ and there exists a compact attractor $A \subset \operatorname{int} \mathbf{R}_+^n$ for ϕ whose basin of attraction is $\operatorname{int} \mathbf{R}_+^n$. F is C^r *robustly permanent* if there exists a neighborhood $\mathcal{N} \subseteq \mathcal{P}_n^r$ of F such that every vector field $G \in \mathcal{N}$ is permanent.

Schreiber in [41] proved the following theorem.

Theorem 3.15. *Let $F \in \mathcal{P}_n^1$ be such that $\dot{x} = F(x)$ generates a dissipative flow*

ϕ. Let $\Lambda \subset \partial \mathbf{R}^n_+$ be the maximal compact invariant set for $\phi | \partial \mathbf{R}^n_+$. If Λ admits an unsaturated Morse decomposition, then F is C^1 robustly permanent.

The proof of the above theorem is much involved. However, for our system (3.2.10), the above theorem implies (see also [42])

Theorem 3.16. *Let $\Lambda \subset \partial \mathbf{C}$ be the maximal compact invariant set for the flow of (3.2.10) restricted to $\partial \mathbf{C}$. If Λ admits a Morse decomposition $M_1, \ldots, M_k$ such that for all $1 \le j \le k$ there exists an i, $1 \le i \le 3$ satisfying*

$$\inf_{x \in M_j} \liminf_{T \to \infty} \frac{1}{T} \int_0^T g_i(\phi_t x)\, dt > 0 \tag{3.2.11}$$

then (3.2.10) is C^∞ robustly permanent.

Proof. We first show that (3.2.10) is dissipative. It is easy to see that $\partial g_i / x_i < 0$ when x_i is large enough for $i = 1, 2$. So there exists a $K > 0$ such that all the solutions to (3.2.10) eventually enter $[0, K] \times [0, K] \times [0, \infty)$. Define $S(x) = B_1 x_1 + B_2 x_2 + x_3$. Then

$$\begin{aligned} \dot{S} &= \sum_{i=1}^{3} x_1 g_1 \\ &= B_1 r_1 x_1 (1 - \frac{x_1 + B_{12} x_2}{K_1}) + B_2 r_1 x_2 (1 - \frac{x_2 + B_{21} x_1}{K_2}) - D x_3. \end{aligned}$$

Hence for sufficiently small $\epsilon > 0$ and sufficiently large $E > 0$ we get $\dot{S} + \epsilon S \le E$ for any $x \in [0, K] \times [0, K] \times [0, \infty)$. Thus, all the solutions eventually enter and remain in the compact set $S^{-1}([0, 2E/\epsilon])$.

If condition (3.2.11) is satisfied, then the ergodic decomposition theorem, see [32], implies that Λ admits an unsaturated Morse decomposition. By Theorem 3.15, the system (3.2.10) is C^1 robustly permanent. And C^1 robust permanence implies C^∞ robust permanence. □

Now we investigate how condition (3.2.11) can be satisfied. First, we have the following lemma.

Lemma 3.17. *Let $E_3^* = (K_1, K_2)$. Then E_3^* is the unique globally stable equilibrium in the interior of the first quadrant of the $(x_1, x_2)-$ plane iff $K_1 > \beta_{12}K_2$ and $K_2 > \beta_{21}K_1$.*

Proof. This is a standard result on the Lotka-Volterra model. For example, see [19]. □

From the proof of Theorem 3.16, we see that the system (3.2.10) is dissipative. So are the subsystems in $(x_1, x_3)-$ plane, $(x_2, x_3)-$ plane and $(x_1, x_2)-$ plane. Then from Theorem 3.9 we have that E_0, E_{K_1} and E_{K_2} are saddles iff $F_1(K_1) > D/B_1$ and $F_2(K_2) > D/B_2$. Let Λ be the attractor for $\phi|\partial\mathbf{C}$ which is contained in $[0, K] \times [0, K] \times [0, \infty)$. From the analysis above we see that a Morse Decomposition for Λ is given by $(0, 0, 0)$, $(K_1, 0, 0)$, $(0, K_2, 0)$, $(K_1, K_2, 0)$, A_1 and A_2 where A_i is the compact attractor in the interior of the first quadrant of $(x_i, x_3)-$ plane for $i = 1, 2$.

Lemma 3.18. *Assume that $K_1 > \beta_{12}K_2$, $K_2 > \beta_{21}K_1$, $F_1(K_1) > D/B_1$ and $F_2(K_2) > D/B_2$, then we have*

$$g_1(0, 0, 0) > 0, g_1(0, K_2, 0) > 0, g_2(K_1, 0, 0) > 0, \text{ and } g_3(K_1, K_2, 0) > 0.$$

Proof. This follows from a simple calculation. □

Lemma 3.19. *Assume that $h_1 > 2\mu_1$. Let $(x_1^*, 0, x_3^*)$ be the unique equilibrium in the interior of first quadrant of the $(x_1, x_3)-$ plane. If $x(t) = (x_1(t), 0, x_3(t))$ is a solution of the system (3.2.10) with $x_1(0) > 0$ and $x_3(0) > 0$, then*

$$\liminf_{T\to\infty} \frac{1}{T} \int_0^T g_2(x(t))\, dt > 0$$

provided that

$$\left[\frac{C_2 r_1}{C_1 K_1 \left(1 - \frac{\mu_1}{h_1}\right)} - \frac{r_2 \beta_{21}}{K_2}\right] x_1^* + \left[r_2 - \frac{C_2 r_1}{C_1 \left(1 - \frac{\mu_1}{h_1}\right)}\right] > 0.$$

Proof. Let $x(t) = (x_1(t), 0, x_3(t))$ be a solution of the system (3.2.10) with $x_1(0) > 0$ and $x_3(0) > 0$. With the analysis in Section 3.1, we have that each orbit is either attracted to the equilibrium $(x_1^*, 0, x_3^*)$ or to a limit cycle which contains it. So for any continuous function $g : \mathbf{C} \to \mathbf{R}$, the limit

$$\overline{g(x(t))} := \liminf_{T\to\infty} \frac{1}{T} \int_0^T g(x(t))\, dt$$

exists. Since A_1 is compact and is contained in the positive quadrant of the (x_1, x_3)–plane,

$$\overline{g_i(x(t))} = \overline{\frac{x_i'(t)}{x_i(t)}} = \lim_{T\to\infty} \frac{1}{T} \ln\left(\frac{x_i(T)}{x_i(0)}\right) = 0$$

for $i = 1, 3$. With the assumption that $h_1 > 2\mu_1$, we have that $g_3(x_1, 0, 0)$ is strictly concave. Then by Jensen's inequality, $\overline{g_3(x_1(t), 0, 0)} < g_3(\overline{x_1(t)}, 0, 0)$. Also, $g_3(x_1, 0, 0)$ is strictly increasing in x_1 and $g_3(x_1^*, 0, 0) = 0$, so

$$g_3(\overline{x_1(t)}, 0, 0) > \overline{g_3(x_1(t), 0, 0)} = 0 = g_3(x_1^*, 0, 0)$$

from which it follows $\overline{x_1(t)} > x_1^*$. On the other hand, $\overline{g_1(x_1(t))} = 0$ implies

$$\begin{aligned} & r_1\left(1 - \frac{1}{K_1}\overline{x_1(t)}\right) \\ = \;& \liminf_{T\to\infty} \frac{1}{T} \int_0^T \frac{C_1 x_3}{A}\left(1 - \mu_1 \frac{C_1 x_1}{A}\right) dt \\ \geq \;& \liminf_{T\to\infty} \frac{1}{T} \int_0^T \frac{C_1 x_3}{A}\left(1 - \frac{\mu_1}{h_1}\right) dt. \end{aligned}$$

So

$$\overline{\left(\frac{x_3}{A}\right)} \leq \frac{r_1\left(1 - \frac{\overline{x_1(t)}}{K_1}\right)}{C_1\left(1 - \frac{\mu_1}{h_1}\right)}.$$

Since

$$\begin{aligned} \overline{g_2(x(t)} &= r_2\left(1 - \frac{\beta_{21}}{K_2}\overline{x_1(t)}\right) - C_2\overline{\left(\frac{x_3}{A}\right)} \\ &\geq r_2\left(1 - \frac{\beta_{21}}{K_2}\overline{x_1(t)}\right) - \frac{C_2 r_1\left(1 - \frac{\overline{x_1(t)}}{K_1}\right)}{C_1\left(1 - \frac{\mu_1}{h_1}\right)}, \end{aligned}$$

it follows from a simple algebraic calculation that $\overline{g_2(x(t)} > 0$ if

$$\left[\frac{C_2 r_1}{C_1 K_1 \left(1 - \frac{\mu_1}{h_1}\right)} - \frac{r_2 \beta_{21}}{K_2}\right] \overline{x_1(t)} + \left[r_2 - \frac{C_2 r_1}{C_1 \left(1 - \frac{\mu_1}{h_1}\right)}\right] > 0.$$

Since $\overline{x_1(t)} > x_1^*$, it follows that

$$\left[\frac{C_2 r_1}{C_1 K_1 \left(1 - \frac{\mu_1}{h_1}\right)} - \frac{r_2 \beta_{21}}{K_2}\right] x_1^* + \left[r_2 - \frac{C_2 r_1}{C_1 \left(1 - \frac{\mu_1}{h_1}\right)}\right] > 0.$$

□

Analogously, we have

Lemma 3.20. *Assume that* $h_2 > 2\mu_2$. *Let* $(0, x_2^{**}, x_3^{**})$ *be the unique equilibrium in the interior of the first quadrant of the* (x_2, x_3)*– plane. If* $x(t) = (0, x_2(t), x_3(t))$ *is a solution to the system (3.2.10) with* $x_2(0) > 0$ *and* $x_3(0) > 0$*, then*

$$\liminf_{T \to \infty} \frac{1}{T} \int_0^T g_1(x(t))\, dt > 0.$$

provided that

$$\left[\frac{C_1 r_2}{C_2 K_2 \left(1 - \frac{\mu_2}{h_2}\right)} - \frac{r_1 \beta_{12}}{K_1}\right] x_2^{**} + \left[r_1 - \frac{C_1 r_2}{C_2 \left(1 - \frac{\mu_2}{h_2}\right)}\right] > 0.$$

Now we can give the main theorem about the robust permanence of the system (3.2.10).

Theorem 3.21. *Let* $(x_1^*, 0, x_3^*)$ *be the unique equilibrium in the interior of first quadrant of the* (x_1, x_3)*– plane and* $(0, x_2^{**}, x_3^{**})$ *the unique equilibrium in the interior of first quadrant of the* (x_2, x_3)*– plane. Assume that the following are satisfied:*

(**L1**) $h_1 > 2\mu_1$,

(**L2**) $h_2 > 2\mu_2$,

(**L3**) $\left[\frac{C_2 r_1}{C_1 K_1 \left(1 - \frac{\mu_1}{h_1}\right)} - \frac{r_2 \beta_{21}}{K_2}\right] x_1^* + \left[r_2 - \frac{C_2 r_1}{C_1 \left(1 - \frac{\mu_1}{h_1}\right)}\right] > 0,$

(**L4**) $\left[\frac{C_1 r_2}{C_2 K_2\left(1-\frac{\mu_2}{h_2}\right)} - \frac{r_1\beta_{12}}{K_1}\right] x_2^{**} + \left[r_1 - \frac{C_1 r_2}{C_2\left(1-\frac{\mu_2}{h_2}\right)}\right] > 0,$

(**L5**) $K_1 > \beta_{12} K_2,$

(**L6**) $K_2 > \beta_{21} K_1,$

(**L7**) $F_1(K_1) > D/B_1,$

(**L8**) $F_2(K_2) > D/B_2.$

Then the system (3.2.10) is C^∞ robustly permanent.

Proof. From Lemma 3.17, 3.18, 3.19 and 3.20, the Morse Decomposition for Λ satisfies the condition (3.2.11). So Theorem 3.16 implies that the the system (3.2.10) is C^∞ robustly permanent. □

CHAPTER 4

SIMULATION STUDIES OF THE MODEL

We note that all the simulations in this chapter are done by using Matlab and especially its ODE suite package. Matlab is a numeric computation software made by Mathworks Inc.

4.1 The 2-D Model

All the simulations in this section are for the monotonic functional response. Recall from Theorem 3.9 that, if $F_1(K_1) \leq D/B_1$, then all the solutions will be attracted to E_{K_1}. In this simulation, we choose $r_1 = 0.0005$, $K_1 = 75000$, $C_1 = 0.0005$, $h_1 = 1/16$, $\mu_1 = 1/64$, $B_1 = 0.000012$, and $D = 0.0002$. The result is plotted in Fig. 4.1.

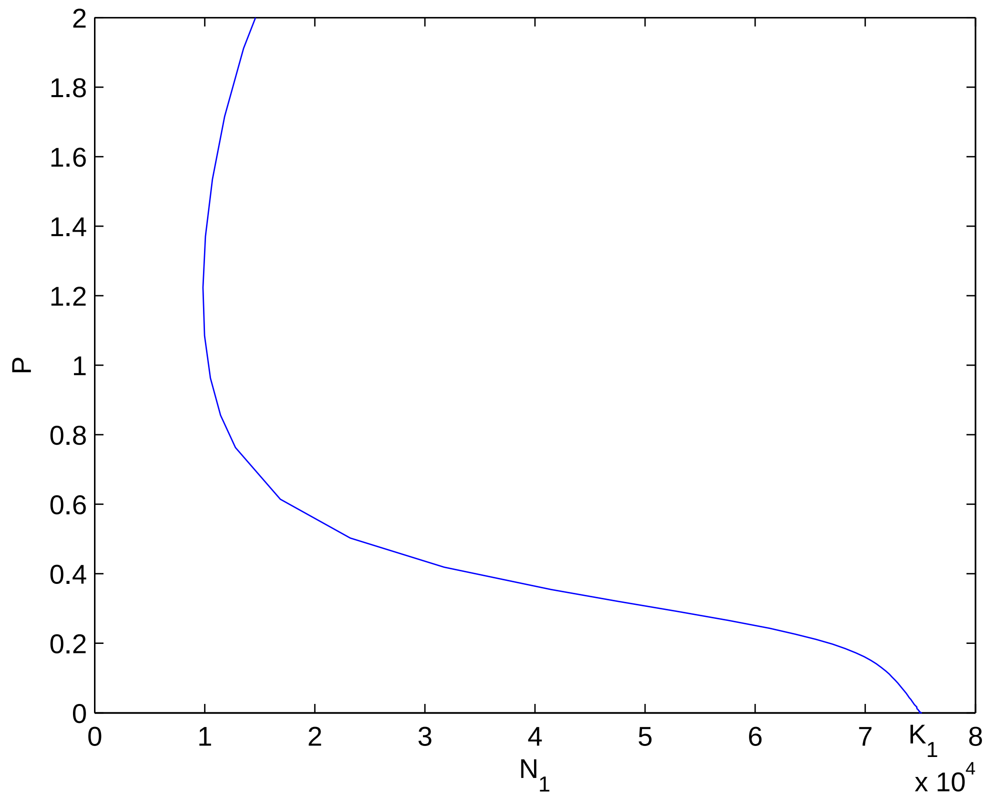

Figure 4.1: 2-D model: no interior equilibrium ($F_1(K_1) <= D/B_1$)

If $F_1(K_1) > D/B_1$ and $K_1 < K_1^0$, then there exists a unique interior equilibrium, which is a stable node or a stable focus. In this simulation, we choose $r_1 = 0.0005$, $K_1 = 50000$, $C_1 = 0.0005$, $h_1 = 1/16$, $\mu_1 = 1/64$, $B_1 = 0.000034$, and $D = 0.0002$. A simple calculation tells us that $K_1^0 = 68745 > K_1$. The simulation result is plotted in Fig. 4.2.

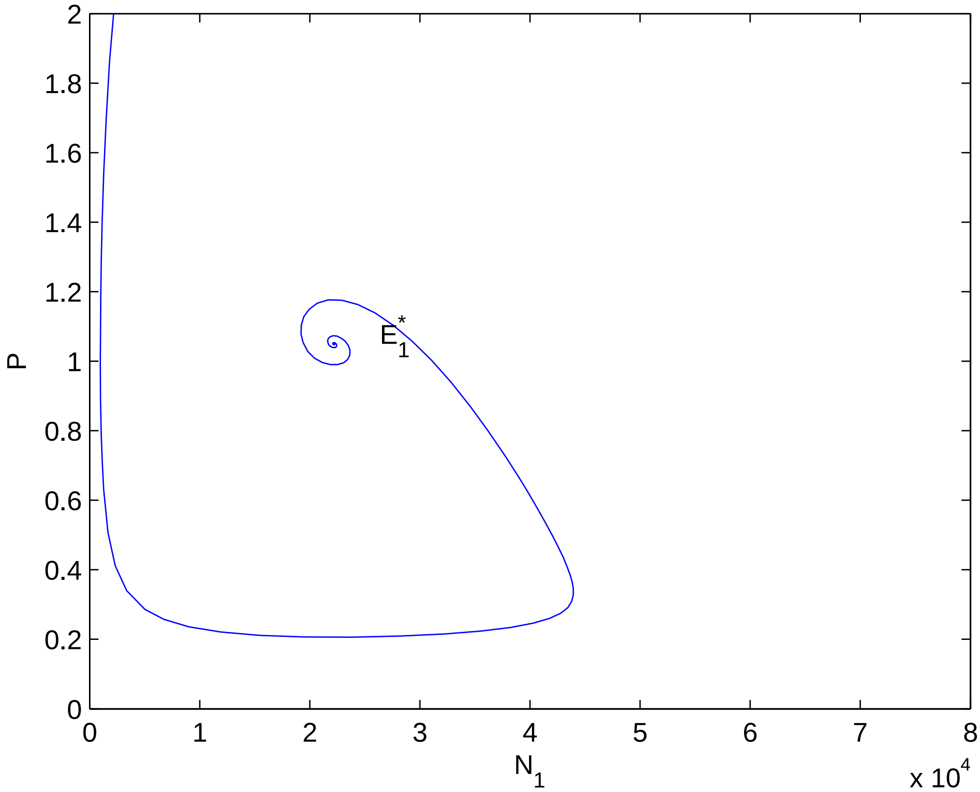

Figure 4.2: 2-D model: stable interior equilibrium ($F_1(K_1) > D/B_1$ and $K_1^0 > K_1$)

If $F_1(K_1) > D/B_1$ and $K_1 > K_1^0$, then there exists at least one stable limit cycle. In this simulation, we choose $r_1 = 0.0005$, $K_1 = 75000$, $C_1 = 0.0005$, $h_1 = 1/16$, $\mu_1 = 1/64$, $B_1 = 0.000034$, and $D = 0.0002$. So $K_1^0 = 68745 < K_1$. The simulation result is plotted in Fig. 4.3.

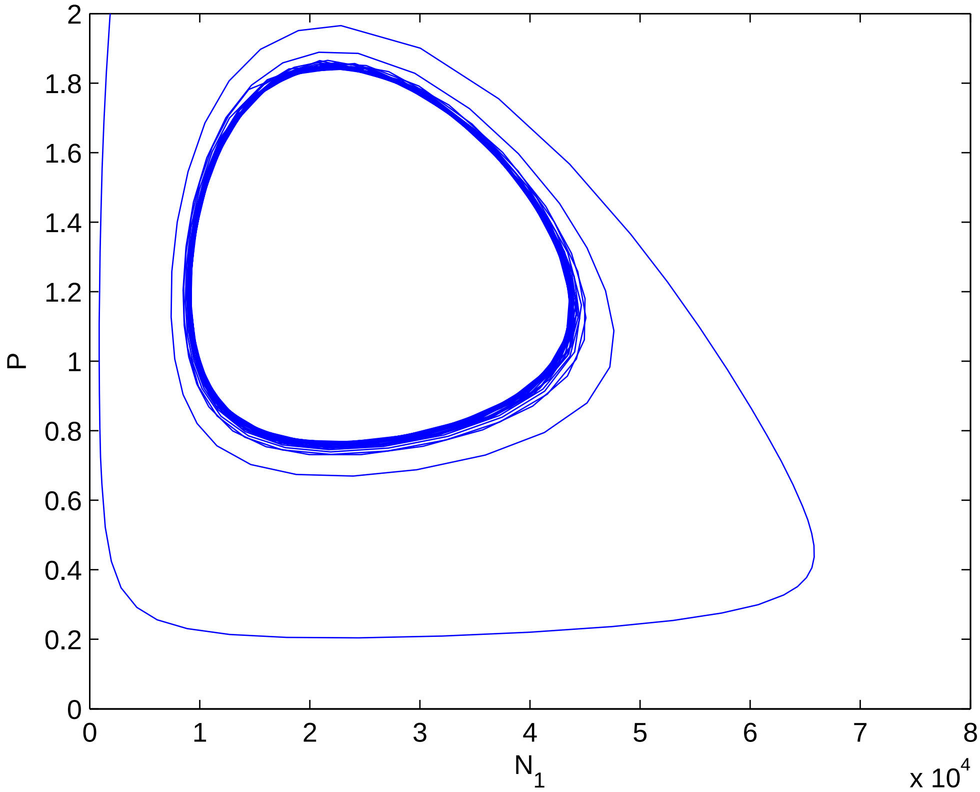

Figure 4.3: 2-D model: stable limit cycle ($F_1(K_1) > D/B_1$ and $K_1^0 < K_1$)

If $F_1(K_1) > D/B_1$ and $K_1 = K_1^0$, then there is a supercritical Hopf bifurcation. Here we choose $r_1 = 0.0005$, $K_1 = K_1^0 = 68745$, $C_1 = 0.0005$, $h_1 = 1/16$, $\mu_1 = 1/64$, $B_1 = 0.000034$, and $D = 0.0002$. We used *Matcont*—a continuation package of Matlab (see [15]) to detect a Hopf bifurcation at the equilibrium $(N_1, P) = (22199.3989771.27760668744.806040)$. The first Lyapunov coefficient is computed as $-3.166062e^{-13}$, which shows that it is supercritical. The simulation result is plotted in Fig. 4.4.

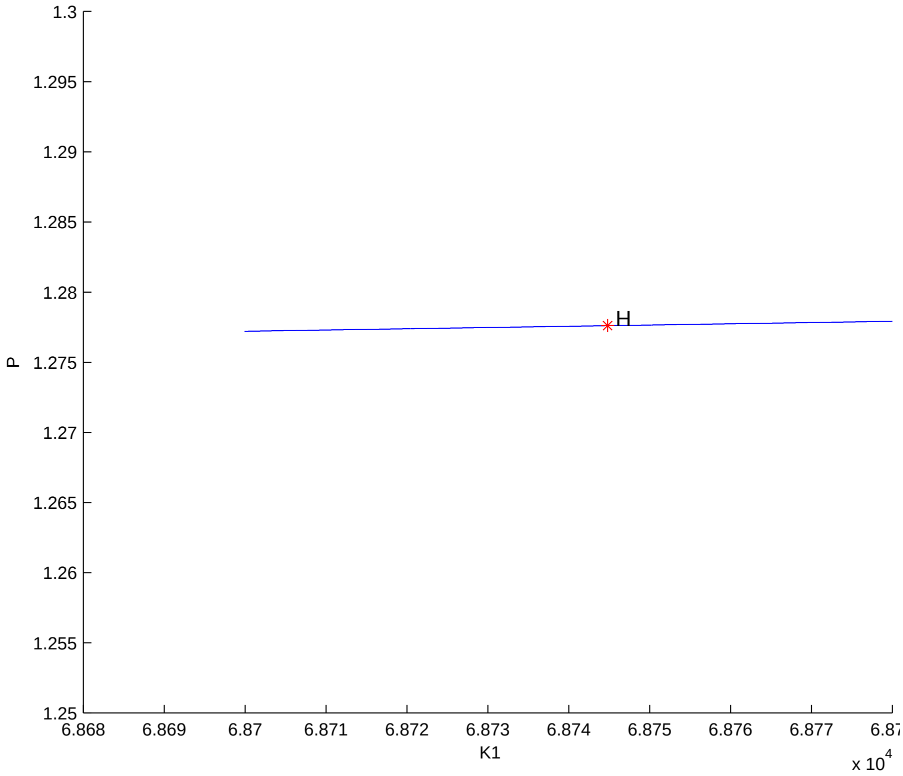

Figure 4.4: 2-D model: Hopf bifurcation ($F_1(K_1) > D/B_1$ and $K_1 = K_1^0$)

4.2 The 3-D Model

The 3-D model is much more complicated. We found some new phenomena which we were unable to analyze analytically.

First we demonstrate that the bistable phenomenon can occur. The parameters we choose are: $r_1 = 0.0005$, $r_2 = 0.011$, $K_1 = 79000$, $K_2 = 40000$, $C_1 = 0.0005$, $C_2 = 0.005$, $h_1 = 1/16$, $h_2 = 1/200$, $\mu_1 = 1/32$, $\mu_2 = 1/400$, $B_1 = 0.000034$, $B_2 = 0.000056$, $\beta_{12} = 0.55$, $\beta_{21} = 1$, $D = 0.0002$. Then we vary the initial point. We fix $N_1(0) = 26040$ and $P(0) = 1.8117$. As we slowly increase $N_2(0)$ from 0.1 to 10, we found that if $N_2(0) \leq 1.04$, then the orbit is attracted to the stable limit cycle in the (N_1, P)– plane, and if $N_2(0) \geq 1.05$, then the orbit is attracted to the stable limit cycle in the (N_2, P)– plane. See Fig. 4.5.

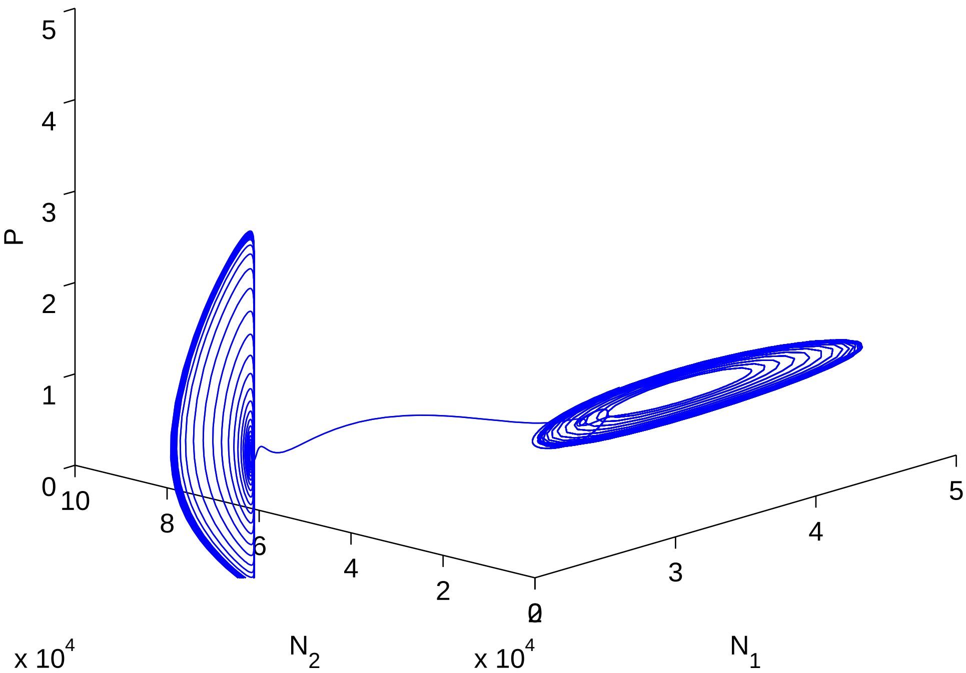

Figure 4.5: 3-D model: a bistable phenomenon

Also we found that a similar phenomenon occurs if all the parameters other than r_2 are fixed. The parameters we chose are: $r_1 = 0.0005, K_1 = 79000, K_2 = 40000, C_1 = 0.0005, C_2 = 0.005, h_1 = 1/16, h_2 = 1/200, \mu_1 = 1/32, \mu_2 = 1/400, B_1 = 0.000034, B_2 = 0.000056, \beta_{12} = 0.55, \beta_{21} = 1, D = 0.0002$. We set $N_1(0) = 26040, N_2(0) = 100$ and $P(0) = 1.8117$. As we slowly increase r_2 from 0.010 to 0.011, we found that if $r_2 \leq 0.0101$, then the orbit is attracted to the stable limit cycle in the (N_1, P)– plane, and if $r_2 \geq 0.0102$, then the orbit is attracted to the stable limit cycle in the (N_2, P)– plane. See Fig. 4.6.

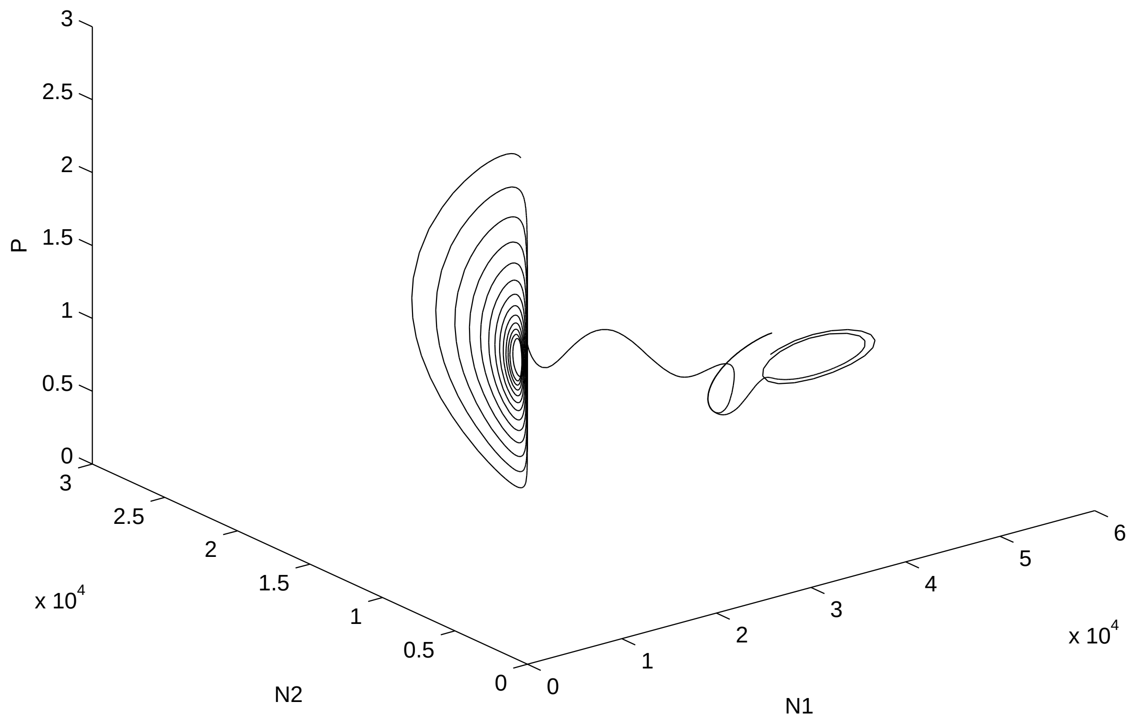

Figure 4.6: 3-D model: bifurcation about r_2

However, if we modify the parameter values above to satisfy the conditions in Theorem 3.21, we found chaotic dynamics in the interior of the first octant of the (N_1, N_2, P) space. Here are the parameters we chose: $r_1 = 0.0005, r_2 = 0.011, K_1 = 79000, K_2 = 40000, C_1 = 0.0005, C_2 = 0.005, h_1 = 1/10, h_2 = 1/150, \mu_1 = 1/32, \mu_2 = 1/400, B_1 = 0.00034, B_2 = 0.000056, \beta_{12} = 0.55, \beta_{21} = 0.3, D = 0.0002$. We set $N_1(0) = 26040, N_2(0) = 1000$ and $P(0) = 1.8117$. Then we see that a strange attractor occurs. Fig. 4.7 shows the strange attractor and Fig. 4.8 shows the frequency of N_1 and N_2 versus time t.

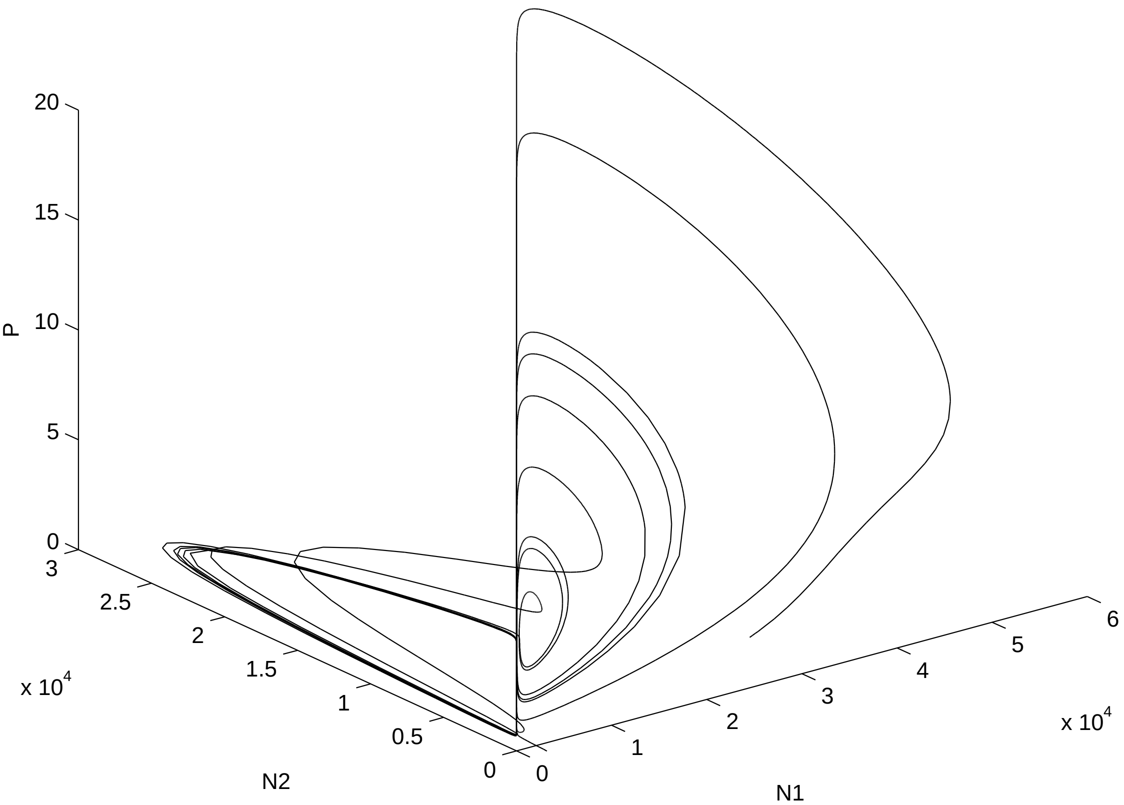

Figure 4.7: 3-D model: the strange attractor

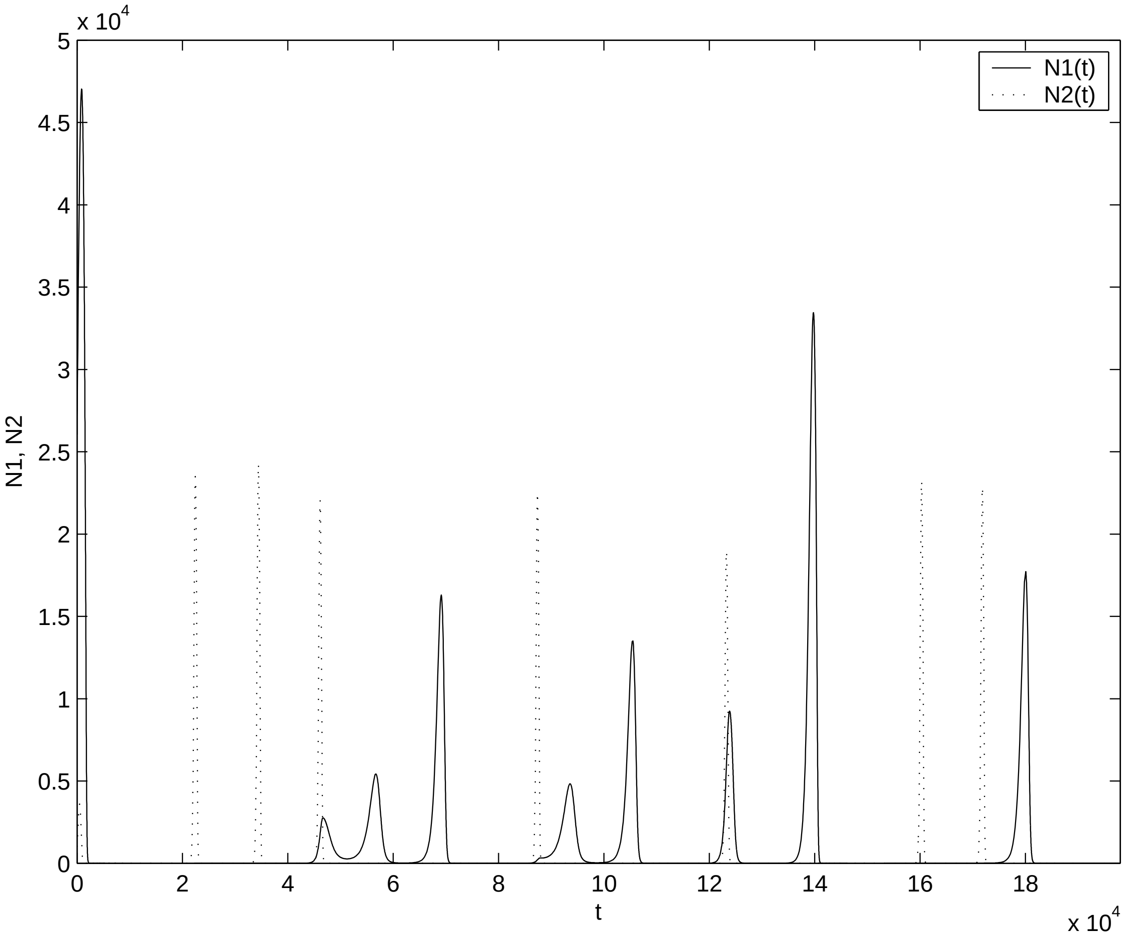

Figure 4.8: 3-D model: the frequency of N_1 and N_2 over time t

CHAPTER 5

DISCUSSIONS AND FUTURE DIRECTIONS

5.1 Nonmonotonic Functional Response

As discussed in Section 2 for the one prey, one predator subsystem, if $h_1 < 2\mu_1$, then F_1 defined in (3.1.2) is monotonically increasing when $N_1 \in [0, \frac{1}{C_1(2\mu_1 - h_1)}]$ and is monotonically decreasing when $N_1 \in [\frac{1}{C_1(2\mu_1 - h_1)}\ ,\ +\infty)$. It attains maximum at $N_1 = \frac{1}{C_1(2\mu_1 - h_1)}$. As N_i approaches positive infinity, F_1 approaches $\frac{h_1 - \mu_1}{h_1^2}$. By some algebraic calculation, we can rewrite F_1 as

$$F_1(N_1) = \frac{h_1 - \mu_1}{h_1^2} + \frac{\frac{1}{h_1^2}\left[h_1 C_1(2\mu_1 - h_1)N_1 + (\mu_1 - h_1)\right]}{(1 + h_i C_1 N_1)^2}. \tag{5.1.1}$$

In [50], the authors conducted a detailed bifurcation analysis about the type of functional response of type

$$p(x) = \frac{mx}{a^2x^2 + bx + 1}. \tag{5.1.2}$$

And they found, "The bifurcation sequences involving Hopf bifurcations, homoclinic bifurcations, as well as the saddle-node bifurcations of limit cycles are determined using information from the complete study of the Bogdanov-Takens bifurcation point of codimension 3 and the geometry of the system". If we let $\mu_1 < h_1 < 2\mu_1$ and set the numerical part in (5.1.1) as a new variable n_1, then we can transform $F_1(N_1)$ into $F_1(n_1)$ which is of the type in (5.1.2). Consequently, complicated bifurcation phenomena are expected.

However, note that the limit in (5.1.1) can be negative as N_1 approaches infinity if $0 < h_1 < \mu_1$. This is another new type of functional response which means that the predator can have positive impacts on their prey. "These positive feedbacks have the potential to change predictions based on food web theory, such as the assertion that enrichment is destabilizing," see [5].

To our knowledge, not much is known for the case of two prey−one predator system with nonmonotonic functional response.

5.2 Future Directions

As we have seen, the exploration of the dynamical behavior of this system is far from being completed. The following three aspects seem to be interesting for future study.

The first is to study higher order bifurcations in the 2D model. This is because, in the planar case, we have tools to apply.

The second is to study the relationship between the parameters. As we can see, in the 2D model, we have seven parameters, of course, by some substitutions of variable we may delete at least three of them. However, if one reduces the parameters, the structure of the original system may not be the same as the old one. That is why in this thesis we never did a structural reduction. And normal form theory also is not applied here.

The third is to extend the model to a wider setting. For example, we may consider the following spatiotemporal model:

$$\begin{aligned} \frac{dN_i}{dt} &= d_i \nabla^2 N_i + \quad r_i N_i \left(1 - \frac{N_i + \sum\limits_{j=1, j\neq i}^{n} \beta_{ij} N_j}{K_i} \right) - P f_i(\mathbf{N}) \left(1 - \frac{a_i T_i f_i(\mathbf{N})}{M_i} \right), \\ \frac{dP}{dt} &= \quad d_p \nabla^2 P + P \left(\sum_{j=1}^{n} B_j f_j(\mathbf{N}) \left(1 - \frac{a_j T_j f_j(\mathbf{N})}{M_j} \right) - D \right), \end{aligned} \tag{5.2.1}$$

for $i = 1, 2, \ldots, n$. Under some reasonable boundary conditions, we expect the system to exhibit some interesting patterns.

Bifurcations, Normal Forms and their Applications

PART II
First Integrals and Normal Forms for Germs of Analytic Vector Fields

by

Jian Chen

CHAPTER 6

INTRODUCTION

6.1 Motivation

Normal form is a technique for transforming ordinary differential equations describing nonlinear vector fields into certain standard forms. Using a particular class of coordinate transformations, one can remove the inessential part of higher-order nonlinearities. The investigation of the normal form theory can be traced back to Poincaré and even earlier. This tool or method is extremely useful in the studies of bifurcation of periodic orbits, KAM theory, (see for instance , [3, 8, 29] and the references therein), stability problems and so on.

The reductions to normal forms is realized by means of power series in the deviation from the equilibrium position or periodic motion. The series is not always convergent. So the existence of the convergent normalization is a fundamental problem in normal form theory. In particular, the existence of analytic normalizations is highly related to the existence of analytic first integrals and resonances of the dynamical system, see [22, 23, 29, 52].

For a standard Hamiltonian flow on a symplectic manifold with a Hamiltonian $H(x, y)$ starting from the second order terms, with n degrees of freedom, let $\lambda = (\lambda_1, \ldots, \lambda_n, \lambda_{n+1}, \ldots, \lambda_{2n})$ be the $2n$-tuple of eigenvalues of the linear part of the Hamiltonian flow. Without loss of generality, we can set $\lambda_k = -\lambda_{n+k}$. Assume that $\lambda_1, \ldots, \lambda_n$ are non-resonant in the sense that $n_1\lambda_1 + \ldots + n_n\lambda_n \neq 0$ for $(n_1, \ldots, n_n) \in \mathbb{Z}^n \setminus \{0\}$. This implies that the $2n$-tuple has exactly n independent resonances. It is well-known that if the symplectic transformation reducing H to the Birkhoff normal form, leaving unchanged the Hamiltonian character of the flow, is convergent, then the Hamiltonian system has exactly n functionally independent convergent first integrals. However, the reverse is not true. Let Ω_H be the set of Hamiltonians having the same second order terms as those of H. Siegel [43] proved that there exists a dense subset of Ω_H endowed with the coefficient topology, in which every Hamiltonian only has itself as the functionally independent convergent first integral, and consequently it cannot be reduced to the Birkhoff normal form by an analytic symplectic transformation. Of course, any Hamiltonian vector field in Ω_H has exactly n functionally independent formal first integrals. For the eigenvalues $\lambda_1, \ldots, \lambda_n$ not resonant or simple resonant, Ito, using the fast convergent method in [22] and [23] respectively, proved that if the Hamiltonian is integrable, i.e. having n functionally independent first integrals in involution, then it is analytically symplectically equivalent to the Birkhoff normal form. Recently, Zung [52] proved that any analytically integrable Hamiltonian system, without any restriction on the resonance of $\lambda_1, \ldots, \lambda_n$, is analytically symplectically equivalent to the Birkhoff normal form using a geometrical method involving holomogical cycles and torus actions. Also Pérez-Marco [36, 35] obtained some sharp results on the convergence and generic divergence of the normalizations for Hamiltonian and non-Hamiltonian flows.

For non-Hamiltonian flows, the existence of first integrals is much more involved. In [28], Llibre and coauthors proved that for an analytic, or a formal, autonomous system with a singularity, if one of the eigenvalues vanishes and others are non-resonant then the system has a formal first integral in a vicinity of the singularity if and only if the singularity is non-isolated. In the planar setting the result is in the analytic category. For a planar analytic vector field having a singularity, if the eigenvalues, denoted by λ_1, λ_2, are resonant and non-zero, then the vector field is locally analytically integrable if and only if it is analytically equivalent to

$$\dot{x} = \lambda_1 x(1 + g(z)), \qquad \dot{y} = \lambda_2 y(1 + g(z)),$$

where g is an analytic function in $z = x^r y^s$ with $r, s \in \mathbb{N}$ relatively prime and $r/s = -\lambda_2/\lambda_1$ (see for instance, [29, 49]).

On the relation between the existence of analytic first integrals and the convergence of normalizations for an analytic vector field, Zung [51] proved the following result: *Let* $\mathbf{X}$ *be a locally analytic vector field in* $(\mathbb{F}^n, 0)$ *with* $\mathbf{X}(0) = 0$. *Suppose that there are* m, $1 \leq m \leq n$, *locally analytic vector fields* $\mathbf{X}_1 = \mathbf{X}, \mathbf{X}_2, \ldots, \mathbf{X}_m$ *commuting pairwise, i.e.* $[\mathbf{X}_j, \mathbf{X}_k] = 0$, *and linearly independent almost everywhere, i.e.* $\mathbf{X}_1 \Lambda \ldots \Lambda \mathbf{X}_m \neq 0$. *If there are* $(n - m)$ *locally analytic and functionally independent functions* $f_1, \ldots, f_{n-m}$ *which are the common first integrals of* $\mathbf{X}_1, \ldots, \mathbf{X}_m$, *i.e.* $\mathbf{X}_j(f_k) = 0$, $j = 1, \ldots, m$, $k = 1, \ldots, n - m$, *then the vector field* $\mathbf{X}$ *has a locally analytic normalization in* $(\mathbb{F}^n, 0)$.

6.2 Main Results

We would like to estimate the number of first integrals for the following vector field:

$$\begin{aligned} \dot{\theta} &= \omega + \Omega(\theta, x), \\ \dot{x} &= Ax + f(\theta, x), \end{aligned} \qquad (\theta, x) \in \mathbb{F}^m \times \mathbb{F}^n, \quad \mathbb{F} = \mathbb{R} \text{ or } C \tag{6.2.1}$$

where $\Omega = O(\|x\|)$ and $f = O(\|x\|^2)$ are analytic functions in their variables, and 2π periodic in θ . In what follows we denote by $\mathcal{X}$ the vector field defined in (6.2.1).

A non-constant function $H(\theta, x)$ is an *analytic first integral* (respectively, a *formal first integral*) of $\mathcal{X}$ if it is analytic (respectively, a formal power series) in its variables and 2π periodic in θ, and the derivative of $H(\theta, x)$ along the flow of $\mathcal{X}$ vanishes, i.e. $\mathcal{X}(H) \equiv 0$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the n-tuple of eigenvalues of the matrix A, and γ the rank of the set $\{(k, l) : i\langle k, \omega\rangle + \langle l, \lambda\rangle = 0, k \in \mathbb{Z}^m, l \in \mathbb{Z}_+^n\}$, where $\mathbb{Z}$ stands for the group of integers, $\mathbb{Z}_+$ the set of non-negative integers, $i := \sqrt{-1}$, and $\langle \cdot, \cdot \rangle$ the usual inner product of two vectors. We have the following

Theorem 6.1. *For the vector field (6.2.1), the number of functionally independent analytic first integrals in a neighborhood of the constant solution $x = 0$ is less than or equal to γ.*

We mention here that this number γ can be reached, for example, in completely integrable non-resonant Hamiltonian vector fields. Theorem 6.1 is an extension of the following classical result due to Poincaré [37] (for a proof, see for instance [17]).

Theorem (H. Poincaré). *For an autonomous system defined by the second equation of (6.2.1), if the n-tuple λ of eigenvalues of the matrix A does not satisfy any resonant conditions, i.e. $\langle l, \lambda \rangle \neq 0$ for all $l = (l_1, \ldots, l_n) \in \mathbb{Z}_+^n$ and $|l| = l_1 + \ldots + l_n \neq 0$, then the system does not have any analytic first integrals in a neighborhood of $x = 0$.*

We note that Theorem 6.1 also generalizes the results given in Theorem 5 of [28] on periodic vector fields of the type $\dot{x} = A(t)x + f(t, x)$ for $x \in \mathbb{C}^n$. However, the condition of Theorem 6.1 is not a necessary condition. For instance, if a germ of planar analytic systems has a pair of pure imaginary eigenvalues at the origin, it may have no analytic first integrals in some neighborhood of the origin.

Related to the above results, we have the following

Theorem 6.2. *Given an analytic vector field $\widetilde{\mathcal{X}}$ in $\mathbb{C}^{2n}$ having the origin as a singularity. Let $(\lambda, \mu) = (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$ be the $2n$-tuple of eigenvalues of $\widetilde{\mathcal{X}}$ at the origin. Assume that λ_j, μ_j are non-zero and pairwise resonant for $j = 1, \ldots, n$, and $\lambda_1, \ldots, \lambda_n$ are non-resonant. If $\widetilde{\mathcal{X}}$ has n analytically functionally independent first integrals in a neighborhood of the origin, then the following holds.*

(a) *The vector field $\widetilde{\mathcal{X}}$ is formally equivalent to*

$$\begin{aligned} \dot{u}_j &= \lambda_j u_j (1 + W_j(z_1, \ldots, z_n)), \\ \dot{v}_j &= \mu_j v_j (1 + W_j(z_1, \ldots, z_n)), \end{aligned} \qquad j = 1, \ldots, n \tag{6.2.2}$$

where W_j is a formal power series in $z_1, \ldots, z_n$ with $z_s = u_s^{\bar{n}_s} v_s^{\bar{m}_s}$, where $\bar{n}_s, \bar{m}_s \in \mathbb{N}$ relatively prime and $\bar{m}_s / \bar{n}_s = -\lambda_s / \mu_s$.

(b) *If either the n-tuple of eigenvalues λ belong to the Poincaré domain, or the formal power series W_j are all equal and $|\langle k, \lambda \rangle - \lambda_j| \geq \epsilon > 0$ for some constant ϵ and $k \in \mathbb{Q}^n$ the field of rational numbers, then the equivalence in the statement* (a) *is analytic.*

(c) *A formal power series is a first integral of (6.2.2) in $u_1, \dots, u_n, v_1, \dots, v_n$ if and only if it is a power series in the n variables $z_1, \dots, z_n$. (This kind of first integral is called universal.)*

(d) *Let $\mathcal{V}$ be the set of vector fields having the same linear part as that of $\widetilde{\mathcal{X}}$. If there exists a vector field in $\mathcal{V}$ with the divergent distinguished normal form* (*respectively, normalization*)*, then generic vector fields in $\mathcal{V}$ have this property.*

The *distinguished normal form* will be defined in the proof of Theorem 6.1. We recall that $\lambda = (\lambda_1, \dots, \lambda_n)$ is in the *Poincaré domain* if the convex hull of the n points $\lambda_1, \dots, \lambda_n$ in $\mathbb{C}$ does not contain the origin of the complex plane. Two vector fields in $\mathbb{C}^m$ are *formally equivalent* if they can be exchanged into each other by a formal series transformation f satisfying $f(0) = 0$ and $Df(0) = I$, and *analytically equivalent* if the transformation is analytic. In the statement (d), the *genericity* is in the sense of Lemma 7.5 below.

We note that for planar vector fields the conditions in the statement (b) hold naturally. Consequently, in this case the vector field $\widetilde{\mathcal{X}}$ is analytically equivalent to (6.2.2).

In order to prove the statement (b) we need to use the majorant series. In the proof of the statement (d) we will get the help from pluripotential theory in the complex domain.

For a given vector field $\mathcal{Z}$ in $\mathbb{C}^n$ with a singularity at the origin, similar to the statement (c) of the last theorem we have the following. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be the n-tuple of eigenvalues of $\mathcal{Z}$ at the origin. Denote by $\mathcal{M}$ the sublattice of $k \in \mathbb{Z}_+^n$

satisfying $\langle k, \lambda\rangle = 0$ and g.c.d.$(k_1, \ldots, k_n) = 1$. We note that even when $\lambda \neq 0$ it is possible that $\#\mathcal{M} = n$. For instance $\lambda = (1, 1, -2)$, $\mathcal{M} = \{(1, 1, 1), (2, 0, 1), (0, 2, 1)\}$.

Proposition 6.3. *If $\mathcal{Z}$ is in the distinguished normal form, then its formal first integral is a formal power series in the $\#\mathcal{M}$ variables $z_j = x^k$, $k \in \mathcal{M}$, where we have used the multi-index $x^k = x_1^{k_1} \ldots x_n^{k_n}$.*

This proposition can be proved easily by combining some linear algebra, the details will be omitted.

Theorem 6.4. *For a planar analytic flow with a singularity, if the eigenvalues of the flow at the singularity satisfy a unique linearly independent resonant condition and the flow has an analytic first integral in a neighborhood of the singularity, then either the singularity is non-isolated or the flow is analytically orbitally equivalent to a linear one.*

A planar Hamiltonian system is always completely integrable in the conventional sense. In the case that the Hamiltonian system has a center, related to the periods of closed orbits in the central annulus, the following result is well known.

Theorem 6.5. *A planar analytic Hamiltonian system has an isochronous center if and only if it is analytically linearizable.*

We will provide a new proof of the theorem by using the Euler-Lagrange equation. For the characterization of isochronous centers, we refer to [7, 9, 24] and the references therein.

The next chapter will be solely devoted to the proofs of the main theorems above. Particularly, in Section 7.1 we prove Theorem 6.1. The proofs of Theorems 6.2 and 6.4 are given in 7.2 and 7.3, respectively. In Section 7.4 we give a new proof of Theorem 6.5.

CHAPTER 7

PROOF OF THE MAIN THEOREMS

7.1 Proof of Theorem 6.1

For the vector field $\mathcal{X}$ given in (6.2.1) we say that the n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$ of eigenvalues of the matrix A is *non-resonant* if for all $k \in \mathbb{Z}^m$, $l \in \mathbb{Z}_+^n$ and $|l| = l_1 + \ldots + l_n > 1$, the following hold

$$\langle l, \lambda \rangle \neq i \langle k, \omega \rangle, \quad \langle l, \lambda \rangle - \lambda_j \neq i \langle k, \omega \rangle, \qquad j = 1, \ldots, n. \tag{7.1.1}$$

The n-tuple λ is *weakly non-resonant* if the conditions (7.1.1) hold except for the case $k = 0$.

Set $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_h$, where $\mathcal{X}_1 = \langle \omega, \partial_\theta \rangle + \langle Ax, \partial_x \rangle$ and $\mathcal{X}_h$ are the higher order terms. Since the algebra of linear vector fields in $\mathbb{F}^n$, under the standard Lie bracket, is nothing but the reductive algebra $gl(n, \mathbb{F}) = sl(n, \mathbb{F}) \oplus \mathbb{F}$, we write $A = A_1 + A_2$ with A_1 being semisimple and A_2 nilpotent. Correspondingly we separate $\mathcal{X}_1 = \mathcal{X}_1^s + \mathcal{X}_1^n$ with $\mathcal{X}_1^s = \langle \omega, \partial_\theta \rangle + \langle A_1 x, \partial_x \rangle$ called the *semisimple part* of $\mathcal{X}_1$ and $\mathcal{X}_1^n = \langle A_2 x, \partial_x \rangle$ called the *nilpotent part* of $\mathcal{X}_1$. Without loss of generality, we assume that

$$\mathcal{X}_1^s = \langle \omega, \partial_\theta \rangle + \langle \lambda x, \partial_x \rangle,$$

where $\lambda x = (\lambda_1 x_1, \ldots, \lambda_n x_n)$.

The vector field $\mathcal{X}$ is in *normal form* if the Lie bracket of $\mathcal{X}_1^s$ and $\mathcal{X}_h$ vanishes, i.e. $[\mathcal{X}_1^s, \mathcal{X}_h] = 0$. We note that for a vector field of type (6.2.1) in normal form, all *pseudomonomials* $e^{i\langle k, \theta \rangle} x^l$, are resonant. Specifically, if $e^{i\langle k, \theta \rangle} x^l$ is in the component ∂_{θ_j}, then $i\langle k, \omega \rangle + \langle l, \lambda \rangle = 0$ is called *in the first resonant.* And if in the component ∂_{x_j}, then $i\langle k,\omega\rangle+\langle l,\lambda\rangle=\lambda_j$ is called *in the second resonant.* A pseudomonomial $e^{i\langle k,\theta\rangle}x^l$ of an analytic or a formal quasi-periodic function is called *resonant* if $i\langle k,\omega\rangle+\langle l,\lambda\rangle=0$.

Usually, a transformation reducing a vector field to its normal form is not unique. In what follows, we call such a transformation *distinguished normalization* if the transformation contains non-resonant terms only. The distinguished normalization is unique. Correspondingly, the normal form is called a *distinguished normal form.*

The following result due to Bibikov [4] is the key to prove the following Lemma 7.2.

Lemma 7.1. *Denote by $\mathcal{G}^r(\mathbb{F})$ the linear space of n-dimensional vector-valued homogeneous polynomials of degree r in n variables with coefficients in $\mathbb{F}$. Let A and B be two nth square matrix with entries in $\mathbb{F}$, and their n-tuple of eigenvalues be λ and κ, respectively. Define a linear operator L on $\mathcal{G}^r(\mathbb{F})$ as follows,*

$$Lh=\langle\partial_x h, Ax\rangle-Bh,\qquad h\in\mathcal{G}^r(\mathbb{F}).$$

Then the spectrum of the operator L is

$$\{\langle l,\lambda\rangle-\kappa_j;\ l\in\mathbb{Z}_+^n,\ |l|=r,\ j=1,\dots,n\}$$

.

Our next result will be used in the proof of Theorem 6.1. We note that it is an extension to the classical Poincaré-Dulac normal form theorem on autonomous systems, and to Lemma 6 in [27] on periodic systems.

Lemma 7.2. *The vector field $\mathcal{X}$ defined in (6.2.1) can be formally normalized by a distinguished normalization.*

Proof. Assume that the vector field $\mathcal{X}$ is transformed to

$$\dot\beta=\omega+\Lambda(\beta,y),\qquad \dot y=Ay+g(\beta,y),\tag{7.1.2}$$

under the transformation

$$\theta = \beta + \phi(\beta, y), \qquad x = y + \psi(\beta, y), \tag{7.1.3}$$

where $\Lambda, \phi = O(\|y\|)$ and $g, \psi = O(\|y\|^2)$ are 2π-periodic in β. Then ϕ, ψ satisfy the following

$$\begin{aligned} \langle \partial_\beta \phi, \omega \rangle + \langle \partial_y \phi, Ay \rangle &= \Omega(\beta + \phi, y + \psi) - \Lambda(\beta, y) - \langle \partial_\beta \phi, \Lambda \rangle - \langle \partial_y \phi, g \rangle, \\ \langle \partial_\beta \psi, \omega \rangle + \langle \partial_y \psi, Ay \rangle - A\psi &= f(\beta + \phi, y + \psi) - g(\beta, y) - \langle \partial_\beta \psi, \Lambda \rangle - \langle \partial_y \psi, g \rangle. \end{aligned} \tag{7.1.4}$$

Expanding the considered functions in Taylor series in y

$$V(\beta, y) = \sum_r V_r(\beta, y) \qquad \text{for } V \in \{\Lambda, g, \phi, \psi, \Omega, f\} \tag{7.1.5}$$

where V_r is a homogeneous polynomial of degree r in y with 2π periodic coefficients in β. The system of equations (7.1.4) is equivalent to

$$\begin{aligned} \langle \partial_\beta \phi_r, \omega \rangle + \langle \partial_y \phi_r, Ay \rangle &= \Omega_r - \Lambda_r - p_r, \\ \langle \partial_\beta \psi_{r+1}, \omega \rangle + \langle \partial_y \psi_{r+1}, Ay \rangle - A\psi_{r+1} &= f_{r+1} - g_{r+1} - q_{r+1}, \end{aligned} \qquad r = 1, 2, \dots \tag{7.1.6}$$

where p_r, q_{r+1} are known inductively. In precisely, p_r is a polynomial in $\phi_s, \Lambda_s, g_{s+1}$ with $s = 1, \dots, r-1$; q_{r+1} is a polynomial in $\psi_s, \Lambda_{s-1}, g_s$ with $s = 2, \dots, r$.

Make the Fourier expansions on V_r,

$$V_r(\beta, y) = \sum_{k \in \mathbb{Z}^m} V_r^k(y) e^{i\langle k, \beta \rangle}, \qquad \text{for } V \in \{\Lambda, g, \phi, \psi, \Omega, f\} \tag{7.1.7}$$

From (7.1.6) we obtain

$$\begin{aligned} \mathcal{A}_0 \phi_r^k &= \Omega_r^k - \Lambda_r^k - p_r^k, \\ \mathcal{A}_1 \psi_{r+1}^k &= f_{r+1}^k - g_{r+1}^k - q_{r+1}^k, \end{aligned} \qquad r = 1, 2, \dots \tag{7.1.8}$$

where $\mathcal{A}_s = i\langle k, \beta \rangle + L_s$, $s = 0, 1$, and L_0 and L_1 are the linear operators on $\mathcal{G}^r(y)$ and $\mathcal{G}^{r+1}(y)$ respectively, defined by

$$\begin{aligned} L_0 h(y) &= \langle \partial_y h, Ay \rangle, \quad h \in \mathcal{G}^r(y) \\ L_1 h(y) &= \langle \partial_y h, Ay \rangle - Ah, \quad h \in \mathcal{G}^{r+1}(y). \end{aligned}$$

Applying Lemma 7.1 to the operators $\mathcal{A}_0$ and $\mathcal{A}_1$ in (7.1.8), we obtain

$$\begin{aligned}\text{the spectrum of } \mathcal{A}_0 &= \{i\langle k,\beta\rangle + \langle l,\lambda\rangle;\ l\in\mathbb{Z}_+^n, |l|=r\},\\ \text{the spectrum of } \mathcal{A}_1 &= \{i\langle k,\beta\rangle + \langle l,\lambda\rangle - \lambda_j;\ l\in\mathbb{Z}_+^n, |l|=r+1, j=1,\dots,n\}.\end{aligned}$$

According to the operator $\mathcal{A}_0$ (respectively, $\mathcal{A}_1$), we separate the space $\mathcal{G}^r(y)$ into the direct sum $\mathcal{G}^r(y) = \mathcal{G}^r_{0,1}(y)\oplus\mathcal{G}^r_{0,2}(y)$ (respectively, $\mathcal{G}^{r+1}(y) = \mathcal{G}^{r+1}_{1,1}(y)\oplus\mathcal{G}^{r+1}_{1,2}(y)$) in such a way that $\mathcal{A}_0$ restricted to $\mathcal{G}^r_{0,1}(y)$, denoted by $\mathcal{A}_0^1$, is invertible, and restricted to $\mathcal{G}^r_{0,2}(y)$, denoted by $\mathcal{A}_0^2$, is degenerated, i.e. having only zero spectrum (respectively, $\mathcal{A}_1$ restricted to $\mathcal{G}^{r+1}_{1,1}(y)$, denoted by $\mathcal{A}_1^1$, is invertible, and restricted to $\mathcal{G}^{r+1}_{1,2}(y)$, denoted by $\mathcal{A}_1^2$, is degenerated). Decompose the right hand side of (7.1.8) as $\Omega_r^k - \Lambda_r^k - p_r^k = \left(\Omega_r^{k,1}-\Lambda_r^{k,1}-p_r^{k,1}\right)\oplus\left(\Omega_r^{k,2}-\Lambda_r^{k,2}-p_r^{k,2}\right)\in\mathcal{G}^r_{0,1}(y)\oplus\mathcal{G}^r_{0,2}(y)$, and $f^k_{r+1}-g^k_{r+1}-q^k_{r+1} = \left(f^{k,1}_{r+1}-g^{k,1}_{r+1}-q^{k,1}_{r+1}\right)\oplus\left(f^{k,2}_{r+1}-g^{k,2}_{r+1}-q^{k,2}_{r+1}\right)\in\mathcal{G}^{r+1}_{1,1}(y)\oplus\mathcal{G}^{r+1}_{1,2}(y)$. Then equations (7.1.8) are the same as the following

$$\begin{aligned}\mathcal{A}_0^s\phi_r^{k,s} &= \Omega_r^{k,s}-\Lambda_r^{k,s}-p_r^{k,s},\\ \mathcal{A}_1^s\psi_{r+1}^{k,s} &= f_{r+1}^{k,s}-g_{r+1}^{k,s}-q_{r+1}^{k,s},\end{aligned}\qquad r=1,2,\dots, s=1,2 \tag{7.1.9}$$

For $s=1$, since the operators in (7.1.9) are invertible, for any choice of $\Lambda_r^{k,s}$ and $g_{r+1}^{k,s}$ the equations have a unique solution. In order for obtaining the distinguished normal form, we choose $\Lambda_r^{k,1} = g_{r+1}^{k,1} = 0$, then we obtain a unique solution $\phi_r^{k,1}$ and $\psi_{r+1}^{k,1}$ corresponding to the two equations in (7.1.9), respectively. For $s=2$, since the operators in (7.1.9) are degenerated, choosing $\Lambda_r^{k,2} = \Omega_r^{k,2}-p_r^{k,2}$ and $g_{r+1}^{k,2} = f_{r+1}^{k,2}-q_{r+1}^{k,2}$, we have $\phi_r^{k,2} = \psi_{r+1}^{k,2} = 0$.

Summarizing the above process, we obtain a formal transformation

$$\theta = \beta + \sum_{k\in\mathbb{Z}_+^m,\, r\geq 1}\phi_r^{k,1}(y)e^{i\langle k,\beta\rangle},\qquad x = y + \sum_{k\in\mathbb{Z}_+^m,\, r\geq 2}\psi_r^{k,1}(y)e^{i\langle k,\beta\rangle},$$

where all the components in the summations are non-resonant, under which the vector field $\mathcal{X}$ is transformed into

$$\dot\beta = \omega + \sum_{k\in\mathbb{Z}^m,\, r\geq 1}\Lambda_r^{k,2}(y)e^{i\langle k,\beta\rangle},\qquad \dot y = Ay + \sum_{k\in\mathbb{Z}^m,\, r\geq 2} g_r^{k,2}(y)e^{i\langle k,\beta\rangle},$$

where each component in the summations is resonant.

Denote by $\mathcal{Y}$ the last vector field, and write it in the form $\mathcal{Y} = \mathcal{Y}_1 + \mathcal{Y}_h$ with $\mathcal{Y}_1 = \langle \omega, \partial_\beta \rangle + \langle Ay, \partial_y \rangle$ and

$$\mathcal{Y}_h = \sum_{p=1}^{m} \left(\sum_{k \in \mathbb{Z}^m,\, l \in \mathbb{Z}_+^n} \xi_p^{k,l} y^l e^{i\langle k,\beta \rangle} \right) \partial_{\beta_p} + \sum_{q=1}^{n} \left(\sum_{k' \in \mathbb{Z}^m,\, l' \in \mathbb{Z}_+^n} \eta_q^{k',l'} y^{l'} e^{i\langle k',\beta \rangle} \right) \partial_{y_q},$$

where $\xi_p^{k,l}, \eta_q^{k',l'} \in \mathbb{F}$, and k, l, k', l' satisfy $|l| \neq 0$, $|l'| > 1$, $i\langle k, \omega \rangle + \langle l, \lambda \rangle = 0$ and $i\langle k', \omega \rangle + \langle l', \lambda \rangle = \lambda_q$. Then the Lie bracket

$$\begin{aligned} [\mathcal{Y}_1^s, \mathcal{Y}_h] \;=\; & \sum_{p=1}^{m} \left(\sum_{k \in \mathbb{Z}^m,\, l \in \mathbb{Z}_+^n} \xi_p^{k,l} (i\langle k, \omega \rangle + \langle l, \lambda \rangle) y^l e^{i\langle k,\beta \rangle} \right) \partial_{\beta_p} \\ & + \sum_{q=1}^{n} \left(\sum_{k' \in \mathbb{Z}^m,\, l' \in \mathbb{Z}_+^n} \eta_q^{k',l'} (i\langle k', \omega \rangle + \langle l', \lambda \rangle - \lambda_q) y^{l'} e^{i\langle k',\beta \rangle} \right) \partial_{y_q} = 0, \end{aligned}$$

where $\mathcal{Y}_1^s$ is the semisimple part of $\mathcal{Y}_1$. This proves the lemma. □

Corollary 7.3. *If the n-tuple of eigenvalues of the matrix A is non-resonant, then the vector field $\mathcal{X}$ is formally equivalent to its linear part $\mathcal{X}_1$. If the n-tuple is weekly non-resonant, then the vector field $\mathcal{X}$ is formally equivalent to an autonomous system.*

Proof. If the n-tuple of eigenvalues of the matrix A is non-resonant, then the operators $\mathcal{A}_0$ and $\mathcal{A}_1$ in (7.1.8) are both invertible. So for any choice of Λ_r^k and g_{r+1}^k the equations (7.1.8) has a unique solution. By choosing all $\Lambda_r^k = g_{r+1}^k = 0$, we get the desired normal form.

Assume that the n-tuple of eigenvalues of the matrix A is weakly non-resonant. For $k \neq 0$, the equations (7.1.8) have a unique solution for any given Λ_r^k and g_{r+1}^k. In these cases, set $\Lambda_r^k = g_{r+1}^k = 0$. For the terms related to $k = 0$, they are independent of β. Hence, we get a normal form which is autonomous. □

Lemma 7.4. *Assume that $\mathcal{H}(\theta, x)$ is an analytic (or a formal) first integral, with 2π period in θ, of the vector field $\mathcal{X}$. Let $\mathcal{Y}$ be the distinguished normal form associated* *to $\mathcal{X}$, and let $\overline{\mathcal{H}}(\beta, y)$ be $\mathcal{H}(\theta, x)$ written in the normalized coordinates β, y. Then $\overline{\mathcal{H}}(\beta, y)$ is a first integral of $\mathcal{Y}$, and it contains resonant terms only, i.e. if we expand $\overline{\mathcal{H}}$ in Fourier series*

$$\overline{\mathcal{H}}(\beta, y) = \sum_{\mu \in \mathbb{Z}^m,\, \nu \in \mathbb{Z}_+^n} \bar{h}^{\mu,\nu} y^\nu e^{i\langle \mu, \beta\rangle},$$

then we should have $i\langle \mu, \omega\rangle + \langle \nu, \lambda\rangle = 0$.

Proof. Here we still use the notations given in the proof of Lemma 7.2. The first statement is obvious. Without loss of generality, in what follows we assume that $\mathcal{X}$ is in the distinguished normal form. To prove the second statement, we expand $\mathcal{H}$ into Taylor series in x,

$$\mathcal{H}(\theta, x) = \sum_{p=r}^{\infty} H_p(\theta, x),$$

where H_r is the first non-zero terms with $r \geq 0$, and H_p is homogeneous in x. Then we have

$$\mathcal{L} H_p = -\sum_{q=1}^{p-r} \left(\langle \partial_\theta H_{p-q}, \Omega_q \rangle + \langle \partial_x H_{p-q}, f_{q+1} \rangle \right), \quad p = r, r+1, \ldots \tag{7.1.10}$$

where $\mathcal{L}$ is the linear operator defined by $\mathcal{L} H_p = \langle \partial_\theta H_p, \omega\rangle + \langle \partial_x H_p, Ax\rangle$.

Equation (7.1.10) with $p = r$ is a linear homogeneous equation. It follows from the spectrum of the linear operator that its non-trivial solution $H_r(\theta, x)$ should be composed of the resonant terms.

Consider the equation (7.1.10) with $p = r + 1$. From the construction of the distinguished normal form, we know that each pseudomonomial in Ω_q, e.g. $\Omega_q^{k,l} x^l e^{i\langle k, \theta\rangle}$, is in the first resonant, and that each pseudomonomial in the jth component of f_{q+1} for $j = 1, \ldots, n$, e.g. $f_{q+1,j}^{k,l} x^l e^{i\langle k,\theta\rangle}$, is in the second resonant. Hence, all the terms in the right hand side of (7.1.10) with $p = r + 1$ is in the first resonant. Thus, the terms in the left hand side, consequently the solution H_{r+1} of (7.1.10), should be in the first resonant.

By induction we can prove that for each p the solution H_p of (7.1.10) is composed of the resonant terms. This completes the proof. $\square$

Proof of Theorem 6.1. Working in a similar way to the proof of Lemma 7.4, we can assume that the vector field $\mathcal{X}$ is in the distinguished normal form, and its functionally independent analytic (or formal) first integrals are $\mathcal{H}_1, \ldots, \mathcal{H}_\tau$. Since all pseudomonomials in each of $\mathcal{H}_j$ for $j = 1, \ldots, \tau$ are resonant, we have that $\mathcal{X}_1^s(\mathcal{H}_j) = 0$, i.e. each $\mathcal{H}_j$ is also a first integral of $\mathcal{X}_1^s$. Obviously, the set of analytic and formal first integrals of $\mathcal{X}_1^s$ is generated by $\left\{x^l e^{i\langle k,\theta\rangle};\ i\langle k,\omega\rangle + \langle l,\lambda\rangle = 0, l \in \mathbb{Z}_+^m, k \in \mathbb{Z}^n\right\}$, denote S. Then, the number of functionally independent elements of S is exactly γ. This proves that the maximum number of functionally independent first integrals of $\mathcal{X}$ is less than or equal to γ.

7.2 Proof of Theorem 6.2

(a) From the assumption on the resonance of eigenvalues of the vector field $\widetilde{\mathcal{X}}$ at the origin and the proof of Theorem 6.1, it follows that the vector field $\widetilde{\mathcal{X}}$ is formally equivalent to the following

$$\begin{aligned} \dot{u}_j &= \lambda_j u_j(1 + F_j(z_1, \ldots, z_n)), \\ \dot{v}_j &= \mu_j v_j(1 + G_j(z_1, \ldots, z_n)), \end{aligned} \qquad j = 1, \ldots, n \tag{7.2.1}$$

where z_s is defined in Theorem 6.2, and F_j, G_j are formal power series in $z_1, \ldots, z_n$. Indeed, for each monomial $u_1^{\alpha_1} \ldots u_n^{\alpha_n} v_1^{\beta_1} \ldots v_n^{\beta_n}$ in the component ∂_{u_j} we have $\langle \alpha, \lambda\rangle + \langle \beta, \mu\rangle = \lambda_j$. By the resonant relations it follows that $\sum\limits_{s=1}^{n} \left(\alpha_s - \frac{\bar{m}_s}{\bar{n}_s}\beta_s - \sigma_{sj}\right)\lambda_s = 0$ with $\sigma_{sj} = 1$ if $s = j$, or $\sigma_{sj} = 0$ if $s \neq j$. So, for $s \neq j$ there exists a $k_s \in \mathbb{Z}_+$ for which $\alpha_s = k_s \bar{m}_s$ and $\beta_s = k_s \bar{n}_s$; for $s = j$ there exists $k_s \in \mathbb{Z}_+$ for which $\alpha_s = k_s \bar{m}_s + 1$ and $\beta_s = k_s \bar{n}_s$. This proves the claim.

Since $\widetilde{\mathcal{X}}$ has n functionally independent analytic first integrals, the vector field (7.2.1) has n functionally independent formal first integrals. Lemma 7.4 tells us that the first integrals of (7.2.1) contain resonant terms only. So, if $\mathcal{H}$ is a first integral of (7.2.1), we can assume without loss of generality that $\mathcal{H} = \mathcal{H}(z_1, \ldots, z_n)$. Then direct calculations show that the first integral $\mathcal{H}$ satisfies

$$\bar{n}_1\lambda_1 z_1(F_1 - G_1)\frac{\partial H}{\partial z_1} + \ldots + \bar{n}_n\lambda_n z_n(F_n - G_n)\frac{\partial H}{\partial z_n} \equiv 0.$$

This implies that every first integral of (7.2.1) is a first integral of the vector field

$$\mathcal{X}^* = \bar{n}_1\lambda_1 z_1(F_1 - G_1)\frac{\partial}{\partial z_1} + \ldots + \bar{n}_n\lambda_n z_n(F_n - G_n)\frac{\partial}{\partial z_n},$$

in the n dimensional space. It is well known that if a vector field in an n dimensional space is non-trivial, it has at most $n-1$ functionally independent first integrals. But $\mathcal{X}^*$ has n functionally independent first integrals, it should be trivial. Hence, we have $F_j = G_j$, $j = 1, \ldots, n$. This proves the statement (a).

(b) In order to prove the statement, we need to refine the normalization process. Under the assumption of the theorem, without loss of generality we set the vector field $\widetilde{\mathcal{X}}$ in the form

$$\dot{x}_j = \lambda_j x_j + p_j(x, y), \quad \dot{y}_j = \mu_j y_j + q_j(x, y), \qquad j = 1, \ldots, n$$

where p_j, q_j are analytic functions in x, y. We assume that it is already reduced, by the formal transformation

$$x_j = u_j + \phi_j(u, v), \quad y_j = v_j + \psi_j(u, v), \qquad j = 1, \ldots, n$$

to the following formal vector vector

$$\dot{u}_j = \lambda_j u_j + \alpha_j(u, v), \quad \dot{v}_j = \mu_j v_j + \beta_j(u, v), \qquad j = 1, \ldots, n.$$

Using the multi-index notation, for $w \in \{p, q, \phi, \psi, \alpha, \beta\}$ we denote

$$w_j(u, v) = \sum_{k,l} w_j^{(k,l)} u^k v^l,$$

where $u^k = u_1^{k_1} \ldots u_n^{k_n}$ and $v^l = v_1^{l_1} \ldots v_n^{l_n}$. Then from the proof of Theorem 6.1 we have that

$$
\begin{aligned}
(\langle k,\lambda\rangle+\langle l,\mu\rangle-\lambda_j)\phi_j^{(k,l)} &= [p_j(u+\phi,v+\psi)]^{(k,l)}-\alpha_j^{(k,l)} \\
&\quad -\sum_{s=1}^{n}\sum_{(a,b)}\phi_j^{(a,b)}\left(a_s\alpha_s^{(k+e_s-a,l-b)}+b_s\beta_s^{(k-a,l+e_s-b)}\right), \\
(\langle k,\lambda\rangle+\langle l,\mu\rangle-\mu_j)\psi_j^{(k,l)} &= [q_j(u+\phi,v+\psi)]^{(k,l)}-\beta_j^{(k,l)} \qquad (7.2.2)\\
&\quad -\sum_{s=1}^{n}\sum_{(a,b)}\psi_j^{(a,b)}\left(a_s\alpha_s^{(k+e_s-a,l-b)}+b_s\beta_s^{(k-a,l+e_s-b)}\right),
\end{aligned}
$$

where $[p_j(u+\phi,v+\psi)]^{(k,l)}$ and $[q_j(u+\phi,v+\psi)]^{(k,l)}$ are obtained after we re-expand $p_j(u+\phi,v+\psi), q_j(u+\phi,v+\psi)$ in power series in u and v, e_s is the n dimensional vector equal to 1 at the sth entry and equal to 0 for otherwise, and in the summation (a,b) are taken over all the vectors in $\mathbb{Z}_+^{2n}$ for which $(k-a,l-b)\in\mathbb{Z}_+^{2n}$. For simplicity we denote $[p_j]^{(k,l)}=[p_j(u+\phi,v+\psi)]^{(k,l)}$ and $[q_j]^{(k,l)}=[q_j(u+\phi,v+\psi)]^{(k,l)}$

For the resonant cases of (k,l), i.e. $\langle k,\lambda\rangle+\langle l,\mu\rangle-\lambda_j=0$ or $\langle k,\lambda\rangle+\langle l,\mu\rangle-\mu_j=0$, in order for the normalization transformation to be distinguished, we choose $\phi_j^{(k,l)}=0$ or $\psi_j^{(k,l)}=0$. Correspondingly we have

$$
\alpha_j^{(k,l)}=[p_j]^{(k,l)}-\sum_{s=1}^{n}\sum_{(a,b)}\phi_j^{(a,b)}\left(a_s\alpha_s^{(k+e_s-a,l-b)}+b_s\beta_s^{(k-a,l+e_s-b)}\right), \qquad (7.2.3)
$$

or

$$
\beta_j^{(k,l)}=[q_j]^{(k,l)}-\sum_{s=1}^{n}\sum_{(a,b)}\psi_j^{(a,b)}\left(a_s\alpha_s^{(k+e_s-a,l-b)}+b_s\beta_s^{(k-a,l+e_s-b)}\right). \qquad (7.2.4)
$$

If (k,l) is not in resonance, we choose $\alpha_j^{(k,l)}=\beta_j^{(k,l)}=0$. Then equation (7.2.2) has a unique solution

$$
\phi_j^{(k,l)} = \frac{[p_j]^{(k,l)}-\sum_{s=1}^{n}\sum_{(a,b)}\phi_j^{(a,b)}\left(a_s\alpha_s^{(k+e_s-a,l-b)}+b_s\beta_s^{(k-a,l+e_s-b)}\right)}{\langle k,\lambda\rangle+\langle l,\mu\rangle-\lambda_j}, \qquad (7.2.5)
$$

$$
\psi_j^{(k,l)} = \frac{[q_j]^{(k,l)}-\sum_{s=1}^{n}\sum_{(a,b)}\psi_j^{(a,b)}\left(a_s\alpha_s^{(k+e_s-a,l-b)}+b_s\beta_s^{(k-a,l+e_s-b)}\right)}{\langle k,\lambda\rangle+\langle l,\mu\rangle-\mu_j}. \qquad (7.2.6)
$$

We claim that $\alpha_j^{(k,l)}=[p_j]^{(k,l)}$ in (7.2.3) and $\beta_j^{(k,l)}=[q_j]^{(k,l)}$ in (7.2.4). Indeed, since $\alpha_s^{(k+e_s-a,l-b)}$ and $\beta_s^{(k-a,l+e_s-b)}$ are the coefficients of the resonant terms (otherwise, they are zero by the construction), we have $\langle k+e_s-a,\lambda\rangle+\langle l-b,\mu\rangle = \lambda_s$. This is equivalent to $\langle k-a,\lambda\rangle + \langle l-b,\mu\rangle = 0$. Using this equality and the fact that (k,l) in resonance, we have that $0 = \langle k,\lambda\rangle + \langle l,\mu\rangle - \lambda_j = \langle a,\lambda\rangle + \langle b,\mu\rangle - \lambda_j$. This proves that (a,b) is also in resonance. Therefore, we should have $\phi_j^{(a,b)} = 0$. Working in a similar way we have that $\psi_j^{(a,b)} = 0$. This proves the claim.

Summarizing the above construction, we obtain a distinguished formal transformation

$$x_j = u_j + \sum_{(k,l)} \phi_j^{(k,l)} u^k v^l, \qquad y_j = v_j + \sum_{(k,l)} \psi_j^{(k,l)} u^k v^l,$$

with all (k,l) in non-resonant cases and $\phi_j^{(k,l)}$ and $\psi_j^{(k,l)}$ given in (7.2.5) and (7.2.6), respectively. Under the action of this transformation, the distinguished normal form satisfies

$$\alpha_j = \sum_{(k,l)} [p_j]^{(k,l)} u^k v^l, \qquad \beta_j = \sum_{(k,l)} [q_j]^{(k,l)} u^k v^l,$$

where (k,l) are in resonant cases.

Now we prove the convergence of the distinguished transformation. Comparing with the formal normal form (6.2.2), we have

$$\alpha_s^{(k+e_s-a,l-b)} = \lambda_s w_s^{(k-a,l-b)}, \quad \beta_s^{(k-a,l+e_s-b)} = \mu_s w_s^{(k-a,l-b)},$$

where $w_s^{(k-a,l-b)}$ are the coefficients of monomials in W_s, and $\langle k-a,\lambda\rangle + \langle l-b,\mu\rangle = 0$. Then

$$\sum_{s=1}^{n} \left(a_s \alpha_s^{(k+e_s-a,l-b)} + b_s \beta_s^{(k-a,l+e_s-b)} \right) = \sum_{s=1}^{n} (k_s\lambda_s + l_s\mu_s) w_s^{(k-a,l-b)}. \tag{7.2.7}$$

If the eigenvalues $\lambda = (\lambda_1,\ldots,\lambda_n)$ belong to the Poincaré domain, then there exists a δ_2 such that

$$(\langle k,\lambda\rangle + \langle l,\mu\rangle - \lambda_j)^{-1},\ (\langle k,\lambda\rangle + \langle l,\mu\rangle - \mu_j)^{-1} \ge \delta_2.$$

Moreover, since λ_j and μ_j for $j = 1,\ldots,n$ are pairwise resonant, there exists a constant C_1 for which

$$\frac{k_s + l_s}{\langle k,\lambda\rangle + \langle l,\mu\rangle - \mu_j},\ \frac{k_s + l_s}{\langle k,\lambda\rangle + \langle l,\mu\rangle - \lambda_j} \le C_1.$$

If all W_j are equal to W, then there exists a $C_2 = \max_j \left\{1 + \frac{|\lambda_j|+|\mu_j|}{\epsilon}\right\}$ such that

$$\left| \frac{\sum_{s=1}^{n} (k_s\lambda_s + l_s\mu_s) w_s^{(k-a,l-b)}}{\langle k, \lambda\rangle + \langle l, \mu\rangle - \nu_j} \right| \le C_2 \sum_{s=1}^{n} w^{(k-a,l-b)},$$

where $\nu \in \{\lambda, \mu\}$.

Thus, in any cases there exist $\delta > 0$ and $C > 0$ such that

$$\left|\phi_j^{(k,l)}\right| \le \delta \left|[p_j(u+\phi, v+\psi)]^{(k,l)}\right| + C \sum_{s=1}^{n} \sum_{(a,b)} \left|\phi_j^{(a,b)} w_s^{(k-a,l-b)}\right|, \qquad (7.2.8)$$

$$\left|\psi_j^{(k,l)}\right| \le \delta \left|[q_j(u+\phi, v+\psi)]^{(k,l)}\right| + C \sum_{s=1}^{n} \sum_{(a,b)} \left|\psi_j^{(a,b)} w_s^{(k-a,l-b)}\right|. \qquad (7.2.9)$$

For p_j and q_j to be analytic in a neighborhood of the origin, there exists a polydisc $\mathcal{D}$: $|x_s|, |y_s| \le r$ on which p_j and q_j are analytic. From the Cauchy inequality we have

$$\left|p_j^{(k,l)}\right|, \ \left|q_j^{(k,l)}\right| \le M r^{-|k|-|l|},$$

where $M = \max_j \sup_{\partial\mathcal{D}} \{|p_j|, |q_j|\}$. Set

$$\hat{p} = M \sum_{|k|+|l|=2}^{\infty} r^{-|k|-|l|} x^k y^l.$$

Then $\hat{p}$ is an analytic function in the interior of $\mathcal{D}$, and it is a majorant series of p_j, q_j for $j = 1, \ldots, n$. Consider the following majorant relations

$$\begin{aligned} \sum_{j=1}^{n} (\phi_j + \psi_j + \alpha_j + \beta_j) &\preccurlyeq \sum_{j=1}^{n} \left(\hat{\phi}_j + \hat{\psi}_j + \hat{\alpha}_j + \hat{\beta}_j\right) \\ &\preccurlyeq 2n(1+\delta)\hat{p}(u+\hat{\phi}, v+\hat{\psi}) \\ &\quad + C \sum_{j=1}^{n} \sum_{s=1}^{n} (\hat{\phi}_j + \hat{\psi}_j)\hat{W}_s, \end{aligned} \qquad (7.2.10)$$

where $\hat{\omega}$ denotes the corresponding majorant series of ω with $\omega \in \{\alpha, \beta, \phi, \psi, W\}$, and $\preccurlyeq$ shows the majorant relations between two power series (see for instance, [18]).

Let

$$\Pi(u, v) = \sum_{j=1}^{n} \left(\hat{\phi}_j + \hat{\psi}_j + \hat{\alpha}_j + \hat{\beta}_j\right).$$

Since all coefficients in Π are non-negative, it is sufficient to consider the case $u_1 = \ldots = u_n = v_1 = \ldots = v_n = \theta$. Let $\Pi(u,v) = R(\theta)\theta$ with R being a function in the single variable θ. Then by the construction we have $R(0) = 0$. From the relation (7.2.10), we have

$$R(\theta)\theta \preccurlyeq 2n(1+\delta)\theta^2 \hat{p}^*(1+R(\theta), 1+R(\theta)) + CR^2(\theta)\theta, \tag{7.2.11}$$

where we have used $W_s = \alpha_s/u_s$ or β_s/v_s, $\hat{p}(u+\hat{\phi}, v+\hat{\psi}) \preccurlyeq \hat{p}(u_1 + \Pi(u,v), \ldots, u_n + \Pi(u,v), v_1 + \Pi(u,v), \ldots, v_n + \Pi(u,v))$, and $\hat{p}^* = \hat{p}(\theta+\Pi, \ldots, \theta+\Pi, \theta+\Pi, \ldots, \theta+\Pi)/\theta^2$ a power series.

Consider the following function

$$\Phi(\theta, h) = h - 2n(1+\delta)\theta\hat{p}^*(1+h, 1+h) - Ch^2. \tag{7.2.12}$$

Clearly, the function Φ is analytic in θ, h. Since $\Phi(0,0) = 0$ and $(\partial\Phi/\partial h)|_{(0,0)} = 1$, the Implicit Function Theorem implies that the equation $\Phi(\theta, h) = 0$ has an analytic solution $h(\theta)$ in a neighborhood of the origin.

Comparing (7.2.11) and (7.2.12), we have that $h(\theta)$ majorizes $R(\theta)$. This proves that $R(\theta)$ is analytic in some neighborhood of the origin. Therefore, $\Pi(u,v)$ is analytic, and consequently the power series $\phi_j, \psi_j, \alpha_j, \beta_j$ are analytic. Thus, we have proved that the vector field $\widetilde{\mathcal{X}}$ is analytically equivalent to the distinguished analytic normal form by the analytic distinguished normalization.

(c) Obviously, $z_1, \ldots, z_n$ are the first integrals of (6.2.2). So, any formal power series in $z_1, \ldots, z_n$ is a formal first integral. Conversely, if H is a formal first integral of (6.2.2), then working in a similar way to the proof of (7.2.1) we obtain the desired form of H.

(d) In order to prove this statement, we need some elementary facts on pluripolar set. A set $\mathbf{E} \subset \mathbb{C}^m$ is called *pluripolar* if for each $z \in \mathbf{E}$, there exists a neighborhood U of z and a plurisubharmonic function u on U for which $\mathbf{E} \cap U \subset u^{-1}(-\infty)$ (see for instance, [25, 45]). Given an open subset Ω in $\mathbb{C}^m$. A function $u : \Omega \to [-\infty, \infty)$ is *plurisubharmonic* if it is upper semicontinuous, i.e. $\{z \in \Omega : u(x) < c\}$ is open for each $c \in \mathbb{R}$ and not identically $-\infty$ on any connected component of Ω, and for any $x \in \Omega$ we have

$$u(x) \leq \frac{1}{2\pi} \int_0^{2\pi} u(x + e^{it}y)dt,$$

where y is any number in $\mathbb{C}^m$ satisfying $x + \delta y \in \Omega$ and $|\delta| \leq 1$. A pluripolar set in $\mathbb{C}^m$ has $2n$-dimensional Lebesgue measure 0. The countable union of pluripolar subsets is again pluripolar.

Let $\mathcal{L}$ be the set of plurisubharmonic functions in $\mathbb{C}^m$ with minimal growth in the sense that $u(z) - \log \|z\|$ is bounded above for $\|z\| \to \infty$. For any given subset $\mathbf{E} \subset \mathbb{C}^m$, define

$$V_{\mathbf{E}}(z) = \sup\{u(z);\, u \in \mathcal{L}, u \leq 0 \text{ on } \mathbf{E}\}.$$

Then we have

Lemma (Bernstein-Walsh). If $\mathbf{E} \subset \mathbb{C}^m$ is not pluripolar and $P(z)$ is a polynomial of degree d, then for $z \in \mathbb{C}^m$

$$|P(z)| \leq \|P\|_{\mathbf{E}} \exp\left(dV_{\mathbf{E}}(z)\right).$$

This lemma is the key to prove the following result, its proof follows from the idea of Pérez-Marco [36].

Lemma 7.5. *Each vector field in any affine finite dimensional subspace $\mathcal{F}$ of $\mathcal{V}$ has a convergent distinguished normal form (respectively, normalization), or only the vector fields in an exceptional pluripolar subset of $\mathcal{F}$ have this property.*

Proof. If the second statement of Lemma 7.5 holds, we are done. So, we assume that there is a subset of $\mathcal{F}$ not pluripolar in which every vector field has a convergent distinguished normal form. Let $\mathcal{X}_1, \ldots, \mathcal{X}_m$ be the $2n$-dimensional vector fields with the starting terms of order at least two, and let $\widetilde{\mathcal{X}}_0$ be the linear part of $\widetilde{\mathcal{X}}$. Consider $\mathcal{F}$ to be the m-dimensional vector space $\{\widetilde{\mathcal{X}}_0 + t_1\mathcal{X}_1 + \ldots + t_m\mathcal{X}_m;\, t = (t_1, \ldots, t_m) \in \mathbb{C}^m\}$. Then $\mathcal{F} \subset \mathcal{V}$ is isomorphic to $\mathbb{C}^m$. Denote by $\mathcal{X}_t$ the vector field in $\mathcal{F}$.

Let $\mathcal{C}$ be the set of $t \in \mathbb{C}^m$ for which the corresponding vector field $\mathcal{X}_t \subset \mathcal{F}$ has a convergent distinguished normalization, and assume that it is not pluripolar. Write $\mathcal{C} = \cup_{r\geq 1}\mathcal{C}_r$, where $\mathcal{C}_r$ is set of t for which the vector field $\mathcal{X}_t$ has a convergent distinguished normalization Φ_t at least in the polydisc D_r of the radius $1/r$, and the normalization is bounded by 1 in D_r. By the assumption there exists a $\mathcal{C}_r$ which is non-pluripolar (otherwise $\mathcal{C}$ should be pluripolar).

According to the proof of Statement (b), we write the normalization to be the form

$$\Phi_t(u,v) = \sum_{j\in\mathbb{Z}_+^{2n}} \Phi_j(t)(u,v)^j,$$

where $(u,v)^j$ is the multi-index, and $\Phi_j(t)$ are $2n$-dimensional vector-valued functions. From the construction of the normalization, especially the formulae (7.2.3)-(7.2.6), it follows that $\Phi_j(t)$ is a vector-valued polynomial of degree at most $|j|$. Since Φ_t is analytic in $\mathcal{C}_r$ by the construction, it follows from the Cauchy inequality that there exists a $\rho_0 > 0$ for which

$$\Psi(t) = \sup_j \|\Phi_j(t)\|_\infty \rho_0^{-|j|} < \infty, \qquad t \in \mathcal{C}_r$$

where the norm $\|\cdot\|_\infty$ denotes the summation of the absolute values of components of a vector. Now the non-pluripolar set $\mathcal{C}_r$ can be represented as the union of the subsets $\{t \in \mathcal{C}_r;\ \Psi(t) \leq s, s \in \mathbb{N}\}$, in which there is a non-pluripolar set. Denote by $\mathcal{D}_s$ one of the non-pluripolar subsets. Choose $\Omega \subset \mathcal{D}_s$ to be a non-pluripolar compact set for which there exists $\rho_1 > 0$ such that for all $t \in \Omega$ and all j we have

$$\|\Phi_j(t)\|_\infty \leq \rho_1^{|j|}.$$

So, it follows from the Bernstein-Walsh Lemma that for any compact subset $\mathcal{C} \subset \mathbb{C}^m$ and $|j| \geq 2$ there exists a $\rho_2 > 0$ depending on $\mathcal{C}$ only for which the following holds

$$\|\Phi_j\|_{\mathcal{C}} \leq \|\Phi_j\|_\Omega \exp\left(|j| \max_{t\in\mathcal{C}} V_\Omega(t)\right) \leq \rho_1^{|j|}\rho_2^{|j|},$$

where $\|\Phi_j\|_\Omega = \max_{t\in\Omega} \|\Phi_j(t)\|_\infty$. This implies that for arbitrary t on any compact subset of $\mathbb{C}^m$, $\Phi_t(u,v)$ is convergent on the polydisc $\{(u,v) : |u_i|, |v_i| \le \min\{\frac{1}{\rho_1}, \frac{1}{\rho_2}\}, i = 1, \ldots, n\}$. Consequently, it is an analytic diffeomorphism in a neighborhood of the origin in the (u,v) space for all $t \in \mathbb{C}^m$. The proof is now completed. □

Now the proof of statement (d) follows from Lemma 7.5 and the assumption that $\mathcal{V}$ contains a vector field having the divergent distinguished normalization or normal form.

7.3 Proof of Theorem 6.4

Denote by $\mathcal{Y}$ the planar analytic flow. Under the assumption of the theorem, the flow $\mathcal{Y}$ has an analytic first integral. We have the following three cases.

Case 1. One of the eigenvalues is zero. Then the other does not vanish, it follows from [28] that the singularity is non-isolated.

Case 2. The two eigenvalues are a pair of pure imaginary numbers. The classical result of Poincaré's, see for instance [4], tells us that the vector field $\mathcal{Y}$ is analytically equivalent, with a possible time rescaling by a non-zero constant, to

$$\dot{x} = x(i + g(xy)), \qquad \dot{y} = -y(i + g(xy)).$$

Obviously, the above is a Hamiltonian, and is orbitally equivalent to the linear vector field.

Case 3. The two eigenvalues are real, and their ratio is a negative rational number. Then we get from Theorem 1 of [49] that the vector field is analytically equivalent, with a possible time rescaling by a non-zero constant, to

$$\dot{x} = nx(1 + g(x^m y^n)), \qquad \dot{y} = -my(1 + g(x^m y^n)).$$

This proves the theorem.

7.4 Proof of Theorem 6.5

Recall that for a given smooth function $L(x, y, t)$ of three variables, a curve $\gamma : x = x(t)$ for $t \in [t_0, t_1]$ is an extremal of the functional $\Phi(\gamma) = \int_{t_0}^{t_1} L(x, \dot{x}, t)dt$ on the space consisting of a smooth class of curves passing through the points $x(t_0) = x_0$ and $x(t_1) = x_1$, if and only if $x(t)$ satisfies the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0. \tag{7.4.1}$$

Given a planar Hamiltonian vector field $\mathcal{X}$ with the Hamiltonian function $H(x, y)$. Assume that the origin is a linear center of $\mathcal{X}$. For otherwise, it is not isochronous. We now construct the action-angle coordinates (I, ϕ) according to the method of Arnold [2]. In the neighborhood of the origin, every closed orbit is a level curve $H = h$, denoted by C_h, for $h \in (0, h_0)$ with h_0 finite or infinite. Set

$$I = \frac{1}{2\pi}\Pi(h) = \frac{1}{2\pi}\int_{C_h} y dx.$$

We remark that $\Pi(h)$ is the area of the domain enclosed by C_h. Choosing ϕ as the usual angle variable. Clearly, the transformation from (x, y) to (I, ϕ) is analytic. The Hamiltonian vector field $\mathcal{X}$ under this action-angle coordinates is of the form

$$\dot{I} = 0, \qquad \dot{\phi} = \partial_I H(I).$$

The period of the closed orbit C_h is

$$T(h) = \int_0^{2\pi} \frac{d\phi}{\partial_I H(I)}.$$

Set $L(I, \dot{I}, \phi) = (\partial_I H(I))^{-1}$. The center is isochronous if and only if $T(h)$ is constant, and if and only if all the closed orbits C_h are the extremal of the functional $T(h)$. So on all the closed orbits the Euler-Lagrange equation holds, i.e.

$$\frac{\partial L}{\partial I} = \frac{d}{d\phi}\left(\frac{\partial L}{\partial \dot{I}}\right) = 0,$$

because in this case L is independent of $\dot{I}$. This last equation means that L is independent of I, too. Therefore, $H(I)$ is a linear function in I. This proves the theorem.

Remark 7.6. Let $I(h)$ be the inverse function of $H(I) = h$. Since

$$T(h) = \frac{2\pi}{\dot{\phi}} = \frac{2\pi}{\partial_I H(I)} = 2\pi \partial_h I(h) = \partial_h \Pi(h),$$

the origin is an isochronous center if and only if the area of the domain enclosed by the closed orbit C_h is a linear function of h.

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INDEX

VITA

I was born in Shanghai, the largest city in China. At age of about six, my family moved to Changzhou, Jiangsu Province, a middle-sized city about 100 miles west of Shanghai. I spent 12 years there to finish my elementary and middle school education. At age 18 I enrolled in Shanghai Jiaotong University, major in Mathematics. I then continued to study in its graduate school until I obtained my master degree. In 1999, I came to Georgia Tech to work on my Ph.D. During all the time in Georgia Tech, I have been working under the direction of Prof. Yingfei Yi in the field of dynamical systems.