Conley-Morse Chain Maps

A Thesis
Presented to
The Academic Faculty

by

Todd Moeller

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

School of Mathematics
Georgia Institute of Technology
August 2005

Conley-Morse Chain Maps

Approved by:

Konstantin Mischaikow
School of Mathematics
Georgia Institute of Technology

Margaret Symington
School of Mathematics
Georgia Institute of Technology

Guillermo Goldsztein
School of Mathematics
Georgia Institute of Technology

Greg Turk
College of Computing
Georgia Institute of Technology

William Green
School of Mathematics
Georgia Institute of Technology

Date Approved: July 14,2005

ACKNOWLEDGEMENTS

With thanks to···

- **Konstantin** for your patience, intuition, and guidance. These last few months have been fun and I look forward to seeing what's next.
- **My family** for the support and encouragement to finish the degree (even though it wasn't as a geneticist).
- **Margaret** for an absolutely incredible amount of help at crunch time. And to the rest of my committee members for their time and effort.
- **Greg** and **Eugene** for the educational lunches.
- **John** and **Rob** for the superb instruction, patience, and insight.
- **Andrew** for more than two years of friendship. Here's to our next roommates being young, cute, and nubile.
- **Victor,**a kindred spirit. You have been sorely missed.
- **Erik** for, amongst other things, the format of these acknowledgements.
- **Kyle, Andy, Craig, Sean, Chad** and the rest of the gamers for your friendship and the diversions.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

SUMMARY

We consider the problem of comparing the qualitative structure of two sets of scientific data measured, perhaps, at different resolutions. This data could either be experimentally measured or simulated.

1. What is meant by the qualitative behavior of a data set?
2. How do we measure the qualitative behavior?
3. How do we the compare qualitative behaviors of 2 data sets?
4. Since the behavior is often scale dependent how do we account for this in the comparison process?

Let us consider how we can address these questions when the data is presented as a discrete intensity function, ι over a cubical domain. In such a case the domain of ι is a set of pixels. In applications the function ι may estimate terrain height, radiation intensity, temperature, or the like. An example of terrain data over a 2-D domain is presented in 1 (a). The intensity function corresponds to the elevation of terrain in an area surrounding Nashville, Tennessee. The terrain was stored as a matrix of intensities in the data bases of the National Oceanic and Atmosheric Administration [16] which was graphed in Matlab.

The illustration in 1(b) presents a toy example of 1-D intensity data. The smooth curve represents the actual height of the terrain on the interval $[0, 10]$. Suppose a digital satellite took a picture of this terrain. The boxes represent a digital representation of the terrain provided by a satellite whose resolution is $1 - unit$.

Edelsbrunner, Harer, and Zomorodian presented a Morse theoretic approach for studying intensity data in [4]. They constructed a **gradient flow** on the data in order to analyze its features.

Definition 1. *Let X be a compact subset of $\mathbb{R}^n$ Let $F : X \subset \mathbb{R}^n \to \mathbb{R}$. The function $\phi : X \times \mathbb{R} \to X$ is said to be a* **gradient flow** *of F on X if ϕ satisfies*

$$\frac{d\phi}{dt} = -\nabla F(\phi)$$

In Morse theory the qualitative structure of a gradient flow is determined by the set of **fixed points** (where $\nabla F = 0$) of the flow and the structure of the trajectories connecting the fixed points. Given experimental (cf. terrain) data in the form of a intensity function Edelsbrunner *et.al.* used an Euler characteristic-based method to find the fixed points of the constructed flow. Steepest descent algorithms were then implemented to construct connections between the fixed points thus capturing the qualitative behavior of the data.

In $2 - D$ Morse theory a saddle and sink, or saddle and source, that share a connecting orbit can be cancelled in a manner that preserves the global behavior of the flow. In [1] and [4] an algorithmic method for performing such cancellations on 2-D gradient data was presented. This provides a means of removing small-scale structures from the data and can be use to construct a hierarchical description of the flow. This allows for the analysis of the

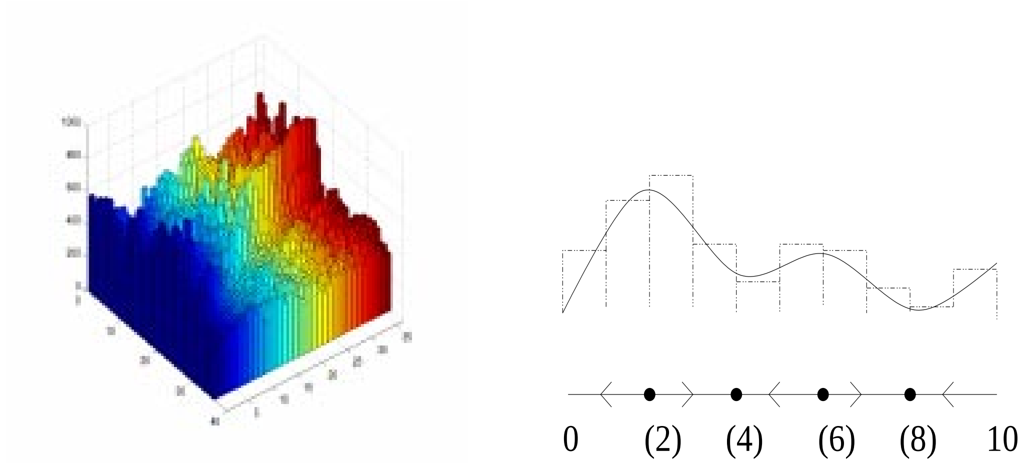


Figure 1: (a)Nashville by Satellite (b) Toy Example

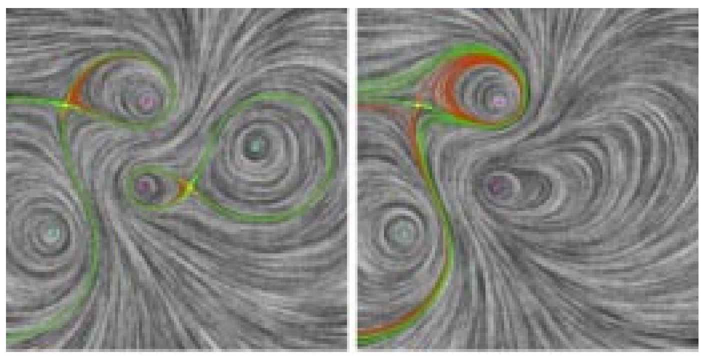


Figure 2: Reduction of a flow (Courtesy of Eugene Zhang)

flow, and hence the data set, on multiple scales. Preliminary work on $3-D$ intensity data is discussed in [3]

Morse theory, while an effective tool for studying gradient systems, is not sufficient to tackle flows in general (see Figure 2). Conley index theory ([2],[15], [14]), however, provides a series of tools for stuyding the dynamics of general flows. We will adopt Conley index theory as the basis for all the work to follow.

Let $\phi : X \times \mathbb{R} \to X$ be a flow (Definition 2 on some metric space X. A set $S \subset X$ is said to be **invariant** under the flow if S, as a set, remains stationary for all time (see Definition 6. Invariant sets are generalizations of fixed points. In fact the fixed points of ϕ are invariant sets. However, when we leave the realm of gradient vector fields we can encounter more exotic invariant sets. Classification of an invariant set S can be performed by examining the nature of trajectories in a small neighborhood of S. Such analysis can be done only if we can **isolate** S from all other invariant sets with a compact set (see Definition 8).

A given isolated invariant set S can often be decomposed into a collection of smaller isolated invariant sets, $\{M(p)|p \in P\}$, where P is an indexing set. If we can assign a partial ordering to the component sets that, in some sense, preserves the action of the flow off of the individual components we refer to $\{M(p)|p \in P\}$ as a **Morse decomposition** and each $M(p)$ as a **Morse set.** A formal definition is presented as Definition 15

Theorem 2, known as *Conley's decomposition theorem,* lends another interpretation of a Morse decomposition. In essence the decomposition theorem states that a flow, off of its Morse sets, always move downhill. Therefore, given a Morse decomposition, the flow looks/acts like a gradient flow except on the Morse sets.

A given invariant set can possess many Morse decompositions. Trivially S is always a Morse decomposition of itself. The finer the Morse decomposition, however, the more we can learn about the system. Consider the system in 1 (b). In this case $S = [2, 8]$ is an invariant set.

We consider the Morse decompsition $M_1(S) = \{2, 4, 6, 8\}$. In this case our Morse sets are all fixed points. Furthermore if we examine small neighborhoods around $4, 8$ we see that trajectories are moving into such regions. Conversely trajectories are leaving neighborhoods of 2 and 6.

We note that $M_2(S) = \{[2, 6], [8]\}$ is also a Morse decomposition. In this case we note that all trajectories are leaving small neighborhoods of the Morse set $[2, 6]$. Furthermore all trajectories are entering 8.

Given a Morse decomposition the qualitative behavior of the flow is determined by the nature of the individual Morse sets and how the Morse sets interact. For instance the qualitative behavior of $M_2(S)$ is characterized by following features:

1. All trajectories leave small neighborhoods of $[2, 6]$
2. All trajectories enter small neighborhoods of 8
3. Trajectories leaving neighborhoods of $[2, 6]$ flow into 8.

It is obvious but important to note that this behavior is dependent on the Morse decomposition. Throughout this paper references to the qualitative behavior of a system will always assume a fixed Morse decomposition.

Conley constructed a topological index on (isolated) invariant sets which classified their behavior by measuring how trajectories enter and leave a small neighborhood of the set. The index is presented in Definition 20 but we will present a rough description of the process of computing the index of a Morse set, $M(p)$, here.

1. Choose a compact neighborhood, N, of $M(p)$ such that $M(p)$ is the only invariant set in N
2. Choose a compact subset of N, labelled L, such that all trajectories that leave N must do so through L

Then we define the (homological) Conley index as $CH_*(M(p)) \approx H_*(N, L)$.

Let us compute the Conley index of the Morse set 2 from our previous example. The process of computing $CH_*(2)$ is illustrated in Figure 3. We begin by choosing $N = [1, 3]$, drawn in purple. The exit set L is labelled in green and given by $L = [1, 1.25] \cup [2.75, 3]$. When we collapse L to a point we receive a topological circle. Hence $CH_*(2) \approx \widetilde{H}_*(S^1)$. So the Conley index of the Morse set 2 is equivalent to the reduced homology of a cirlce.

In order to determine the qualitative structure we also need to consider how the Morse sets are connected by trajectories. In Chapter 1 we will see that connection information between two Morse sets is stored in long exact sequences. In [6] Franzosa constructed a **homology index braid** to store these long exact sequences.

Conley speculated that it would be possible to store this same information as a matrix of maps between the Conley indices of the Morse sets. In [7], [8], and [5] Franzosa proved

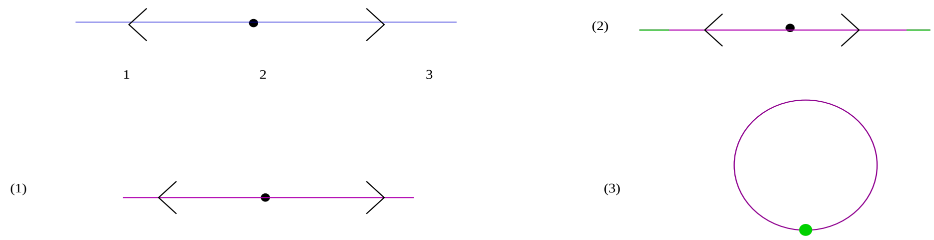


Figure 3: Computing the Conley Index

the existence of these **connection matrices** (see Theorem 10). Franzosa showed that information concerning the qualitative structure of a system can be stored as a collection of a homology groups and a matrix of maps between those homology groups, the connection matrices. Furthermore the homology groups form a natural chain complex for which a given connection matrix, $\triangle$, serves as a boundary map.

Given two such chain complexes it seems intuitively reasonable to believe that a chain map, T, between the complexes might be able to compare the qualitative dynamics of the two systems. **Algebraic transition matrix** theory, presented in [9], showed that, as long as two Morse decompositions were related by **continuation**(see 27) they could be compared by an invertible chain map. Unfortunately this constraint is too rigid for use in comparing data measured at multiple scales.

The purpose of this work is to develop a purely algebraic approach for comparing Morse decompositions. Given two systems with Morse decompositions $M(S_1, <_1)$ and $M(S_2, <_2)$ with connection matrices $\triangle$ and $\triangle'$ which agree algebraically on some scale we can compare them via an order preserving chain map. The existence of these Conley-Morse chain maps is proven in Chapter 2. In Chapter 3 we present an algorithm to reduce a refined algebraic description of the qualitative structure of a flow to a coarser description. In Chapter 4 we present examples of how they can be used to compare data sets.

CHAPTER I

A BRIEF INTRODUCTION TO CONLEY INDEX THEORY

1.1 Invariant Sets

We begin by introducing the class of objects we wish to study, flows.

Definition 2. *Let X be a compact metric space. A continuous map $\phi : \mathbb{R} \times X \to X$ is said to be a flow if*

1. $\phi(0, x) = x$
2. $\phi(t, \phi(s, x)) = \phi(t + s, x)$

We can also consider parameter dependent flows.

Definition 3. *Let $\Lambda \subset \mathbb{R}^n$ be open. A map $\phi : \mathbb{R} \times X \times \Lambda \to X$ is said to be a Λ-parameterized flow if*

1. *ϕ is continuous*
2. *The restriction map $\phi_\lambda = \phi : \mathbb{R} \times X \times \lambda \to X$ is a (continuous) flow for all $\lambda \in \Lambda$.*

Let $x \in X$. Over the course of time x traces out a path in X which we will call the **trajectory of x.** More formally the trajectory of a point is given by $\phi\big((-\infty, \infty), x\big)$. We can also follow how sets evolve under the flow. In particular, given a set $A \subset$ of X we would like to see where A flows towards in forwards and backwards time.

Definition 4. *The* **α-limit set of** *A is given by $\alpha(A) = \cap_{t>0} cl\Big(\phi\big((-\infty, -t], A\big)\Big)$.*

Definition 5. *The* **ω-limit set of** *A is given by $\omega(A) = \cap_{t>0} cl\Big(\phi\big([t, \infty), A\big)\Big)$.*

We next consider those sets, S, which remain stationary for all time.

Definition 6. *S is said to be an* **invariant set** *for the flow ϕ if $\bigcup_{t\in R}\phi(t,S)=S$.*

Definition 7. *Let $N\subset X$. Then $Inv(N,\phi)=\{x\in N:\phi\big((-\infty,-\infty],x\big)\subset N\}$ is referred to as the* **maximal invariant set in N** *.*

Fixed points are fundamental examples of invariant sets. To capture the structure of the flow near an invariant set, S, we examine the trajectories in a small neighborhood of S. Therefore it is crucial to find a neighbordhood of S that contains no other invariant set.

Definition 8. *An invariant set S is said to be* **isolated** *if there exists a compact neighborhood, N, of S so that $Inv(N,\phi)=S\subset int(N)$*

Note that in order for S to be isolated by N, S cannot intersect the boundary of N. This allows for the following continuation theorem.

Theorem 1. *Let $\phi_\lambda : R\times X\to X$ be a continuous family of parameterized flows where $\lambda\in\Lambda$. Let N be an isolating neighborhood for ϕ_{λ_O}. Then there exists $\delta>0$ so that if $\mid\lambda-\lambda_0\mid<\delta$ then N is isolating for ϕ_λ.*

The result follows trivially from the continuity of the parameterized flow and compactness. Hence isolating neighborhoods are robust with respect to perturbations, i.e. they persist or "continue" as the parameter changes. We now formalize the notion of continuation.

Definition 9. *Let $N\subset X$ be compact and consider a parameterized flow $\phi:\Lambda\times X\times R\to X$. Suppose $S_\lambda=Inv(N,\phi_\lambda)$ Then S_{λ_0}* **continues to** *S_{λ_1} if there exists a simply connected set $U\in\Lambda$ that contains λ_0 and λ_1 such that N is isolating for each $\lambda\in U$.*

Note that while S_λ continues over U, S_λ may change structure. We shall see that such a change must take place in a controlled manner.

1.2 Partial Orderings

We begin this section by defining a partial order.

Definition 10. *Let P be an (indexing) set. A* **partial order,** *$<$, on P is a relation that compares elements in P and satisfies:*

1. *(Strictness)* $p < p$ *is not true for any* $p \in P$

2. *(Transitivity)* $q < p$ *and* $r < q$ *implies* $r < p$ *for all* $p, q, r \in Q$

Given P *and* $<$ *we say that* $(P, <)$ *is a* **poset** *with indexing set* P *and partial order* $<$.

Given a partial order $<$ on P we can create other partial ordering $<'$ by adding relations. In such a case we refer to $<'$ as an **extension of** $<$. Given a partial ordering it makes sense to discuss intervals.

Definition 11. *Let* $(P, <)$ *be a poset.* $I \subset P$ *is said to be an* **interval** *if* $p, q \in I$ *and* $r \in P$ *with* $q < r < p$ *implies* $r \in I$. *The set of intervals in* P *is labelled* $\Im(P)$. J *is said to be an* **attracting interval** *if* $p \in J$ *and* $q < p$ *implies that* $q \in J$. *The set of attracting intervals is denoted* $\mathcal{A}(<)$.

Given two intervals we would like to know when their union is an interval.

Definition 12. *For* $1 \leq i \leq n$ *let* $I_i \in \Im(<)$. *Suppose that*

1. $\cup_{i=m_1}^{m_2}\{I_i\}$ *is an interval when* $1 \leq m_1 < m_2 \leq n$;

2. *For each* i *and* j *we have* I_i *and* I_j *disjoint;*

3. $q \not< p$ *whenever* $p \in I_j$ *and* $q \in I_k$ *with* $j < k$.

Then we refer to $(I_1, \cdots, I_n) \in \Im_n(P)$ *as an* **adjacent n-tuple of intervals.**

Definition 13. *If* $I, J \in \Im(<)$ *then* J *is said to be* **partially greater than** I**, (I¡J),** *if there exists* $a \in I$ *and* $b \in J$ *such that* $a < b$. I *and* J *are said to be* **not comparable** *if for each* $a \in I$ *and* $b \in J$ $a \not< b$ *and* $b \not< a$.

Definition 14. *An adjacent n-tuple of intervals,* $D = (I_1, I_2, \ldots, I_n) \in \Im_n(P)$, *is said to be a* **coarsening of P** *if*

- $\cup_{i=1}^{n}\{I_i\} = P$

- *When* $i < j$ *either* $I_i < I_j$ *or* I_i *and* I_j *are not comparable.*

Note that not every partial order admits a nontrivial coarsening. For example if $<$ is a partial order on $\{p, q, r\}$ with the relations $r < p$ and $q < p$ it is not possible to further coarsen P. However, if we add a relation between r and q then we can receive a new partial order $<'$. For example if we add $r <' q <' p$ then we can coarsen P over $<'$ with $D' = \{\{r, q\}, p\}$.

By adding relations it is always possible to extend the partial order into one that admits a coarsening.

1.3 Morse Decompositions

Given a gradient flow $\phi : \mathbb{R} \times X \to X$ every $x \in X$ remains stationary (if x is a fixed point) or flows downhill (otherwise). Conley showed that every flow can be deconstructed in a similar manner.

Definition 15. *Let S be an isolated invariant set. Let $M(S) = \{M(p) : p \in P\}$ be a collection of disjoint compact isolated invariant sets, $M(p) \subset S$, indexed by P. $M(S)$ is a* **Morse decomposition** *of S if there exists a partial ordering, $<$, on P so that for all $x \in S - \bigcup_{p \in P} M(p)$ there exists $p, q \in P$, $q < p$, so that $\omega(x) \in M(q)$ and $\alpha(x) \in M(p)$. Each $M(p)$ is referred to as a* **Morse set.** *Any partial ordering on P that satisfies the above condition is said to be an* **admissable partial ordering.**

So a Morse decomposition of S is a collection of isolated invariant sets in S such that every point in S lies in a Morse set or is on a trajectory that connects distinct Morse sets. We now define the set of trajectories linking two Morse sets.

Definition 16. *Given two Morse sets, $M(p)$ and $M(q)$ the* **set of connecting orbits** *between $M(p)$ and $M(q)$ is given by*

$$C(M(q), M(p); S) := \{x \in S : \omega(x) \in M(p) \text{ and } \alpha(x) \in M(q)\}.$$

We will associate an invariant set with each interval in P.

Definition 17. *If $I \subset P$ is an interval then $M(I) = \left(\bigcup_{p \in I} M(p)\right) \bigcup \left(\bigcup_{p,q \in I} C(M(p), M(q))\right)$ is said to be the corresponding* **Morse interval.**

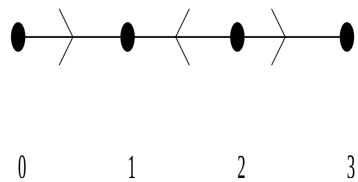


Figure 4: 1-D System

It can be shown that $M(I)$ is an isolated invariant set for all $I \in \Im(P, <)$. (See [15].) It is significant to note that $M(I)$, by definition, stores the connection data between its component Morse sets.

We consider the coarsest type of non-trivial Morse decomposition, an attractor repeller pair decomposition, as an example.

Definition 18. *Given a compact invariant set, S, $A \subset S$ is said to be an* **attractor** *if there is a neighborhood U of A so that $\omega(U \cap S) = A$. The* **dual repeller** *of A in S is the compact set $R = \{x \in S | \omega(x) \cap A = \emptyset\}$. We refer to (A, R) as an* **attractor repeller pair** *for S.*

It is easily shown that $S = A \cup R \cup C(R, A)$. We consider an example where we can define several different Morse decompositions including an attractor-repeller pair decomposition.

Example 1. *Consider the one dimesional gradient system depicted in Figure 4. The system has two sources at 0 and 2 and two sinks at 1 and 3. The interval $[0, 3]$ is an invariant set. The coarsest nontrivial Morse decomposition is the attractor repeller pair $A = [0, 1]$ and $R = [2, 3]$. The finest decomposition is $M(S) = \{0, 1, 2, 3\}$. Several admissable partial order are allowed for the Morse decomposition $M(S)$. For example $1 < 3 < 2 < 0$ is an admissable parial order that totally orders $\{0, 1, 2, 3\}$. The flow defined partial order is given by $1 <_F 0$, $1 <_F 2$ and $3 <_F 2$.*

Later it will be necessary to assume that we can use the same partial order as the parameter values change. Thus we make the following definition.

Definition 19. *Consider a parameterized flow ϕ where $M(S_\lambda) = \{M_\lambda(p) : p \in P\}$ is a Morse decomposition for each λ in the simply connected set $U \subset \Lambda$. An partial order $<$ on P is said to* **continue** *over U if $<$ is admissable for each $\lambda \in U$.*

Not every collection of invariant sets produces a Morse decomposition. Conley's Decomposition Theorem, which follows, characterizes Morse decompositions by their ability to yield Lyapunov functions.

Theorem 2. *Let S be an isolated invariant set. Let $\{M(p) : p \in P\}$ be a collection of disjoint compact invariant sets contained in S. Then $\{M(p) : p \in P\}$ is a Morse decomposition of S iff there exists a continuous $V : S \to [0, 1]$ so that*

1. $x, y \in M(p) \Rightarrow V(x) = V(y)$
2. $x \in S - \bigcup_{p \in P} M(p) \Rightarrow V(x) > V(\phi(t, x))$ *when* $t > 0$

V is referred to as a Lyapunov function.

Conley's proof is constructive. Hence, any Morse decomposition returns an *explicit* Lyapunov functions. An algorithm for constructing Lyapunov functions is presented in [13].

1.4 Conley Index

The topics discussed below are the heart of classical Conley index theory. A more detailed presentation is given in Conley's monograph [2].

The qualitative behavior of the flow (with respect to a Morse decomposition) is determined by the nature of the Morse sets and the way they connect. We will begin characterizing the flow by algebraically indexing the Morse sets using tools from homology.

As discussed in the introduction we will want to index the invariant set, S, in such a way as to encode the local dynamics. Conley captures the dynamics in a pair of compact sets N and L.

Theorem 3. *Let S be an isolated invariant set. Then there exists a pair of compact sets (N, L) where $L \subset N$ (called an* **index pair***) so that*

1. $N - L$ *is a neighborhood of* S *and* $S = Inv(cl(N - L))$
2. $x \in L$ *and* $\phi([0, t], x) \subset N \Rightarrow \phi([0, t], x) \subset L$

3. $x \in N$ *and* $t_1 > 0$ *with* $\phi(t_1, x) \in N$*, implies that there exists* $t_0 \in [0, t_1]$ *so that* $\phi([0, t_0], x) \subset N$ *and* $\phi(t_0, x) \in L$*.*

We summarize the above theorem by noting that isolation implies the existence of a pair of compact sets such that:

1. $N - L$ is a neighborhood of S and S does not touch L

2. Once in L there is no return to $N - L$ (Positive Invariance)

3. The only way to leave N is through L. (Exit Set)

Note that index pairs store local dynamic information by keeping track of how trajectories are entering and leaving the area. Since every isolated invariant set yields an index pair we are now in position to define the Conley Index.

Definition 20. *Let* S *be an isolated invariant set with index pair* (N, L)*. The (homological) Conley Index of* S *is defined as follows:*

$CH_*(S) \approx \widetilde{H}_*(N/L, [L])$ *where* N/L *is the quotient space received by collapsing* L *to a single point* $[L]$*.*

The reader interested in a quick introduction to homology theory is directed to [10]. The following theorem shows that we can define the index in terms of relative homology groups.

Theorem 4. *(see [2]) Let* S *be an isolated invariant set. Then there exists an index pair* (N, L) *so that* $CH_*(S) \approx \widetilde{H}_*(N/L) \approx \widetilde{H}_*(N, L)$*.*

The reader interested in a quick introduction to homology theory is directed to [10]. Note that $CH_*(S)$ is defined in terms of an index pair of S. To truly be an index of S we must show that $CH_*(S)$ is independent of the index pair used. Given two index pairs, (N_1, L_1) and (N_2, L_2), of S Conley showed that (N_1/L_1) and (N_2/L_2) are homotopic. Thus we have the following result.

Theorem 5. *see [2] Let* (N_1, L_1) *and* (N_2, L_2) *be index pairs for the isolated invariant set* S*. Then* $\widetilde{H}_*(N_1/L_1) \approx \widetilde{H}_*(N_2/L_2)$.

Thus $CH_*(S)$ is well defined. Moreover, since isolating neighborhoods continue, the Conley Index is robust. This final statement is formalized as follows.

Theorem 6. *Suppose the isolated invariant sets* S_{λ_1} *and* S_{λ_2} *are related by continuation. Then* $CH(S_{\lambda_1}) \approx CH(S_{\lambda_2})$.

We have now established that the Index is well-defined. We now examine how the index detects invariant sets with a result referred to as Wazewski's property.

Proposition 1. *Suppose that* N *is an isolating neighborhood such that* $CH_*(Inv(N)) \not\approx 0$ *then* $Inv(N) \neq \emptyset$.

Proof: It is sufficient to prove the contrapositive. Suppose N is an isolating neighborhood with $Inv(N) = \emptyset$. Then, trivially, $(\emptyset, \emptyset)$ is an index pair for $Inv(N)$. So $CH_*(Inv(N)) \approx 0$. □

Therefore, the Conley Index detects invariant sets. A non-trivial index is a sufficient but not necessary condition for the existence of a non-trivial invariant set.

1.5 Detecting Connections

Given an attractor repeller pair decomposition of S we will show that the existence of connecting orbits between R and A can be proven via the the Conley indices of A, R, and S.

Theorem 7. *(see [2]) Let* $M(S) = \{A, R\}$ *be an attractor-repeller pair decomposition of* S*. Then there exists compact sets* $N_0 \subset N_1 \subset N_2$ *so that*

1. (N_2, N_0) *is an index pair for* S
2. (N_2, N_1) *is an index pair for* R
3. (N_1, N_0) *is an index pair for* A

N_0, N_1, N_2 *is referred to as an* **index triple** *for* $M(S)$.

Let $C_n(N_i, N_j)$ denote the space of relative singular $n-$chains for (N_i, N_j). Since $N_0 \subset N_1 \subset N_2$ we have that $C_n(N_1, N_0) \subset C_n(N_2, N_0)$ and $C_n(N_2, N_1) \subset C_n(N_2, N_0)$. Given the inclusion and projection maps ι and ρ we can produce the following short exact sequence.

$$0 \to C_n(N_1, N_0) \xrightarrow{\iota} C_n(N_2, N_0) \xrightarrow{\rho} C_n(N_2, N_1) \to 0$$

When we pass to homology this short exact sequence becomes the following long exact sequence.

$$\cdots \to \widetilde{H}_n(N_1, N_0) \to \widetilde{H}_n(N_2, N_1) \to \widetilde{H}_n(N_2, N_1) \xrightarrow{\partial_n} \widetilde{H}_{n-1}(N_1, N_0) \to \cdots$$

By definition this is just...

$$\cdots \to CH_n(A) \to CH_n(S) \to CH_n(R) \xrightarrow{\partial_n} CH_{n-1}(A) \to \cdots$$

Therefore, index triples yield long exact sequences of Conley indices. Furthermore, these sequences contain information on the gradient-like part of the flow.

Theorem 8. *Given the attractor repeller pair decomposition $M(S) = \{A, R\}$ we have that $\partial_n = 0$ whenever $C(R, A; S) = \emptyset$.*

Proof. Given that $C(R, A; S) = \emptyset$ we have $S = A \cup R$ and hence, since A and R are isolated, can find neighborhoods U_A and U_R that separate A and R. Let (N_1^A, N_0^A) and (N_2^B, N_0^B) be index pairs of A and B with $N_1^A, N_0^A \in U_A$ and $N_2^B, N_0^B \in U_B$. Then $(N_1^A \cup N_2^R, N_1^A \cup N_0^R, N_0^A \cup N_0^R)$ is an index triple for the attractor repeller pair. The triple yields the long exact sequence

$$\cdots \to \widetilde{H}_n(N_1^A \cup N_0^R, N_0^A \cup N_0^R) \to \widetilde{H}_n(N_1^A \cup N_2^R, N_1^A \cup N_0^R) \to$$
$$\widetilde{H}_n(N_1^A \cup N_2^R, N_1^A \cup N_0^R) \xrightarrow{\partial_n} \widetilde{H}_{n-1}(N_1^A \cup N_0^R, N_0^A \cup N_0^R) \to \cdots$$

We apply the exision principle to receive

$$\cdots \to \widetilde{H}_n(N_1^A, N_0^A) \to \widetilde{H}_n(N_1^A, N_0^a) \oplus \widetilde{H}_n(N_2^B, N_0^R) \to \widetilde{H}_n(N_1^A, N_1^A) \xrightarrow{\partial_n} \widetilde{H}_{n-1}(N_1^A, N_0^A) \to \cdots$$

Exactness implies that ∂_n is trivial.

□

Therefore, a *nontrivial boundary operator indicated the presence of connecting orbits.* We have seen that, given an index triple for an attractor repeller decomposition, we can compare the Conley indices of A, R, and S via a long exact sequence. We are able to do this because the index triple produces a consistent set of chains. Can we do the same for any Morse decomposition? The answer is yes.

Definition 21. *A family of compact sets* $\mathcal{N} = \{N(I)\}_{I \in \mathcal{A}(<)}$ *is said to be an* **index filtration** *for* $M(S,<)$ *if*

1. *For each* $I \in \mathcal{A}(<)$ *we have that* $(N(I), N(\emptyset))$ *is an index pair for* $M(I)$

2. *Given* $I_2, I_1 \in \mathcal{A}(<)$ *we have* $N(I_1 \cap I_2) = N(I_1) \cap N(I_2)$ *and* $N(I_1 \cup I_2) = N(I_1) \cup N(I_2)$

It is shown in [6] that index filtrations can be constructed for any Morse decomposition. Furthermore, it is shown that an index filtrations contain index pairs for all intervals $I \in \Im(<)$. In particular, if the adjacent pair of intervals I, J is a decomposition of the attracting interval $K \in \mathcal{A}(<)$ then $(N(K), N(I))$ is an index pair for $M(J)$.

Given an adjacent pair of intervals $I, J \in \Im_2(<)$ we have that $M(I)$, $M(J)$ is an attractor repeller pair decomposition of the $M(IJ)$. Using the index pairs provided by the filtration [6] constructed the following long exact sequence for any adjacent pair of intervals.

$$\cdots \to CH_n(M(I)) \to CH_n(M(IJ)) \to CH_n(M(R)) \xrightarrow{\partial_n} CH_{n-1}(M(A)) \to \cdots$$

We will now begin the process of connecting the long exact sequences generated by an index filtration with homomorphisms.

Definition 22. *Let* $C = \{C(p) | p \in P\}$ *be a set of chain complexes with* $<$ *a partial order on* P*. A* **chain complex braid** *generated by* C *over the partial order* $<$ *is a collection* $\mathcal{C}(<)$ *of chain complexes and chain maps satifying:*

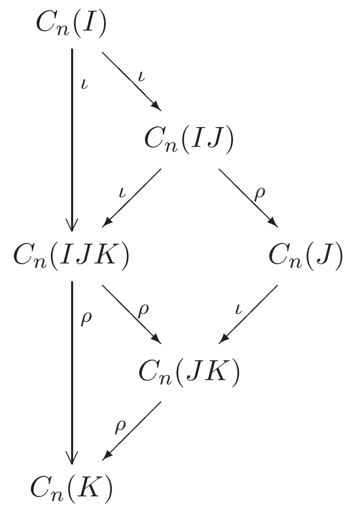
1. *For each* $I \in \Im(<)$ *there exists a chain complex* $C(I)$

2. *For each* $(I, J) \in \Im(<)$ *there exist chains maps* $\iota(I, IJ) : C(I) \to C(IJ)$ *and* $\rho(IJ, I) : C(IJ) \to C(J)$ *so that*

- $C(I) \xrightarrow{i} C(IJ) \xrightarrow{\rho} C(J)$ *is exact*

- $\rho(IJ, I)\iota(I, IJ) = id|_{\gamma(I)}$ *whenever* I *and* J *are noncomparable.*

- $(I, J, K) \in \Im_3(<)$ *yields the commutative diagr*



Franzosa showed in [6] that index filtrations produce chain complex braids. Given an index filtration , $\mathcal{N}$, its corresponding braid will be denoted $\mathcal{C}(\mathcal{N})$.

The Conley index of a Morse set $M(p)$, written in field coefficients, is an infinite collection of (possible trivial) ordered vector spaces. The Conley index of an isolated invariant set, written in field coefficients, is an example of a **graded vector space.**

Definition 23. *G is said to be a* **graded vector space** *if $G = \oplus_i\big(G(i)\big)$ where each $G(i)$ is a vector space. G is said to be* **finite** *if $G(i) \neq 0$ for only a finite number of $i \in \mathbb{Z}$ and each $G(i)$ is finite.*

The graded vector spaces (GVSs) we will work with are finite.

Definition 24. *Let $\mathcal{G} = \{G(p)|p \in P\}$ be a set of graded vector spaces (GVSs) with $<$ a partial order on P. The* **graded vector space braid (GVSB)** *generated by G over $<$ is a collection $\mathcal{G}(<)$ of graded vector spaces (GVSs) and maps such that*

1. *For each $I \in \Im(<)$ there exists a graded vector space $G(I)$*

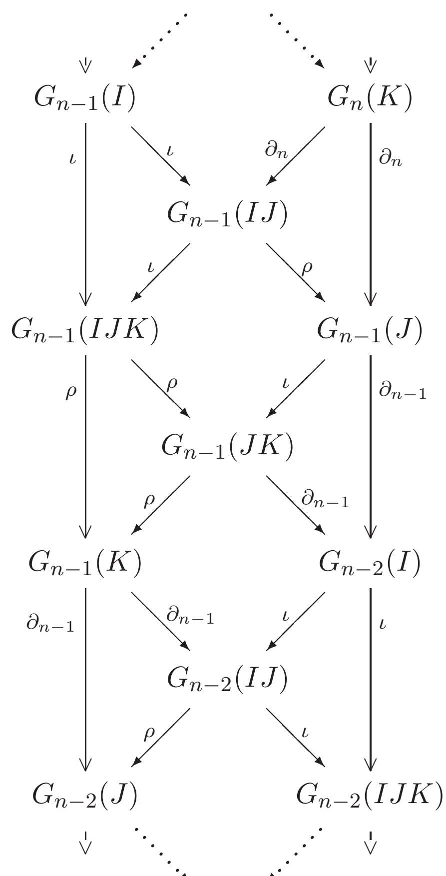
2. *If $(I, J) \in \Im_2(<)$ then there exist the following maps:*

- $\iota(I, IJ) : G(I) \to G(IJ)$ *(An inclusion map of degree 0)*
- $\rho(IJ, I) : G(IJ) \to G(J)$ *(A projection map of degree 0)*
- $\partial(J, I) : G(J) \to G(I)$ *(A boundary map)*

So that $\cdots$

1. $\cdots \xrightarrow{\partial_n} G(I) \xrightarrow{\iota(I,IJ)} G(IJ) \xrightarrow{\rho(IJ,J)} G(J) \xrightarrow{\partial_{n-1}} \cdots$ *is a long exact sequence*

2. *Given* $(I, J, K) \in \Im_3(<)$ *the following diagram commutes*



Chain complex braids generate GVSBs. Let $\mathcal{C}(<)$ be a chain complex braid. Let $(I, J) \in \Im(<)$. Then the short exact sequence

$$C(I) \xrightarrow{i} C(IJ) \xrightarrow{\rho} C(J) \tag{1}$$

yields the long exact sequence

$$\cdots \xrightarrow{\partial} H(I) \xrightarrow{\iota} H(IJ) \xrightarrow{\rho} H(J) \xrightarrow{\partial} \cdots \tag{2}$$

when we pass to homology (in field coefficients) with ι, ρ and ∂ inherited from the chain complex maps. It is shown in [6] that the set of GVSs $\{G(I) | I \in \Im(<)\}$ and the maps ι, ρ and ∂ form a GVSB. We refer to this braid as the **homology braid** , $H\mathcal{C}(<)$, generated by $\mathcal{C}(<)$.

Of particular importance are those GVSBs generated by CCBs resulting from index filtrations. Given a Morse decomposition $(M(S), <)$ with index filtraion $\mathcal{N}$ we refer to the GVSB generated by $\mathcal{C}(\mathcal{N})$ as the homology braid and label it $\mathcal{H}(<)$. Franzosa showed in [6] that the homology braid is independent of the filtration chosen to construct it.

We now build maps between braids.

Definition 25. *Consider two chain complex braids $\mathcal{C}$ and $\mathcal{C}'$ over $<$. A chain map $\phi : \mathcal{C} \to \mathcal{C}'$ is a family of chain maps $\{\phi(I) | I \in \Im(<)\}$ so that $\phi(I) : C(I) \to C'(I)$ is a chain map for each $I \in \Im(<)$ and such that the following diagram commutes.*

$$\begin{array}{ccccc} C(I) & \xrightarrow{\iota} & C(IJ) & \xrightarrow{\rho} & C(J) \\ \downarrow{\scriptstyle \phi(I)} & & \downarrow{\scriptstyle \phi(IJ)} & & \downarrow{\scriptstyle \phi(J)} \\ C'(I) & \xrightarrow{\iota} & C'(IJ) & \xrightarrow{\rho} & C'(J) \end{array}$$

As we pass to homology each $\phi(I)$ induces a map $\Phi(I) : H\mathcal{C}(\mathcal{I}) \to H\mathcal{C}'(\mathcal{I})$. We refer to Φ as the homology map induced by ϕ. Moreover, $\Phi : H\mathcal{C}(<) \to H\mathcal{C}'(<)$ is a **braid isomorphism** if each $\Phi(I)$ is an isomorphism.

Let us consider a collection of graded vector spaces $G = \{g(p) : p \in P\}$ partially ordered by $<$. Let $I \in \Im(<)$ and $\triangle : \oplus_{p \in I} G(p) \to \oplus_{p \in I} g(p)$ be a homomorphism. We can consider this map as a matrix with entries $\triangle(p, p') : G(p) \to G(p')$. We say $\triangle$ is **upper triangular** if $\triangle(p, p') \neq 0$ implies that $p = p'$ or $p' < p$. We say it is **strictly upper triangular** if $\triangle(p, p') \neq 0$ implies that $p' < p$.

In [8] it was shown that given an upper triangular boundary map

$$\triangle : \oplus_{p \in P} G(p) \to \oplus_{p \in P} G(p)$$

then the map restricted to an interval I given by

$$\triangle(I) : \oplus_{p\in I} G(p) \to \oplus_{p\in I} G(p)$$

is also an upper triangular boundary map. Therefore, for each $I \in \Im(<)$,

$$\big(C(I) = \{G(p) : p \in I\}, \triangle(I)\big)$$

is a chain complex. Furthermore given an adjacent pair of intervals, I and J, the obvious inclusion and projection maps produce the following short exact sequence

$$C(I) \xrightarrow{\iota(I,IJ)} C(IJ) \xrightarrow{\rho(IJ,I)} C(J)$$

This collection of maps and chain complexes produces a chain complex braid. The result is stated below.

Theorem 9. *(see [8]) Let $G = \{G(p) : p \in P\}$ be a collection of GVSs ordered by $<$. Let $\triangle \oplus_{p\in P} G(p) \to \oplus_{p\in P} g(p)$ be a $<$-upper triangular boundary map. Then the set of chain complexes $\{C(I) = \{G(p) : p \in I\}, \triangle(I))$ for $I \in \Im(<)$ together with the maps $\iota(I, IJ)$ and $\rho(IJ, I)$ for a chain complex braid, $\mathcal{C}\triangle(<)$ over $<$. We label the correseponding GVSB by $H\triangle(<)$ and refer to it as the* **homology braid generated by** $\triangle$.

The structure for the connecting homomorphisms of $H\triangle(<)$ are of interest. Let I and J be adjacent intervals. Then the following long exact sequence is contained in the homology braid $H\triangle(<)$.

$$\cdots \to H_n\triangle(I) \to H_n\triangle(IJ) \to H_n\triangle(J) \xrightarrow{\triangle_n(J,I)} H_{n-1}\triangle(I) \to \cdots$$

Franzosa showed the the connecting homomorphism, $\triangle_n(J, I)$, acts as follows. Let

$$[\alpha] \in H_n\triangle_n(J)$$

then

$$\triangle_n(J, I)[\alpha] = [\triangle_n(J, I)\alpha].$$

The end result is that given a set of $GVSs$ and an upper triangular boundary map, $\triangle$, we can build a GVSB where $\triangle$ defines the necessary boundary operators and connecting homomorphisms. This begs the following question. *Is every GVSB isomorphic to $\mathcal{H}\triangle(<)$ for some $<$-upper triangular boundary map $\triangle$? The answer is yes and is one of the main results of [8].*

Theorem 10. *Suppose that $\mathcal{G}(<)$ is a GVSB generated by $G = \{G(p)|p \in P\}$ over $<$. Then there exists a $<$-upper triangular boundary map $\triangle : G(P) \to G(P)$ so that $\mathcal{H}\triangle(<)$ is isomorphic to $\mathcal{G}$. We refer to $\triangle$ as a* **connection matrix** *for $\mathcal{G}$. The set of connection matrices is labelled $CM(<)$.*

We consider the significance of connection matrices in the context of a homology index braid. Let $M(S) = \{M(p)|p \in P\}$ be a Morse decomposition. As previously discussed $\mathcal{H}(<)$ is a GVSB generated by $\{H(p) = CH(M(P))|p \in P\}$. Theorem 10 proposes that there exists a $<$-upper triangular boundary map $\triangle : \oplus_p H(p) \to \oplus_p H(p)$ such that $\mathcal{H}\triangle(<) \cong \mathcal{H}(<)$. Recall that the boundary operators ∂ of $\mathcal{H}(<)$ store connecting orbit information. Since $\mathcal{H}\triangle(<)$ and $\mathcal{H}(<)$ are isomorphic this information must be stored in $\triangle$.

We label the connection matrices for the flow defined partial order by $CM(M(S))$. Every other admissable partial order is an extension of $<_F$ and hence $CM(M(S)) \subset CM(<)$.

Theorem 11. *If $\triangle \in CM(M)$ with p and q adjacent with respect to $<_F$ then $\triangle(p,q) \neq 0$ implies that $C(M(p), M(q)) \neq \emptyset$.*

We will summarize what has been discussed so far. Given a Morse decomposition of an invariant set we can now algebraically index the invariant part of the flow via the Conley index and algebraically encode the gradient-like part via connection matrices.

1.6 Conley-Morse Chain Maps

We have established that qualitative information about a flow, for a given Morse decomposition, can be stored algebraically. We turn to the question of comparing two such structures. We assume that the two flows we wish to compare are part of a continuously parameterized family of flows.

Let Λ be a path connected subset of R^n and let

$$\{\phi_\lambda : \mathbb{R} \times X \times \to X : \lambda \in \Lambda\}$$

be a continuously parameterized family of flows. Then if we define

$$\phi(t, x, \lambda) = (\phi_\lambda(t, x), \lambda)$$

we have that

$$\phi : \mathbb{R} \times X \times \Lambda \to X$$

is a continuous flow. Given U, a path connected subset of Λ, we let ϕ_U be the restriction of ϕ to $\mathbb{R} \times X \times U$.

Let S be an isolated invariant set of ϕ_U. Let $M(S) = \{M(p) : p \in P\}$ be a Morse decomposition for S with order $<$. Then, by intersecting $M(S)$ with the $X \times \{\lambda\}$ we receive a Morse decomposition, $M(S_\lambda) = \{M_\lambda(p) : p \in P\}$ for ϕ_λ. We are, therefore, allowed the following definition.

Definition 26. *Let $U \subset \Lambda$ be path connected. $M(S)$ continues over U with order $<$ if $M(S)$ is a Morse decomposition for ϕ_U. If $\lambda_1, \lambda_2 \in U$ then $M(S_{\lambda_1})$ and $M(S_{\lambda_2})$ are said to be* **related by continuation.**

We now present an overview of Franzosa and Mischaikow's algebraic transition matrix theory for parameterized flows(see [9]). At the second author's request we will refer to algebraic transition matrices as Conley-Morse chain maps. They found that given connection matrices for two Morse decompositions related by continuation there exists an invertible, upper triangular chain map between the resulting chain complexes. The existence of such chain maps ensures that the following set of chain maps is non-empty.

Definition 27. *Let $M(S)$ be a Morse decomposition which continues with partial order $<$ over $U \in \Lambda$, simply connected. Given $\lambda_0, \lambda_1 \in U$, T is in the set of* **Conley Morse chain maps,** *$T^U_{\lambda_1,\lambda_0}$, from $CM(M(S_{\lambda_0}))$ to $CM(M(S_{\lambda_1}))$ if*

1. *T is 0-degree*

2. T *is* $<$*-upper triangular*

3. $\triangle_{\lambda_1} = T\triangle_{\lambda_0}T^{-1}$ *where* $\triangle_i \in CM(M(S_{\lambda_i}))$

Definition 28. *A partial order* $<$*, on* P*, is said to be* **stackable** *if there exists a decomposition* $(I_1, \cdots, I_n)$ *of* P *so that* $<$ *is trivial on each component of the decomposition and if* $i < j$ *then for each* $a \in I_i$ *and* $b \in I_j$ *we have* $a < b$*.*

It was shown that if a Morse decomposition continues with respect to a stackable order then Conley Morse chain maps can be used to detect bifurcations. The interested reader is referred to [9] for examples and applications.

Theorem 12. *Let* $<$ *be a stackable partial order with coarsening* $\{I_1, \cdots, I_n\}$*. Suppose the Morse decomposition* $M(S)$ *continues with partial order* $<$ *over* $U \in \Lambda$*, path connected. Let* $\lambda_0, \lambda_1 \in U$ *and suppose for all* $T \in T^U_{\lambda_1,\lambda_0}$*.* $T(I_j, I_{j-1}) \neq 0$*. Then there exists a set* $V \subset U$ *such that* $U - V$ *has two componensts* U_0 *and* U_1 *so that* $\lambda_0 \in U_0$ *and* $\lambda_1 \in U_1$*. Furthermore for all* $\lambda \in V$ *we have* $C(M_\lambda(I_j), M_\lambda(I_{j-1})) \neq \emptyset$*.*

Therefore, given a topological condition, continuation, two Morse decompositions can be compared via chain maps. When end this chapter by reiterating the primary question raised in this paper. Can two data sets be qualitatively compared in spite of differences in resolution? In this setting it is unreasonable to discuss continuation. We will instead rely on purely algebraic means for comparing the decompositions. We dedicate the remainder of the dissertation to this problem.

CHAPTER II

CONLEY-MORSE CHAIN MAPS

2.1 Overview

Our goal is to recover the results similar to the last section when the Morse decomposition does not continue. We will relax the continuation assumption by instead insisting that the homology braids agree on some coarse scale. Therefore we switch from a topological condition for similarity to a purely algebraic condition.

2.2 Definitions

The following definitions are written in the context of graded vector space braids. Although we are primarily interested in homology braids the following definitions are written in the context of abstract GVSBs.

Definition 29. *Let $G = \{G(p) : p \in P\}$ be a GVSB over $<$. Let D be a coarsening. We define $G|_D$*, **the GVSB coarsened by** D, *to be the subbraid of G generated by the graded vector spaces $\{G(J)\}_{J\in D}$ over the order $<_D$.*

Definition 30. *Let G and G' be GVSBs over $<$ and $<'$ with coarsenings D and D'. We say that G and G' are* **related by coarsening** *if $G|_D$ is isomorphic to $G'|_{D'}$.*

Definition 31. *Let $G = \{G(p) : p \in P\}$ and $G' = \{G'(p') : p' \in P'\}$ be GVSBs isomorphic under the coarsenings $D = \{I_1, \cdots, I_n\}$ and $D' = \{I'_1, \cdots, I'_n\}$. We say that a homomorphism $T : G(P) \to G'(P')$* **preserves the coarsening** *if given $I_j \in D$ and $I'_k \in D'$ with $T(I_j, I'_k) \neq 0$ then $I_k < I_j$*

When two homology braids, $\mathcal{H}(<)$ and $\mathcal{H}(<')$, related by coarsening and induced by Morse decompositions $M(S, <)$ and $M(S', <')$ we abuse notation to say that $M(S)$ and $M(S')$ are related by coarsening. We are now in position to define the set of **Conley-Morse chain maps.**

Definition 32. *Let $M(S_0)$ and $M(S')$ be Morse decompositions that are related by the coarsenings $D = \{I_1, \cdots, I_n\}$ and $D' = \{I'_1, \cdots, I'_n\}$. Then the set of* **coarsening defined Conley-Morse chain maps ,** *$\mathcal{T}(D, D')$ is defined as follows. $T \in \mathcal{T}(\mathcal{D}, \mathcal{D}')$ if and only if*

1. *$T_* = \theta(P)$ for some braid isomorphism $\theta : H(<)|_D \to H(<')|_{D'}$*

2. *$T(I_i, I'_i)_* = \theta(I)$ for $i \in \{1, \cdots, n\}$.*

3. *T preserves the coarsening*

4. *$\triangle' T = T\triangle$ for some $\triangle \in CM(M(S))$ and $\triangle' \in CM(M(S'))$*

Our goal is to show that $\mathcal{T}(\mathcal{D}, \mathcal{D}')$ is non-empty. To this end we need to prove the following existence theorem.

Theorem 13. *Let $M(S) = \{M(p) : p \in P\}$ and $M(S') = \{M(p') : p' \in P'\}$ be Morse decompositions of the isolated invariant sets S and S', respectively. Let $<$ and $<'$ be admissable orders for the respective decompositions. Then, if there exist coarsening D and D' of P and P' so that $\mathcal{H}(<)|_{\mathcal{D}} \cong \mathcal{H}(<')|_{\mathcal{D}'}$ we have that $\mathcal{T}(\mathcal{D}, \mathcal{D}') \neq \emptyset$.*

Theorem 13 follows immediately from the following result concerning GVSBs.

Theorem 14. *Let $G = \{G(p)\}_{p \in P}$ and $G' = \{G'(q)\}_{q \in P'}$ be collections of GVSs with boundary maps $\triangle : G(P) \to G(P)$ and $\triangle' : G'(P') \to G'(P')$, respectively. Assume $\triangle$ and $\triangle'$ are $<$ and $<'$ upper triangular. Suppose $\theta : H\triangle(<)|_D \to H\triangle'(<')|_{D'}$ is a braid isomorphism under the coarsenings D and D'. Then there exists a coarsening preserving homomorphism $T : G(P) \to G'(P')$ so that:*

1. *$T_* = \theta(P)$*

2. *$T_* = \theta(I)$ for each $I \in D$*

3. *$\triangle' T = T\triangle$*

The proof of Theorem 14 will be presented at the end of the chapter. We will begin the next section discussing two results used in the proof of Theorem 13.

2.3 Existence of Conley-Morse Chain Maps

The first proof details the process of concatenating chain maps. It closely follows the constructive proof presented by Franzosa and Mischaikow in [9].

Theorem 15. *Let $G = \{G(p)\}_{p\in P}$ and $G' = \{G'(q)\}_{q\in P'}$ be collections of GVSs with a $<$ −upper triangular boundary map $\triangle : G(P) \to G(P)$ and a $<'$ −upper triangular boundary map $\triangle' : G'(P') \to G'(P')$, respectively. Suppose $D = \{I_1, I_2\}$ and $D' = \{I'_1, I'_2\}$ are coarsenings of P and P', respectively, so that $\theta : H\triangle(<)|_D \to H\triangle'(<')|_{D'}$ is a braid isomorphism. Further suppose that there exist homomorphisms $T(1)$, $T(2)$ so that*

1. *For $i \in \{1, 2\}$, $T(i) : G(I_i) \to G'(I'_i)$ satisfies $T(i)\triangle(I_i) = \triangle(I'_i)T(i)$*
2. *$T(I_i)_* = \theta(I_i)$*

Then there exists $T : G(P) \to G'(P')$ such that:

1. *$T\triangle = \triangle' T$*
2. *$T_* = \theta(P)$*
3. *T preserves the coarsening*

Proof. We define $T(I_1, I'_2) : G(I_1) \to G'(I'_2)$ by $T(I_1, I'_2) = 0$.

We must now define $T(I_2, I'_1) : G(I_2) \to G'(I'_1)$. We consider two cases.

When I_2 and I_1 are non-comparable define $T(I_2, I'_1) = 0$. We now consider the case when $I_2 > I_1$.

We define the following submodules of $G(I_2)$:

$$
\begin{aligned}
A &= \Big(Im\triangle(I_2)\Big) \\
B &= \Big((\triangle(I_2, I_1)^{-1}(Im\triangle(I_1)) \cap ker\triangle(I_2)\Big) - A \\
C &= ker\triangle(I_2) - B \\
D &= G(I_2) - C
\end{aligned}
$$

We now define $T(I_2, I_1')$ module-wise.

Define $T(I_2, I_1')d = 0$ for all $d \in D$.

Let $c \in C$. By definition $c \in ker\triangle(I_2)$. Since $T(2)$ covers $\theta(I_2)$ we must have $T(2)c \in ker\triangle'(I_2')$. So we have the following equivalence on the homology level.

$$\begin{aligned}
[\triangle'(I_2', I_1')T(2)c] &= \partial'(I_2', I_1')[T(2)c] && (\triangle' \text{ is a connection matrix}) \\
&= \partial'(I_2', I_1')\theta_2[I_2] && (T(I_2)_* = \theta(I_1)\ ) \\
&= \theta(I_1)\partial(I_2, I_1)[c] && (\theta \text{ commutes between braids}) \\
&= \theta(I_1)[\triangle(I_2, I_1)c] && (\triangle \text{ is a connection matrix}) \\
&= [T(1)\triangle(I_2, I_1)c] && (T(I_1)_* = \theta(I_1)\ )
\end{aligned}$$

Since $[\triangle'(I_2', I_1')T(2)c] = [T(1)\triangle(I_2, I_1)c]$ there exists $t_c \in G'(I_1)$ such that $\triangle'(I_1)t_c = T(1)\triangle(I_2, I_1)c - \triangle'(I_2', I_1')T(2)c$. We define $T(I_2, I_1')c = t_c$.

Let $b \in B$. By definition there exists $z_b \in G(I_1)$ such that $\triangle(I_1)z_b = \triangle(I_2, I_1)b$. We now show that there exists $s_b \in G'(I_1')$ such that $[s_b \oplus T_2 b] \in H(P)$. We do this by showing that there exists $s_b \in G'(I_1')$ such that $[s_b + T_2 b] = \theta(P)[-z_b \oplus b]$ and use the fact that $\theta(P)$ maps kernels to kernels. Let $\nu \oplus \mu \in G'(I_1') \oplus G'(I_2') = G'$ be such that $[\nu \oplus \mu] = \theta(P)[-z_b \oplus b]$. Since $[\mu] = p'(P', I_2')[\nu \oplus \mu] = p'(P', I_2')\theta(P)[-z_b \oplus b] = \theta(I_2)p(P, I_2)[-z_b \oplus b] = \theta(I_2)[b] = [T(2)b]$ there exists $\rho \in G'(I_2')$ so that $\triangle'(I_2')\rho = T(2)\beta - \mu$. Define $s_b = \nu + \triangle'(I_2', I_1')\rho$. Then $\triangle'(0 \oplus \rho) = \triangle'(I_2', I_1')\rho \oplus \triangle'(I_2')\rho = (s_b - \nu) \oplus (T(2)b - \mu) = (s_b \oplus T(2)b) - (\nu \oplus \mu)$. Therefore $[s_b \oplus T(2)b] = \theta(P)[-z_b \oplus b]$. Our claim is proved and we define $T(I_2, I_1') = s_b + T(1)z_b$.

We now complete our definition of $T(I_2, I_1')$. If $a \in A$ then $a = \triangle(I_2)d$ for a unique $d_a \in D$. Define $T(I_2, I_1)a = \triangle'(I_2', I_1')T(2)d_a - T(1)\triangle(I_2, I_1')d_a$.

Since $T(1)$ is a similarity isomorphism it is clear that $\triangle' T = T\triangle$ on the subspace $G(I_1) \oplus 0 \subset G(P)$. We examine the subspaces $0 \oplus A$, $0 \oplus B$, $0 \oplus C$, $0 \oplus D$, to verify the chain condition on $0 \oplus G(I_2)$.

Let $a \in A$ so that $a = \triangle(I_2)d_a$ for $d_a \in D$.

$$\begin{aligned}
\triangle' T(0 \oplus a) &= \triangle'\Big(T(I_2, I_1')a \oplus T(2)a\Big) \\
&= \triangle'\bigg(\Big(\triangle'(I_2', I_1')T(2) - T_1\triangle(I_2, I_1)\Big)d_a \oplus T(2)a\bigg) \\
&= \Big(\triangle'(I_2', I_1')T(2)\triangle(I_2) + \triangle'(I_1')\triangle'(I_2', I_1')T(2) - \triangle'(I_1')T(1)\triangle(I_2, I_1)\Big)d_a \oplus \triangle'(I_2)T(2)a \\
&= \Big(\triangle'(I_2', I_1)\triangle'(I_2)T(2) + \triangle'(I_1)\triangle'(I_2', I_1')T(2) - T(1)\triangle(I_1)\triangle(I_2, I_1)\Big)d_a \oplus T(2)\triangle(I_2)a \\
&= -T(1)\triangle(I_1)\triangle(I_2, I_1)d_a \oplus T(2)\triangle(I_2)a \\
&= T(1)\triangle(I_2, I_1)a \oplus 0 \\
&= T\triangle(0 \oplus a)
\end{aligned}$$

Let $b \in B$ and $s_b \in G'(I_1')$ be as above.

$$\begin{aligned}
\triangle' T(0 \oplus b) &= \triangle'\Big(T(I_2, I_1')b \oplus T(2)b\Big) \\
&= \triangle'\Big((s_b + T_1 z_b) \oplus T(2)b\Big) && \text{(Definition of } T(I_2, I_1')b \text{)} \\
&= \triangle'(I_1')T(1)z_b \oplus 0 && (s_b \oplus T(2)b \in ker\triangle') \\
&= T(1)\triangle(I_1)z_b \oplus 0 && (T(I_1) \text{ and } \triangle(I_1) \text{ commute}) \\
&= T(1)\triangle(I_2, I_1)b \oplus 0 \\
&= T\triangle(0 \oplus b)
\end{aligned}$$

Let $c \in C$ with $t_c \in \triangle'(I_1')$ as defined earlier.

$$\begin{aligned}
\triangle' T(0 \oplus c) &= \triangle'(T(I_2, I_1')c \oplus T(2)c) \\
&= \triangle'(t_c \oplus T(2)c) && \text{(Definition of } T(I_2, I_1')c \text{)} \\
&= (\triangle'(I_2', I_1')T(2)c + \triangle'(I_1')t_c) \oplus \triangle'(I_2')T(2)c) \\
&= T(1)\triangle(I_2, I_1)c \oplus T(2)\triangle(I_2)c && \text{(Definition of } t_c) \\
&= T(1)\triangle(I_2, I_1)c \oplus 0 && (c \in ker\triangle(I_2)) \\
&= T\triangle(0 \oplus c)
\end{aligned}$$

Let $d \in D$ and $a = \triangle(I_2)d$.

$$\begin{aligned}
\triangle' T(0 \oplus d) &= \triangle'(0 \oplus T(2)d) \\
&= \triangle'(I_2', I_1')T(2)d \oplus \triangle'(I_2')T(2)d \\
&= \Big(T(I_2, I_1')d + T_1\triangle(I_2, I_1)a\Big) \oplus T(2)\triangle(I_2)d \\
&= \Big(T(I_2, I_1')\triangle(I_2)d + T(1)\triangle(I_2, I_1')d\Big) \oplus T(I_2)\triangle(I_2)d \\
&= T\Big(\triangle(I_2, I_1')d \oplus \triangle(I_2)d\Big) \\
&= T\triangle(0 \oplus d)
\end{aligned}$$

Hence $\triangle' T = T\triangle$.

We will now show that $T_* = \theta(P)$. Let $\omega \in H(P)$. Then $\omega = [\lambda \oplus \kappa]$ where $\lambda \oplus \kappa \in G(I_1) \oplus G(I_2)$. We consider two cases. First we suppose $\triangle(I_2, I_1)\kappa = 0$. Then $\lambda \in ker\triangle(I_1)$.

$$\begin{aligned}
T_*\omega &= [T(1)\lambda] \\
&= \iota'(I_1', P')[T(1)\lambda] \\
&= \iota'(I_1', P')\theta(I_1)[\lambda] \\
&= \theta(P)\iota(I_1, P)[\lambda] \\
&= \theta(P)[\lambda \oplus 0] \\
&= \theta(P)\omega
\end{aligned}$$

If we assume $\kappa \neq 0$ then $k \in ((\triangle(I_2, I_1)^{-1}(Im\triangle(I_1)) \cap ker\triangle(I_2))$. Hence we can consider $\kappa \in B$ and $\lambda \oplus \kappa = -z_b \oplus \beta$ as above.

$$
\begin{aligned}
T_*\omega &= [T(-z_b \oplus \beta)] \\
&= [(T(I_2, I_1')\beta + T(1)(-z_b)) \oplus T(2)\beta] \\
&= [((\sigma + T(1)z_b) + T(1)(-z_b)) \oplus T(2)\beta] \\
&= [\sigma \oplus T(2)\beta] \\
&= \theta(P)[-z_b \oplus \beta] \\
&= \theta(P)\omega
\end{aligned}
$$

Hence $T_* = \theta(P)$. □

We now show that given two homologically isomorphic chain complexes there exists a chain map that induces the isomorphism.

Theorem 16. *Let $G = \{G(p)\}_{p\in J}$ and $G' = \{G'(q)\}_{q\in J'}$ be finite collections of free and finitely generated graded modules with $<$-boundary map $\triangle : G(J) \to G(J)$ and $<'$- boundary map $\triangle' : G'(J') \to G'(J')$, respectively. Suppose $\theta : H\triangle(<) \to H\triangle'(<')$ is an isomorphism. Then there exists a homomorphism $T : G(J) \to G'(J')$ such that $T_* = \theta(J)$ and $\triangle' T = T\triangle$.*

Proof. For all positive k we make the following definitions.

Let $M_k = Image(\triangle_k)$. Let $B_k = \{[w_1], [w_2], \cdots, [w_{d(k)}]\}$ be a basis for $H_k\triangle(J)$. Let K_k denote the subspace of $ker(\triangle_k)$ generated by $span(w_1, w_2, \cdots, w_{d(k)})$. Note that K_k is complementary to M_k. Let L_k be a subspace of $G_k(J)$ complementary to $M_k \oplus K_k$. Note that $G_k(J) = L_k \oplus K_k \oplus M_k$. Let $B'_k = \{[w'_1], [w'_2], \cdots, [w'_k]\}$ be a basis for $H_k\triangle'(J')$. Without loss of generality we may assume $\theta : [w_i] \to [w'_i]$.

We decompose $G'_k(J')$ in an identical manner to receive $C'_k = L'_k \oplus K'_k \oplus M'_k$.

We use induction on the grade of homology groups.

For $k \geq N$ our assumption holds. Set $T_k = 0$. Trivially $(T_k)_{*}* = 0 = \theta_n$ and $\triangle'_{k+1}T_{k+1}w = 0 = T_k\triangle_{k+1}$.

Now assume that for all $j > k$ there exists a homomorphism $T_j : G_j(J) \to G'_j(J')$ such that $(T_j)_* = \theta_j$ and $\triangle'_{j+1}T_{j+1} = T_j\triangle_{j+1}$. We wish to show that there exists a homomorphism $T_k : G_k(J) \to G'_k(J')$ such that $(T_k)_* = \theta_k$ with $\triangle'_{k+1}T_{k+1}w = T_k\triangle_{k+1}$.

Let $\iota \in M_k$. Then there exists $d \in G_{k+1}(J)$ so that $\triangle d = \iota$.

$$\begin{aligned} T_k\iota &= T_k\triangle_{k+1}d \\ &= \triangle'_k T_{k+1}d \end{aligned}$$

We define $T_k\iota = \triangle'_k T_k d$. Now choose $w \in B$. Let $w' \in B'$ be such that $\theta_k([w]) = [w']$. Define $T_k(w) = w'$.

Finally let $\Phi : L_k \to L'_k$. be any homomorphism. Then define $T_k(l) = \Phi(l)$ for all $l \in L_k$.

Clearly $(T_k)_* = \theta_k(J)$. By construction we have $\triangle'_{k+1}T_{k+1} = T_k\triangle_{k+1}$.

□

Note that $\mathcal{CMCM}$s are not unique. This non-uniqueness results from two choices:

1. The choice of the set of homology generators

2. The homomorphism taking L_k to L'_k

We will consider the meaning and consquences of these choices in the next section. We are now in position to give an inductive proof of the existence theorem

Proof. Theorem 13 We induct on the number of intervals.

Suppose $D = \{I_1, I_2, \cdots, I_n\}$ and $D' = \{I'_1, I'_2, \cdots, I'_n\}$ where $I_i < I_{i+1}$. We induct on the number of intervals. By previous theorem the result holds for I_1 and I'_1.

Define $P_i = P_{i-1} \cup (I_i)$ and $P'_i = P'_{i-1} \cup (I'_i)$. Our induction statement is as follows. Assume there exists coarsening preserving chain map $T : G(P_{k-1}) \to G'(P'_{k-1})$ which preserves $\theta(D_{k-1})$. We must show there exists a coarsening preserving chain map $T : G(P_k) \to G'(P'_k)$ which preserves $\theta(D_k)$.

By assumption there exists a coarsening preserving chain map $T : C(I_k) \to C(I'_k)$ and by the induction assumption there exists a coarsening preserving chain map $T : G(P_{k-1}) \to G'(P'_{k-1})$. Hence, by the previous theorem there exists a coarsening preserving chain map $T : G(P) \to G'(P')$ that preserves the coarsening and such that $T_* = \theta(P)$. $\square$

2.4 Properties

We begin this section with a pair of composition lemmas. The first proposition is barely worth mention save that the result will often be used implicitly.

Proposition 2. *Let $\mathcal{G}_0$, $\mathcal{G}_1$ and $\mathcal{G}_2$ be graded vector space braids ordered by $<_0$, $<_1$, and $<_2$, respectively. Suppose that D_i is a coarsening of $\mathcal{G}_i$, $1 \leq i \leq 3$, and that $\mathcal{G}|_{D_0} \cong \mathcal{G}|_{D_1}$ and $\mathcal{G}|_{D_0} \cong \mathcal{G}|_{D_2}$. Then $\mathcal{G}|_{D_1} \cong \mathcal{G}|_{D_2}$*

Proof. Given $\phi_{0,1}$, a braid isomorphism between $\mathcal{G}|_{D_0}$, and $\mathcal{G}|_{D_1}$ and $\phi_{\in,\prime}$, a braid isomorphism between $\mathcal{G}|_{D_2}$, and $\mathcal{G}|_{D_0}$ we have that $\phi_{2,0} \circ \phi_{0,1}$ is a braid isomorphism between $\mathcal{G}|_{D_2}$ and $\mathcal{G}|_{D_1}$. $\square$

Given three Morse decompositions related by coarsening we can relate their Conley-Morse chain maps.

Proposition 3. *Let $M(S_i) = \{M_i(p) : p \in P_i\}$ be related by the coarsening D_i for $1 \leq i \leq 3$. Let $T \in \mathcal{T}(D_0, D_1)$ and $S \in \mathcal{T}(D_2, D_0)$. Then there exists an invertible chain map $R : \oplus CH(M_0(p_0)) \to \oplus CH(M_0(p_0))$, invertible, such that $TRS \in \mathcal{T}(D_2, D_1)$*

Proof. Let $T \in \mathcal{T}(D_0, D_1)$ and $S \in \mathcal{T}(D_2, D_0)$. We need to show the following.

1. $(TRS)_* = \theta(P)$ for some braid isomorphism $\theta : H(<_2)|_{D_2} \to H(<_1)|_{D_1}$

2. $(TRS)_*(I_i, I'_i) = \theta(I)$ for $i \in \{1, \cdots, n\}$.

3. TRS preserves the coarsening

4. $\triangle_1(TRS) = (TRS)\triangle_2$ for some $\triangle_1 \in CM(M(S_1))$ and $\triangle_2 \in CM(M(S_2))$

Since $T \in \mathcal{T}(D_0, D_1)$ there exists $\triangle_0 \in CM(M(S_0))$ and $\triangle_1 \in CM(M(S_1))$ so that $T\triangle_0 = \triangle_1 T$. Similarly, since $S \in \mathcal{T},(D_2, D_0)$ there exists $\triangle_0' \in CM(M(S_0))$ and $\triangle_2 \in CM(M(S_2))$ so that $\triangle_0' S = S\triangle_1$.

Since $\triangle_0, \triangle_0' \in CM(M(S_0))$ there exists a Conley-Morse chain map (in the sense of Franzosa and Mischaikow), R, so that $\triangle_0 = R^{-1}\triangle_0' R$.

(1) and (2) follow immediately from Proposition 2. Since S, R, and T preserve the coarsening TRS preserves the coarsening and (3) follows. We are left to show that TRS is a chain map.

$$\begin{aligned} \triangle_0 = R^{-1}\triangle_0' R \text{ and } \triangle_0' S = S\triangle_1 \quad &\Rightarrow \\ \triangle_0 RS = RS\triangle_2 \quad &\Rightarrow \\ T\triangle_0 RS = TRS\triangle_2 \quad &\Rightarrow \\ \triangle_1 TRS = TRS\triangle_2 \quad & \end{aligned}$$

Hence, $TRS \in \mathcal{T}(D_2, D_1)$. □

Corollary 1. *Let $M(S_i) = \{M_i(p) : p \in P_i\}$ be related by the coarsening D_i for $0 \le i \le 1$. Let $T \in \mathcal{T}(D_0, D_1)$ and $S \in \mathcal{T}(D_1, D_0)$. Let $\triangle_i \in CM(M(S_i))$ with $\triangle_0 T = T\triangle_1$. Then $TS \in \mathcal{T}(D_0, D_0)$ and $ST \in \mathcal{T}(D_1, D_1)$*

We show that the continuation based Conley Morse chain maps of Franzosa and Mischaikow fit into our purely algebraic regime.

Proposition 4. *Let $M(S)$ be a Morse decomposition which continues with partial order $<$ over $U \in \Lambda$, simply connected. Given $\lambda_0, \lambda_1 \in U$ let $T \in T^U_{\lambda_1,\lambda_0}$. Then $T \in \mathcal{T}(\mathcal{D}_\infty, \mathcal{D}_\prime)$ where $D_i = \{p : p \in P\}$ are trivial coarsenings of P.*

Proof. Since $M(S_{\lambda_1}) = \{M_{\lambda_1}(p) : p \in P\}$ and $M(S_{\lambda_0}) = \{M_{\lambda_0}(p) : p \in P\}$ are related by continuation over $<$ the corresponding homology braids $\mathcal{H}(<_0)$ and $\mathcal{H}(<_1)$ are isomorphic. Hence, $\mathcal{T}(D_1, D_0)$ is well defined and non-empty. T is $<$-upper triangular, by definition, and, hence, is coarsening preserving. Also, T is a chain map. Since T is invertible, $T_*(p) : CH(M_{\lambda_1}(p)) \to CH(M_{\lambda_0}(p))$ is an isomorphism for each $p \in P$. Similarly $T_*(P) : CH(M(S_{\lambda_1})) \to M(S_{\lambda_0}))$ is an isomorphism. Hence, by Definition 32, $T \in \mathcal{T}(D_1, D_0)$. ☐

Finally, we show that we can find homological inverses of Conley-Morse chain maps that are Conley-Morse chain maps.

Proposition 5. *Let $M(S_i, <_i) = \{M_i(p) : p \in P_i\}$ be related by the coarsening D_i for $i \in \{0, 1\}$. Let $T \in \mathcal{T}(D_0, D_1)$ with $T\triangle = \triangle' T$. Then there exists $S \in \mathcal{T}(D_1, D_0)$, so that $(TS)_* : CH(M_1(S_1)) \to CH(M_1(S_1))$ and $(ST)_* : CH(M_0(S_0)) \to CH(M_0(S_0))$ are identity maps.*

Proof. Let $D_0 = \{I_1, \cdots, I_n\}$ and $D_1 = \{I'_1, \cdots, I'_n\}$. For each i we know that $(T(I_i))_* : CH(M_0(I_i)) \to CH(M_1(I'_i)$ is an isomorphism. Let $M_k = Im(\triangle_k(I_i))$. Let K_k be the subspace of $ker(\triangle_k(I_i))$ complementary to L_k. Let L_k be the subspace of $\oplus_{p \in I_i} CH_k(M_0(p))$ complementary to K_k.

We decompose $\oplus_{p \in I'_i} CH_k(M_1(p))$ similarly into L'_k, M'_k, and K'_k where $K'_k = Image(T(I_i)(K_k))$. Since $T(I_i)$ is an isomorphism on homology for each $k' \in k'_k$ let $S(I'_i)k' = k$ for $k \in K_k$ such that $T(I_i)k = k'$. Let $l' \in L'_k$. We consider two cases. If $l' \in Im(T(I_i))$ then let $S(I'_i)l' = 0$. If not then there exists $l \in L_k$ so that $T(I_i)l = l'$. Define $S(I'_i)l' = l$.

If $m' \in M'_k$ then there exists $l' \in L'_k$ so that $\triangle'(I'_i)l' = m'$. Let $S(I'_i)l' = \triangle(I_i)S(I'_i)l'$. By construction, $S(I'_i)$ is the homological inverse of $T(I_i)$ and is a chain map.

We now consider a pair of adjacent intervals $I', J' \in \Im_2(< |_{D_0})$. We consider how to define $S(I, J')$. Let $S(I, J') = S(I)T(I, J')S(J)$. Simple calculation shows that $T(IJ)$ and $S(IJ)$ are inverses on the level of homology.

The construction of S is done by inducting on the number of intervals in the decomposition.

Let $P_i = P_{i-1} \cup (I_i)$ and $P'_i = P'_{i-1} \cup (I'_i)$

Our induction statement is as follows. We note that $S(I'_1)$ and $T(I_1)$ are homological inverses. Assume that for $i \leq k-1$, $S(P'_i)$ and $T(P_i)$ are homological inverses. We must show that $S(P'_k)$ and $T(P_k)$ are inverses on the homology level. But P'_{k-1} and I'_k are adjacent intervals so, as above we can construct $S(P'_k)$, a homological inverse of $T(P_k)$. Our result follows.

□

CHAPTER III

REDUCTION ALGORITHM

3.1 Introduction

In this chapter we present an algorithm for producing a connection matrix for a coarsening. Given a chain complex braid $\mathcal{C}\triangle(<)$ we produce a chain complex braid $\mathcal{C}\triangle^{1}(<^{1})$ so that the corresponding homology braids $\mathcal{H}\triangle(<)$ and $\mathcal{H}\triangle^{1}(<^{1})$ are related by coarsening. The algorithm presented is a generalization of the CCR algorithm (see [12]). A detailed discussion of the CCR algorithm can be found in [11].

The chain complexes of interest will be generated from the Conley indices of Morse sets with a connection matrix, $\triangle$, as the boundary operator. Given two such chain complexes $(C, \triangle)$ and $(C', \triangle')$ corresponding to fine and coarse Morse decompositions, respectively we have seen that we can define a $\mathcal{CMCM}$ between the chain complexes.

In this section we turn the table (partially) and consider how to algebraically simplyfy $(C, \triangle)$ to $(C', \triangle')$. We will start off with a chain complex braid generated by $G = \{G(p) | p \in P\}$ with P ordered by $<$ and a $<$-upper triangular boundary map $\triangle$.

Recall that $G(I) = \oplus_{p\in I} G(p)$. Then $W_k(J) = \{G_k(p) \neq 0 : p \in P\}$ is a basis for the k chains, $G_k(J)$.

We decompose $G_k(P)$ by letting M_k denote $Im(\triangle_{k+1})$. We then choose N_k to be a basis for the subspace complementary to $M_k \subset G_k(P)$.

Now that we have finished decomposing $G_k(P)$ we consider an interval $J \in I(<)$. We choose an orthogonal basis $M_k(J) = \{a_{1,k}, \cdots, a_{d(k),k}\}$ for $Im(\triangle_{k+1}(J))$. Let $K_k(J)$ be a basis for the subspace of $ker(\triangle_{k+1}(J))$ complimentary to $M_k(J)$. Finally we define $L_k(J) = \{b_{1,k}, \cdots, b_{d(k-1),k}\}$ as a basis for the subspace of $G_k(J)$ complimentary to $M_k(J) \oplus K_k(J)$. We may make this construction assuming that $\triangle_{k+1}(J)(b_{i,k+1}) = a_{i,k}$ for $1 \leq i \leq d(k)$ and $< a_{i,k}, a_{i,k} >= 1$ where $<,>$ is the inner product on the vector space $G_k(P)$.

Definition 33. *$J \in I(<)$ is said to be a reduction interval if*

$$H_*\triangle(G(J)) \ncong \oplus_{p\in J} H_*\triangle(G(p))$$

J is said to be a minimal reduction interval (MRI) if there does not exist proper subinterval, $J_s \subset J$, so that J_s is a reduction interval.

J is a reduction interval only if there are connecting orbits between components of J. If J is not a reduction interval then the algorithm that follows will return the original complex. Given this definition we are now in position to define our first projection operator. For ease of notation for all $k \in \mathbb{Z}$ let $R(c_1, c_2) =< c_1, c_2 >$ for all $c_1, c_2 \in C_k$ where $<,>$ denotes the inner product on the underlying vector space.

Definition 34. *Let $J \in I(<)$ be a reduction interval. Then we define the* **J-projection operator** *as the collection of homomorphisms $\pi : G(P) \to G(P)$ where:*

$$\pi_k c = \begin{cases} c - \sum[R(c, a_{(i,k)})\triangle_{k+1}(b_{(i,k+1)})] & c \in M_k \\ c - \sum[R(\triangle_k c, a_{(i,k-1)})(b_{(i,k)})] & c \in N_k \end{cases} \tag{3}$$

Let us consider the action of π_k on $G_k(P)$. Note that if $c \in L_k(J)$ then $\pi_k c = 0$. Furthermore, we note that for each $c \in M_k$, $c \notin Image(\pi_k)$. Therefore, π_k annihilates M_k. We will now show that π is a chain map and each π_k is a projection.

Theorem 17. *$\pi : G(P) \to G(P)$ is a chain map defined by the homomorphisms $\{\pi_k\}_{k\in Z}$. If we define $G'_k = Im(\pi_k)$ then π_k projects $G_k(P)$ onto C'_k. Then $C' = \{C'_k, \triangle_k\}$ is a chain complex.*

Proof. Let $c \in M_k$.

$$\begin{aligned} \triangle_k\pi_k(c) &= \triangle_k\Big(c - \sum[R(c, a_{(i,k)})\triangle_{k+1}(b_{(i,k+1)})]\Big) \\ &= \triangle_k c - \sum[R(c, a_{(i,k)})\triangle_k\triangle_{k+1}(b_{(i,k+1)})] \\ &= 0 \qquad \text{(Since } \triangle^2 = 0\text{)} \\ &= \pi_{k-1}\triangle_k(c) \qquad (c \in M_k) \end{aligned}$$

Now we consider $c \in L_k$.

$$\begin{aligned}
\triangle_k \pi_k(c) &= \triangle_k(c - \sum[R(\triangle_k c, a_{(i,k-1)})(b_{(i,k)})]) \\
&= \triangle_k(c) - \sum[R(\triangle_k c, a_{(i,k-1)})\triangle_k(b_{(i,k)})] \\
&= \pi_{k-1}\triangle_k(c) \qquad \text{(Since } \triangle_k(c) \in M_k\text{)}
\end{aligned}$$

The last equality holds since $\triangle_k(c) \in M_k$. Hence, we have shown that π is a chain map.

We now show that each π_k is a projection onto its image. Let $c \in M_k$.

$$\begin{aligned}
\pi_k^2(c) &= \pi_k\Big(c - \sum_i [R(c, a_{i,k})\triangle_{k+1} b_{(i,k+1)}]\Big) \\
&= \pi_k(c) - \sum_i [R(c, a_{i,k})\pi_k(\triangle_{k+1} b_{(i,k+1)})] \\
&= \pi_k(c) - \sum_i \Big[[R(c, a_{(i,k)})(\triangle_{k+1}(b_{(i,k+1)}))] - \sum_j [R(\triangle_{k+1} b_{(j,k+1)}, a_{(i,k)})\triangle_{k+1} b_{(i,k+1)}]\Big] \\
&= \pi_k(c) - \sum_i \Big[[R(c, a_{(i,k)})(\triangle_{k+1}(b_{(i,k+1)}))] - [R(\triangle_{k+1} b_{(i,k+1)}, a_{(i,k)})\triangle_{k+1} b_{(i,k+1)}]\Big] \\
&= \pi_k(c)
\end{aligned}$$

It can be shown, with obvious changes, that $\pi_k^2(c) = \pi_k(c)$ for all $c \in N_k$. Therefore, π_k projects onto its image. □

The fact that π is a projection is fundamental in showing that G and C' are homologically equivalent.

Theorem 18. $H_*(G) \sim H_*(C')$

Proof. The proof will demonstrate that π is a chain equivalence and the inclusion map $\iota : C' \to G$ is its homotopical inverse. Since π is a projection we note that $\pi\iota = Id_{C'}$ and hence $\pi\iota$ is trivially chain homotopic to $Id_{C'}$. We are left to show that there exists a chain homotopy between $\iota\pi$ and Id_C. For each k let $D_k : G_k(P) \to C'_{k+1}$ be defined as:

$$D_k(c) = \begin{cases} \sum[R(c, a_{i,j})b_{i,k}] & c \in M_k \\ 0 & c \in N_k \end{cases} \tag{4}$$

We consider two cases. We start with $c \in N_k$.

$$\begin{aligned} Id_{C_k} - \iota_k \pi_k c &= c - (c - \sum[R(c, a_{i,j})b_{i,k}]) \\ &= \sum[R(c, a_{i,j})b_{i,k}] \\ &= 0 + D_{k-1}\triangle_k(c) \qquad (\text{Since } \triangle_k(c) \in M_k) \\ &= \triangle_{k+1}D_k + D_{k-1}\triangle_k(c) \qquad (\text{Since } c \in N_k) \end{aligned}$$

For case two let $c \in M_k$. Then

$$\begin{aligned} Id_{C_k} - \iota_k \pi_k c &= c - \Big(c - \sum[R(\triangle_k c, a_{i,j})\triangle_k(b_{i,k})]\Big) \\ &= \sum[R(\triangle_k c, a_{i,j})\triangle_k(b_{i,k})] \\ &= \triangle_{k+1}D_k + 0 \qquad (\text{Since } c \in M_k) \\ &= \triangle_{k+1}D_k + D_{k-1}\triangle_k(c) \qquad (\text{Since } \triangle_k(c) = 0) \end{aligned}$$

So for all k, $(\pi_k)_*$ is an isomorphism between $H_k(C)$ and $H_k(C')$.

□

A natural basis, W_k', for C_k' is given by $W_k' = \{\pi_k(w) : w \in W - J\} \cup K_k(J)$.

Since our goal is to cancel of basis elements that don't generate the kernel we wish to use the following basis. For each k consider the following set $W_k^1 = [W_k - W_k(J)] \cup K_k(J)$. Let C_k^1 be the GVS spanned by W_k^1. Since $ker(\pi_k) = L_k(J) \oplus M_k(J)$ the following is a collection of isomorphisms from $C_k^1 \to C_k'$.

Definition 35. *For each k the* **(k)-canonical restructuring map (CRM) from C_k^1 to** C_k' *is given by the isomorphism $\eta_k(c) = \pi_k|_{C_k'}(c)$ for all $c \in C_k^1$.*

Note that since π is a projection $\eta(c) = c$ for all $c \in C_k^1 \cap (M_k)$. These isomorphisms send basis elements of W_k^1 to basis elements of W_k'. Furthermore, since η is an isomorphism, we can use it to define a boundary map, $\triangle^1$ on C^1.

Definition 36. *The map $\triangle^1 : C^1 \to C^1$ given $\triangle^1 := \eta^{-1}\triangle\eta$ is referred to as the* **canonical boundary map (CBM).**

Of course we need to show that $\triangle^1$ is a boundary map.

Theorem 19. *$\triangle^1$ is a boundary map on C^1. Furthermore $H_*(G) \cong H_*(C') \cong H_*(C^1)$.*

Proof. For each k

$$\triangle^1_{k-1}\triangle^1_k = \eta^{-1}_{k-2}\triangle_{k-1}\eta_{k-1}\eta^{-1}_{k-1}\triangle_k\eta_k = \eta^{-1}_{k-2}\triangle_{k-1}\triangle_k\eta_k = 0$$

We have previously shown that $H_*(G) \cong H_*(C')$. Also, since η is a chain isomorphism $H_*(C') \cong H_*(C^1)$. The result immediately follows. □

We can construct an explicit formula for $\triangle^1$.

Theorem 20.

$$\triangle^1_k = \begin{cases} \triangle_k(c), & c \in span(K_k(J)) \\ 0, & c \in C^1_k - span(K_k(J)) \text{ and } c \in M_k \\ \pi_{k-1}\Big(\triangle_k(c)\Big), & c \in C^1_k - span(K_k(J)) \text{ and } c \in N_k \end{cases} \tag{5}$$

Proof. Recall that our basis, W^1 for C^1 is given by $W^1_k = W_k - W_k(J) \cup K_k(J)$. We consider three cases.

CASE 1

Assume $c \in K_k(J)$. Then, since $\pi_k(c) = c$, $\triangle^1_k(c) = \triangle_k(c)$.

CASE II

Assume that $c \in C^1_k - K_k(J)$ and furthermore that $c \in M_k$. Then

$$\begin{aligned} \triangle^1(c) &= \eta^{-1}\triangle\eta(c) \\ &= \eta^{-1}\Big(\triangle(c - \textstyle\sum[R(c, a_{i,k})\triangle_{k+1}(b_{(i,k+1)})]\Big) \\ &= \eta^{-1}(0) \qquad \text{(Since } \triangle^2 = 0) \\ &= 0 \end{aligned}$$

CASE III

Assume that $c \in C_k^1 - K_k(J)$ and more precisely that $c \in N_k$. Then

$$\begin{aligned}
\triangle^1(c) &= \eta^{-1}\triangle\eta(c) \\
&= \eta^{-1}\triangle\Big(c - \sum[R(\triangle_k c, a_{(i,k-1)})(b_{(i,k)})]\Big) \\
&= \eta^{-1}\Big(\triangle_k c - \sum[R(\triangle_k c, a_{(i,k-1)})\triangle_k(b_{(i,k)})]\Big) \\
&= \eta^{-1}\Big(\pi_{k-1}(\triangle_k(c))\Big) \\
&= \pi_{k-1}\Big(\triangle_k(c)\Big) \qquad \text{(Since } \pi_{k-1}\triangle_k(c) \in C_k^1 \cap M_k \text{)}
\end{aligned} \tag{6}$$

□

We consider the action of $\triangle^1$. Note that for each $c \in N_k$ any connections from c to elements of $M_k(J)$ are removed. Also, c has the possibility of inheriting connections from elements in $L_k(J)$. Such cancellations and inheritences are well displayed in the examples presented at the end of the section.

We now have a boundary map on our complex C^1. Our next step is to define a chain map between C and C^1. Since π is a chain map between C and C' and η is an isomorphism between C^1 and C' $\pi^1 = \eta^{-1}\pi$ is a chain map between C and C'. We refer to π^1 as the the **canonical transition map** between C and C^1. Before we continue let us examine the reduction process through two examples.

3.2 Examples

Example 2. *1-D Gradient System*

Consider the system in Figure 5. Our Morse decomposition takes the following form $M(S) = \{M(1), M(2), M(3), M(4), M(5)\}$*. We define our total ordering as follows:* $M(i) < M(j)$ *iff* $i < j$*. Then the* i^{th} *module takes the form* $G(i) = \oplus_j[CH_j(M(i))]$*. Then* $W_1 = \{G_1(5), G_1(4)\}$ *and* $W_0 = \{G_0(1), G_0(2), G_0(3)\}$*.*

Let $J = \{4, 5\}$ *be our reduction interval. Our canoncial reduction complex* $C^1 = \{G(1), G(2), G(5)\}$*.* $W_1^1 = \{G_1(5)\}$*.* $W_0^1 = \{G_0(1), G_0(2)\}$*.*

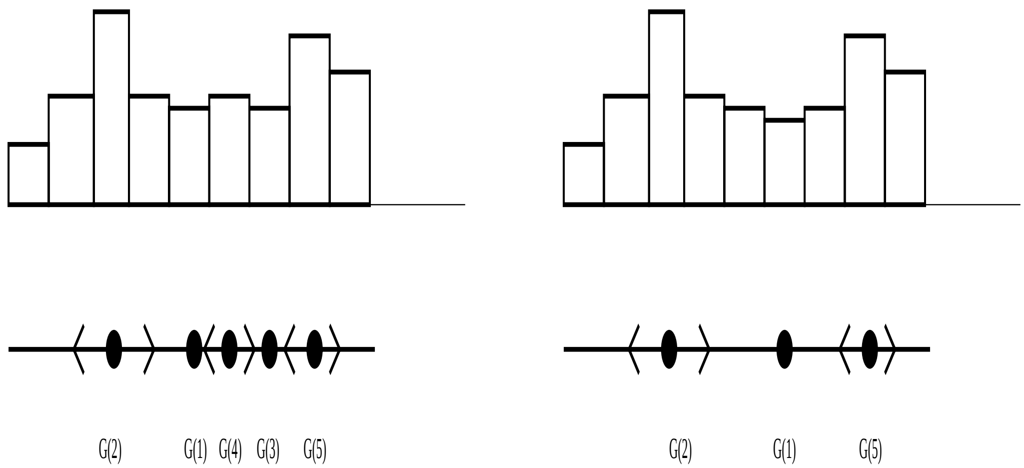


Figure 5: (a)Pre-Reduction (b) Post-Reduction

Using Z_2 coefficients $\triangle$ is a $<-$upper triangular connection matrix.

$$\triangle = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Also,

$$\triangle(J) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Tables 1 and 2 contain our decompositions of C and $C(J)$, resp.

Table 1: Decomposition of C

M_1	0	M_0	$G_0(2), G_0(1)$
N_1	$G_1(6), G_1(5)$	N_0	$G_0(4)$

We are now in position to consider the action of the canonical boundary operator $\triangle^1$.

Table 2: Decomposition of C(J)

$M_1(J)$	0	$M_0(J)$	$G_0(4)$
$K_1(J)$	0	$K_0(J)$	0
$L_1(J)$	$G_1(5)$	$L_0(J)$	0

$$\triangle_1^1 G_1(5) = \triangle_1 g_1(5) - R(\triangle_1 G_1(5), G_0(3))\triangle_1 G_1(4) = G_0(2) + G_0(1)$$

$$\triangle_0^1(G_0(2)) = 0$$

$$\triangle_0^1(G_0(1)) = 0$$

Such reductions on 1 and 2 dimensional gradient flows have been constructed *ad hoc* by Edelsbrunner, Harer and Zomorodian in [17] and [4]. We consider a reduction of the chain complex in a more general case. The following is an reduction representing an algebraic Hopf bifurcation.

Example 3. *Hopf bifurcation*

Consider the system S in Figure 1(a). Our Morse sets are

$$M(S) = \{M(1), M(2), M(3), M(4), M(5), M(*)\}.$$

Then $G(P) = \oplus_j[CH_j(M(p)]$ are the graded vector spaces associated with $M(S)$. We note that $1 < 2 < * < 3 < 4 < 5$ is an admissable total order for the system. If we work with Z_2 coefficient then the following connection matrix $\triangle$, is $<$ upper triangular.

$$\triangle = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Table 3: Decomposition of C

M_2	0	M_1	$G_1(1), G_1(2)+G_1(3)$	M_0	$G_0(1)$
N_2	$G_2(4), G_2(*), G_2(5)$	N_1	$G_1(2)$	N_0	0

Table 4: Decomposition of C(J)

$M_2(J)$	0	$M_1(J)$	$g_1(1)$	$M_0(J)$	0
$K_2(J)$	0	$K_1(J)$	0	$K_0(J)$	$g_0(1)$
$L_2(J)$	$G_2(*)$	$L_1(J)$	0	$L_0(J)$	0

The bases of C are as follows: $W_2 = \{G_2(*), G_2(5), G_2(4)\}$, $W_1 = \{G_1(1), G_1(2), G_1(3)\}$, $W_0 = \{g_0(1)\}$. *We would like to reduce the interval* $J = \{1, *\}$. *Using* Z_2 *coefficients we have*

$$\triangle(J) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We now decompose C_k *for* $1 \leq k \leq 2$ *as* $C_k = M_k \oplus N_k$. *The resulting groups are listed in Table 1. We also decompose* $C_k(J) = L_k(J) \oplus K_k(J) \oplus M_k(J)$. *The results are listed in Table 2.*

We note that $W_2^1 = \{G_2(5), G_2(4)\}$, $W_1^1 = \{G_1(2), G_1(3)\}$,$W_0^1 = \{G_0(1)\}$ *are the canonical bases for* C_2^1, C_1^1, C_0^1, *resp. We consider the action of the canonical boundary operator,* $\triangle^1$ *on these bases.*

$$\triangle_2^1 G_2(5) = \triangle_2 G_2(5) - R(\triangle_2 G_2(5), G_1(1))\triangle_2 G_2(5) = \triangle_2 G_2(5)$$

$$\triangle_2^1 G_2(4) = \triangle_2 G_2(4) - R(\triangle_2 G_2(4), G_1(1))\triangle_2 G_2(5) = \triangle_2 G_2(4)$$

$$\triangle_1^1(G_1(2) + G_1(3)) = 0$$

$$\triangle_1^1(G_1(2)) = \triangle G_1(2) = G_0(1)$$

$$\triangle_0^1(G_0(1)) = 0$$

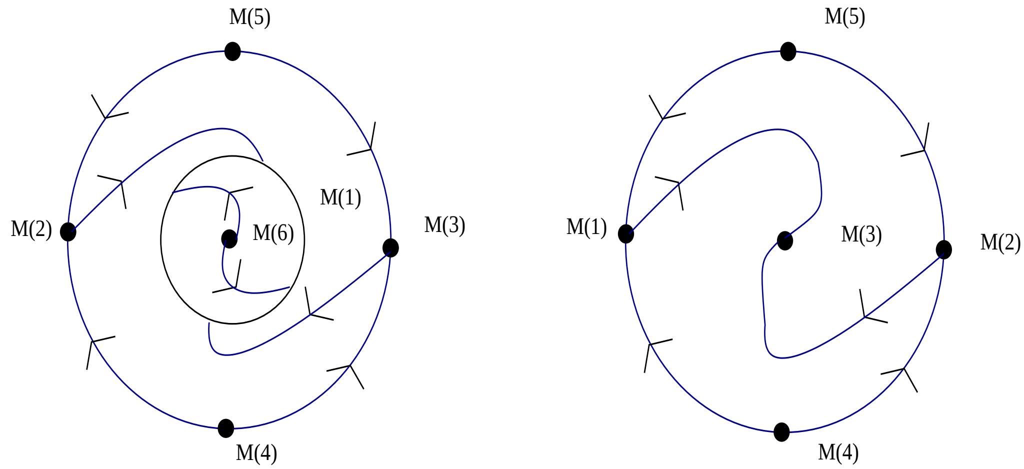


Figure 6: (a)Pre-Reduction (b) Post-Reduction

It is immediate from the above that $\triangle_1^1(G_1(3)) = G_0(1)$. *A representation of this reduction is pictured in Figure 1.*

3.3 Constructing the Reduced Homology Braid

We have already seen that $H\triangle(G(P)) \cong H\triangle^1(C^1)$. We will now use $\triangle^1$ to generate a chain complex braid , $\mathcal{H}\triangle^1(<^1)$ for some partial order $<^1$. This braid will be shown to be isomorphic to a coarsening of $\mathcal{H}\triangle(<)$. We will need to do the following.

- Interpret the chains of C^1 as a collection of GVSs
- Define an order $<^1$ on the braided modules that is derived from $<$

Let $p \in P \setminus J$. Since π^1 is the identity map off of J we define $G^1(p) = \pi^1 G(p) = G(p)$. Let $G(p^*) = \oplus_{p\in J}\pi^1 G(p)$. Let $P^1 = \{p : p \in P \setminus J\} \cup p^*\}$. We order P^1 by $<'$ where when $p, q \in P \cap P^1$ then $p <^1 q$ if and only if $p < q$. And if $p \in P \cap P^1$ then $p <^1 p^*$ if and only if there exists $q \in J$ so that $p < q$. Furthermore, $p^* <^1 p$ iff and only if there exists $q \in J$ so that $q < p$.

Then the canonical projection vector space (CPVS) is given by $G^1(P^1) = span\{G^1\}$In order for us to guarantee $\triangle^1(J^1)$ is boundary operator for $G(J^1)$ for all $J^1 \in I(<^1)$ we will show that $\triangle^1$ is $<^1$ −upper triangular.

Theorem 21. $\triangle^1$ *is* $<^1$ −*upper triangular.*

Proof. We must show that given $p, q \in P^1$ then $p \not<^1 q$ implies that $< \triangle^1(G(q), G(p)) >= 0$. If $q = p^*$ the result clearly holds. If $q \neq p^*$ we consider two cases.

Suppose $p \neq p^*$. Then $p, q \in P$ and $p \not< q$. Since $p \neq p^*$ for each $k \in \mathbb{Z}$ we can decompose $G_k(p) = m_k \oplus n_k$ where $m_k \in (C^1_k - span(K_k(J))) \cap M_k$ and $n_k \in (C^1_k - span(K_k(J))) \cap N_k$. So

$$\begin{aligned} \triangle^1_k(G_k(p)) &= \triangle^1_k(m_k \oplus n_k) \\ &= \pi_{k-1}(\triangle_k n_k) \\ &= \triangle_k n_k - \sum [R(\triangle_k n_k, a_{(i,k)}) \triangle_{k+1}(b_{(i,k+1)})] \end{aligned}$$

But since $p \not< q$ and $\triangle$ is $<$ −upper triangular $< \triangle_k n_k, p_k >= 0$. Suppose $< \sum [R(\triangle_k n_k, a_{(i,k)}) \triangle_{k+1}(b_{(i,k+1)})], p_k > \neq 0$. Then $p <^1 p^* <^1 q$. A contradiction due to the transitivity of the partial order. Hence $< \triangle G(q), G(p) >= 0$.

For case two let $p = p^*$. Since $p^* \not<^1 q$ we have that $r < q$ for all $r \in J$. Therefore, as above, $< \triangle_k n_k, r_k >= 0$ for all $r \in J$. Also, $< \sum [R(\triangle_k n_k, a_{(i,k)}) \triangle_{k+1}(b_{(i,k+1)})], p_k >= 0$ since $R(\triangle_k n_k, a_{(i,k)}) = 0$ for all i.

□

Since $\triangle^1$ is $<^1$-upper triangular the homology braid $\mathcal{H}\triangle^1(<^1)$ is well defined. We coarsen P with $D = \{p : p \in P \setminus J\} \cup J$. We are in position to prove the following theorem.

Theorem 22. $H\triangle(<)|_D \cong H\triangle^1(<^1)$

Proof. A chain complex homomorphism between $G|_D$ and C^1 is defined by π^1. When we pass to homology $\pi^1_*(p)$ is an isomorphism for each $p \in P \setminus J$. Also, by construction, $\pi^1_*(J)$ is an isomorphism. Hence, by the 5-lemma, $H\triangle(<)|_D \cong H\triangle^1(<^1)$.

□

Theorem 23. *Let $M(S) = \{M(p) : p \in P\}$ be a Morse decomposition of S partially ordered by $<$. Let $\triangle \in CM(M(S))$. Let $D = \{I_1, \cdots, I_n\}$ be a coarsening of P. Then, given $M_D(S) = \{M(I_i) : 1 \leq i \leq n\}$ let $D' = \{I_1, \cdots, I_n\}$ be the trivial coarsening. Then* *there exists a* $<_D$ *preserving boundary map* $\triangle' \in CM(M_D(S))$ *and* $T \in \mathcal{CMCM}(D, D')$ *such that* $T(I_i, I_j) = 0$ *if* $i \neq j$.

Proof. Use the previous theorem to induct on the number of components of the coarsening. ∎

Therefore, given a Morse decomposition $M(S) = \{M(p) : p \in P\}$ with a coarsening $D = \{I_1, \cdots, I_n\}$ we can build a connection matrix $\triangle^1$ for $H\triangle(<)|_D$ and a CMCM, π^1, between the two complexes.

CHAPTER IV

EXAMPLES

In this chapter we begin by constructing $\mathcal{CMCM}$s for a simple $1-d$ gradient system. The example sheds light on the information stored in the entries of a $\mathcal{CMCM}$.

4.1 Gradient Example

Consider the two 1-dimensional systems depicted in Figure (7). Such dynamics result from the flow

$$\dot{x} = \lambda x - x^3, \quad \lambda \in \mathbb{R}$$

When $\lambda = 0$ there exists one possible Morse decomposition, $M_0 = \{M_0(0)\}$, where

$$CH(M_0(0)) \cong \widetilde{H}(S^1).$$

When $\lambda = 1$ we $M_1 = \{M_1(-1), M_1(0), M_1(1)\}$ is a Morse decomposition. Then $0 < -1 < 1$ is an admissable order. In this case

$$CH(M_1(\pm 1)) \cong \widetilde{H}(S^1)$$

and

$$CH(M_1(0)) \cong \widetilde{H}(S^0).$$

For $\lambda = 1$ we can define a coarsening $D_1 = \{I\}$ where I is the interval $\{0, -1, 1\}$.

Since $CH(M(I)) \cong CH(M_0(0))$ the homology braids trivially agree under coarsening. Therefore, we are guaranteed the existence of a Conley Morse chain map between the two systems.

The following is a connection matrix for M_1

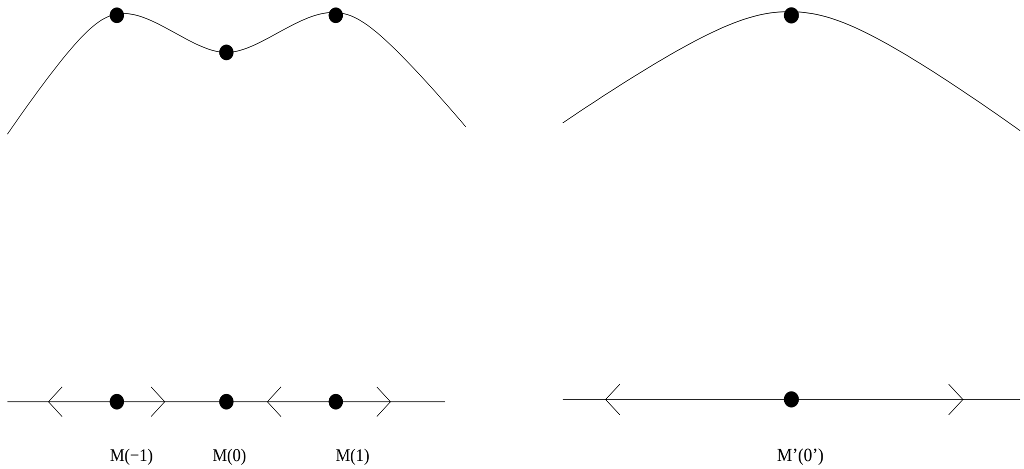


Figure 7: (a)$\lambda = 1$ (b) $\lambda = 0$

$$\triangle = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We note that there is only one possible connection matrix, $\triangle' = [0]$, for M_0. We now try to compute the $\mathcal{CMCM}$s between these two systems. We begin by decomposing the associated chain complexes.

Clearly $L'_1 = M'_1 = 0$ and $K'_1 = CH_1(M_0(0))$. We must set $L_1 = 0$. Furthermore $L_0 = K_0 = 0$ and M_0 is spanned by $CH_0(M_1(0))$. $K_1 = CH_1(M_1(-1)) + CH_1(M_1(1))\}$. We are left with a choice for L_1. First let us consider the case when we choose L_1 so that it generates $CH_1(M_1(1))$.

Then

$T_0(CH_0(M_1(0))) = 0$

$T_1(CH_1(M_1(1))) = 0$

$T_1(CH_1(M_1(-1)) + CH_1(M_1(1))) = K'_1 = CH_1(M_0(0))$

Which implies

$T_1(CH_1(M_1(-1))) = CH_1(M_0(0))$

If we choose L_1 so that it generates $CH_1(M_1(-1))$ then

$T_0(CH_0(M_1(0))) = 0$

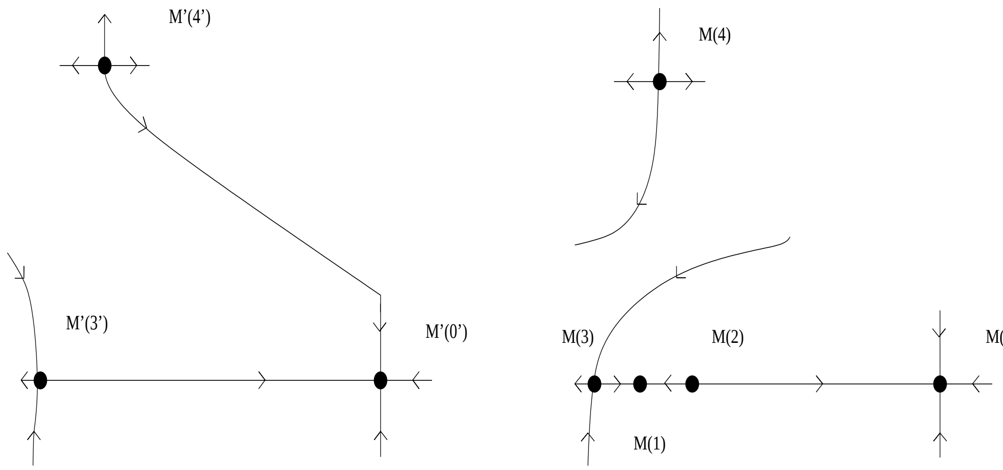


Figure 8: Comparing Morse Decompositions

$T_1(CH_1(M_1(-1))) = 0$

$T_1(CH_1(M_1(-1)) + CH_1(M_1(1))) = K_1' = CH_1(M_0(0))$

Which implies

$T_1(CH_1(M_1(1))) = CH_1(M_0(0))$

Therefore, our $\mathcal{CMCM}$s are of the form [0 0 1] and [0 1 0].

In the first case $CH_1(M_1(1))$ is annihilated and in the second case $CH_1(M_1(-1))$ is annihilated. In each case $CH_0(M_1(0))$ is annihilated. This is exactly what we would expect. It is well known that our system undergoes a pitchfork bifurcation. It is therefore no surprise that a source-sink pair is annihilated. It is expected that we receive two $\mathcal{CMCM}$s since there are 2 sources that could potentially be removed.

4.2 Comparing Morse Decompositions

The following example motivates how Conley-Morse chain maps could be used to compare data sets. It is clear by inspection of Figure 8 that the systems discussed below are not equivalent except when we use the coarsest Morse decomposition. Given high dimension data or data ripe with complicated structure, however, differentiating between the flows by eye is impossible. It seems fitting to end a dissertation with an example that motivates possible future work.

Consider the systems displayed in figure 8. We would like address whether or not the first system is simply a more refined version of the second.

We begin by coarsening the second system. A Morse decomposition is given by $M(S_{\lambda_0}) = \{M_{\lambda_0}(p) | p \in P\}$ where $P = \{0, 1, 2, 3, 4\}$. Then $J = \{1, 2, 3\}$ is an interval and $D = \{0, J, 4\}$ is a coarsening where $<_D$ is given by $0 <_D J <_D 4$. $M(S_{\lambda_1}) = \{M_{\lambda_1}(q) | q \in P'\}$ where $P' = \{0', 3', 4'\}$ is a Morse decomposition for the second system. We trivially coarsen the second system as $D' = P' = \{0', 3', 4'\}$ with order $0' <' 3' < 4'$.

A connection matrix for $M(S_{\lambda_0})$ is given by

$$\triangle = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The following is a connection matrix for $M(S_{\lambda_1})$.

$$\triangle' = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Lets consider the set of $\mathcal{CMCM}$s between the two systems. Such a matrix, T, must satisfy $\triangle' T = T \triangle$. Furthermore the $T(4, 4')$ and $T(0, 0')$ entries must be isomorphisms. Working in Z_2 coefficients we will label these entries with a 1. We will label non-significant unknown entries of T with a (*). T must have the form

$$T = \begin{pmatrix} 1 & * & * & * & * \\ 0 & * & * & * & T(4,3') \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The chain map property of T forces $T(4, 3')$ to be non-zero. But such maps are not coarsening preserving. Hence, these systems are not related by coarsening. Therefore, $\mathcal{CMCM}$s can be used to compare Morse sets on an a purely algebraic level.

REFERENCES

[1] Bremer, P.-T., Pascucci, V., Edelsbrunner, H., and Hamann, B., "A topological hierarchy for functions on triangulated surfaces.," *IEEE Trans. Vis. Comput. Graphics*, no. 10, pp. 385–396, 2004.

[2] Conley, C., "Isolated invariant sets and the Morse index," 1978.

[3] Edelsbrunner, H., Harer, J., Natarajan, V., and Pascucci, V., "Morse-Smale complexes for piecewise linear 3-manifolds," in *Proc. 19th Ann. Sympos. Comput. Geom. 2003*, 2003.

[4] Edelsbrunner, H., Harer, J., and Zomorodian, A., "Hierarchical Morse complexes in piecewise linear 2-manifolds," *Discrete and Comp. Geo.*, no. 30, pp. 87–107, 2003.

[5] Franzosa, R., *Index Filtrations and connection matrices for partially ordered Morse decompositions*. PhD dissertation, University of Wisconsin-Madison, Department of Mathematics, 1984.

[6] Franzosa, R., "Index filtrations and the homology index braid for partially ordered Morse decompositions," *Trans. Amer. Math. Soc.*, no. 298, pp. 193–213, 1986.

[7] Franzosa, R., "The continuation theory for Morse decompositions and connection matrices," *Trans. Amer. Math. Soc.*, no. 310, pp. 781–803, 1988.

[8] Franzosa, R., "The connection matrix theory for Morse decompositions," *Trans. Amer. Math. Soc.*, no. 311, pp. 561–592, 1989.

[9] Franzosa, R. and Mischaikow, K., "Algebraic transition matrices in the Conley index theory," *Trans. Amer. Math. Soc.*, no. 350, pp. 889–912, 1998.

[10] Hatcher, A., *Algebraic Topology*. Cambridge University Press, 2002.

[11] Kaczynski, T., Mischaikow, K., and Mrozek, M., *Computational Homology*, vol. 157 of *Applied Math Series*, section 4.3-4.4, pp. 155–171. New York: Springer-Verlag, first ed., 2004.

[12] Kaczynski, T., Mrozek, M., and Slusarek, M., "Homology computation by reduction of chain complexes," *Computers and Math. App.*, no. 35, pp. 59–70, 1998.

[13] Kalies, B., Mischaikow, K., and van der Vorst, R., "An algorithmic approach to chain recurrence." Submitted for publication in 2004.

[14] Mischaikow, K., "The Conley index theory: A brief introduction." The final version can be found in Conley Index Theory, Banach Center Publications, 47, 1999.

[15] Mischaikow, K. and Mrozek, M., *Handbook of Dynamical Systems II: Towards Applications,*, chapter Conley Index Theory. North Holland, 2002.

[16] NOAA. http://www.noaa.gov/index.html.

[17] Zomorodian, A., *Computing and Comprehending Topology: Persistence and Hierarchical Morse Complexes*. PhD dissertation, University of Illinois: Urbana-Champaign, Department of Computer Science, 2001.