ADAPTIVE ESTIMATION FOR CONTROL OF UNCERTAIN NONLINEAR SYSTEMS WITH APPLICATIONS TO TARGET TRACKING

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ADAPTIVE ESTIMATION FOR CONTROL OF UNCERTAIN NONLINEAR SYSTEMS WITH APPLICATIONS TO TARGET TRACKING

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"A mind all logic is like a knife all blade. It makes the hand bleed that uses it"

"If you shut your door to all errors, truth will be shut out"

- Rabindranath Tagore
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<td>LTI</td>
<td>Linear Time Invariant</td>
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<tr>
<td>SSKF</td>
<td>Steady State Kalman Filter</td>
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<tr>
<td>EKF</td>
<td>Extended Kalman Filter</td>
</tr>
<tr>
<td>NN</td>
<td>Neural Network</td>
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<tr>
<td>SHL</td>
<td>Single Hidden Layer</td>
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<tr>
<td>LOS</td>
<td>Line-of-sight</td>
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<td>Proj</td>
<td>Projection operator</td>
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<tr>
<td>tr</td>
<td>Trace operator</td>
</tr>
<tr>
<td>sup</td>
<td>Supremum</td>
</tr>
<tr>
<td>vec</td>
<td>Kronecker vec operator</td>
</tr>
<tr>
<td>ZEM</td>
<td>Zero effort miss</td>
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<tr>
<td>PN</td>
<td>Proportional navigation</td>
</tr>
<tr>
<td>APN</td>
<td>Augmented proportional navigation</td>
</tr>
<tr>
<td>t</td>
<td>Time</td>
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<tr>
<td>$f(\cdot)$</td>
<td>Plant dynamics</td>
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<tr>
<td>$x(t)$</td>
<td>State vector of the modelled plant dynamics</td>
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<tr>
<td>$f_{x}(:,:,\cdot)$, $f_{x_{1}}(:,:,\cdot)$, $f_{x_{2}}(:,:,\cdot)$</td>
<td>Unmodeled dynamics function</td>
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<td>$z(t)$</td>
<td>State vector of the unmodeled dynamics</td>
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<td>$y(t)$</td>
<td>System output</td>
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<td>$u(t)$</td>
<td>Control input</td>
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<td>$\hat{x}(t)$</td>
<td>Estimated state vector</td>
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<td>$A$, $B$, $C$</td>
<td>Linear system matrices</td>
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<td>$K$, $\bar{K}$</td>
<td>Observer and Error observer gains respectively</td>
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<td>$e(t)$</td>
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<td>$\hat{e}(t)$</td>
<td>Error observer vector</td>
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\( \hat{e}(t) \)  
Difference between observer error and error observer

\( \mu(t) \)  
Input to the neural network

\( \sigma(\cdot) \)  
Neural network basis function

\( \sigma'(\cdot) \)  
Jacobian matrix

\( M, N \)  
Output layer and input layer neural network weight matrices

\( \epsilon_f(\mu(t)), \epsilon_g(\mu(t)) \)  
Neural network reconstruction error

\( \nu_{ad}(t) \)  
Neural network output

\( F_f, F_g, G_f, G_g, \Gamma_M \)  
Neural network learning rates

\( k_f, k_g, k_\sigma, k_M \)  
\( \sigma \) – modification gains

\( \alpha \)  
Activation potential

\( d \)  
Positive time delay

\( n \)  
Dimension of the system

\( \mathbb{R}^n \)  
n-dimensional Euclidean space

\( r_d \)  
Relative degree

\( B_r \)  
Ball of radius \( r \) centered around the origin

\( \Delta \)  
Difference quotient operator

\( \Omega, \mathcal{D} \)  
Open and Compact sets respectively

\( Q, \tilde{Q} \)  
Positive definite matrices of Lyapunov equations

\( P, \tilde{P} \)  
Positive definite solutions of Lyapunov equations

\( \lambda_{\max}(\cdot), \lambda_{\min}(\cdot) \)  
Maximum and minimum eigen values

\( V(\cdot) \)  
Candidate Lyapunov function

\( d_1(t), d_2(t) \)  
Process and Sensor disturbances

\( \nu \)  
Band limited measurement white noise

\( q_{\nu} \)  
Power spectral density of process noise

\( I \)  
Identity matrix

\( E[\cdot] \)  
Expectation operator

\( \varphi(\cdot) \)  
Higher order Taylor series terms

\( t_{go} \)  
Time-to-go
SUMMARY

Design of nonlinear observers has received considerable attention since the early development of methods for linear state estimation. The most popular approach is the extended Kalman filter (EKF), that goes through significant degradation in the presence of nonlinearities, particularly if unmodeled dynamics are coupled to the process and the measurement. For uncertain nonlinear systems, adaptive observers have been introduced to estimate the unknown parameters along with the state variables where no priori information about the unknown parameters is available. While establishing global results, these approaches are applicable only to systems transformable to output feedback form. Over the recent years, neural network (NN) based identification and estimation schemes have been proposed that relax the assumptions on the system at the price of sacrificing on the global nature of the results. However, most of the NN based adaptive observer approaches in the literature require knowledge of the full dimension of the system, therefore may not be suitable for systems with unmodeled dynamics.

We first propose a novel approach to nonlinear state estimation from the perspective of augmenting a linear time invariant observer with an adaptive element. The class of nonlinear systems treated here are finite but of otherwise unknown dimension. The objective is to improve the performance of the linear observer when applied to a nonlinear system. The approach relies on the ability of the NNs to approximate the unknown dynamics from finite time histories of the available measurements.

Next we investigate nonlinear state estimation from the perspective of adaptively augmenting an existing time varying observer, such as an EKF. EKFs find their applications mostly in target tracking problems. The proposed approaches are robust to unmodeled dynamics, including unmodeled disturbances.
Lastly, we consider the problem of adaptive estimation in the presence of feedback control for a class of unknown multivariable nonlinear systems with unmodeled dynamics and disturbances coupled to the process. The states from the adaptive EKF are used as inputs to the control law, which in target tracking usually takes the form of a guidance law. The applications of this approach lie in the areas of missile-target tracking, formation flight control and obstacle avoidance.
CHAPTER 1

INTRODUCTION

The availability of all state variables for direct measurement is a rare occasion in practice. In physical systems, some components of the state are inaccessible internal variables, which either cannot be measured or the measurements require the use of very costly measurement devices. Therefore it is not feasible, or it is very expensive to measure all the state variables. Hence, in most practical scenarios, there is a true need to construct estimates of state variables which are not available by direct measurement, especially when they are used in applications such as implementation of state feedback controllers, monitoring of nonlinear processes and missile-target tracking to name a few.

1.1 Observers for Linear Systems

In the case of linear dynamical systems with white process and measurement noise, some of the early methods for observer design, which date as far back as the early 1940’s, involved the Wiener-Hopf filter, which is a linear system [1,2], and the Kalman-Bucy filter [3,4]. The major contribution of the Wiener filter to observer theory was in the derivation of the steady state optimal filter and predictor for stochastic stationary scalar processes using spectral factorization techniques. On the other hand, in the development of the Kalman filtering theory, the input-output signal Wiener model was replaced by a state-space model, and a recursive solution of the optimal state estimator is obtained for stationary and non stationary cases. The Wiener and Kalman filters are optimal filters in the least squares sense and are equivalent to each other in steady state [5]. In the aforementioned filter designs it is assumed that the spectral properties of the signal and noise processes in case of the Wiener filter, and the covariance matrices of the process and measurement, are completely known. For the case of linear dynamical systems with deterministic disturbances, the Luenberger observer offers a complete and comprehensive answer to the problem a state estimation [6]. Although the
Kalman filtering technique has gained wide acceptance over the past four decades, and has proven to be extremely useful in a wide variety of applications, it has become apparent that modelling errors and nonlinearity can lead to entirely unacceptable results [7]. In particular, the usefulness of the filter may be nullified by the phenomenon known as divergence, i.e., after an extended period of operation of the filter, the errors in the estimates eventually diverge to values entirely out of proportion to the root mean square values predicted by the equations of the filtering procedure [8, 9]. The explanation most offered for this phenomenon is that the calculated covariance matrix becomes unrealistically small, so that undue confidence is placed in the estimates and the subsequent measurements are effectively ignored. This effect is due to a variety of causes such as sensitivity to system nonlinearities, biases and modelling error. The need to be able to effectively reconstruct state estimates of nonlinear systems has paved the way for research in nonlinear estimation theory.

1.2 Observers for Nonlinear Systems

The design of nonlinear observers is a very challenging problem and has received a considerable amount of attention in the literature over the past couple of decades. Of the numerous attempts being made for the development of nonlinear observer theory, the most popular one is the extended Kalman filter (EKF). The design of the EKF is based on a local linearization of the system around a reference trajectory thereby restricting the validity of the approach within a small region in the state space [10]. Recently, particle filters, which are recursive implementations of Monte Carlo based statistical signal processing, have been used to treat non Gaussian processes as reported in [11–14]. In [15, 16], the authors developed a method called the unscented Kalman filter (UKF), which relies on approximating a Gaussian distribution rather than an arbitrary nonlinear function. Instead of linearizing using Jacobian matrices, the UKF uses a deterministic sampling approach to capture the mean and covariance estimates with a minimal set of sample points. In [17] a nonlinear observer design method is proposed that places the eigenvalues of the linearized error dynamics at certain values that are locally invariant with respect to the operating point of the system. However the first systematic approach for the development of the theory of nonlinear observers was proposed by Krener and Isidori in [18] and by Zeit and Bestle in [19]. These authors made
use of a nonlinear state transformation to linearize the original system up to an additional output injection term. Linear methods were then employed to complete the observer design procedure. The approach can be summarized as follows: A nonlinear system

\[
\begin{align*}
\dot{\xi}(t) &= f(\xi(t)) \\
y(t) &= h(\xi(t))
\end{align*}
\]

can be conceived as the result of applying output injection to a linear system yielding the following system

\[
\begin{align*}
\dot{x}(t) &= A x(t) + \varphi(y(t)) , \\
y(t) &= c x(t),
\end{align*}
\]

followed by a nonlinear change of coordinates

\[
\xi(t) = \xi(x(t))
\]

Thus this approach relies upon finding an output injection, \( \varphi(y(t)) \) and a state transformation, \( \xi(x(t)) \) in order to construct an observer for (1). However in general output injection is not physically realizable if the nonlinear system under consideration models a physical plant. A later attempt was Zeitz’s extended Luenberger observer [20], which is in the same spirit as the EKF, based upon a local linearization technique around the reconstructed state. In [21, 22], nonlinear coordinate transformations were introduced to transform the nonlinear system to a suitable ”observer canonical form”, like a condensed dual Brunovský form, for which the linear observer theory can be generalized. All of these approaches impose restrictive assumptions on the problem formulation. In [23] a state observer is proposed for nonlinear continuous time system by extending the Luenberger observer. This method does not require a preliminary nonlinear change of coordinates. In [24] an observer for nonlinear systems is proposed under certain technical assumptions such as global Lipschitz conditions and for systems with bounded input. In [25], a Lyapunov-like sufficient condition for the existence of a nonlinear observer is proposed. Other methods for the design of observers for nonlinear systems are reported in [26–28].

One of the most important contributions to nonlinear observer theory is the high gain observer by Khalil et al. introduced in [29]. High gain observers are attractive because of
their ability to estimate the unmeasured states while rejecting the effect of disturbances. However, since the high gain observer is basically an approximate differentiator, its application is limited to a class of systems that do not have measurement noise or unmodeled high frequency sensor dynamics. Another characteristic of the high gain observer is the peaking phenomenon of its transient response which is examined in [30]. Peaking occurs in the observer variables and propagates to the state variables through the control law. This peaking could be destabilizing in the case of finite regions of attraction, which only allows recovery of local asymptotic stability as in [31, 32]. However, due to peaking, global asymptotic stability results can be obtained at the expense of imposing restrictive global Lipschitz conditions and tolerating unacceptable transient response. To remedy this problem, the idea of saturating the control law outside a region of interest was introduced in [30]. In [33] the authors extend the work of [34] to encompass the case of systems that are not linearly observable. However, this still only provides for local stability around a critical point. The high gain observer design by Khalil et al. in [30], though it employs judicious scaling of the observer gain matrix, and saturation of the peaking observer signals before they are being fed to the controller, it still requires a global Lipschitz condition for global convergence and the usage of ”high gain”. In [35], the authors eliminate the global Lipschitz restriction and avoid high gain but require a linear matrix inequality (LMI) to be feasible thus implying a strictly positive real (SPR) property for the linear part of the observer error system. In addition, this approach is limited for a particular class of nonlinearities. In [36], the authors present a new nonlinear robust disturbance observer by using only the input-output information for minimum phase dynamical systems with arbitrary relative degrees. However, in the presence of measurement noise in the output the precision of the disturbance observer declines. All the above approaches to nonlinear observer design, like the extended Kalman filter approach which suffers from the lack of guaranteed stability, or the linearizable error dynamics approach [19], [22], [18] which is applied to a limited class of nonlinear systems, require the full degree of the system to be known. In order to be able to deal with a more general class of systems and also handle systems with probably unknown dimension (unknown general multivariable nonlinear system), we turn our attention to the design of adaptive observers.
1.3 Adaptive State Estimation

The design of adaptive observers has been the most intensively addressed topic in the recent literature. These approaches simultaneously estimate the plant state variables and parameters by processing the plant input-output measurement online. This is an important aspect in the problems of state estimation, system identification or output feedback control, e.g. [37–41]. In [42], the authors develop a global adaptive observer for a class of single output nonlinear systems. However, this class of systems is linear with respect to an unknown constant parameter vector. Most of the available results on adaptive observers impose assumptions that limit their domain of applicability, such as to systems that are linear with respect to unknown parameters or to systems that can be transformed to output feedback form. In [43] Besancon presents a unified "adaptive observer form" approach which which requires a "passivity like" condition, of the estimation error system with respect to unknown inputs, be satisfied.

In recent years, adaptive observers have been used in conjunction with neural networks (NNs) to perform adaptive estimation of nonlinear system. The universal approximation property of neural networks (NNs) has motivated NN based identification and estimation schemes, like the ones reported in [44–48], that relax the assumptions contained therein. The main challenge lies in defining an error signal for updating the NN weights. In [47], an adaptive observer is introduced for the following class of nonlinear systems in Brunovský form:

\[
\begin{align*}
\dot{x}(t) &= A x(t) + b \left[ f(x(t)) + d(t) \right], \quad x \in \mathbb{R}^n, \quad b \in \mathbb{R}^n \\
y(t) &= c^T x(t), \quad y \in \mathbb{R}, \quad c \in \mathbb{R}^n
\end{align*}
\]

where \( f(x(t)) : \mathbb{R}^n \rightarrow \mathbb{R} \) is an unknown continuous function, \( d(t) \) is a bounded disturbance. This approach introduces an SPR filter that enables writing the NN weights adaptive laws in terms of only the available measurement error signal. However, the filter needed to satisfy the SPR condition may not always exist, particularly for systems with multiple outputs. In [48] this restriction has been relaxed, and an approach is laid out for general nonlinear processes. However, a major difference is that the approach in [47] augments an existing linear observer, whereas the approach in [48] does not. The adaptive laws developed in both
approaches are limited to adapting only the NN output layer weights. In [49] an augmenting approach similar to the approach in [47] is presented. However this approach does not impose an SPR condition. Instead, a simple linear filter to generate a teaching signal for the adaptive laws, is designed. Also, both the input and output layer weights are adapted.

1.4 Extended Kalman Filtering and Applications to Target Tracking Problems

Of the numerous attempts being made for the development of nonlinear estimator theory, the most popular one is the extended Kalman filter (EKF), whose design is based on a first order local linearization of the system around a reference trajectory at each time step [50–52]. Among the many areas where EKFs have been successfully applied are adaptive filtering [53], state estimation [54, 55], parameter estimation, target tracking [56, 57], training of neural networks, and many others. The EKF approach to the estimation of parameters in dynamical systems has a rather long history and a considerable number of applications of this method have been reported [58–66]. In [65–67], the authors provide a systematic and comprehensive treatment of the EKF when applied to parameter estimation for linear stochastic systems with correlated noise, while in [68], a method for analyzing the behavior of the EKF for nonlinear deterministic systems is provided. The analysis here is based on the methods of [69]. While parameter estimation of linear and nonlinear systems using EKFs has received a fair amount of attention, nonlinear state estimation using EKFs has become one of the most researched problems [69–78].

EKFs are extensively applied to problems related to target tracking, to target rendezvous and interception, formation flight control [79] and obstacle avoidance [80–83]. In [84], the authors have developed a set of tracking algorithms that are applicable for ballistic reentry vehicles, tactical missiles and airplanes. In [85–87], the feasibility of target tracking is studied from a point of view of range-only measurements. However in some situations it may be impractical to measure the range, and state estimation using measurements of the line-of-sight or bearing angle is highly desirable. Hence, designing EKFs for target trackers with bearings-only measurement has been a widely studied subject [88–94]. In the case of bearing measurements the process may be unobservable unless the sensing vehicle executes
a maneuver [89], which further complicates the bearings-only problem.

A key to successful target tracking lies in the effective extraction of useful information about the target’s state from observations. In the setting of estimation, this necessitates adding additional states to model the target dynamics. Consequently, the accuracy of the estimator depends on the accuracy to which the target behavior has been characterized. Target behavior not captured by modelling introduces estimation bias, and can even cause divergence in the estimate.

To account for modelling errors in the process, neural network (NN) based adaptive identification and estimation schemes have been proposed in [47, 49, 95–98]. In [47], an approach is developed that augments a linear time invariant filter with an NN while in [96–98] schemes for augmenting an EKF with an NN are provided. However the approaches in [47, 96–98] all require knowledge of the full dimension of the system. In [49], an approach that does not require knowledge of the full dimension of the system is developed. However, this approach only permits augmentation of a linear time invariant state estimator with an NN.

1.5 Observer Based Controllers

Full state measurement of complex uncertain nonlinear systems or processes is generally not available. Hence we need to reconstruct the unknown state vector, especially in applications that involve control. Some typical applications include missile-target tracking, formation flight control, and problems involving chemical and biochemical reactors. In [99,100], the synthesis of a dynamic controller based on the estimates of the state vector is proposed for linear time invariant systems. In this case, the ease of closed-loop system design is facilitated by the inherent separation of the observer-based controller scheme into the two independent problems of observer design and feedback controller design. This is called as the Separation Theorem. In [101–105], the authors address the problem of observer-based control design for nonlinear systems. However, the above mentioned approaches are not applicable to cases when the dimension of the system is unknown (presence of unmodeled dynamics) or when the system dynamics consists of unknown nonlinearities which are linearly or nonlinearly parameterized. The past several years has witnessed progress in the area of adaptive observer
based control as reported in [106–109]. In [106] it is assumed that the uncertainty is linearly parameterized, while the methods in [107–109] treat the case of systems that are nonlinear with respect to unknown parameters. These approaches further assume that the full dimension of the system is known. In effect, it is assumed that the parameterization of the model error is known.

In this thesis we develop an adaptive design in a form that is useful for augmenting an EKF. The estimated states from the adaptive EKF are then used as inputs to the controller. The approach developed here is adaptive to unmodeled dynamics and disturbances, and does not require knowledge of the full dimension of the system.

1.6 Thesis Outline

Chapter 2 presents a background on NNs as universal approximators.

Chapter 3 presents an approach for adaptive state estimation of a class of bounded nonlinear processes by augmenting existing linear time invariant observers with an NN based adaptive element. Boundedness of the error signals is shown for two different types of adaptive laws via Lyapunov’s direct method. Numerical simulation results are included to demonstrate the viability of this approach to state estimation.

Chapter 4 presents an approach for adaptive state estimation of bounded nonlinear processes by augmenting an extended Kalman filter with an NN based adaptive element. The importance of using EKFs has been described in chapter 1 and the salient feature of the approach presented in chapter 4 originates from being able to augment an EKF with an NN based adaptive element. The adaptive laws are trained directly using the residuals of the EKF. Boundedness of the error signals is shown via Lyapunov’s direct method. Numerical simulation results are included to illustrate the viability of this approach to state estimation.

Chapter 5 presents an approach for adaptive state estimation for controlled multivariable systems with unmodeled dynamics coupled to the process. The design of the adaptive element employs a linearly parameterized neural network. The states of the adaptive estimator
are used to form the feedback control signal. The network weights are adjusted on line using a linear error observer for the nominal system’s error dynamics. Boundedness of the error signals is proven using Lyapunov’s direct method. The significance of this approach is that it can be applied to problems such as missile-target intercept, formation flight control and obstacle avoidance. Simulations illustrate the theoretical results.

Chapter 6 illustrates the approach developed in chapter 5 to the problem of missile-target intercept for low speed and high speed maneuvering targets. In this chapter, we formulate the relative dynamics of the missile-target tracking problem and discuss the manner in which the NN based adaptive element is augmented to the EKF. Simulation results illustrate the improvement in the performance of the EKF when augmented with an NN.

Chapter 7 summarizes the contributions of this thesis work and presents directions for future research work.

Throughout the manuscript bold symbols are introduced for vectors, capital letters for matrices, small letters for scalars, $\| \cdot \|$ is introduced for 2-norm and $\| \cdot \|_F$ is introduced for Frobenius norm. That is, $\| A \|_F = \sqrt{\text{tr}(AA^T)}$.
CHAPTER 2

BACKGROUND ON NEURAL NETWORKS

2.1 What is a Neural Network?

Work on artificial neural networks, commonly referred to as neural networks (NNs), has been motivated right from its inception by the recognition that the human brain computes in an entirely different way from the conventional digital computer. In its most general form, an NN is a machine that is designed to model the way in which the brain performs a task or function of interest. The network is usually implemented by using electronic computers or is simulated in software on a digital computer. To achieve good performance, NNs employ a massive interconnection of simple computing cells referred to as neurons or processing units. In [110], a definition of an NN viewed as an adaptive machine is given as follows:

"A neural network is a massively parallel distributed processor made up of simple processing units, which has a natural propensity for storing experiential knowledge and making it available for use. It resembles the brain in two respects, (1) knowledge is acquired by the network from its environment through a learning process, and (2) interneuron connection strengths, known as synaptic weights, are used to store the acquired knowledge."

The procedure used to perform the learning process is called a learning algorithm, the function of which is to modify the synaptic weights of the network in an orderly fashion to attain the desired design objective. Some of the benefits that NNs offer are functional approximation, Input-Output Mapping, Adaptivity, Neurobiological Analogy and pattern recognition, to name a few.

2.2 Neural Network Architecture

There are three fundamentally different classes of network architectures, which are multi-layer neural networks (MLNN), single-layer neural networks (SLNN), and recurrent networks (RN).
2.2.1 Multi-Layer Neural Networks

MLNNs consist of multiple layers of neurons comprising of an input, an output and multiple number of inner layers, also called hidden layers, whose computation nodes are correspondingly called hidden neurons or hidden units. A commonly used NN of this type is a three-layer-network, which has two sets of weights, \( M \) for the output layer and \( N \) for the input layer, which represent the interconnection between the three layers. The nodes in the input layer supply the respective elements of the activation pattern (input vector) to the hidden neurons which form the second layer. The output signals of this layer are used as inputs to the third layer, which constitutes the overall response of the network to the activation pattern supplied by the nodes of the input layer. The input-output map of the MLNN is given by

\[
\nu_{ad} = b_m \theta_m + \sum_{j=1}^{n_2} m_{jk} \sigma \left( b_n \theta_n + \sum_{i=1}^{n_1} n_{ij} x_i \right), \quad k = 1, \ldots, n_3
\]

where \( n_1, n_2 \) and \( n_3 \) denote the number of neurons in the input, hidden and output layers respectively, the parameters \( n_{ij} \) and \( m_{jk} \) represent the interconnection weights between the input to the hidden layer and the hidden to the output layer, respectively. The parameters \( b_n \) and \( b_m \) represent the bias terms in the inputs to the hidden and the output layers respectively, and their weights are represented by \( \theta_n \) and \( \theta_m \) respectively. The scalar function \( \sigma(\cdot) \) is called the activation function, the function of which is to limit the amplitude of the output of a neuron. The activation function is also referred to as a squashing function in that it squashes (limits) the permissible amplitude range of the output signal to some finite value. Some commonly used choices for activation functions include:

1. Logarithmic-sigmoidal (sigmoid) functions which are defined as

\[
\sigma(x) = \frac{1}{1 + e^{-ax}}
\]

where \( a \) is the slope parameter of the sigmoid function. By varying the parameter \( a \), sigmoid functions of different slopes are obtained.

2. Hyperbolic tangent functions which are defined as

\[
\sigma(x) = \tanh(ax) = \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}
\]
Other basic types of activation functions are

1. Threshold functions which are defined as

\[ \sigma(x) = \begin{cases} 
1 & \text{if } x \geq 0 \\
0 & \text{if } x < 0
\end{cases} \]  

(5)

In the engineering literature, this form of the threshold function is commonly referred to as a **Heaviside function**.

2. Piecewise-linear functions which are defined as

\[ \sigma(x) = \begin{cases} 
1 & \text{if } x \geq +\frac{1}{2} \\
x & \text{if } -\frac{1}{2} < x < +\frac{1}{2} \\
0 & \text{if } x \leq -\frac{1}{2}
\end{cases} \]  

(6)

where the amplification factor inside the linear region of operation is assumed to be unity. This form of an activation function may be viewed as an approximation to a nonlinear amplifier. The piecewise-linear function reduces to a **threshold function** if the amplification factor of the linear region is made infinitely large.

The logarithmic-sigmoidal function is used extensively for this research work since it is computationally less expensive.

### 2.2.2 Single-Layer Neural Networks

In single layer neural networks (SLNNs) there is only one set of tunable weights. As a result, these particular types of network architecture are considerably simpler to analyze. In the simplest form, SLNNs have an input layer of nodes that project onto an output layer of neurons (computational nodes), but not vice versa. In other words these are strictly feedforward networks, as shown in Fig. 1.

The output of an SLNN is given by

\[ \nu_{ad} = M^T \psi(\bar{x}) \]  

(7)

where \( M \) is a matrix containing the tunable weights and \( \psi : \mathcal{R}^n \rightarrow \mathcal{R}^m \) is called the basis. In [111], the requirements for an activation function to form a basis is specified.
2.2.3 Recurrent Neural Networks

A recurrent neural network (RNN) distinguishes itself from a feedforward NN in that it has at least one feedback loop. For example, an RNN may consist of a single layer of neurons with each neuron feeding its output signal back to the inputs of all the other neurons as illustrated in Fig. 2.

The architecture shown in Fig. 2 has no hidden neurons. There are other classes of recurrent networks with hidden neurons. The presence of feedback loops has a profound impact on the learning capability of the network and on its performance. Moreover, the feedback loops involve the use of particular branches composed of unit-delay elements, denoted by $z^{-1}$, which result in a nonlinear dynamical behavior, assuming that the neural network contains nonlinear units.

2.3 Neural Network as a Universal Approximator

In this thesis, we use NNs to approximate unknown continuous functions of the dynamic states. In order to use NNs for online adaptation, it is important to ensure that these NNs are
Figure 2: Recurrent Network Architecture.

capable of uniformly approximating continuous functions over some predetermined compact domains. In this section, we introduce some key definitions and theorems to help understand the NN property of convergence and approximation.

Definition 1. \[112\] A function \( g(x) : D_x \subset \mathcal{R}^n \rightarrow \mathcal{R}^n \) approximates a function \( f(x) : D_x \subset \mathcal{R}^n \rightarrow \mathcal{R}^n \) uniformly on \( D_x \subset \mathcal{R}^n \), to within an arbitrary accuracy \( \epsilon > 0 \) if

\[
\|g(x) - f(x)\| \leq \epsilon, \quad \forall \ x \in D_x
\]  

(8)

or equivalently if

\[
\sup_{x \in D_x} \|g(x) - f(x)\| \leq \epsilon, \quad \forall \ x \in D_x
\]  

(9)

The sup norm, called as the supremum (or a least upper bound), is also sometimes referred to as the \( \mathcal{L}^\infty \) norm since \( \mathcal{L}^\infty \) is the space of piecewise continuous, uniformly bounded functions.

Theorem 1. \[112\] (Stone-Weierstrass Theorem). Let \( \mathcal{K} \) be a compact subset of \( \mathcal{R}^n \) and let \( \mathcal{A} \) be a collection of continuous functions on \( \mathcal{K} \) to \( \mathcal{R} \) with the following properties:

1. The constant function \( \epsilon(x) = 1, \ x \in \mathcal{K} \), belongs to \( \mathcal{A} \).

2. If \( f, g \) belong to \( \mathcal{A} \), then \( \alpha f + \beta g \) belongs to \( \mathcal{A} \) for all \( \alpha, \beta \) in \( \mathcal{R} \).
3. If $f$, $g$ belong to $A$, then $fg$ belongs to $A$.

4. If $x \neq y$ are two points of $K$, there exists a function $f$ in $A$ such that $f(x) \neq f(y)$.

Then any continuous function on $K$ to $R$ can be uniformly approximated on $K$ by functions in $A$.

**Theorem 2.** [113] (NN Universal Approximation Theorem). Let $\phi(x)$ be a nonconstant, bounded and monotone increasing continuous function. Let $K$ be a compact subset (bounded closed subset) of $R^n$ and $f(x_1, x_2, \cdots, x_n)$ be a real valued continuous function on $K$. Then for an arbitrary $\epsilon > 0$, there exists an integer $N$ and real constants $c_i$, $\theta_i (i = 1, 2, \cdots, N)$, $w_{ij} (i = 1, 2, \cdots, N, j = 1, 2, \cdots, n)$ such that

$$f(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{N} c_i \phi_i \left( \sum_{j=1}^{n} w_{ij} x_j - \theta_i \right)$$

satisfies

$$\max_{x \in K} |f(x_1, x_2, \cdots, x_n) - \hat{f}(x_1, x_2, \cdots, x_n)| < \epsilon$$

In other words Theorem 2 states that for an arbitrary $\epsilon > 0$, there exists a three layered network whose output functions for the hidden layer are $\phi(x)$, whose output functions for input and output layers are linear and which has an input-output function $\hat{f}(x_1, x_2, \cdots, x_n)$ such that $\max_{x \in K} |f(x_1, x_2, \cdots, x_n) - \hat{f}(x_1, x_2, \cdots, x_n)| < \epsilon$. In Theorem 2 we observe the following about the form of (10):

1. The NN has $n$ input modes and a single hidden layer consisting of $N$ neurons; the inputs are denoted by $x_1, x_2, \cdots, x_n$.

2. The hidden neuron $i$ has synaptic weights $w_{i1}, w_{i2}, \cdots, w_{in}$, and bias $\theta_i$.

3. The NN output is a linear combination of the outputs of the hidden neurons, with $c_1, c_2, \cdots, c_N$ defining the synaptic weights of the output layer.

Theorem 2 is an existence theorem in the sense that it provides a mathematical justification for the approximation of an arbitrary continuous function as opposed to an exact representation. In effect, the theorem states that a single hidden layer NN is sufficient for a multilayer
perceptron to compute a uniform $\epsilon$ approximation to a given training set represented by the set of inputs $x_1, x_2, \cdots, x_n$ and a desired output $f(x_1, x_2, \cdots, x_n)$. However, the theorem does not say that a single hidden layer is optimum in the sense of learning time, ease of implementation or generalization.

**Remark 1.** [113] Usual output functions such as the sigmoidal function

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

(12)

used for back-propagation NNs satisfy the conditions for $\phi(x)$, i.e., nonconstant, bounded and monotone increasing continuous function.

**Definition 2.** [114] A subset $U$ of $S$ is said to be dense in $S$ if, for any $\epsilon > 0$ and any point $s$ in $S$, there is a point $u$ in $U$ such that

$$|u - s| < \epsilon$$

(13)

**Definition 3.** Let $S$ be a compact, simply connected subset of $R^n$, and $\phi(\cdot) : S \rightarrow R^n$ be integrable and bounded. Then $\phi(\cdot)$ is said to form a basis for $C^m(S)$ if

1. a constant function on $S$ can be expressed as the output of the NN for a finite number of neurons,

2. the span of $\phi(\cdot)$ is dense in $C^m(S)$ for countable number of neurons.
CHAPTER 3

ADAPTIVE AUGMENTATION OF NONLINEAR TIME INvariant SYSTEMS

The design of nonlinear observers from the perspective of augmenting a linear observer with an adaptive element is addressed in this chapter. The adaptive element employs two nonlinearly parameterized neural networks (NNs), the input and output layer weights of which are adapted on line. The objective is to improve the performance of the linear observer when applied to a nonlinear system. A linear filter is used to generate the error signal needed in the adaptive laws. The approach presented here is robust to unmodeled dynamics and unmodeled disturbances. Finally we present some simulation results that illustrate the theoretical results.

Before presenting the main results in this chapter, it is important to understand certain definitions and theorems that are used to arrive at the main result. These definitions and theorems are presented in the following section.

3.1 Mathematical Preliminaries

Consider the nonlinear autonomous interconnected dynamical system:

\[
\begin{align*}
\dot{x}(t) &= f(x(t), z(t)), \quad x(0) = x_0 \\
\dot{z}(t) &= f_z(x(t), z(t)), \quad z(0) = z_0
\end{align*}
\]

where \( x \in \Omega_x \subset \mathbb{R}^{n_x} \) is an open set such that \( x_0 \in \Omega_x, \ z \in \Omega_z \subset \mathbb{R}^{n_z} \) is an open set such that \( z_0 \in \Omega_z, \ f : \Omega_x \times \Omega_z \rightarrow \mathbb{R}^{n_x} \) is such that, for every \( z \in \Omega_z, \ f(\cdot, z) \) is locally Lipschitz in \( x, \ f_z : \Omega_x \times \Omega_z \rightarrow \mathbb{R}^{n_z} \) is such that, for every \( x \in \Omega_x, \ f_z(x, \cdot) \) is locally Lipschitz in \( z \).

Under the above assumptions the solution \((x, z)\) to (14), (15) exists and is unique.

**Definition 4.** [115] The nonlinear system (14), (15) is ultimately bounded with respect to \( x(t) \) uniformly in \( z_0 \) with ultimate bound \( \varepsilon > 0 \) if there exists \( \gamma > 0 \) such that, for every
\( \delta \in (0, \gamma) \), there exists \( T = T(\delta, \varepsilon) > 0 \) such that \( \| x_0 \| < \delta \) implies \( \| x(t) \| < \varepsilon \), \( t \geq T \).

**Definition 5.** [116] A continuous function \( \tilde{\alpha} : [0, a) \to [0, \infty) \) is said to belong to class \( \mathcal{K} \) if it is strictly increasing and \( \tilde{\alpha}(0) = 0 \). It is said to belong to \( \mathcal{K}_\infty \) if \( a = \infty \) and \( \tilde{\alpha}(\bar{r}) \to \infty \) as \( \bar{r} \to \infty \).

**Definition 6.** [116] A continuous function \( \tilde{\beta} : [0, a) \times [0, \infty) \to [0, \infty) \) is said to belong to class \( \mathcal{KL} \) if, for each fixed \( \bar{s} \), the mapping \( \tilde{\beta}(\bar{r}, \bar{s}) \) belongs to class \( \mathcal{K} \) with respect to \( \bar{r} \) and for each fixed \( \bar{s} \), the mapping \( \tilde{\beta}(\bar{r}, \bar{s}) \) is decreasing with respect to \( \bar{s} \) and \( \tilde{\beta}(\bar{r}, \bar{s}) \to 0 \) and \( \bar{s} \to \infty \).

**Theorem 3.** [115] Consider the nonlinear system (14), (15). Assume there exists a continuously differentiable function \( V : \Omega_x \times \Omega_z \to \mathcal{R} \), class \( \mathcal{K} \) functions \( \alpha(\cdot) \) and \( \beta(\cdot) \), a continuous, positive-definite function \( W : \mathcal{D} \to \mathcal{R} \) such that \( W(x(t)) > 0 \), \( \| x(t) \| > \xi \), and

\[
\alpha(\| x(t) \|) \leq V(x(t), z(t)) \leq \beta(\| x(t) \|), \quad x(t) \in \Omega_x \subset \mathcal{R}^n, \quad z(t) \in \Omega_z \subset \mathcal{R}^n \tag{16}
\]

\[
V'(x(t), z(t)) \leq -W(x(t)), \quad x(t) \in \Omega_x \subset \mathcal{R}^n, \quad \| x(t) \| > \xi, \quad z(t) \in \Omega_z \subset \mathcal{R}^n \tag{17}
\]

where \( \xi > 0 \) is such that \( \mathcal{B}_{\alpha^{-1}(\eta)}(0) \overset{\Delta}{=} \left\{ x(t) : \| x(t) \| < \alpha^{-1}(\eta) \right\} \subset \Omega_x \) with \( \eta > \beta(\xi) \). Then the nonlinear system (14), (15) is ultimately bounded with respect to \( x(t) \) uniformly in \( z_0 \) with ultimate bound \( \bar{e} \overset{\Delta}{=} \alpha^{-1}(\eta) \).

**Definition 7.** [116] The system (15) is input-to-state stable, with \( x(t) \) viewed as input, if there exists a class \( \mathcal{KL} \) function \( \beta \), a class \( \mathcal{K} \) function \( \gamma \), and positive constants \( k_1 \) and \( k_2 \) such that for any initial state \( z(t_0) \) with \( \| z(t_0) \| < k_1 \) and any \( x(t) \) with \( \sup_{t \geq t_0} \| x(t) \| < k_2 \), the solution \( z(t) \) exists and satisfies

\[
\| z(t) \| \leq \beta(\| z(t_0) \|, t - t_0) + \gamma \left( \sup_{t_0 \leq \tau \leq t} \| x(\tau) \| \right) \tag{18}
\]

for all \( t \geq t_0 \geq 0 \).

**Theorem 4.** [115] Consider the nonlinear system (14), (15). If (15) is input-to-state stable with \( x(t) \) viewed as the input and (14), (15) is ultimately bounded with respect to \( x(t) \) uniformly in \( z_0 \), then the solution \( (x(t), z(t)), t \geq 0 \), of the interconnected system (14), (15) is ultimately bounded.
Theorem 5. [113] Given arbitrary $\epsilon^* > 0$, a continuous function $f(x)$, $f : \mathbb{R}^n \to \mathbb{R}^m$ and a suitably chosen set of basis functions $\sigma(\cdot)$, defined on a compact set $x \in D \subset \mathbb{R}^n$, there exists a set of bounded constant weights $M, N$, such that the following representation holds $\forall x \in D$:

$$f(x) = M^T \sigma(N^T x) + \epsilon(x), \quad ||\epsilon(x)|| \leq \epsilon^* \quad (19)$$

Here, the structure $M^T \sigma(N^T x)$ is called a single hidden layer neural network (SHL) NN, $\sigma(\cdot)$ is a vector of the basis functions $\sigma(\cdot)$, its dimension specifying the number of nodes in the hidden layer, its $i^{th}$ component being defined as $[\sigma(N^T x)]_i = \sigma \left( [N^T x]_i \right)$, $|\sigma_i(\cdot)| \leq 1$, and $\epsilon(x)$ is the function reconstruction error. In [117,118], it has been shown that for an observable system such an approximation can be achieved using a finite sample of the output history. We recall the main theorem from [118] in the form of the following existence theorem.

Theorem 6. [118] Assume that an $n$-dimensional state vector $x(t)$ of an observable time-invariant system

$$\dot{x}(t) = f(x(t))$$
$$y(t) = h(x(t)) \quad (20)$$

evolves on an $n$-dimensional ball of radius $r$ in $\mathbb{R}^n$, $B_r = \{x(t) \in \mathbb{R}^n, ||x(t)|| \leq r\}$. Also assume that the system output $y(t) \in \mathbb{R}^m$ and its derivatives up to the order $(n - 1)$ are bounded. Then given arbitrary $\epsilon^* > 0$, there exists a set of bounded weights $M, N$ and a positive time delay $d > 0$, such that the function $f(x(t))$ in (20) can be approximated over the compact set $B_r$ by an SHL NN

$$f(x(t)) = M^T \sigma(N^T \mu(t)) + \epsilon(\mu(t)), \quad ||M||_F \leq M^*, \quad ||N||_F \leq N^*, \quad ||\epsilon(\mu(t))|| \leq \epsilon^*$$

using the input vector:

$$\mu(y(t), d) = \begin{bmatrix} \Delta_d^{(0)} y^T(t) & \Delta_d^{(1)} y^T(t) & \cdots & \Delta_d^{(n-1)} y^T(t) \end{bmatrix}^T \in \mathbb{R}^{nm}, \quad ||\mu(t)|| \leq \mu^* \quad (21)$$
where the finite difference quotients are given as

\[
\Delta_d^{(0)} y^T(t) \triangleq y^T(t) \\
\Delta_d^{(1)} y^T(t) \triangleq \frac{y^T(t) - y^T(t - d)}{d} \\
\vdots \\
\Delta_d^{(k)} y^T(t) \triangleq \frac{\Delta_d^{(k-1)} y^T(t) - \Delta_d^{(k-1)} y^T(t - d)}{d}, \quad k = 2, 3, \ldots
\]  

(22)

and \( \mu^* > 0 \) is a uniform bound on \( B_r \).

**Remark 2.** Notice that when the dimension \( n \) of the system is not known, and only an upper bound \( n_1 > n \) for its dimension is available, then, provided that the \( (n_1 - 1) \) derivatives of the output are bounded, one can use an input vector, comprised of \( (n_1 - 1) \) quotients, while not sacrificing on the bound of the approximation.

**Definition 8.** [119] Consider a convex compact set with a smooth boundary, as shown in Fig. 3, given by:

\[
D_c \triangleq \left\{ \theta \in \mathbb{R}^n \mid g(\theta) \leq c \right\}, \quad 0 \leq c \leq 1
\]

where \( g(\theta) : \mathbb{R}^n \rightarrow \mathbb{R} \) is the following smooth convex function:

\[
g(\theta) = \frac{\theta^T \theta - \theta_{\text{max}}^2}{\epsilon_\theta}
\]

\( \theta_{\text{max}} \) is the norm bound imposed on the parameter vector \( \theta \) and \( \epsilon_\theta \) denotes the convergence tolerance of our choice. The projection operator is defined as:

\[
\text{Proj}(\theta, \xi) \triangleq \left\{ \begin{array}{ll}
\xi & \text{if } g(\theta) < 0 \\
\xi & \text{if } g(\theta) \geq 0 \text{ and } \nabla g^T(\theta) \xi \leq 0 \\
\xi - \frac{\nabla g(\theta)^T \xi}{\|\nabla g(\theta)\|^2} \xi & \text{if } g(\theta) \geq 0 \text{ and } \nabla g^T(\theta) \xi > 0
\end{array} \right.
\]

(23)

where \( \nabla g(\theta) = \left[ \frac{\partial g(\theta)}{\partial \theta_1} \frac{\partial g(\theta)}{\partial \theta_2} \ldots \frac{\partial g(\theta)}{\partial \theta_n} \right]^T \).

**Property 1.** [119] The projection operator \( \text{Proj}(\theta, \xi) \) as defined in (23) does not alter \( \xi \) if \( \theta(t) \) belongs to the set \( D_0 = \left\{ \theta \in \mathbb{R}^n \mid g(\theta) \leq 0 \right\} \). In the set \( \left\{ 0 \leq g(\theta) \leq 1 \right\} \), \( \text{Proj}(\theta, \xi) \) subtracts a vector normal to the boundary of \( D_c = \left\{ \theta \in \mathbb{R}^n \mid g(\theta) = c \right\} \) so that we get a smooth transformation from the original vector field \( \xi \) to an inward or tangent vector field for \( c = 1 \). Thus the projection operator ensures that once \( \theta(0) \in D_c, \theta \) will never leave \( D_c \).
Property 2. Given the matrices

\[
Y = [ \mathbf{y}_1 \cdots \mathbf{y}_n ] \in \mathcal{R}^{p \times n}, \quad \Theta = [ \mathbf{\theta}_1 \cdots \mathbf{\theta}_n ] \in \mathcal{R}^{p \times n}
\]

we have:

\[
\text{tr}\left[ (\Theta - \Theta^*)^T \left( \text{Proj}(\Theta, Y) - Y \right) \right] = \sum_{i=1}^{n} (\mathbf{\theta}_i - \mathbf{\theta}_i^*)^T \left( \text{Proj}(\mathbf{\theta}_i, \mathbf{y}_i) - \mathbf{y}_i \right) \leq 0
\]

where \( \Theta^* \) is the true value of the parameter \( \Theta \).

Fact 1. Let \( \hat{M}, \bar{M}, M \in \mathcal{R}^{n \times m} \) and the following representation hold

\[
\hat{M} \triangleq M - \bar{M}
\]

Then the following is true

\[
\text{tr}\left( \hat{M}^T \hat{M} \right) = \frac{1}{2} \| M \|_F^2 - \frac{1}{2} \| \bar{M} \|_F^2 - \frac{1}{2} \| \hat{M} \|_F^2
\]

Proof. Begin by expanding the left hand side as follows:

\[
\text{tr}\left( \hat{M}^T \hat{M} \right) = \text{tr}\left( \hat{M}^T (M - \hat{M}) \right)
\]

\[
= \text{tr}\left( \hat{M}^T M \right) - \| \hat{M} \|_F^2
\]

\[
= \text{tr}\left( (M - \hat{M})^T \hat{M} \right) - \| \hat{M} \|_F^2
\]

\[
= \| M \|_F^2 - \| \bar{M} \|_F^2 - \text{tr}\left( \hat{M}^T \hat{M} \right)
\]

\[
= \| M \|_F^2 - \| \bar{M} \|_F^2 - \| \hat{M} \|_F^2 - \text{tr}\left( \hat{M}^T \hat{M} \right)
\]
Thus, taking $\text{tr}(\tilde{M}^T \tilde{M})$ to the left hand side and further simplifying we have

$$\text{tr}(\tilde{M}^T \tilde{M}) = \frac{1}{2}\|M\|_F^2 - \frac{1}{2}\|\tilde{M}\|_F^2 - \frac{1}{2}\|\hat{M}\|_F^2$$ (29)

\[\square\]

**Remark 3.** The equality in (29) leads to the following inequality

$$\text{tr}[\tilde{M}^T \tilde{M}] \leq \frac{1}{2}\|M\|_F^2 - \frac{1}{2}\|\tilde{M}\|_F^2$$ (30)

### 3.2 System Description and Problem Formulation

Let the dynamics of an observable and bounded nonlinear process be given by the following equations\(^1\):

\[
\begin{align*}
\dot{x}(t) &= A x(t) + f(x(t), z(t)), \quad x(0) = x_0 \quad (31) \\
\dot{z}(t) &= f_z(x(t), z(t)), \quad z(0) = z_0 \quad (32) \\
y(t) &= C x(t) + g(x(t), z(t)) \quad (33)
\end{align*}
\]

where $x \in \Omega_x \subset \mathbb{R}^{n_x}$, $z \in \Omega_z \subset \mathbb{R}^{n_z}$ are the states of the system, $n_z$ need not be known, however an upper bound $n$ for the dimension of the process dynamics (31), (32) is known, $\Omega_x$ and $\Omega_z$ are open sets such that $x_0 \in \Omega_x$, $z_0 \in \Omega_z$, $y \in \mathbb{R}^m$ is the vector of available measurements, $f(x, z): \Omega_x \times \Omega_z \to \mathbb{R}^{n_x}$, $g(x, z): \Omega_x \times \Omega_z \to \mathbb{R}^m$ are unknown continuous functions representing the modelling uncertainties, $f(x, z): \Omega_x \times \Omega_z \to \mathbb{R}^{n_x}$ is such that, for every $z \in \Omega_z$, $f(\cdot, z)$ is locally Lipschitz in $x$, $f_z(x, z): \Omega_x \times \Omega_z \to \mathbb{R}^{n_z}$ is such that, for every $x \in \Omega_x$, $f_z(x, \cdot)$ is locally Lipschitz in $z$, so that the solution to the system (31)-(33) exists for all $t \geq 0$ and is unique. The objective is to design an adaptive observer for the system in (31)-(33) ensuring bounded estimation errors.

### 3.3 Adaptive Observer and Adaptive Laws

Using Theorem 6, consider the NN approximations of $f(x(t), z(t))$ and $g(x(t), z(t))$ defined on the compact set $\mathcal{D} = \{(x, z) : x \in \mathcal{D}_x \subset \Omega_x, z \in \mathcal{D}_z \subset \Omega_z\}$, where $\mathcal{D}_x$ and $\mathcal{D}_z$

\(^1\)For the definition on observability of nonlinear systems, refer to [120]
and compact sets in $\Omega_x$ and $\Omega_z$ respectively

$$f(x(t), z(t)) = M_f^T \sigma (N_f^T \mu(t)) + \epsilon_f(\mu(t)), \|M_f\| \leq M_f^*, \|N_f\| \leq N_f^*, \|\epsilon_f(\mu(t))\| \leq \epsilon_f^* \quad (34)$$

$$g(x(t), z(t)) = M_g^T \sigma (N_g^T \mu(t)) + \epsilon_g(\mu(t)), \|M_g\| \leq M_g^*, \|N_g\| \leq N_g^*, \|\epsilon_g(\mu(t))\| \leq \epsilon_g^* \quad (35)$$

where $M_f^*, N_f^*, M_g^*, N_g^*$ denote known upper bounds for the Frobenius norms of the weights in (34) and (35), $\mu(t)$ is a vector of the difference quotients of the measurement $y(t)$ as defined in (21) and $\|\mu(t)\| \leq \mu^*$. We propose the following adaptive observer for the dynamics in (31)-(33):

$$\begin{align*}
\dot{\hat{x}}(t) &= A\hat{x}(t) + M_f^T(t)\sigma(N_f^T(t)\mu(t)) + K(y(t) - \hat{y}(t)), \quad \hat{x}(0) = \hat{x}_0 \quad (36) \\
\dot{\hat{y}}(t) &= C\hat{x}(t) + M_g^T(t)\sigma(N_g^T(t)\mu(t))
\end{align*}$$

where $\hat{x}_0$ is the initial condition, $\hat{M}_f(t), \hat{N}_f(t), \hat{M}_g(t)$ and $\hat{N}_g(t)$ denote the estimates of the weights that are adjusted online, and $K \neq 0$ is a design matrix ensuring that $\hat{A} \triangleq A - KC$ is Hurwitz.

**Remark 4.** In the absence of modelling errors in (31)-(33), i.e.

$$f(x(t), z(t)) = g(x(t), z(t)) = 0$$

the observer in (36) reduces to

$$\begin{align*}
\dot{\hat{x}}(t) &= A\hat{x}(t) + K(y(t) - \hat{y}(t)), \quad \hat{x}(0) = \hat{x}_0 \\
\dot{\hat{y}}(t) &= C\hat{x}(t)
\end{align*}$$

which is a standard linear observer for the dynamics in (31)-(33).

The conceptual layout of the proposed observer along with the neural network architecture is shown in Fig. 4, where $d_1(t)$ denotes the exogenous disturbance to the process dynamics, while $d_2(t)$ denotes the sensor disturbance. These disturbances can be viewed as parts of the unmodeled dynamics vectors that are coupled to the process and the measurement dynamics respectively.
Figure 4: Augmented Linear Observer.

Denote the observation error vectors $\mathbf{e}(t) \triangleq \hat{\mathbf{x}}(t) - \mathbf{x}(t), \mathbf{y}(t) \triangleq \hat{\mathbf{y}}(t) - \mathbf{y}(t)$. Then the observation error dynamics can be written as:

$$
\begin{align*}
\dot{\mathbf{e}}(t) &= \bar{A}\mathbf{e}(t) + \hat{\mathbf{M}}_f^T(t)\mathbf{\sigma}(\hat{\mathbf{N}}_f^T(t)\mathbf{\mu}(t)) - M_f^T\mathbf{\sigma}(N_f^T\mathbf{\mu}(t)) - \mathbf{e}_f(\mathbf{\mu}(t)) \\
&\quad - K\left[\hat{\mathbf{M}}_g^T(t)\mathbf{\sigma}(\hat{\mathbf{N}}_g^T(t)\mathbf{\mu}(t)) - M_g^T\mathbf{\sigma}(N_g^T\mathbf{\mu}(t)) - \mathbf{e}_g(\mathbf{\mu}(t))\right] \\
\dot{\mathbf{y}}(t) &= C\mathbf{e}(t) + \hat{\mathbf{M}}_g^T(t)\mathbf{\sigma}(\hat{\mathbf{N}}_g^T(t)\mathbf{\mu}(t)) - M_g^T\mathbf{\sigma}(N_g^T\mathbf{\mu}(t)) - \mathbf{e}_g(\mathbf{\mu}(t))
\end{align*}
$$

(38)

Introduce the following linear observer for the observer error dynamics in (38) [121]:

$$
\dot{\hat{\mathbf{e}}}(t) = \bar{A}\hat{\mathbf{e}}(t) + \bar{K}(\hat{\mathbf{y}}(t) - C\hat{\mathbf{e}}(t)), \quad \hat{\mathbf{e}}(0) = \hat{\mathbf{e}}_0
$$

(39)

where $\hat{\mathbf{e}}_0$ is the initial condition of the error observer and $\bar{K} \neq 0$ is chosen such that $\bar{A} \triangleq \bar{A} - \bar{K}C$ is Hurwitz. This observer is used only to generate an error signal needed in adapting the NN weights in (36). Let $\hat{\mathbf{e}}(t) \triangleq \hat{\mathbf{e}}(t) - \mathbf{e}(t)$. Then

$$
\begin{align*}
\dot{\hat{\mathbf{e}}}(t) &= \bar{A}\hat{\mathbf{e}}(t) - \hat{\mathbf{M}}_f^T(t)\mathbf{\sigma}(\hat{\mathbf{N}}_f^T(t)\mathbf{\mu}(t)) + M_f^T\mathbf{\sigma}(N_f^T\mathbf{\mu}(t)) + \mathbf{e}_f(\mathbf{\mu}(t)) \\
&\quad + \bar{K}\left[\hat{\mathbf{M}}_g^T(t)\mathbf{\sigma}(\hat{\mathbf{N}}_g^T(t)\mathbf{\mu}(t)) - M_g^T\mathbf{\sigma}(N_g^T\mathbf{\mu}(t)) - \mathbf{e}_g(\mathbf{\mu}(t))\right]
\end{align*}
$$

(40)

where $\bar{K} \triangleq \bar{K} + K$. 
3.4 Adaptive Laws and Boundedness Analysis

At this point we have set up the problem along with the error dynamics and the error observer error dynamics and are ready to lay down the adaptive laws to train the NN weights on line. Here we propose two types of adaptation laws utilizing the error observer introduced in Section 3.3. These are:

1. $\sigma-$ modification based and
2. projection based.

Proofs of boundedness of error signals for both these laws are given using Lyapunov’s direct method.

3.4.1 Neural Network Adaptation with $\sigma-$ modification

The update law which we will use here is a modification of back propagation. The algorithm was first proposed by Lewis et. al. in [122] in a state feedback setting with $\epsilon$-modification. The $\sigma-$ modification based adaptation laws for $\dot{M}_f(t), \dot{N}_f(t), \dot{M}_g(t)$ and $\dot{N}_g(t)$ are chosen to be the following:

\begin{align*}
\dot{N}_f(t) &= -G_f \left[ 2\mu(t)\dot{e}_f^T(t)P\dot{M}_f(t)\hat{\sigma}_f(t) + k_f\dot{N}_f(t) \right] \quad (41) \\
\dot{M}_f(t) &= -F_f \left[ 2\left(\sigma_f(t) - \hat{\sigma}_f(t)\dot{N}_f^T(t)\mu(t)\right)\dot{e}_f^T(t)P + k_f\dot{M}_f(t) \right] \quad (42) \\
\dot{N}_g(t) &= -G_g \left[ 2\mu(t)\dot{e}_g^T(t)P\dot{K}\dot{M}_g(t)\hat{\sigma}_g(t) + k_g\dot{N}_g(t) \right] \quad (43) \\
\dot{M}_g(t) &= -F_g \left[ 2\left(\hat{\sigma}_g(t) - \hat{\sigma}_g(t)\dot{N}_g^T(t)\mu(t)\right)\dot{e}_g^T(t)P\dot{K} + k_g\dot{M}_g(t) \right] \quad (44)
\end{align*}

where $\hat{\sigma}_f(t) \triangleq \sigma(\dot{N}_f^T(t)\mu(t))$, $\hat{\sigma}_g(t) \triangleq \sigma(\dot{N}_g^T(t)\mu(t))$, the notations $\hat{\sigma}_f(t) \triangleq \sigma'(\dot{N}_f^T(t)\mu(t))$, $\hat{\sigma}_g(t) \triangleq \sigma'(\dot{N}_g^T(t)\mu(t))$ are introduced for the Jacobians, evaluated at the estimates and are
given by

$$\dot{\sigma}'(t) \triangleq \left. \frac{d\sigma_f(t)}{dz_f(t)} \right|_{z_f(t)=\hat{N}_f(t)\mu(t)} = \left[ \begin{array}{ccc} 0 & \cdots & 0 \\ \frac{\partial \sigma_{1_f}(t)}{\partial z_{1_f}(t)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\partial \sigma_{n_{2_f}}(t)}{\partial z_{n_{2_f}}(t)} \end{array} \right] \in \mathcal{R}^{(n_{2_f}+1) \times n_{2_f}} \quad (45)$$

$$\dot{\sigma}'(t) \triangleq \left. \frac{d\sigma_g(t)}{dz_g(t)} \right|_{z_g(t)=\hat{N}_g(t)\mu(t)} = \left[ \begin{array}{ccc} 0 & \cdots & 0 \\ \frac{\partial \sigma_{1_g}(t)}{\partial z_{1_g}(t)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\partial \sigma_{n_{2_g}}(t)}{\partial z_{n_{2_g}}(t)} \end{array} \right] \in \mathcal{R}^{(n_{2_g}+1) \times n_{2_g}} \quad (46)$$

where $n_{2_f}$ and $n_{2_g}$ denote the number of hidden layer neurons of the two NNs respectively, $F_f > 0$, $G_f > 0$, $F_g > 0$, $G_g > 0$ specify the learning rates of the two NNs and the $\sigma$-modification gains are specified by $k_f > 0$, $k_g > 0$. The matrices $P$ and $\hat{P}$ are the solutions of the Lyapunov equation

$$\bar{A}^T P + P \bar{A} = -Q \quad (47)$$

$$\bar{A}^T \hat{P} + \hat{P} \bar{A} = -\hat{Q} \quad (48)$$

for some $Q > 0$, $\hat{Q} > 0$. The adaptive laws in (41)-(44) are based on a Lyapunov like stability analysis, and partially cancel the uncertainties in the error dynamics, as detailed in the next section.

### 3.4.1.1 Stability Analysis

In this section we show through Lyapunov’s direct method that the observation errors $e^T(t), \hat{e}^T(t)$ and the NN weight errors $\hat{M}_f(t), \hat{N}_f(t), \hat{M}_g(t), \hat{N}_g(t)$ are ultimately bounded. Define the NN weight errors as

$$\hat{M}_f(t) \triangleq \hat{M}_f(t) - M_f, \quad \hat{N}_f(t) \triangleq \hat{N}_f(t) - N_f \quad (49)$$

$$\hat{M}_g(t) \triangleq \hat{M}_g(t) - M_g, \quad \hat{N}_g(t) \triangleq \hat{N}_g(t) - N_g \quad (50)$$

Consider the following composite error vector

$$\zeta(t) = \left[ e^T(t) \quad \hat{e}^T(t) \quad \text{vec}Z_f^T(t) \quad \text{vec}Z_g^T(t) \right]^T \quad (51)$$
where

\[
\tilde{Z}_f(t) \triangleq \begin{bmatrix} M_f(t) & 0 \\ 0 & \tilde{N}_f(t) \end{bmatrix}, \quad \tilde{Z}_g(t) \triangleq \begin{bmatrix} M_g(t) & 0 \\ 0 & \tilde{N}_g(t) \end{bmatrix}
\]  

(52)

Notice that the following upper bounds can be immediately derived:

\[
\| \tilde{M}_f(t) \|_F < \| \tilde{M}_f(t) \|_F + M^*_f, \quad \| \tilde{N}_f(t) \|_F < \| \tilde{N}_f(t) \|_F + N^*_f
\]

(53)

\[
\| \tilde{M}_g(t) \|_F < \| \tilde{M}_g(t) \|_F + M^*_g, \quad \| \tilde{N}_g(t) \|_F < \| \tilde{N}_g(t) \|_F + N^*_g
\]

(54)

where \( M^*_f, N^*_f, M^*_g \) and \( N^*_g \) are the Frobenius norms of the weights in (34) and (35). For the stability proof we refer to the following representations:

\[
\tilde{M}_f^T(t) \sigma(\tilde{N}_f^T(t) \mu(t)) - M_f^T \sigma(N_f^T \mu(t)) - \epsilon_f(\mu(t)) = \tilde{M}_f^T(t) \left( \sigma_f(t) - \sigma_f(t) \tilde{N}_f^T(t) \mu(t) \right) + \tilde{M}_f^T(t) \sigma'_f(t) \tilde{N}_f^T(t) \mu(t) + w_f(t) - \epsilon_f(\mu(t))
\]

(55)

where the disturbance term \( w_f \) is given as

\[
w_f(t) = \tilde{M}_f^T(t) \sigma'_f(t) \tilde{N}_f^T(t) \mu(t) - M_f^T \sigma'_f(t)
\]

(56)

and \( \sigma'_f(t) \) is given by

\[
\sigma'_f(t) = \sigma_f(t) - \sigma_f(t) + \sigma'_f(t) \tilde{N}_f^T(t) \mu(t)
\]

Similarly we obtain a representation for \( \tilde{M}_g^T(t) \sigma(\tilde{N}_g^T(t) \mu(t)) - M_g^T \sigma(N_g^T \mu(t)) - \epsilon_g(\mu(t)) \)

\[
\tilde{M}_g^T(t) \sigma(\tilde{N}_g^T(t) \mu(t)) - M_g^T \sigma(N_g^T \mu(t)) - \epsilon_g(\mu(t)) = \tilde{M}_g^T(t) \left( \sigma_g(t) - \sigma_g(t) \tilde{N}_g^T(t) \mu(t) \right) + \tilde{M}_g^T(t) \sigma'_g(t) \tilde{N}_g^T(t) \mu(t) + w_g(t) - \epsilon_g(\mu(t))
\]

(57)

where

\[
w_g(t) = \tilde{M}_g^T(t) \sigma'_g(t) \tilde{N}_g^T(t) \mu(t) - M_g^T \sigma'_g(t) \tilde{N}_g^T(t) \mu(t) - \epsilon_g(\mu(t)) \]

The representations in (55) and (57) are achieved via a Taylor series expansion of \( \sigma(N_f^T \mu(t)) \) and \( \sigma(N_g^T \mu(t)) \) around the estimates \( \tilde{N}_f^T(t) \mu(t) \) and \( \tilde{N}_g^T(t) \mu(t) \) respectively [122]. With (53) and (54) the forcing term in (38) and (40) allows for the following upper bound in terms of computable values of \( \alpha_1, \alpha_2, \alpha_3 \) and \( \alpha_4 \):

\[
\| \tilde{M}_f^T(t) \sigma(\tilde{N}_f^T(t) \mu(t)) - M_f^T \sigma(N_f^T \mu(t)) - \epsilon_f(\mu(t)) \| \leq \alpha_1 \| \tilde{Z}_f(t) \|_F + \alpha_2, \quad \alpha_1, \alpha_2 > 0
\]

(59)

\[
\| \tilde{M}_g^T(t) \sigma(\tilde{N}_g^T(t) \mu(t)) - M_g^T \sigma(N_g^T \mu(t)) - \epsilon_g(\mu(t)) \| \leq \alpha_3 \| \tilde{Z}_g(t) \|_F + \alpha_4, \quad \alpha_3, \alpha_4 > 0
\]

(60)
where
\[
\begin{align*}
\alpha_1 &= \sqrt{n_{2_f}}, \quad \alpha_2 = 2\sqrt{n_{2_f}}M_f^* + \epsilon_f^* \\
\alpha_3 &= \sqrt{n_{2_g}}, \quad \alpha_4 = 2\sqrt{n_{2_g}}M_g^* + \epsilon_g^*
\end{align*}
\] (61)

With the bound in (21), over the compact set \( D \subset \Omega_x \times \mathcal{R}^m \) the following upper bounds can be derived \([122]\):
\[
\begin{align*}
\|w_f(t) - e_f(\mu(t))\| &\leq \gamma_1 \|\hat{Z}_f(t)\| + \gamma_2, \quad \gamma_1 > 0, \ \gamma_2 > 0 \quad (63) \\
\|w_g(t) - e_g(\mu(t))\| &\leq \gamma_3 \|\hat{Z}_g(t)\| + \gamma_4, \quad \gamma_3 > 0, \ \gamma_4 > 0 \quad (64)
\end{align*}
\]

where \( \gamma_1, \gamma_2, \gamma_3 \) and \( \gamma_4 \) are computable constants, such that \( \gamma_1 \) and \( \gamma_3 \) depend upon the unknown constant \( \mu^* \), while \( \gamma_2 \) and \( \gamma_4 \) depend upon the \( \epsilon_f^*, \epsilon_g^* \) respectively. Notice that the error vector, introduced in (51) can be viewed as a function of the state variables \( x(t), \dot{x}(t), \dot{e}(t), \hat{Z}_f(t), \hat{Z}_g(t) \) and the constant matrices \( Z_f \) and \( Z_g \):
\[
\zeta(t) = F \left( x(t), \dot{x}(t), \dot{e}(t), \hat{Z}_f(t), \hat{Z}_g(t), Z_f, Z_g \right) 
\] (65)

where
\[
\begin{align*}
\hat{Z}_f(t) &= \begin{bmatrix} \hat{M}_f(t) & 0 \\ 0 & \hat{N}_f(t) \end{bmatrix}, \quad \hat{Z}_g(t) = \begin{bmatrix} \hat{M}_g(t) & 0 \\ 0 & \hat{N}_g(t) \end{bmatrix} \\
Z_f &= \begin{bmatrix} M_f & 0 \\ 0 & N_f \end{bmatrix}, \quad Z_g = \begin{bmatrix} M_g & 0 \\ 0 & N_g \end{bmatrix}
\end{align*}
\]

The relation in (65) represents a mapping from the original domains of the arguments to the space of error variables \( F : \Omega_x \times \Omega_x \times \Omega_{\hat{e}} \times \Omega_{\hat{Z}_f} \times \Omega_{\hat{Z}_g} \rightarrow \Omega_\zeta \).

Recall that (34) and (35) introduce the set \( D \) over which the NN approximation is valid. Introduce the largest ball \( B_R \), which is included in \( \Omega_\zeta \) in the error space:
\[
B_R \triangleq \{ \zeta \mid \|\zeta\| \leq R \}, \quad R > 0
\]

For every \( \zeta \in B_R \), we have \( x \in D \), i.e. the NN approximations in (34) and (35) remain valid. Let \( \alpha \) be the minimum value of the following function
\[
V(\zeta, z) = \zeta^T T \zeta 
\] (66)
on the edge of $B_R$

$$\alpha \triangleq \min_{||\xi||=R} V(\xi, z) = R^2 \lambda_{\min}(T)$$

where

$$T \triangleq \frac{1}{2}
\begin{bmatrix}
2P & 0 & 0 & 0 & 0 & 0 \\
0 & 2\tilde{P} & 0 & 0 & 0 & 0 \\
0 & 0 & F_f^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & G_f^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & F_g^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & G_g^{-1}
\end{bmatrix}
$$

(67)

Introduce the following set:

$$D_\alpha = \left\{ \xi \in B_R \mid V(\xi, z) \leq \alpha \right\}
$$

(68)

**Assumption 1.** Let

$$R > \gamma_0 \sqrt{\frac{\lambda_{\max}(T)}{\lambda_{\min}(T)}} \geq \gamma_0
$$

(69)

where $\lambda_{\max}(T)$ and $\lambda_{\min}(T)$ are the maximum and minimum eigenvalues of the $T$ matrix, introduced in (67), and

$$\gamma_0 = \max \left( \sqrt{\frac{\rho_0 + Z_f + Z_g}{\lambda_{\min}(Q)} - 3}, \sqrt{\frac{\rho_0 + Z_f + Z_g}{\lambda_{\min}(Q)} - 3}, \sqrt{\frac{\rho_0 + Z_f + Z_g}{k_f^2 - \rho_1}}, \sqrt{\frac{\rho_0 + Z_f + Z_g}{k_g^2 - \rho_2}} \right)
$$

(70)
where

\[ \rho_0 = \kappa_5^2 + \kappa_6^2 \]
\[ \rho_1 = \kappa_1^2 + \gamma_1^2 \lambda_{\text{max}}(P) \]
\[ \rho_2 = \kappa_3^2 + \gamma_2^2 \| \tilde{P} \tilde{K} \|^2 \]
\[ \kappa_1 = \Theta_1 \alpha_1 + \lambda_{\text{max}}(P) \gamma_1 \]
\[ \kappa_2 = \Theta_1 \alpha_2 + \lambda_{\text{max}}(P) \gamma_2 \]
\[ \kappa_3 = \Theta_2 \alpha_3 + \| \tilde{P} \tilde{K} \| \gamma_3 \]
\[ \kappa_4 = \Theta_2 \alpha_4 + \| \tilde{P} \tilde{K} \| \gamma_4 \]
\[ \kappa_5 = \kappa_2 + \| \tilde{P} \tilde{K} \| \gamma_4 \]
\[ \kappa_6 = \kappa_4 + \lambda_{\text{max}}(P) \gamma_2 \]
\[ \Theta_1 \Delta \lambda_{\text{max}}(P) + \lambda_{\text{max}}(\tilde{P}) \]
\[ \Theta_2 \Delta \| P K \| + \| \tilde{P} \tilde{K} \| \]
\[ \tilde{Z}_f = \frac{k_f}{2} \left[ \| \tilde{M}_f - M_{f_0} \|_{F}^2 + \| N_f - N_{f_0} \|_{F}^2 \right] \]
\[ \tilde{Z}_g = \frac{k_g}{2} \left[ \| \tilde{M}_g - M_{g_0} \|_{F}^2 + \| N_g - N_{g_0} \|_{F}^2 \right] \]  

(71)

with minimum eigenvalues \( \lambda_{\text{min}}(Q) > 3, \lambda_{\text{min}}(\tilde{Q}) > 3 \) for \( Q \) and \( \tilde{Q} \) introduced in (47) and (48), and the sigma modification gains satisfying the lower bounds \( k_f > 2 \rho_1, k_g > 2 \rho_2 \).

The significance of this assumption is discussed in Section 3.5. We are now ready to state the main result.

**Theorem 7.** Let Assumption 1 hold. Then, if the initial value \( \zeta_0 = \zeta(0) \) belongs to the set \( D_n \), defined in (68) and shown in Fig. 5, the observer in (36) along with (39) and (41)-(44), guarantees that \( \zeta(t) \) is ultimately bounded.

**Proof.** Consider the function \( V(\zeta(t), z(t)) \), introduced in (66), as a candidate Lyapunov function for the dynamics in (38), (40) and (41)-(44):

\[
V(\zeta(t), z(t)) = e^T(t) Pe(t) + \tilde{e}^T(t) \tilde{P} \tilde{e}(t) \\
+ \frac{1}{2} \text{tr} \left( M_f^T(t) F_f^{-1} M_f(t) \right) + \frac{1}{2} \text{tr} \left( N_f^T(t) G_f^{-1} N_f(t) \right) \\
+ \frac{1}{2} \text{tr} \left( M_g^T(t) F_g^{-1} M_g(t) \right) + \frac{1}{2} \text{tr} \left( N_g^T(t) G_g^{-1} N_g(t) \right) 
\]  

(72)

30
Figure 5: Geometric representation of the sets in the error space.

Taking the derivative of \( V(\zeta(t), z(t)) \) along (38), (40) and (41)-(44) and recalling that

\[
\dot{\sigma}_f(t) \triangleq \sigma(\hat{N}_f^T(t)\mu(t)), \quad \dot{\sigma}_g(t) \triangleq \sigma(\hat{N}_g^T(t)\mu(t))
\]

(73)

\[
\dot{\sigma}'_f(t) \triangleq \sigma'(\hat{N}_f^T(t)\mu(t)), \quad \dot{\sigma}'_g(t) \triangleq \sigma'(\hat{N}_g^T(t)\mu(t))
\]

(74)

we obtain

\[
\dot{V}(\zeta(t), z(t)) = -e^T(t)Qe(t) - \dot{e}^T(t)\dot{Q}e(t)
\]

\[
+2e^T(t)P\left(\hat{M}_f^T(t)\dot{\sigma}_f(t) - M_f^T\sigma_f(t) - \epsilon_f(\mu(t))\right)
\]

\[
-K\left(\hat{M}_g^T(t)\dot{\sigma}_g(t) - M_g^T\sigma_g(t) - \epsilon_g(\mu(t))\right)
\]

\[
+2\dot{e}^T(t)\dot{P}\left(-\hat{M}_f^T(t)\dot{\sigma}_f(t) + M_f^T\sigma_f(t) + \epsilon_f(\mu(t))\right)
\]

\[
+K\left(\hat{M}_g^T(t)\dot{\sigma}_g(t) - M_g^T\sigma_g(t) - \epsilon_g(\mu(t))\right)
\]

\[
+\text{tr}\left(\hat{M}_f^T(t)F_f^{-1}\hat{M}_f(t)\right) + \text{tr}\left(\hat{N}_f^T(t)G_f^{-1}\hat{N}_f(t)\right)
\]

\[
+\text{tr}\left(\hat{M}_g^T(t)F_g^{-1}\hat{M}_g(t)\right) + \text{tr}\left(\hat{N}_g^T(t)G_g^{-1}\hat{N}_g(t)\right)
\]

(75)

With the definition of \( \dot{e}(t) \triangleq \dot{e}(t) - e(t) \) and substituting the adaptive laws in (41)-(44) into
(75), \( \dot{V}(\zeta(t), z(t)) \) can be written as
\[
\dot{V}(\zeta(t), z(t)) = -e^T(t)Qe(t) - \dot{e}(t)^T \dot{Q} \dot{e}(t)
\]
\[
+ 2(e^T(t) + \dot{e}(t))^T P \left( w_f(t) - \epsilon_f(\mu(t)) \right)
\]
\[
- K \left( \dot{M}_g^T(t) \sigma_g(t) - M_g^T \sigma_g(t) - \epsilon_g(\mu(t)) \right)
\]
\[
- 2e^T(t) P \left( \dot{M}_f^T(t) \sigma_f(t) - M_f^T \sigma_f(t) - \epsilon_f(\mu(t)) \right)
\]
\[
+ 2(e^T(t) + \dot{e}(t))^T \dot{P} \left( - \dot{M}_f^T(t) \sigma_f(t) + M_f^T \sigma_f(t) + \epsilon_f(\mu(t)) \right)
\]
\[
+ K \left( w_f(t) - \epsilon_g(\mu(t)) \right)
\]
\[
- 2e^T(t) \dot{P} \left( - \dot{M}_f^T(t) \sigma_f(t) + M_f^T \sigma_f(t) + \epsilon_f(\mu(t)) \right)
\]
\[
- k_f \text{tr}(\dot{M}_f^T(t) \dot{M}_f(t)) - k_f \text{tr}(\dot{N}_f^T(t) \dot{N}_f(t))
\]
\[
- k_g \text{tr}(\dot{M}_g^T(t) \dot{M}_g(t)) - k_g \text{tr}(\dot{N}_g^T(t) \dot{N}_g(t))
\]  
(76)

Using the upper bounds from (59), (60), (63), (64), Fact 1 and rearranging the terms, the derivative of the Lyapunov function can be upper bounded as in [121,123]
\[
\dot{V}(\zeta(t), z(t)) \leq -\lambda_{\min}(Q) \| e(t) \|^2 - \lambda_{\min}(\tilde{Q}) \| \tilde{e}(t) \|^2
\]
\[
- \frac{k_f}{2} \| \dot{Z}_f(t) \|^2_F + \frac{k_f}{2} \| \dot{Z}_g(t) \|^2_F + \tilde{Z}_g
\]
\[
+ 2 \left( \lambda_{\max}(P) + \lambda_{\max}(\tilde{P}) \right) \| e(t) \| \left( \alpha_1 \| \dot{Z}_f(t) \|_F + \alpha_2 \right)
\]
\[
+ 2 \| e(t) \| \left( \| \tilde{P}K \| + \| \tilde{P} \tilde{K} \| \right) \left( \alpha_3 \| \tilde{Z}_g(t) \|_F + \alpha_4 \right)
\]
\[
+ 2 \left( \| e(t) \| + \| \dot{e}(t) \| \right) \lambda_{\max}(P) \left( \gamma_1 \| \dot{Z}_f(t) \|_F + \gamma_2 \right)
\]
\[
+ 2 \left( \| e(t) \| + \| \dot{e}(t) \| \right) \| \tilde{P} \tilde{K} \| \left( \gamma_3 \| \tilde{Z}_g(t) \|_F + \gamma_4 \right)
\]  
(77)

Using the notations in (71) and regrouping terms, we arrive at
\[
\dot{V}(\zeta(t), z(t)) \leq -\lambda_{\min}(Q) \| e(t) \|^2 - \lambda_{\min}(\tilde{Q}) \| \tilde{e}(t) \|^2 - \frac{k_f}{2} \| \dot{Z}_f(t) \|^2_F + \frac{k_f}{2} \| \dot{Z}_g(t) \|^2_F
\]
\[
+ \tilde{Z}_g + 2 \| \tilde{e}(t) \| \left( \alpha_1 \Theta_1 + \lambda_{\max}(P) \gamma_1 \right) \| \dot{Z}_f(t) \|_F + (\alpha_2 \Theta_1 + \lambda_{\max}(P) \gamma_2) \right)
\]
\[
+ 2 \| e(t) \| \left( \alpha_3 \Theta_2 + \| \tilde{P} \tilde{K} \| \gamma_3 \right) \| \dot{Z}_g(t) \|_F + (\alpha_4 \Theta_2 + \| \tilde{P} \tilde{K} \| \gamma_4) \right)
\]
\[
+ 2 \| e(t) \| \lambda_{\max}(P) \left( \gamma_1 \| \dot{Z}_f(t) \|_F + \gamma_2 \right)
\]
\[
+ 2 \| \tilde{e}(t) \| \| \tilde{P} \tilde{K} \| \left( \gamma_3 \| \dot{Z}_g(t) \|_F + \gamma_4 \right)
\]  
(78)
Further, the expression in (78) can be put in the following form:

$$
\dot{V}(\zeta(t), z(t)) \leq -\lambda_{\min}(Q) \|e(t)\|^2 - \lambda_{\min}(\bar{Q}) \|\bar{e}(t)\|^2 - \frac{k_f}{2} \|\bar{Z}_f(t)\|_F^2 + \bar{Z}_f - \frac{k_g}{2} \|\bar{Z}_g(t)\|_F^2 \\
+ \bar{Z}_g + 2\|\bar{e}(t)\| \left(\kappa_1 \|\bar{Z}_f(t)\|_F + \kappa_5\right) \\
+2\|\bar{e}\| \left(\kappa_3 \|\bar{Z}_g\|_F + \kappa_6\right) + 2\|\bar{e}(t)\| \|\bar{P}\bar{K}\|_{\gamma_3} \|\bar{Z}_g(t)\|_F \\
+2\|\bar{e}(t)\| \lambda_{\max}(P)_{\gamma_1} \|\bar{Z}_f(t)\|_F
$$

(79)

Completion of squares, further upper bounding the Lyapunov derivative and rearranging the terms, we obtain

$$
\dot{V}(\zeta(t), z(t)) \leq -\left(\lambda_{\min}(Q) - 3\right)\|e(t)\|^2 - \left(\lambda_{\min}(\bar{Q}) - 3\right)\|\bar{e}(t)\|^2 \\
-\frac{k_f}{2} \|\bar{Z}_f(t)\|_F^2 + \bar{Z}_f - \frac{k_g}{2} \|\bar{Z}_g(t)\|_F^2 + \bar{Z}_g + \left(\kappa_1^2 + \lambda_{\max}(P)\gamma_1^2\right) \|\bar{Z}_f(t)\|_F^2 \\
+ \left(\kappa_3^2 + \|\bar{P}\bar{K}\|_{\gamma_3}^2\right) \|\bar{Z}_g(t)\|_F^2 + \kappa_5^2 + \kappa_6^2
$$

(80)

and can finally be put in the following form:

$$
\dot{V}(\zeta(t), z(t)) \leq -\left(\lambda_{\min}(Q) - 3\right)\|e(t)\|^2 - \left(\lambda_{\min}(\bar{Q}) - 3\right)\|\bar{e}(t)\|^2 \\
-\left(\frac{k_f}{2} - \rho_1\right) \|\bar{Z}_f(t)\|_F^2 - \left(\frac{k_g}{2} - \rho_2\right) \|\bar{Z}_f(t)\|_F^2 + \left(\rho_0 + \bar{Z}_f + \bar{Z}_g\right)
$$

(81)

Thus either of the following conditions

$$
\|e\| > \sqrt{\frac{\rho_0 + \bar{Z}_f + \bar{Z}_g}{\lambda_{\min}(Q) - 3}}
$$

(82)

$$
\|\bar{e}\| > \sqrt{\frac{\rho_0 + \bar{Z}_f + \bar{Z}_g}{\lambda_{\min}(Q) - 3}}
$$

(83)

$$
\|\bar{Z}_f\|_F > \sqrt{\frac{\rho_0 + \bar{Z}_f + \bar{Z}_g}{\frac{k_f}{2} - \rho_1}}
$$

(84)

$$
\|\bar{Z}_g\|_F > \sqrt{\frac{\rho_0 + \bar{Z}_f + \bar{Z}_g}{\frac{k_g}{2} - \rho_2}}
$$

(85)

will render $\dot{V}(\zeta(t), z(t)) < 0$ outside a compact set. To complete the proof on ultimate boundedness of the error signals, consider the ball

$$
B_{\gamma_0} = \{\zeta \in B_R \mid \|\zeta\| \leq \gamma_0\}
$$

(86)
in the space of the error vector $\zeta$ outside of which $\dot{V}(\zeta, z) < 0$. Notice from (69), that $B_{\gamma_0} \subset B_R$. Let $\Gamma$ be the maximum value of the function $V(\zeta)$ on the edge of $B_{\gamma_0}$:

$$\Gamma = \max_{\|\zeta\|=\gamma_0} V(\zeta, z) = \gamma_0^2 \lambda_{\max}(T)$$

(87)

Introduce the level set of $V(\zeta, z)$, that touches the ball $B_{\gamma_0}$ as shown in Fig. 5

$$D_{\gamma_0} = \{ \zeta \mid V(\zeta, z) = \Gamma \}$$

(88)

The condition in (69) ensures that $D_{\gamma_0} \subset D_{\alpha}$. Thus, if the initial error $\zeta_0 = \zeta(0)$ belongs to $D_{\alpha}$ then it follows from Theorem 3 that the solution $(\zeta(t), z(t))$ to (32), (38), (40) and (41)-(44) is ultimately bounded with respect to $\zeta(t)$ uniformly in $z_0$. In addition since (32) is input-to-state stable with $x(t)$ viewed as the input, it follows from Theorem 4 that the solution $z(t)$ to (32) is ultimately bounded, thus completing the proof. \hfill \Box

3.4.2 Neural Network Adaptation with projection

The update law which we will use here is based on projections. The idea of projections was first proposed by Pomet and Praly in [119]. The projection based adaptation laws for $\hat{M}_f(t)$, $\hat{N}_f(t)$, $\hat{M}_g(t)$ and $\hat{N}_g(t)$ are defined as:

$$\hat{M}_f(t) = F_f \text{ Proj } \left( \hat{M}_f(t), (-\hat{\sigma}_f(t)e^T(t)P) \right), \quad i = 1, \ldots, n_3_f$$

(89)

$$\hat{N}_f(t) = G_f \text{ Proj } \left( \hat{N}_f(t), (-\mu(t)e^T(t)P\hat{M}_f(t)\hat{\sigma}_f(t)) \right), \quad i = 1, \ldots, n_2_f$$

(90)

$$\hat{M}_g(t) = F_g \text{ Proj } \left( \hat{M}_g(t), (-\hat{\sigma}_g(t)e^T(t)\hat{P}\hat{K}) \right), \quad i = 1, \ldots, n_3_g$$

(91)

$$\hat{N}_g(t) = G_g \text{ Proj } \left( \hat{N}_g(t), (-\mu(t)e^T(t)\hat{P}\hat{K}\hat{M}_g(t)\hat{\sigma}_g(t)) \right), \quad i = 1, \ldots, n_2_g$$

(92)

where $\hat{\sigma}_f(t) \triangleq \sigma(\hat{N}_f^T(t)\mu(t))$, $\hat{\sigma}_g(t) \triangleq \sigma(\hat{N}_g^T(t)\mu(t))$, the notations $\hat{\sigma}_f'(t) \triangleq \sigma'(\hat{N}_f^T(t)\mu(t))$, $\hat{\sigma}_g'(t) \triangleq \sigma'(\hat{N}_g^T(t)\mu(t))$ are introduced for the Jacobians, evaluated at the estimates and who’s expressions are given in (45) and (46) respectively, $F_f > 0$, $G_f > 0$, $F_g > 0$, $G_g > 0$ specify the learning rates of the two NNs, $\hat{M}_f(t)$, $\hat{N}_f(t)$, $\hat{M}_g(t)$ and $\hat{N}_g(t)$ denote the $i^{th}$ column vectors of $\hat{M}_f(t)$, $\hat{N}_f(t)$, $\hat{M}_g(t)$ and $\hat{N}_g(t)$ respectively, $n_3$ and $n_2$ are the number of outputs and number of hidden layer neurons of each NN and $P$ and $\hat{P}$ are the solutions of
the Lyapunov equation

\[
\begin{align*}
\bar{A}^T P + P \bar{A} &= -Q \\
\bar{A}^T \bar{P} + \bar{P} \bar{A} &= -\bar{Q}
\end{align*}
\]  

(93)  

(94)

for some \( Q > 0 \), \( \bar{Q} > 0 \). The adaptive laws in (89)-(92) are based on a Lyapunov like stability analysis, and partially cancel the uncertainties in the error dynamics, as detailed in the next section.

3.4.2.1 Stability Analysis

In this section we show through Lyapunov’s direct method that the observation errors \( e^T(t) \) and \( \bar{e}^T(t) \) are ultimately bounded. Before proceeding further we present the following remarks:

**Remark 5.** From Property 1, it follows that the NN weights \( \hat{M}_f(t) \), \( \hat{N}_f(t) \), \( \hat{M}_g(t) \) and \( \hat{N}_g(t) \) are confined to the compact sets \( \mathcal{D}_{\hat{M}_f}, \mathcal{D}_{\hat{N}_f}, \mathcal{D}_{\hat{N}_g} \) and \( \mathcal{D}_{\hat{N}_g} \ \forall t \geq 0 \). Denote

\[
\hat{M}_f^* \triangleq \max_{\hat{M}_f \in \mathcal{D}_{\hat{M}_f}} \| \hat{M}_f(t) \| \tag{95}
\]

\[
\hat{N}_f^* \triangleq \max_{\hat{N}_f \in \mathcal{D}_{\hat{N}_f}} \| \hat{N}_f(t) \| \tag{96}
\]

\[
\hat{M}_g^* \triangleq \max_{\hat{M}_g \in \mathcal{D}_{\hat{M}_g}} \| \hat{M}_g(t) \| \tag{97}
\]

\[
\hat{N}_g^* \triangleq \max_{\hat{N}_g \in \mathcal{D}_{\hat{N}_g}} \| \hat{N}_g(t) \| \tag{98}
\]

**Remark 6.** Notice that from the form of the adaptive observer in (36) and the boundedness of the NN weight estimates as ensured by Remark 5, it follows that \( \hat{x}(t) \in \mathcal{D}_x \), \( \forall t \geq 0 \), \( \mathcal{D}_x \) being compact. This in turn guarantees that \( \hat{e}(t) \in \mathcal{D}_e \), \( \forall t \geq 0 \), \( \mathcal{D}_e \) being compact.

Define the following NN weight errors

\[
\bar{M}_f(t) \triangleq \hat{M}_f(t) - M_f, \quad \bar{N}_f(t) \triangleq \hat{N}_f(t) - N_f, \quad \bar{M}_g(t) \triangleq \hat{M}_g(t) - M_g, \quad \bar{N}_g(t) \triangleq \hat{N}_g(t) - N_g
\]

Consider the following composite error vector

\[
\zeta(t) = \begin{bmatrix}
    e^T(t) & \bar{e}^T(t) & \text{vec}\bar{Z}_f^T(t) & \text{vec}\bar{Z}_g^T(t)
\end{bmatrix}^T
\]

(99)
where

$$\tilde{z}_f(t) \triangleq \text{diag} \left[ \tilde{M}_f(t) \quad \tilde{N}_f(t) \right], \quad \tilde{z}_g(t) \triangleq \text{diag} \left[ \tilde{M}_g(t) \quad \tilde{N}_g(t) \right]$$

We proceed to rewrite the observation error dynamics in (38) as

$$\dot{\epsilon}(t) = \tilde{A} \epsilon(t) + \tilde{M}_f^T(t) \sigma(\tilde{N}_f^T(t) \mu(t)) + \tilde{M}_f^T(t) \sigma'(\tilde{N}_f^T(t) \mu(t)) \tilde{N}_f^T(t) \mu(t) + \tilde{w}_f(t)$$

$$-K \left[ \tilde{M}_g^T(t) \sigma(\tilde{N}_g^T(t) \mu(t)) + \tilde{M}_g^T(t) \sigma'(\tilde{N}_g^T(t) \mu(t)) \tilde{N}_g^T(t) \mu(t) + \tilde{w}_g(t) \right]$$

where

$$\tilde{w}_f(t) = M_f^T \left[ \sigma(\tilde{N}_f^T(t) \mu(t)) - \sigma(N_f^T \mu(t)) \right]$$

$$-\tilde{M}_f^T(t) \sigma'(\tilde{N}_f^T(t) \mu(t)) \tilde{N}_f^T(t) \mu(t) - \epsilon_f(\mu(t))$$

$$\tilde{w}_g(t) = M_g^T \left[ \sigma(\tilde{N}_g^T(t) \mu(t)) - \sigma(N_g^T \mu(t)) \right]$$

$$-\tilde{M}_g^T(t) \sigma'(\tilde{N}_g^T(t) \mu(t)) \tilde{N}_g^T(t) \mu(t) - \epsilon_g(\mu(t))$$

The representation in (100) is achieved via the Taylor series expansion of $\sigma(N_f^T \mu(t))$ and $\sigma(N_g^T \mu(t))$ around the estimates $\tilde{N}_f^T(t) \mu(t)$ and $\tilde{N}_g^T(t) \mu(t)$ respectively [122]. Using (34), (35) and (95) - (98), we obtain the bounds on $\tilde{w}_f(t)$ and $\tilde{w}_g(t)$ as:

$$\|\tilde{w}_f(t)\| \leq 2\sqrt{n_{2_f}} M_f^* + s_f \tilde{M}_f^*(\tilde{N}_f^* + N_f^*) \mu^* + \epsilon_f^*$$

$$\|\tilde{w}_g(t)\| \leq 2\sqrt{n_{2_g}} M_g^* + s_g \tilde{M}_g^*(\tilde{N}_g^* + N_g^*) \mu^* + \epsilon_g^*$$

where $s_f \triangleq \frac{a_f n_{2_f}}{4}$ and $s_g \triangleq \frac{a_g n_{2_g}}{4} = \|\tilde{\sigma}'_f(t)\|$, $a_f$ and $a_g$ are the activation potentials while $n_{2_f}$ and $n_{2_g}$ are the number of hidden layer neurons of the two NNs respectively. Also the terms

$$\left[ \tilde{M}_f^T(t) \sigma(\tilde{N}_f^T(t) \mu(t)) + \tilde{M}_f^T(t) \sigma'(\tilde{N}_f^T(t) \mu(t)) \tilde{N}_f^T(t) \mu(t) \right]$$

$$\left[ \tilde{M}_g^T(t) \sigma(\tilde{N}_g^T(t) \mu(t)) + \tilde{M}_g^T(t) \sigma'(\tilde{N}_g^T(t) \mu(t)) \tilde{N}_g^T(t) \mu(t) \right]$$

can be upper bounded as:

$$\left\| \tilde{M}_f^T(t) \sigma(\tilde{N}_f^T(t) \mu(t)) + \tilde{M}_f^T(t) \sigma'(\tilde{N}_f^T(t) \mu(t)) \tilde{N}_f^T(t) \mu(t) \right\| \leq \sqrt{n_{2_f}}(\tilde{M}_f^* + M_f^*)$$

$$+ s_f \tilde{M}_f^*(\tilde{N}_f^* + N_f^*) \mu^* \quad (105)$$

$$\left\| \tilde{M}_g^T(t) \sigma(\tilde{N}_g^T(t) \mu(t)) + \tilde{M}_g^T(t) \sigma'(\tilde{N}_g^T(t) \mu(t)) \tilde{N}_g^T(t) \mu(t) \right\| \leq \sqrt{n_{2_g}}(\tilde{M}_g^* + M_g^*)$$

$$+ s_g \tilde{M}_g^*(\tilde{N}_g^* + N_g^*) \mu^* \quad (106)$$
Similarly the observer error dynamics (40) can be written as:

\[
\dot{\tilde{e}}(t) = \tilde{A}\tilde{e}(t) - \left[ M_f^T(t)\sigma(N_f^T(t)\mu(t)) + M_g^T(t)\sigma'(N_g^T(t)\mu(t)) N_f^T(t)\mu(t) + \tilde{w}_f(t) \right] \\
+ K \left[ M_g^T(t)\sigma(N_g^T(t)\mu(t)) + M_g^T(t)\sigma'(N_g^T(t)\mu(t)) N_g^T(t)\mu(t) + \tilde{w}_g(t) \right]
\]  

(107)

Recall that (34) and (35) introduce the set \( \mathcal{D} \) over which the NN approximation is valid. Introduce the largest ball \( \mathcal{B}_R \triangleq \left\{ \zeta \mid \|\zeta\| \leq R \right\} \), \( R > 0 \), that lies in

\[
\Omega_\zeta \triangleq \left\{ (e, \tilde{e}, \bar{M}_f, \bar{N}_f, \bar{M}_g, \bar{N}_g) \in \mathcal{R}^{n_e} \times \mathcal{R}^{n_e} \times \mathcal{R}^{n_f} \times \mathcal{R}^{n_g} \times \mathcal{R}^{n_f \times n_g} \times \mathcal{R}^{n_g \times n_f} \\
\times \mathcal{R}^{n_g \times m} \times \mathcal{R}^{n_f \times n_g} : (x, z, \tilde{x}, \tilde{e}) \in \mathcal{D}_x \times \mathcal{D}_z \times \mathcal{D}_\tilde{x} \times \mathcal{D}_\tilde{e} \right\}
\]  

(108)

Let \( \alpha \) be the minimum value of

\[
V(\zeta, z) \triangleq \zeta^T T \zeta
\]  

(109)

on the edge of \( \mathcal{B}_R \), where

\[
T \triangleq \begin{bmatrix}
P & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{P} & 0 & 0 & 0 & 0 \\
0 & 0 & F_f^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & G_f^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & F_g^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & G_g^{-1}
\end{bmatrix}
\]

Then

\[
\alpha \triangleq \min_{\|\zeta\|=R} V(\zeta, z) = R^2 \lambda_{\min}(T)
\]

where \( \lambda_{\min}(T) \) is the minimum eigenvalue of \( T \). Introduce the set

\[
\mathcal{D}_\alpha = \left\{ \zeta \in \mathcal{B}_R \mid V(\zeta, z) \leq \alpha \right\}
\]

**Assumption 2.** Let

\[
R > \gamma_0 \sqrt{\frac{\lambda_{\max}(T)}{\lambda_{\min}(T)}} \geq \gamma_0
\]  

(110)

where \( \lambda_{\max}(T) \) is the maximum eigenvalue of \( T \) and

\[
\gamma_0 = \max \left( \sqrt{\frac{\rho_0}{\lambda_{\min}(Q) - 1}}, \sqrt{\frac{\rho_0}{\lambda_{\min}(Q) - 1}}, \bar{M}_f^*, \bar{N}_f^*, \bar{M}_g^*, \bar{N}_g^* \right)
\]  

(111)
where

\[
\begin{align*}
\rho_0 &= \rho_1^2 + \rho_2^2 \\
\rho_1 &= \| \tilde{P} \bar{K} \| \| \bar{w}_g \| + \lambda_{\max}(\tilde{P}) \| \bar{w}_f \| + \left( \lambda_{\max}(P) + \lambda_{\max}(\tilde{P}) \right) \kappa_1 \\
\rho_2 &= \lambda_{\max}(P) \| \bar{w}_f \| + \| PK \| \| \bar{w}_g \| + \| PK + \tilde{P} \bar{K} \| \kappa_2 \\
\kappa_1 &= \| \tilde{M}_f^T \sigma(\hat{N}_f^T \mu) + \tilde{M}_f^T \sigma'(\hat{N}_f^T \mu) \hat{N}_f^T \mu \| \\
\kappa_2 &= \| \tilde{M}_g^T \sigma(\hat{N}_g^T \mu) + \tilde{M}_g^T \sigma'(\hat{N}_g^T \mu) \hat{N}_g^T \mu \| \\
\tilde{M}_f^* &\triangleq \tilde{M}_f^* + M_f^* \\
\tilde{N}_f^* &\triangleq \tilde{N}_f^* + N_f^* \\
\tilde{M}_g^* &\triangleq \tilde{M}_g^* + M_g^* \\
\tilde{N}_g^* &\triangleq \tilde{N}_g^* + N_g^*
\end{align*}
\] (112)

with \( \| \bar{w}_f \|, \| \bar{w}_g \|, \kappa_1 \) and \( \kappa_2 \) upper bounded as shown in (103)-(106). Further assume that \( P \) and \( \tilde{P} \) in (93) and (94) respectively have been computed with \( \lambda_{\min}(Q) > 1 \) and \( \lambda_{\min}(\tilde{Q}) > 1 \).

**Theorem 8.** Let Assumption 2 hold. Then, if the initial error \( \zeta_0 = \zeta(0) \) belongs to the set \( \mathcal{D}_\alpha \) as shown in Fig. 5, the observer in (36) along with (39) and (89)-(92), guarantees that \( \zeta(t) \) is ultimately bounded.

**Proof.** Consider the function \( V(\zeta(t), z(t)) \), introduced in (109), as a candidate Lyapunov function for the dynamics in (38), (40) and (89)-(92)

\[
V(\zeta(t), z(t)) = e^T(t) P e(t) + \bar{e}^T(t) \tilde{P} \bar{e}(t) + \text{tr} \left( \tilde{M}_f^T(t) F_f^{-1} \tilde{M}_f(t) \right) + \text{tr} \left( \tilde{N}_f^T(t) G_f^{-1} \tilde{N}_f(t) \right) \\
+ \text{tr} \left( \tilde{M}_g^T(t) F_g^{-1} \tilde{M}_g(t) \right) + \text{tr} \left( \tilde{N}_g^T(t) G_g^{-1} \tilde{N}_g(t) \right)
\] (113)

Notice that \( V(\zeta(t), z(t)) \) satisfies (16) with

\[
\begin{align*}
\varphi(t) &= \zeta(t) \\
\alpha(\| \varphi(t) \|) &= \lambda_{\min}(T) \| \varphi(t) \|^2 \\
\beta(\| \varphi(t) \|) &= \lambda_{\max}(T) \| \varphi(t) \|^2
\end{align*}
\] (114)

Differentiating (113) once with respect to time along (93), (94), (100) and (107), and recalling
that
\[
\begin{align*}
\bar{\sigma}_f(t) & \triangleq \sigma(\hat{N}_f^T(t)\bar{\mu}(t)), \quad \bar{\sigma}_g(t) \triangleq \sigma(\hat{N}_g^T(t)\bar{\mu}(t)) \\
\bar{\sigma}'_f(t) & \triangleq \sigma'(\hat{N}_f^T(t)\bar{\mu}(t)), \quad \bar{\sigma}'_g(t) \triangleq \sigma'(\hat{N}_g^T(t)\bar{\mu}(t))
\end{align*}
\] (115)
we obtain
\[
\dot{V}(\zeta(t), z(t)) = -e^T(t)Qe(t) - \bar{e}^T(t)\bar{Q}\bar{e}(t) + 2e^T(t)P\bar{\omega}_f(t) - 2e^T(t)PK\bar{\omega}_g(t) - 2\bar{e}^T(t)\bar{P}\bar{\omega}_f(t) + 2\bar{e}^T(t)\bar{P}\bar{K}\bar{\omega}_g(t)
\]
\[
+ 2\left(\bar{e}^T(t) - \bar{\bar{e}}^T(t)\right)P\left(M_f^T(t)\bar{\sigma}_f(t) + \hat{M}_f^T(t)\bar{\sigma}'_f(t)\hat{N}_f^T(t)\bar{\mu}(t)\right)
- 2\left(\bar{e}^T(t) - \bar{\bar{e}}^T(t)\right)PK\left(M_g^T(t)\bar{\sigma}_g(t) + \hat{M}_g^T(t)\bar{\sigma}'_g(t)\hat{N}_g^T(t)\bar{\mu}(t)\right)
- 2\bar{e}^T(t)\bar{P}\left(M_f^T(t)\bar{\sigma}_f(t) + \hat{M}_f^T(t)\bar{\sigma}'_f(t)\hat{N}_f^T(t)\bar{\mu}(t)\right)
+ 2\bar{e}^T(t)\bar{P}\bar{K}\left(M_g^T(t)\bar{\sigma}_g(t) + \hat{M}_g^T(t)\bar{\sigma}'_g(t)\hat{N}_g^T(t)\bar{\mu}(t)\right)
+ 2\text{tr}\left(M_f^T(t)F_f^{-1}\dot{M}_f(t)\right) + 2\text{tr}\left(N_f^T(t)G_f^{-1}\dot{N}_f(t)\right)
+ 2\text{tr}\left(M_g^T(t)F_g^{-1}\dot{M}_g(t)\right) + 2\text{tr}\left(N_g^T(t)G_g^{-1}\dot{N}_g(t)\right)
\] (117)

The terms in (117) can be rearranged as
\[
\dot{V}(\zeta(t), z(t)) = -e^T(t)Qe(t) - \bar{e}^T(t)\bar{Q}\bar{e}(t) + 2e^T(t)P\bar{\omega}_f(t) - 2e^T(t)PK\bar{\omega}_g(t) - 2\bar{e}^T(t)\bar{P}\bar{\omega}_f(t) + 2\bar{e}^T(t)\bar{P}\bar{K}\bar{\omega}_g(t)
\]
\[
+ 2\bar{e}^T(t)\bar{P}\bar{K}\bar{\omega}_g(t)
+ 2\bar{e}^T(t)\bar{P}\left(M_f^T(t)\bar{\sigma}_f(t) + \hat{M}_f^T(t)\bar{\sigma}'_f(t)\hat{N}_f^T(t)\bar{\mu}(t)\right)
- 2\bar{e}^T(t)\bar{P}\left(M_g^T(t)\bar{\sigma}_g(t) + \hat{M}_g^T(t)\bar{\sigma}'_g(t)\hat{N}_g^T(t)\bar{\mu}(t)\right)
- 2\bar{e}^T(t)PK\left(M_g^T(t)\bar{\sigma}_g(t) + \hat{M}_g^T(t)\bar{\sigma}'_g(t)\hat{N}_g^T(t)\bar{\mu}(t)\right)
+ 2\bar{e}^T(t)PK\left(M_g^T(t)\bar{\sigma}_g(t) + \hat{M}_g^T(t)\bar{\sigma}'_g(t)\hat{N}_g^T(t)\bar{\mu}(t)\right)
- 2\bar{e}^T(t)\bar{P}\left(M_f^T(t)\bar{\sigma}_f(t) + \hat{M}_f^T(t)\bar{\sigma}'_f(t)\hat{N}_f^T(t)\bar{\mu}(t)\right)
+ 2\bar{e}^T(t)\bar{P}\bar{K}\left(M_g^T(t)\bar{\sigma}_g(t) + \hat{M}_g^T(t)\bar{\sigma}'_g(t)\hat{N}_g^T(t)\bar{\mu}(t)\right)
+ 2\text{tr}\left(M_f^T(t)F_f^{-1}\dot{M}_f(t)\right) + 2\text{tr}\left(N_f^T(t)G_f^{-1}\dot{N}_f(t)\right)
+ 2\text{tr}\left(M_g^T(t)F_g^{-1}\dot{M}_g(t)\right) + 2\text{tr}\left(N_g^T(t)G_g^{-1}\dot{N}_g(t)\right)
\] (118)
Substituting (89)-(92) into (118) and rearranging the terms, we obtain

\[
\dot{V}(\zeta(t), z(t)) = -\mathbf{e}^T(t)Q\mathbf{e}(t) - \dot{\mathbf{e}}^T(t)\dot{\mathbf{e}}(t) + 2\dot{\mathbf{e}}^T(t)P\ddot{\mathbf{w}}_f(t) + 2\dot{\mathbf{e}}^T(t)\dot{\mathbf{P}}\dot{\mathbf{K}}\ddot{\mathbf{w}}_g(t)
\]

\[
= -2\mathbf{e}^T(t)PK\ddot{\mathbf{w}}_g(t) - 2\dot{\mathbf{e}}^T(t)\dot{\mathbf{P}}\ddot{\mathbf{w}}_f(t) - 2\dot{\mathbf{e}}^T(t)P\left(\dddot{\mathbf{M}}_f(t)\dddot{\sigma}_f(t) + \dddot{\mathbf{M}}_g(t)\dddot{\sigma}_g(t)\right)N_f^T(t)\mu(t)
\]

\[
+ 2\dot{\mathbf{e}}^T(t)PK\left(\dddot{\mathbf{M}}_f(t)\dddot{\sigma}_f(t) + \dddot{\mathbf{M}}_g(t)\dddot{\sigma}_g(t)\right)N_g^T(t)\mu(t)
\]

\[
+ 2\mathbf{e}^T(t)\dot{\mathbf{P}}K\left(\dddot{\mathbf{M}}_f(t)\dddot{\sigma}_f(t) + \dddot{\mathbf{M}}_g(t)\dddot{\sigma}_g(t)\right)N_f^T(t)\mu(t)
\]

\[
+ 2\mathbf{e}^T(t)PK\left(\dddot{\mathbf{M}}_f(t)\dddot{\sigma}_f(t) + \dddot{\mathbf{M}}_g(t)\dddot{\sigma}_g(t)\right)N_g^T(t)\mu(t)
\]

Using Property 2, upper bounding the derivative of \(V(\zeta(t), z(t))\) and rearranging the terms, we obtain

\[
\dot{V}(\zeta(t), z(t)) \leq -\lambda_{\min}(Q)\|\mathbf{e}(t)\|^2 - \lambda_{\min}(\dot{Q})\|\dot{\mathbf{e}}(t)\|^2
\]

\[
+ 2\|\mathbf{e}(t)\|\left(\lambda_{\max}(P)\|\ddot{\mathbf{w}}_f(t)\| + \|PK\|\|\ddot{\mathbf{w}}_g(t)\|
\]

\[
+ \|PK + \dot{\mathbf{P}}K\|\|\dddot{\mathbf{M}}_f(t)\dddot{\sigma}_f(t) + \dddot{\mathbf{M}}_g(t)\dddot{\sigma}_g(t)\|N_f^T(t)\mu(t)\|
\]

\[
+ 2\|\dot{\mathbf{e}}(t)\|\left(\|PK\|\|\ddot{\mathbf{w}}_g(t)\| + \lambda_{\max}(P)\|\ddot{\mathbf{w}}_f(t)\|
\]

\[
+ (\lambda_{\max}(P) + \lambda_{\max}(\dot{P}))\|\dddot{\mathbf{M}}_f(t)\dddot{\sigma}_f(t) + \dddot{\mathbf{M}}_g(t)\dddot{\sigma}_g(t)\|N_f^T(t)\mu(t)\|
\]

Using the notations in (112), the Lyapunov derivative can be expressed as

\[
\dot{V}(\zeta(t), z(t)) \leq -\lambda_{\min}(Q)\|\mathbf{e}(t)\|^2 - \lambda_{\min}(\dot{Q})\|\dot{\mathbf{e}}(t)\|^2 + 2\|\mathbf{e}(t)\|\rho_2 + 2\|\dot{\mathbf{e}}(t)\|\rho_1
\]

Completion of squares on \(2\|\mathbf{e}(t)\|\rho_2\) and \(2\|\dot{\mathbf{e}}(t)\|\rho_1\) yields

\[
\dot{V}(\zeta(t), z(t)) \leq -\lambda_{\min}(Q)\|\mathbf{e}(t)\|^2 - \left(\|\mathbf{e}(t)\| - \rho_2\right)^2 + \|\mathbf{e}(t)\|^2 + \rho_2^2
\]

\[
- \lambda_{\min}(\dot{Q})\|\dot{\mathbf{e}}(t)\|^2 - \left(\|\dot{\mathbf{e}}(t)\| - \rho_1\right)^2 + \|\dot{\mathbf{e}}(t)\|^2 + \rho_1^2
\]
Hence by regrouping terms and further upper bounding the Lyapunov derivative in (121), we obtain

\[ \dot{V}(\zeta(t), z(t)) \leq -\|e(t)\|^2(\lambda_{\text{min}}(Q) - 1) - \|\dot{e}(t)\|^2(\lambda_{\text{min}}(\bar{Q}) - 1) + \rho_0 \]  

(122)

Thus either of the following conditions

\[ \|e\| > \sqrt{\frac{\rho_0}{\lambda_{\text{min}}(Q) - 1}} \]  

(123)

\[ \|\dot{e}\| > \sqrt{\frac{\rho_0}{\lambda_{\text{min}}(\bar{Q}) - 1}} \]  

(124)

will render \( \dot{V}(\zeta(t), z(t)) < 0 \) outside the compact set

\[ \mathcal{B}_{\gamma_0} = \left\{ \zeta \in \mathcal{B}_R \mid \|\zeta\| \leq \gamma_0 \right\} \]  

(125)

Notice from (110), that \( \mathcal{B}_{\gamma_0} \subseteq \mathcal{B}_R \). Let \( \Gamma \) be the maximum value of the function \( V(\zeta, z) \) on the edge of \( \mathcal{B}_{\gamma_0} \) such that

\[ \Gamma \overset{\Delta}{=} \max_{\|\zeta\| = \gamma_0} V(\zeta, z) = \gamma_0^2 \lambda_{\text{max}}(T) \]  

(126)

Introduce the level set of \( V(\zeta, z) \), that touches the ball \( \mathcal{B}_{\gamma_0} \),

\[ \mathcal{D}_{\gamma_0} = \left\{ \zeta \mid V(\zeta, z) = \Gamma \right\} \]  

(127)

The condition in (110) ensures that \( \mathcal{D}_{\gamma_0} \subseteq \mathcal{D}_\alpha \). Thus, if the initial error \( \zeta_0 = \zeta(0) \) belongs to \( \mathcal{D}_{\alpha} \) then it follows from Theorem 3 that the solution \( (\zeta(t), z(t)) \) to (32), (38), (40) and (89)-(92) is ultimately bounded with respect to \( \zeta(t) \) uniformly in \( z_0 \). In addition since (32) is input-to-state stable with \( \bar{x}(t) \) viewed as the input, it follows from Theorem 4 that the solution \( z(t) \) to (32) is ultimately bounded, thus completing the proof.

\[ \square \]

### 3.5 Comments on the learning rate

1. The NN update laws in (41)-(44) consist of a modified gradient algorithm with the standard \( \sigma- \) modification term to prevent parameter drift [124].
2. Assumption 2 may be interpreted as placing both upper and lower bounds on the adaptation gains. Define

$$\gamma \triangleq \max \left( \lambda_{\text{max}}(F_f), \lambda_{\text{max}}(G_f), \lambda_{\text{max}}(F_g), \lambda_{\text{max}}(G_g) \right)$$  \hspace{1cm} (128)

$$\underline{\gamma} \triangleq \min \left( \lambda_{\text{min}}(F_f), \lambda_{\text{min}}(G_f), \lambda_{\text{min}}(F_g), \lambda_{\text{min}}(G_g) \right)$$  \hspace{1cm} (129)

$$\bar{\lambda} \triangleq \max \left( \lambda_{\text{max}}(P), \lambda_{\text{max}}(\tilde{P}) \right)$$  \hspace{1cm} (130)

$$\underline{\lambda} \triangleq \min \left( \lambda_{\text{min}}(P), \lambda_{\text{min}}(\tilde{P}) \right)$$  \hspace{1cm} (131)

where $\lambda(\cdot)$ denotes the eigenvalue. Then an upper bound for the adaptation gains results when $\bar{\lambda}_\gamma > 1$ and $\underline{\gamma}_\gamma > 1$, for which the relation in (110) reduces to $\gamma < R^2/(\gamma_0^2 \bar{\lambda})$. A lower bound for the adaptation gains results when $\bar{\lambda}_\gamma < 1$ and $\underline{\gamma}_\gamma < 1$, for which the relation in (110) reduces to $\gamma > R^2/(\gamma_0^2 \underline{\lambda})$.

### 3.6 Simulation Results

In this section we show some simulation results that were carried out on two different kinds of systems using NN adaptation based on $\sigma-$ modification and projection. In the first case we considered a nonlinear system with known dimension, but with unmodeled disturbances and nonlinear measurement. For this case we augmented two NNs, one for the process dynamics and one for the measurement, where the NNs are trained via $\sigma-$ modification based adaptation. In the second case we considered a nonlinear system with unmodeled dynamics (hence the system has unknown dimension) and nonlinear measurement. The NNs here are trained via projection based adaptation.

#### 3.6.1 System with Unmodeled Disturbances

Consider the following nonlinear process:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - 10 \sin x_1 - 2(x_1^2 - 1)x_2 + d_1$$
$$y = x_1 + \varepsilon \sin(2x_1) + d_2$$  \hspace{1cm} (132)

where $\varepsilon$ scales the nonlinearity in the measurement. The initial conditions have been set to $x_1(0) = 0.5, x_2(0) = 0.1$. The disturbances $d_1, d_2$ are defined to be the following deterministic
processes and are shown in Fig. 6

\[
d_1 = \begin{cases} 
5 \sin(0.5t) + 0.1 \sin(t) + \sin(0.1t) & t \in [0, 10] \\
0, & t \in [10, 15] \\
15, & t \in [15, 20]
\end{cases}
\]

(133)

\[
d_2 = 0.5 \sin t + 0.2 \sin(0.1t) + 0.2 \sin(2t)
\]

(134)

The initial states for the observer are chosen as: \( \hat{x}_1(0) = 0, \hat{x}_2(0) = 0 \). The linear model used in the design of the linear observer is:

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1, \\
y_1 = x_1
\]

(135)

The matrix \( K \) is chosen to place the poles of \( \tilde{A} \) in (38) at \((-1.5, -0.5)\). The observer dynamics in (39) were designed so that the poles of \( \tilde{A} = \tilde{A} - \tilde{K}C \) in (40) are 6 times faster.

The matrices \( Q \) and \( \bar{Q} \) in (47) and (48) respectively are chosen as follows

\[
Q = \begin{bmatrix} 34.1 & 0 \\ 0 & 3.1 \end{bmatrix}, \quad \bar{Q} = \begin{bmatrix} 40.3 & 0 \\ 0 & 3.1 \end{bmatrix}
\]

(136)

For these simulations 10 hidden neurons were implemented and the sigmoidal basis function \( \sigma(x) = \frac{1}{1+e^{-x}} \). The NN learning rates in (41)-(44) were set to \( F_f = 250I, \quad G_f = I, \quad F_g = 0.5I, \quad G_g = 0.5I \) with the sigma modification gains in (41)-(44) set to \( k_f = k_g = 1 \).

Fig. 7 shows the performance of the observer (36), in the absence of the two NNs, when applied to the system in (132) with \( \varepsilon = 0, d_2 = 0 \). In this case, the second NN is not needed [49]. Fig. 8 shows the performance of the observer in (36) with \( \dot{M}_f(t) \neq 0 \) and \( \dot{M}_g(t) = 0 \). Setting \( \varepsilon = 0.25 \) and introducing \( v \) in the measurement, gives rise to a degradation in the performance of the observer, as shown in Fig. 9. The performance of the observer defined in (36), \( \dot{M}_f(t) \neq 0 \) and \( \dot{M}_g(t) \neq 0 \), is shown in Fig. 10.

From Fig. 7 it is evident that for a nonlinear system with unmodeled disturbances, a linear observer goes through perceptible amount of performance degradation. Fig. 8 shows that when the measurement is linear then it is sufficient to design an NN to approximate the modelling errors in the process dynamics and augment this NN to the existing linear
Figure 6: Disturbance applied to the system dynamics.

observer to achieve better performance. From Fig. 9 it is clear that when nonlinearities are introduced into the measurement, just augmenting the process dynamics with an NN does not help to improve the performance of the linear observer. Fig. 10 shows that by augmenting the measurement with a second NN, the performance of the linear observer is greatly improved.

3.6.2 System with Internal Unmodeled Dynamics

We consider the following nonlinear process consisting of a modified Van der Pol oscillator, containing an unmodeled mode and unmodeled disturbances:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - 0.2 (x_1^2 + 1) x_2 + w + d_1 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -x_3 - 0.5 x_4 + x_1 \\
y &= x_1 + 0.3 x_3 + 0.5 \sin x_3 + 0.3 \sin (2x_3) + \nu + d_2
\end{align*}
\] (137)
with the initial conditions $x_1(0) = 0.5$, $x_2(0) = 0.1$, $x_3(0) = 0$, $x_4(0) = 0$. The process and measurement disturbances, $d_1$ and $d_2$, are given by

$$
\begin{align*}
    d_1 &= \sin t + 0.1 \sin 0.25t + 0.25 \sin 10t \\
    d_2 &= 0.5 \sin t + 0.1 \sin 0.1t + 0.3 \sin 5t
\end{align*}
$$

The states $x_3$ and $x_4$ represent the internal unmodeled dynamics. It is obvious that with $x_1 = 0$, the unique equilibrium $x_3 = 0$, $x_4 = 0$ of these internal dynamics is globally exponentially stable.

### 3.6.3 Steady State Kalman Filter Design

The linear model with process and sensor noise used in the design of the steady state Kalman filter (SSKF) is

$$
\begin{align*}
    \dot{x}_1 &= x_2, \quad \dot{x}_2 = -x_1 - 0.2x_2 + w \\
    y &= x_1 + \nu
\end{align*}
$$
Figure 8: SSKF estimation Performance with one NN: \( \varepsilon = 0, \nu = 0 \).

where the noise intensity matrices for the design of the Kalman filter are

\[
E[\mathbf{w}(t)\mathbf{w}^T(\tau)] = Q_0 \delta(t - \tau), \quad E[\mathbf{\nu}(t)\mathbf{\nu}^T(\tau)] = R \delta(t - \tau), \quad E[\mathbf{w}(t)\mathbf{\nu}^T(\tau)] = 0 \tag{140}
\]

with \( Q_0 = 1 \) and \( R = 0.0001 \). An SSKF design of \( K \) results in \( \tilde{A} \) having poles at \(-7.04\pm7.11j \).

3.6.4 Neural Network Design

We implemented 10 hidden neurons with the sigmoidal basis function \( \sigma(x) = \frac{1}{1+e^{-\alpha x}} \) where \( \alpha \) is the activation potential and was set to 1. The NN learning rates in (89)-(92) were set to \( F_f = G_f = 0.4I, \quad F_g = 5I, \quad G_g = I \) and the number of delays, \( \mu(t) \), to the NN approximator was chosen to be 5.

Fig. 11 shows the estimation performance of the SSKF designed to account for the nonlinearities in the process dynamics of (137). Fig. 12 shows the estimation performance when the SSKF is augmented with two NNs. Fig. 13 shows the state estimation error with the SSKF and the SSKF augmented with two NNs. From Fig. 11 it can be clearly seen that the SSKF suffers from drastic performance degradation. This behavior is expected
Figure 9: SSKF estimation Performance with one NN: $\varepsilon = 0.25$.

since the plant under consideration is nonlinear with unmodeled dynamics and unmodeled disturbances. However, Fig. 12 shows that the SSKF augmented with the NNs produces nearly unbiased estimates for $x_1$ and $x_2$.

3.7 Conclusions

A new approach is proposed for adaptive state estimation of nonlinear processes. The approach involves augmenting a linear observer with two nonlinearly parameterized neural networks. The adaptive laws employ an error signal that is generated by a second linear observer of the error dynamics for the nominal system. Boundedness of the NN weights is ensured through the use of a projection and boundedness of error signals is shown through Lyapunov’s direct method. The approach is applicable to a broad class of uncertain multivariable nonlinear systems, including systems with unmodeled dynamics, unmodeled disturbance processes and imperfect nonlinear measurements.
Figure 10: SSKF estimation Performance with two NNs: $\varepsilon = 0.25$.

Figure 11: SSKF performance without NN augmentation.
Figure 12: SSKF performance with augmentation of 2 NN’s.

Figure 13: State estimation errors - SSKF and SSKF + NNs.
CHAPTER 4

ADAPTIVE AUGMENTATION OF NONLINEAR TIME VARYING SYSTEMS

In chapter 3 the problem of augmenting a linear time invariant observer (LTI) with an adaptive NN element was addressed. However in applications such as tracking a randomly maneuvering target, we do not have a good model to capture the behavior of the target. These kinds of problems are nonlinear and time varying, with rapidly changing dynamics. Hence the design of an LTI observer, which is based on a nominal model, is not a preferred method for such kinds of applications. It is in these applications that extended Kalman filters (EKF}s) are extensively used. In this chapter, a method is presented for augmenting an EKF with an adaptive element. The resulting estimator provides robustness to parameter uncertainty and unmodeled dynamics. The design of the adaptive element employs a linearly parameterized neural network (NN). The NN weights are adjusted on line using the filter error residuals. Boundedness of signals is proven using Lyapunov’s direct method. Simulations illustrate the theoretical results.

Before presenting the main results in this chapter, it is important to understand certain definitions and theorems that are used to arrive at the main result. These definitions and theorems are presented in the following section.

4.1 Mathematical Preliminaries

Consider the nonlinear dynamical system

\[ \dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0 \]  \hspace{1cm} (141)

where \( f : [0, \infty) \times \mathcal{D} \to \mathbb{R}^n \) is continuously differentiable, \( \mathcal{D} = \{ x \in \mathbb{R}^n \mid \|x\| < r \} \), and the Jacobian matrix \( \frac{\partial f}{\partial x} \) is bounded and Lipschitz on \( \mathcal{D} \), uniformly in \( t \).

**Theorem 9.** \([116]\) Let \( x = 0 \) be an equilibrium point for the nonlinear system in (141).
Let $k$, $\lambda$ and $r_0$ be positive constants with $r_0 < \frac{1}{k}$. Let $D_0 = \{ x \in \mathbb{R}^n \mid \|x\| < r_0 \}$. Assume that the trajectory of the system satisfies

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall \ x(t_0) \in D_0, \ \forall \ t \geq t_0 \geq 0 \tag{142}$$

Then, there is a $C^1$ function $V : [0, \infty) \times D_0 \rightarrow \mathbb{R}$ that satisfies the inequalities

$$c_1\|x(t)\|^2 \leq V(t, x(t)) \leq c_2\|x(t)\|^2$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x(t)) \leq -c_3\|x(t)\|^2$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq c_4\|x(t)\|^2 \tag{143}$$

for some positive constants $c_1$, $c_2$, $c_3$ and $c_4$.

**Definition 9.** [116] The equilibrium point $x = 0$ of the nonlinear system (141) is exponentially stable if there exists positive constants $c$, $k$ and $\lambda$, such that

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \quad \|x(t_0)\| < c \tag{144}$$

**Definition 10.** [116] The nonlinear system

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0 \tag{145}$$

where $x \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$, is said to be locally input-to-state stable if there exists a class $\mathcal{K}\mathcal{L}$ function $\beta$, a class $\mathcal{K}$ function $\gamma$, and positive constants $k_0$ and $k_1$ such that for any initial state $x(t_0)$ with $\|x(t_0)\| < k_0$ and any input $u(t)$ with $\sup_{t \geq t_0} \|u(t)\| < k_1$, the solution $x(t)$ exists $\forall t \geq t_0 \geq 0$ and satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma \left( \sup_{t_0 \leq \tau \leq t} \|u(t)\| \right)$$

**Remark 7.** Definition 10 guarantees that for a bounded input $u(t)$, the state $x(t)$ will be bounded.

**Theorem 10.** [113] Given arbitrary $\epsilon^* > 0$, a continuous function $f(x)$, $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and a suitably chosen set of basis functions $\sigma(\cdot)$, defined on a compact set $x \in D \subset \mathbb{R}^n$, there exists a set of bounded constant weights $M$, such that the following representation holds $\forall x \in D$:

$$f(x) = M^T \sigma(x) + \epsilon(x), \quad \|\epsilon(x)\| \leq \epsilon^*.$$
Here, the structure $M^T \sigma(x)$ is called a linearly parameterized neural network, $\sigma(\cdot)$ is a vector of the basis functions $\sigma(\cdot)$, its $i^{th}$ component being defined as $[\sigma(x)]_i = \sigma_i(x)$, $|\sigma_i(\cdot)| \leq 1$, and $\epsilon(x)$ is the function reconstruction error. The next theorem is an existence theorem which states that it is possible to approximate the unknown dynamics of a system by using a finite sample of the available measurement. The NN here is parameterized in a linear manner.

**Theorem 11.** [118] Assume that an $n$-dimensional state vector $x(t)$ of an observable time-invariant system

$$
\dot{x}(t) = f(x(t)) \\
y(t) = h(x(t))
$$

(146)

evolves on an $n$-dimensional ball of radius $\bar{r}$ in $\mathcal{R}^n$, $\mathcal{B}_r = \{ x(t) \in \mathcal{R}^n, \|x(t)\| \leq \bar{r} \}$. Also assume that the system output $y(t) \in \mathcal{R}^m$ and its derivatives up to the order $(n - 1)$ are bounded. Then given arbitrary $\epsilon^* > 0$, there exists a set of bounded weights $M$ and a positive time delay $d > 0$, such that the function $f(x(t))$ in (146) can be approximated over the compact set $\mathcal{B}_r$ by a linearly parameterized NN

$$
f(x(t)) = M^T \sigma(\mu(t)) + \epsilon(\mu(t)), \|M\|_F \leq M^*, \|\epsilon(\mu(t))\|_F \leq \epsilon^*
$$

using the input vector:

$$
\mu(y(t), d) = \left[ \Delta_d^{(0)} y^T(t) \cdots \Delta_d^{(n-1)} y^T(t) \right]^T \in \mathcal{R}^{nm}, \|\mu(t)\| \leq \mu^*
$$

(147)

where the finite difference quotients are given by

$$
\Delta_d^{(0)} y^T(t) \triangleq y^T(t) \\
\Delta_d^{(1)} y^T(t) \triangleq \frac{y^T(t) - y^T(t - d)}{d} \\
\vdots \\
\Delta_d^{(k)} y^T(t) \triangleq \frac{\Delta_d^{(k-1)} y^T(t) - \Delta_d^{(k-1)} y^T(t - d)}{d}, \quad k = 1, 2, \cdots
$$

(148)

and $\mu^* > 0$ is a uniform bound on $\mathcal{B}_r$. 

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Definition 11. [116] The solution of (141) is

1. uniformly bounded if there exists a positive constant $c$, independent of $t_0 \geq 0$, and for every $a \in (0, c)$, there is $\beta = \beta(a) > 0$, independent of $t_0$, such that

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq \beta, \quad \forall \, t \geq t_0$$  \hspace{1cm} (149)

2. uniformly ultimately bounded with ultimate bound $b$ if there exists positive constants $b$ and $c$, independent of $t_0 \geq 0$, and for every $a \in (0, c)$, there is $T = T(a, b) \geq 0$, independent of $t_0$, such that

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b, \quad \forall \, t \geq t_0 + T$$  \hspace{1cm} (150)

Theorem 12. [116] Let $D \subset \mathbb{R}^n$ be a domain that contains the origin and $V : [0, \infty) \times D \rightarrow \mathbb{R}$ be a continuously differentiable function such that

$$\alpha_1(\|x(t)\|) \leq V(t, x(t)) \leq \alpha_2(\|x(t)\|)$$  \hspace{1cm} (151)

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x(t)) \leq W_3(x(t)), \quad \forall \, \|x(t)\| \geq \mu > 0$$  \hspace{1cm} (152)

$\forall \, t \geq 0$ and $\forall \, x \in D$, where $\alpha_1$ and $\alpha_2$ are class $\mathcal{K}$ functions and $W_3(x(t))$ is a continuous positive definite function. Take $r > 0$ such that $B_r \subset D$ and suppose that

$$\mu < \alpha_2^{-1}(\alpha_1(r))$$  \hspace{1cm} (153)

Then, there exists a class $\mathcal{K}$ function $\beta$ and for every initial state $x(t_0)$, satisfying

$$\|x(t_0)\| \leq \alpha_2^{-1}(\alpha_1(r))$$  \hspace{1cm} (154)

there is a $T \geq 0$, dependent on $x(t_0)$ and $\mu$, such that the solution of the differential equation (141) satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall \, t_0 \leq t \leq t_0 + T$$  \hspace{1cm} (155)

$$\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\mu)), \quad \forall \, t \geq t_0 + T$$  \hspace{1cm} (156)
4.2 Problem Formulation

Let the dynamics of an observable and bounded nonlinear process be given by the following equations \(^1\):

\[
\begin{align*}
\dot{x}(t) &= f(x(t)) + B_1 g_1(x(t), z_1(t)), \quad x(0) = x_0 \\
\dot{z}_1(t) &= f_{z_1}(x(t), z_1(t), z_2(t)), \quad z_1(0) = z_{10} \\
\dot{z}_2(t) &= f_{z_2}(x(t), z_1(t), z_2(t)), \quad z_1(0) = z_{10} \\
y(t) &= C x(t)
\end{align*}
\]

(157)

where \(x \in \mathcal{D}_x \subseteq \mathcal{R}^{n_x}\), \(z_1 \in \mathcal{D}_{z_1} \subseteq \mathcal{R}^{n_{z_1}}\) and \(z_2 \in \mathcal{D}_{z_2} \subseteq \mathcal{R}^{n_{z_2}}\) are the states of the system, \(\mathcal{D}_x, \mathcal{D}_{z_1}\) and \(\mathcal{D}_{z_2}\) are compact sets, \(f(x) : \mathcal{D}_x \to \mathcal{R}^{n_x}\) is a known smooth function which can be expressed as a Taylor series expansion for all the values of \(x\) in the domain of interest \(\mathcal{D}_x\), \(B_1\) and \(C\) are known matrices, \(f_{z_1}(x, z_1, z_2) : \mathcal{D}_x \times \mathcal{D}_{z_1} \times \mathcal{D}_{z_2} \to \mathcal{R}^{n_{z_1}}\) and \(f_{z_2}(x, z_1, z_2) : \mathcal{D}_x \times \mathcal{D}_{z_1} \times \mathcal{D}_{z_2} \to \mathcal{R}^{n_{z_2}}\) are unknown functions, constituting the source of unmodeled dynamics, \(g_1(x, z_1) : \mathcal{D}_x \times \mathcal{D}_{z_1} \to \mathcal{D}_{g_1}\) is an unknown function and represents the way in which \(z_1\) is coupled to the process, \(z_1\) has a known upper bound \(\bar{z}_1\) and \(y \in \mathcal{R}^m\) is a vector of available measurements. The dimension \(n_z = n_{z_1} + n_{z_2}\) of the unmodeled dynamics is unknown and hence the dimension \(n = n_x + n_z\) is also unknown. The objective is to design an adaptive observer for the system in (157) ensuring bounded estimation errors.

4.2.1 Taylor Series

The function \(f(x)\) can be expressed as a Taylor series expansion for all the values of \(x\) in the domain of interest \(\mathcal{D}_x\) as

\[
f(x(t)) = f(\check{x}(t)) + \frac{\partial f}{\partial x} |_{\check{x}(t)} (x(t) - \check{x}(t)) + \varphi(x(t), \check{x}(t))
\]

(158)

where \(\check{x}(t)\) is the operating point for the approximation and \(\varphi(x(t), \check{x}(t))\) contains the higher order Taylor series terms. Denoting \(\frac{\partial f}{\partial x} |_{\check{x}(t)} = A(t)\), we can rewrite (158) as

\[
f(x(t)) - f(\check{x}(t)) = A(t)(x(t) - \check{x}(t)) + \varphi(x(t), \check{x}(t))
\]

(159)

\(^1\)For the definition on observability of nonlinear systems, refer to [120].
4.3 Adaptive Estimator and Error Dynamics

Using Theorem 11, consider the following NN approximation of \( g_1(x, z_1) \) defined on the compact set \( D_{g_1} = \{(x, z_1): x \in D_x \subseteq \mathbb{R}^{n_x}, z_1 \in D_{z_1} \subseteq \mathbb{R}^{n_z}\} \), where \( D_x \) and \( D_{z_1} \) are compact sets

\[
g_1(x, z_1) = M^T \sigma(\mu) + \epsilon(\mu), \quad ||M||_F \leq M^*, \quad ||\epsilon(\mu)||_F \leq \epsilon^*
\]  
(160)

where \( M^* \) denotes a known upper bound for the Frobenius norm of the weight in (160), \( \mu \) is a vector of the difference quotients of the measurement \( y(t) \) as defined in (147). We propose the following adaptive estimator for the dynamics in (157)

\[
\begin{align*}
\dot{x}(t) &= f(\hat{x}(t)) + B_1 \nu_{ad}(t) + K(t)(y(t) - \hat{y}(t)), \quad x(0) = \bar{x}_0 \\
\dot{\hat{y}}(t) &= C\hat{x}(t)
\end{align*}
\]  
(161)

where the gain history \( K(t) \) is the Kalman gain history which depends on the history of the past measurements, which are all uniquely defined by \( x_0 \) and \( \hat{x}_0 \). The Kalman gain \( K(t) \) is obtained through the following set of equations [51]:

\[
\begin{align*}
\dot{P}(t) &= A(t)P(t) + P(t)A^T(t) - P(t)C^TR^{-1}CP(t) + Q \\
K(t) &= P(t)C^TR^{-1}
\end{align*}
\]  
(162)  
(163)

where \( Q \geq 0 \) is the process noise matrix, \( R > 0 \) is the measurement noise matrix and the solution \( P(t) \) is bounded, symmetric, positive definite and continuously differentiable. The output of the adaptive element, denoted by \( \nu_{ad}(t) \), is designed as follows:

\[
\nu_{ad}(t) = \hat{M}^T(t)\sigma(\mu(t))
\]  
(164)

where \( \hat{M} \) is the estimate of the weight that is adjusted on line. The NN adaptation law for \( \hat{M} \) is designed as

\[
\dot{\hat{M}}(t) = -\Gamma_M \left( \sigma(\mu(t))\hat{y}^T(t) + k_\sigma \|\hat{y}(t)\|\hat{M}(t) \right)
\]  
(165)

where \( \hat{y}(t) \triangleq y(t) - \hat{y}(t) \), \( k_\sigma \) denotes the \( \sigma \)-modification gain [124] and \( \Gamma_M > 0 \) specifies the learning rate of the NN. The conceptual layout of the proposed observer along with the neural
Figure 14: Extended Kalman Filter Augmented with a Neural Network.

The network architecture is shown in Fig. 14, where \( d(t) \) denotes the exogenous disturbance to the process dynamics. Denoting the estimation error signal \( e(t) = x(t) - \hat{x}(t) \), we can formulate the error dynamics as follows

\[
\begin{align*}
\dot{e}(t) &= \left(A(t) - K(t)C\right)e(t) + B_1\left(g_1(x(t), z_1(t)) - v_{ad}(t)\right) + \varphi(e(t)) \\
\tilde{y}(t) &= Ce(t)
\end{align*}
\]  

(166)

where \( \varphi(e(t)) \) represents the higher order Taylor series terms. The term \( \varphi(e(t)) \) is bounded according to \( \|\varphi(e(t))\| \leq \varphi^* \), for all \( x \in \mathcal{D}_x \).

**Assumption 3.** [125] We assume that \( f(x(t)) \) is such that for \( g_1(x(t), z_1(t)) = 0 \), and in the absence of the NN, the equilibrium point \( e = 0 \) of the error dynamics in (166) is locally exponentially stable regardless of the measurement history.

**Remark 8.** When \( g_1(x(t), z_1(t)) \neq 0 \), and in the absence of the NN, (166) is not necessarily input-to-state stable.
By substituting (160) and (164) into (166) and defining the NN weight error \( \hat{M}(t) \triangleq M(t) - \hat{M}(t) \), the error dynamics along with the adaptation law in (165) can be written as

\[
\begin{align*}
\dot{e}(t) & = (A(t) - K(t)C)e(t) + B_1\hat{M}^T(t)\sigma(\mu(t)) + B_1\epsilon(\mu(t)) + \varphi(e(t)) \\
\dot{\hat{M}}(t) & = \Gamma_M(\sigma(\mu(t))\hat{y}^T(t) + k_\sigma\|\bar{y}(t)\|\hat{M}(t)) \\
\hat{y}(t) & = Ce(t)
\end{align*}
\]

(167)

4.4 Boundedness Analysis

In this section we show through Lyapunov’s direct method that the estimation error \( e(t) \) and the NN weight error \( \hat{M}(t) \) are ultimately bounded. The arguments in the proof will be based on the idea of Lyapunov redesign [116].

Remark 9. By defining \( F_e(t, e(t)) \triangleq (A(t) - K(t)C)e(t) \), and using Assumption 3 and Theorem 9, when \( g_1(\mathbf{x}(t), \mathbf{z}_1(t)) = 0 \), and in the absence of the NN, we are guaranteed the existence of a Lyapunov function \( V_e(t, e(t)) \) that satisfies Theorem 9.

Consider the composite error vector,

\[
\zeta(t) = \begin{bmatrix} e^T(t) & \hat{M}^T(t) \end{bmatrix}^T
\]

(168)

and the vector \( \mathbf{z}(t) = \begin{bmatrix} \mathbf{z}_1^T(t) & \mathbf{z}_2^T(t) \end{bmatrix}^T \). Introduce the largest ball \( B_R \) such that

\[
B_R \triangleq \left\{ \zeta \mid \|\zeta\| \leq R \right\}, \quad R > 0
\]

(169)

that lies in \( \Omega_\zeta \triangleq \left\{ \zeta \triangleq (e, \hat{M}) \in \mathbb{R}^{n_\epsilon} \times \mathbb{R}^{n_\mu \times n_\mu} : (\mathbf{x}, \mathbf{z}_1, \mathbf{z}_2, \hat{\mathbf{x}}) \in \mathcal{D}_x \times \mathcal{D}_{\mathbf{z}_1} \times \mathcal{D}_{\mathbf{z}_2} \times \mathcal{D}_{\hat{\mathbf{x}}} \right\} \).

Using Theorem 9 it is easily verified that

\[
\zeta^T T_1 \zeta \leq V(\zeta, \mathbf{z}) \leq \zeta^T T_2 \zeta
\]

(170)

where

\[
T_1 \triangleq \begin{bmatrix} c_1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad T_2 \triangleq \begin{bmatrix} c_2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}
\]

(171)

Let

\[
\alpha \triangleq \min_{\|\zeta\|=R} \zeta^T T_1 \zeta = R^2 T_{\min}
\]

(172)
where \( T_{\min} \) is the minimum of the matrix \( T_1 \). Introduce the set
\[
\Omega_\alpha = \left\{ \zeta \in \mathcal{B}_R \mid V(\zeta, z) \leq \alpha \right\}
\] (173)

**Assumption 4.** Let
\[
R > \gamma_0 \sqrt{\frac{T_{\max}}{T_{\min}}} \geq \gamma_0
\] (174)
where \( T_{\max} \) is the maximum of the matrix \( T_2 \) and
\[
\gamma_0 = \max(C_1, C_2)
\] (175)

where
\[
C_1 \triangleq \sqrt{\frac{(\kappa M^* + N c_5)^2}{4\kappa c_3} + \frac{c_4 \eta}{c_3}}
\]
\[
C_2 \triangleq \frac{\kappa M^* + N c_5}{2\kappa} + \sqrt{\frac{(\kappa M^* + N c_5)^2}{4\kappa^2} + \frac{c_4 \eta}{\kappa}}
\]
\[
c_5 \triangleq c_4 \|B_1\| + \|C\|
\]
\[
\kappa \triangleq k\sigma \|C\|
\]
\[
\eta \triangleq \|B_1\|e^* + \varphi^*
\] (176)

and \( N \) denotes the number of basis functions.

We are now ready to state and prove the main theorem of this approach.

**Theorem 13.** Let the initial errors, \( e(0) \) and \( \tilde{M}(0) \), belong to the set \( \Omega_\alpha \) in Fig. 15. Let Assumptions 3 and 4 hold. Let the NN adaptation law be given by (165). Then the tracking error \( e(t) \) and the NN weight error \( \tilde{M}(t) \) are uniformly ultimately bounded by \( \alpha_1^{-1}\left(\alpha_2(\gamma_0)\right) \), where \( \gamma_0 \) is given by the right hand side of (184) and (185).

**Proof.** When \( g_1(\varphi(t), z(t)) \neq 0 \), and in the presence of the NN, we choose the following Lyapunov function candidate to arrive at boundedness of the error signals
\[
V(t, \zeta(t), z(t)) = V_e(t, e(t)) + \frac{1}{2} \text{tr} \left( \tilde{M}^T(t) \Gamma_{\tilde{M}}^{-1} \tilde{M}(t) \right)
\] (177)
Notice that from Assumption 3 and Theorem 9, replacing \( x(t) \) with \( e(t), f(t, x(t)) \) with \( F_e(t, e(t)) \) and \( V_e(t, x(t)) \) with \( V_e(t, e(t)) \) the following inequality is immediate:
\[
\frac{\partial V_e}{\partial t} + \frac{\partial V_e}{\partial e} F_e(t, e(t)) \leq -c_3 \|e(t)\|^2
\] (178)
Taking the derivative of $V(t, \zeta(t), z(t))$ along (167), we obtain

$$\dot{V}(t, \zeta(t), z(t)) = \frac{\partial V_e}{\partial t} + \frac{\partial V_e}{\partial e} \left( F_e(t, e(t)) + B_1 \tilde{M}^T(t) \sigma(\mu(t)) + B_1 e(\mu(t)) + \varphi(e(t)) \right) + \text{tr} \left( \tilde{M}^T(t) \left( \sigma(\mu(t)) \tilde{y}^T(t) + k_\sigma ||\tilde{y}(t)|| \tilde{M}(t) \right) \right)$$

$$= \frac{\partial V_e}{\partial t} + \frac{\partial V_e}{\partial e} F_e(t, e(t)) + \frac{\partial V_e}{\partial e} \left( B_1 \tilde{M}^T(t) \sigma(\mu(t)) + B_1 e(\mu(t)) + \varphi(e(t)) \right) + \text{tr} \left( \tilde{M}^T(t) \left( \sigma(\mu(t)) \tilde{y}^T(t) + k_\sigma ||\tilde{y}(t)|| \tilde{M}(t) \right) \right)$$

(179)

Using (160), (178), Theorem 9 and the inequality $||\sigma(\mu(t))|| \leq N$, the Lyapunov derivative can be upper bounded as

$$\dot{V}(t, \zeta(t), z(t)) \leq -c_3 ||e(t)||^2 + c_4 N ||e(t)|| ||B_1|| ||\tilde{M}(t)||_F + c_4 ||e(t)|| ||B_4|| ||e^* + c_4 ||e(t)|| ||\varphi^* + N ||\tilde{y}(t)|| ||\tilde{M}(t)||_F + k_\sigma ||\tilde{y}(t)|| \text{tr} \left( \tilde{M}^T(t) \tilde{M}(t) \right)$$

(180)

Further substituting the inequalities

$$||\tilde{y}(t)|| \leq ||C|| ||e(t)||, \quad \text{tr} \left[ \tilde{M}^T(t) \left( M - \tilde{M}(t) \right) \right] \leq ||\tilde{M}(t)||_F M^* - ||\tilde{M}(t)||_F^2,$$

(181)

in (180), rearranging the terms and using the notations in (176), the Lyapunov derivative can be upper bounded as

$$\dot{V}(t, \zeta(t), z(t)) \leq -||e(t)|| \left( c_3 ||e(t)|| - c_4 \eta + ||\tilde{M}(t)||_F \left( \kappa ||\tilde{M}(t)||_F - \kappa M^* - N c_3 \right) \right)$$

(182)
Completing the squares on \( \| \tilde{M}(t) \|_F \left( \kappa \| \tilde{M}(t) \|_F - \kappa M^* - N \cdot c_5 \right) \) we finally obtain
\[
\dot{V}(t, \zeta(t), z(t)) \leq -\| e(t) \| \left( c_3 \| e(t) \| + \left( \sqrt{\kappa} \| \tilde{M}(t) \|_F - \left( \frac{\kappa M^* + N \cdot c_5}{2\sqrt{\kappa}} \right) \right)^2 \right. \\
\left. - \left( \frac{\kappa M^* + N \cdot c_5}{4\kappa} \right)^2 - c_4 \eta \right) \] (183)

Thus either of the conditions
\[
\| e \| > \frac{(\kappa M^* + N \cdot c_5)^2}{4\kappa c_3} + \frac{c_4 \eta}{c_3} \] (184)
\[
\| \tilde{M} \|_F > \frac{\kappa M^* + N \cdot c_5}{2\kappa} + \sqrt{\frac{(\kappa M^* + N \cdot c_5)^2}{4\kappa^2} + \frac{c_4 \eta}{\kappa}} \] (185)

will guarantee \( \dot{V}(t, \zeta(t), z(t)) < 0 \) outside a compact set. Introduce the set
\[
B_{\gamma_0} = \left\{ \zeta \in B_R \mid \| \zeta \| \leq \gamma_0 \right\} \] (186)
in the space of the error vector \( \zeta \) outside of which \( \dot{V}(\zeta, z) < 0 \). Notice from (174), that \( B_{\gamma_0} \subset B_R \). Let \( \Gamma \) be the maximum value of the function \( V(\zeta, z) \) on the edge of \( B_{\gamma_0} \):
\[
\Gamma \triangleq \max_{\| \zeta \| = \gamma_0} \zeta^T T_2 \zeta = \gamma_0^2 T_{\text{max}} \] (187)

Introduce the level set of \( V(\zeta, z) \), that touches the ball \( B_{\gamma_0} \) as shown in Fig. 15
\[
\Omega_{\gamma_0} = \left\{ \zeta \mid V(\zeta, z) = \Gamma \right\} \] (188)

The condition in (174) ensures that \( \Omega_{\gamma_0} \subset \Omega_{\alpha} \). Thus, if the initial error \( \zeta_0 = \zeta(0) \) belongs to \( \Omega_{\alpha} \) then it follows from Theorem 12 that the solution \( (\zeta(t), z(t)) \) to (167) is ultimately bounded with respect to \( \zeta(t) \) uniformly in \( z_0 \).

\[ \square \]

### 4.5 Simulation Results for Adaptive Estimation

We illustrate a typical application in which the range between two aircrafts is regulated, by feeding back the true values of the target velocity obtained by processing camera images. The goal is to accomplish a task in an unmanned system that is commonly performed in a manned system that relies primarily on visual information. The typical geometric scenario is shown in Fig. 16, where \( (X, Y) \) denotes the inertial frame, \( T \) represents the maneuvering target,
Figure 16: Target Tracking Scenario in 2 Dimension.

$F$ represents the follower, a 2D passive vision sensor, $\beta$ is the bearing angle, $r$ represents the range between the target and the follower, $b$ represents the size of the target and $\alpha$ represents the angle subtended by the target in an image plane.

The simulation runs were performed from the perspective of using $\beta$ and $\alpha$ as the available measurements, which we refer to as the”two-angles case”.

**Remark 10.** The approach developed in this chapter is for adaptive estimation of uncertain nonlinear systems without treating control input in the analysis. Thus in these simulations we treat the case of target-follower formation by feeding back the true values of target velocity and range into the guidance law rather than the estimated values. By this treatment we are able to show the performance of the NN augmented EKF from a pure estimation perspective.

**Remark 11.** For the purpose of simulations, all the dimensional variables except time are nondimensionalized with respect to gravity, $g = 32.2 \text{ ft/s}^2$.

4.5.1 True Plant Dynamics

Let the follower coordinates in the $(X, Y)$ frame be denoted by $(x_F, y_F)$ and the target coordinates be denoted by $(x_T, y_T)$. Let the components of follower velocity be denoted by
\((V_{xT}, V_{yT})\) and the components of target velocity be denoted by \((V_{xT}, V_{yT})\). The relative range between the target and follower in polar coordinates is
\[
r = \sqrt{x^2 + y^2}
\]
where \(x = (x_T - x_F)\) and \(y = (y_T - y_F)\). The bearing angle \(\beta\) is given as
\[
\beta = \tan^{-1}\left(\frac{y}{x}\right)
\]
Then the true plant dynamics for the target-follower tracking problem can be given by the following equations:
\[
\begin{align*}
\dot{x} &= V_x \\
\dot{y} &= V_y \\
\dot{V}_x &= a_x \\
\dot{V}_y &= a_y
\end{align*}
\]
where \(x, y, V_x, V_y\) and \(a_x, a_y\) denote the relative position, velocity and acceleration between target and follower respectively. The true relative accelerations \(a_x\) and \(a_y\) are modelled as follows:
\[
\begin{align*}
a_x &= \left( k_1 V^2 + k_2 \frac{a_1^2 + 1}{V^2} + \frac{V}{\tau_V}\right) \cos(\psi) - a_1 \sin(\psi) \quad (192) \\
a_y &= \left( k_1 V^2 + k_2 \frac{a_1^2 + 1}{V^2} + \frac{V}{\tau_V}\right) \sin(\psi) + a_1 \cos(\psi) \quad (193)
\end{align*}
\]
where \(\psi, V\) are the heading and speed of the maneuvering aircraft with respect to the inertial frame, \(n\) is the load factor of the aircraft, \(k_1, k_2, \tau_V\) are constants and \(a_1 = \tan(\phi)\), where \(\phi\) is the bank angle of the aircraft as shown in Fig. 17.

The time histories of \(\psi\) and \(V\) of the aircraft are obtained through the following equations of motion:
\[
\begin{align*}
mV\dot{\psi} &= L \sin(\phi) \quad (194) \\
m\dot{V} &= T - D \quad (195) \\
m g &= L \cos(\phi) \quad (196)
\end{align*}
\]
where $m$ represents the mass of the aircraft, $L$, $T$, $D$ represent the lift, thrust and drag forces on the aircraft, and $g$ is the acceleration due to gravity. For further details on the dynamics of this problem refer [126].

4.5.1.1 True Process Initial Conditions

All dimensional variables are nondimensionalised using gravity, $g = 32.174$ ft/s$^2$. The true process for the target and follower aircrafts are initialized as follows:

1. The target aircraft was initialized as follows:

   $$ x_T(0) = 0, \quad y_T(0) = 20, \quad V_{x_T}(0) = 0, \quad V_{y_T}(0) = 1.5 \quad (197) $$

2. The follower aircraft was initialized as follows:

   $$ x_F(0) = 5, \quad y_F(0) = 0, \quad V_{x_F}(0) = 0.1, \quad V_{y_F}(0) = 1 \quad (198) $$

From (197) and (198), we see that

$$ r(0) = \sqrt{425} = 20.6155 $$

$$ \beta(0) = \tan^{-1}\left(-\frac{20}{5}\right) = 104 \text{ deg} = 1.8151 \text{ rad} \quad (199) $$
4.5.2 Model for EKF Design and NN Design Parameters

To obtain the model for the design of the EKF we choose the following to be the state vector:

\[
X(t) = \begin{bmatrix}
\dot{\beta} \\
\dot{r} \\
\beta \\
\dot{r}
\end{bmatrix}
\]  

where \( \beta \) and \( r \) are given by (190) and (189). Taking the derivative of (190) once with respect to time we get

\[
\dot{\beta} = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2} = \frac{x\dot{y} - y\dot{x}}{r^2} 
\]  

(201)

Differentiating (201) once with respect to time, using (191), noting that \( x = r \cos(\beta), \ y = r \sin(\beta) \) and further simplifying, we obtain

\[
\ddot{\beta} = \frac{a_y \cos(\beta) - a_x \sin(\beta)}{r} \frac{2\hat{r}(xV_y - yV_x)}{r^3} 
\]  

(202)

Substituting (201) into (202), we obtain

\[
\ddot{\beta} = \frac{a_y \cos(\beta) - a_x \sin(\beta)}{r} \frac{2\dot{\beta}\hat{r}}{r} 
\]  

(203)

Differentiating (189) once with respect to time we obtain

\[
\dot{r} = \frac{d}{dt}(\sqrt{x^2 + y^2}) = \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}} = \frac{x\dot{V}_x + yV_y}{r} 
\]  

(204)

The cartesian coordinates \((x, y)\) can be expressed in polar coordinates as

\[
x = r \cos(\beta), \quad y = r \sin(\beta) 
\]  

(205)

Differentiating each equation in (205) with respect to time we get

\[
\dot{x} = V_x = \dot{r} \cos(\beta) - r\dot{\beta} \sin(\beta) \\
\dot{y} = V_y = \dot{r} \sin(\beta) + r\dot{\beta} \cos(\beta) 
\]  

(206)

Differentiating (204) with respect to time, using (191), (206) and further simplifying, we get

\[
\ddot{r} = \frac{\dot{r}^2 + r^2 \dot{\beta}^2 + ra_x \cos(\beta) + ra_y \sin(\beta)}{r} - \frac{\dot{r}^2}{r} 
\]  

(207)
The derivative of the second state of $\mathbf{X}(t)$ in (200) can be written as

$$
\frac{d}{dt}\left(\frac{\dot{r}}{r}\right) = \frac{\dot{r}}{r} - \frac{\dot{r}^2}{r^2}
$$

(208)

Substituting (207) into (208), we obtain

$$
\frac{d}{dt}\left(\frac{\dot{r}}{r}\right) = \frac{\dot{r}^2 + \ddot{r} + r \dddot{r} + r \dot{a}_x \cos(\beta) + r \dot{a}_y \sin(\beta)}{r^2} - \frac{\dot{r}^2}{r^2} - \frac{\ddot{r}}{r^2}
$$

$$
= \frac{\dot{r}^2}{r^2} + \frac{\ddot{r}}{r^2} + \frac{\dot{a}_x \cos(\beta) + a_y \sin(\beta)}{r} - \dot{\beta} \frac{\dot{r}}{r^2} - \ddot{\beta} \frac{\dot{r}}{r^2}
$$

$$
= \beta^2 - \frac{\dot{r}^2}{r^2} + \frac{a_x \cos(\beta) + a_y \sin(\beta)}{r}
$$

(209)

The target size is modelled as a constant and is given as

$$
\dot{b} = 0
$$

(210)

Thus, using (203), (209) and (210), the model for the design of the EKF can be described in polar coordinates by the following set of nonlinear differential equations:

$$
\frac{d}{dt} \begin{bmatrix} \dot{\beta} \\ \frac{\dot{r}}{r} \\ \dot{\beta} \\ \frac{1}{r} \\ \dot{b} \end{bmatrix} = \begin{bmatrix} -2\dot{\beta}^2 + \frac{1}{r}(a_y \cos(\beta) - a_x \sin(\beta)) \\ \dot{r}^2 - \left(\frac{\dot{r}}{r}\right)^2 + \frac{1}{r}(a_y \sin(\beta) + a_x \cos(\beta)) \\ \dot{\beta} \\ \frac{1}{r} \\ 0 \end{bmatrix}
$$

(211)

and the measurements are given by the following equations:

$$
\beta_m(t) = \beta(t) + \nu_\beta(t)
$$

$$
\alpha_m(t) = 2 \tan^{-1}\left(\frac{b}{2r}\right) + \nu_\alpha(t)
$$

(212)

where $a_x$ and $a_y$ are the horizontal relative acceleration components in a Cartesian frame, and $\nu_\beta$, $\nu_\alpha$ are band limited zero mean white noise processes with a standard deviation of 0.01. The acceleration components $a_x$ and $a_y$ are treated as a source of unmodeled dynamics coupled to the model.

4.5.2.1 Model Initial Conditions

The model, (211), for the design of the EKF was initialized as follows:

$$
\dot{\beta}(0) = -0.0087, \quad \frac{\dot{r}}{r}(0) = 0.0672, \quad \dot{\beta}(0) = 1.8158, \quad \frac{1}{r}(0) = 0.0932, \quad \dot{b}(0) = 0.2407 \quad (213)
$$
4.5.2.2 Initial covariance matrix, \( P(0) \)

The initial covariance matrix was chosen as

\[
P(0) = P_0 = \begin{bmatrix}
0.01 & 0 & 0 & 0 \\
0 & 0.3 & 0 & 0 \\
0 & 0 & 0.01 & 0 \\
0 & 0 & 0 & 0.1 \\
0 & 0 & 0 & 0.001
\end{bmatrix}
\] (214)

4.5.2.3 Measurement noise matrix

The measurement noise matrix, \( R \), corresponds to the variance of the measurement noise channels, \( \beta_m(t) \) and \( \alpha_m(t) \) and is given by

\[
R = \begin{bmatrix}
0.01^2 & 0 \\
0 & 0.01^2
\end{bmatrix}
\] (215)

4.5.2.4 NN design parameters

The adaptive law in (165) was implemented for the simulation runs with the learning rate \( \Gamma_M = 10 \), the gain \( k_\sigma = 3.5 \) and 2 sampled delayed values of each measurement were used. The sigmoidal basis function, \( \sigma(x) = \frac{1}{1+e^{-\alpha x}} \), was used with the activation potential \( \alpha = 1 \). The acceleration components \( a_x \) and \( a_y \) are treated as a source of unmodeled dynamics coupled to the model. The goal of the NN augmentation was to correct for the estimates of the model in (211) in the presence of the unmodeled dynamics.

4.5.3 Discussion of Results

We considered the situation in which a follower aircraft regulates a range of two wing spans from a maneuvering target aircraft with the measurements of line-of-sight angle, \( \beta_m \), and the angle subtended by the target in the image plane, \( \alpha_m \). The target is assumed to perform a sinusoidal maneuver with an acceleration of \( 0.3g \). Figs. 19, 20 and 21 show the performance of the EKF, while Figs. 23, 24 and 25 show the performance of the EKF + NN, in estimating the states of the model in (211). Figs. 18 and 22 show the true and estimated trajectories of the target aircraft for the EKF and EKF + NN respectively.
From the lower subfigure of Fig. 19 we see that the EKF is unable to reduce the drift in the estimates of the LOS range histories. This trend can also be seen from the range estimate error plot of the EKF in Fig. 20. The lower subfigure of Fig. 23 shows that the NN augmented EKF is able to significantly reduce the drift in the LOS range estimates and produce a nearly unbiased estimate as seen from the LOS range estimate error plot in Fig. 24. From Fig. 21, we see that the EKF is unable to correct for the drift in the estimate of the target size. The NN augmented EKF greatly reduces the bias in the estimate of the target size as seen from Fig. 25.

4.6 Conclusions

In this chapter we address the problem of augmenting an EKF with an adaptive element. A key application area is that of tracking a maneuvering target. The approach is applicable to uncertain nonlinear systems with uncertain parameters and unmodeled dynamics coupled to the process. The adaptive law is trained by an error signal that is generated from the residuals of the EKF. Boundedness of error signals is shown through Lyapunov’s direct method. Simulations are used to show that augmenting the EKF with an NN helps in removing the bias in the range estimates and also improves the estimate of the target size.
Figure 18: Target and Follower Trajectory - Performance of EKF

Range between leader aircraft and follower aircraft

Figure 19: True and Estimated LOS Range - Performance of EKF
Figure 20: Estimation error of LOS range - Performance of EKF

Figure 21: Estimation errors of target size, b - Performance of EKF
**Figure 22:** Target and Follower Trajectory - Performance of EKF + NN

Range between leader aircraft and follower aircraft

![Graph showing range between leader and follower aircrafts.](image)

**Figure 23:** True and Estimated LOS Range - Performance of EKF + NN

![Graph showing true and estimated LOS range differences.](image)
**Figure 24:** Estimation error of LOS range - Performance of EKF + NN

**Figure 25:** Estimation error of target size, $b$ - Performance of EKF + NN
CHAPTER 5

ADAPTIVE ESTIMATION FOR CONTROL OF
UNCERTAIN NONLINEAR SYSTEMS

In chapter 4, the problem of augmenting an existing time varying observer, such as an
EKF, with an adaptive NN element was addressed. The analysis developed therein treated
the problem from a purely estimation setting, i.e., the control input of the system was not
included in the analysis. However, in most of the practical applications such as missile-target
tracking, formation flight control and obstacle avoidance, to name a few, the control/guidance
law relies on the estimates of the unknown state vector. In this chapter, an adaptive extended
Kalman filter based control method is presented for a class of multivariable systems with
unmodeled dynamics coupled to the process. The design of the adaptive element employs a
linearly parameterized neural network. The states of the adaptive estimator are used to form
the feedback control signal. The network weights are adjusted on line using a linear observer
for the nominal system’s error dynamics. Boundedness of the error signals is proven using
Lyapunov’s direct method. Simulations illustrate the theoretical results.

5.1 Mathematical Preliminaries

Consider the nonlinear dynamical system

\[ \dot{x}(t) = f(t, x(t), u(t)) \tag{216} \]

where \( f : [0, \infty) \times \mathcal{D}_x \times \mathcal{D}_u \rightarrow \mathbb{R}^n \) is piecewise continuous in \( t \) and locally Lipschitz in \( x, u \)
on \( [0, \infty) \times \mathcal{D}_x \), and \( \mathcal{D}_x \subset \mathbb{R}^n \) is a domain that contains the origin.

**Theorem 14.** [116] Let \( \mathcal{D}_x \subset \mathbb{R}^n \) be a domain that contains the origin and \( V : [0, \infty) \times \mathcal{D}_x \rightarrow \mathbb{R} \) be a continuously differentiable function such that

\[ \alpha_1(\|x(t)\|) \leq V(t, x(t)) \leq \alpha_2(\|x(t)\|) \tag{217} \]

\[ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x(t), u(t)) \leq -W_s(x(t)), \quad \forall \|x(t)\| \geq \gamma > 0 \tag{218} \]
∀ (t, x, u) ∈ [0, ∞) × D_x × D_u, where α_1 and α_2 are class K functions and W_3(x) is a continuous positive definite function. Take r > 0 such that B_r ⊂ D and suppose that
\[ γ < α_2^{-1}(α_1(r)) \]  
(219)

Then, there exists a class KL function β and for every initial state x(t_0), satisfying
\[ \|x(t_0)\| ≤ α_2^{-1}(α_1(r)) \]  
(220)

there is a T ≥ 0, dependent on x(t_0) and γ, such that the solution of the differential equation (216) satisfies
\[ \|x(t)\| ≤ β(\|x(t_0)\|, t - t_0), \quad ∀ \ t_0 ≤ t ≤ t_0 + T \]  
(221)
\[ \|x(t)\| ≤ α_1^{-1}(α_2(γ)), \quad ∀ \ t ≥ t_0 + T \]  
(222)

5.2 Problem Formulation

Consider the dynamics of an observable process as^1:

\[ \dot{x}(t) = f(x(t)) + B_1 g_1(x(t), z_1(t)) + B_2 u(t), \quad x(0) = x_0 \]
\[ \dot{z}_1(t) = f_{z_1}(x(t), z_1(t), z_2(t)), \quad z_1(0) = z_{10} \]
\[ \dot{z}_2(t) = f_{z_2}(x(t), z_1(t), z_2(t)), \quad z_2(0) = z_{20} \]
\[ y(t) = Cx(t) \]  
(223)

where \(x ∈ \mathcal{R}^{n_x}, z_1 ∈ D_{z_1} ⊂ \mathcal{R}^{n_{z_1}}, z_2 ∈ D_{z_2} ⊂ \mathcal{R}^{n_{z_2}}\) are the states of the system, \(f(x) : \mathcal{R}^{n_x} → \mathcal{R}^{n_x}\) is a known smooth function which can be expressed as a Taylor series expansion for all \(x ∈ D_x ⊂ \mathcal{R}^{n_x}\), \(D_x\) is a compact subset of \(\mathcal{R}^{n_x}\), \(B_1, B_2\) and \(C\) are known constant matrices, \(f_{z_1}(x, z_1, z_2) : \mathcal{R}^{n_x} × D_{z_1} × D_{z_2} → \mathcal{R}^{n_{z_1}}\) and \(f_{z_2}(x, z_1, z_2) : \mathcal{R}^{n_x} × D_{z_1} × D_{z_2} → \mathcal{R}^{n_{z_2}}\) are unknown functions, \(g_1(x, z_1) : D_x × D_{z_1} → D_{g_1}\) is an unknown function and represents the way in which the unmodeled dynamics \(z_1\) is coupled to the process, \(u ∈ \mathcal{R}^m\) is a control input vector and \(y ∈ \mathcal{R}^q\) is a vector of available measurements.

^1^For the definition on observability of nonlinear systems, refer to [120].
5.2.1 Taylor Series

The function $f(x)$ can be expressed as a Taylor series expansion for all the values of $x$ in the domain of interest $\mathcal{D}_x$ as

$$f(x(t)) = f(\hat{x}(t)) + \frac{\partial f}{\partial x}\bigg|_{x(t)} (x(t) - \hat{x}(t)) + \varphi(x(t), \hat{x}(t)) \quad (224)$$

where $\varphi(x(t), \hat{x}(t))$ contains the higher order Taylor series terms. Denoting $\frac{\partial f}{\partial x}\bigg|_{\hat{x}(t)} = A(t)$, we can rewrite (224) as

$$f(x(t)) - f(\hat{x}(t)) = A(t)(x(t) - \hat{x}(t)) + \varphi(x(t), \hat{x}(t)) \quad (225)$$

5.3 Adaptive Estimator and Error Dynamics

Consider a linear parameterization of $g_1(x, z_1)$ defined on a compact set of

$$\mathcal{D}_{g_1} = \left\{ \bar{x} \triangleq (x, z_1) : x \in \mathcal{D}_x \subseteq \mathbb{R}^{n_x}, \ z_1 \in \mathcal{D}_{z_1} \subseteq \mathbb{R}^{n_{z_1}} \right\} \quad (226)$$

where $\mathcal{D}_x$ and $\mathcal{D}_{z_1}$ are compact sets

$$g_1(\bar{x}) = \bar{M}^T \bar{\sigma}(\bar{x}) + \bar{\epsilon}(\bar{x}), \quad \|\bar{M}\|_F \leq M^*, \quad \|\bar{\epsilon}(\bar{x})\|_F \leq \epsilon^* \quad (227)$$

where $M^*$ denotes a known upper bound for the Frobenius norm of the weight in (227). In conventional adaptive control methods, the adaptive signal is designed as $\hat{M}^T(t)\sigma(\hat{x}(t))$, where $\hat{M}(t)$ are the weight estimates, the adaptive laws for which are designed via a Lyapunov like analysis. Since our objective is state estimator design, we cannot use the states as inputs to the NN. The analysis that follows here is similar to the one presented in [127]. We build our adaptive element as

$$\nu_{ad}(t) = \hat{M}^T(t)\sigma(\mu(t)) \quad (228)$$

where $\hat{M}(t)$ is the estimate of the weight that is adjusted on line and $\mu(t)$ is a vector of delayed values and is given as:

$$\mu(t) = \begin{bmatrix} \tilde{y}_d^T(t) & \tilde{a}_d^T(t-d) \end{bmatrix}^T, \ d > 0 \quad (229)$$
Here $\mathbf{y}_d^T(t)$ and $\mathbf{u}_d^T(t)$ are vectors of measurement and control variables respectively and are given as:

\[
\begin{align*}
\mathbf{y}_d(t) & = \begin{bmatrix} \Delta_d^{(0)} y_1(t) & \cdots & \Delta_d^{(n-1)} y_1(t) & \cdots & \Delta_d^{(0)} y_q(t) & \cdots & \Delta_d^{(n-1)} y_q(t) \end{bmatrix}^T \\
\Delta_d^{(0)} y_i(t) & \triangleq y_i(t) \\
\Delta_d^{(k)} y_i(t) & \triangleq \frac{\Delta_d^{k-1} y_i(t) - \Delta_d^{k-1} y_i(t - d)}{d}, \quad k = 1, \ldots, n-1, \quad i = 1, \ldots, q \\
\mathbf{u}_d(t) & = \begin{bmatrix} \Delta_d^{(0)} u_1(t) & \cdots & \Delta_d^{(n-r_d-1)} u_1(t) & \cdots & \Delta_d^{(0)} u_m(t) & \cdots & \Delta_d^{(n-r_d-1)} u_m(t) \end{bmatrix}^T
\end{align*}
\]

where $r_d = r_1 + \cdots + r_q < n$ and $r_1, \ldots, r_q$ represent the relative degrees of the regulated outputs of the system. Notice that despite the fact that the dimensions of the input vector in (229) and (227) are different, the number of basis functions are the same. Thus $M$ and $\tilde{M}(t)$ have same dimensions and the term $\left( M^T \boldsymbol{\sigma}(\bar{x}(t)) - \tilde{M}^T(t) \boldsymbol{\sigma}(\mu(t)) \right)$ is represented as

\[
\begin{align*}
M^T \boldsymbol{\sigma}(\bar{x}(t)) - \tilde{M}^T(t) \boldsymbol{\sigma}(\mu(t)) & = M^T \boldsymbol{\sigma}(\bar{x}(t)) - M^T \boldsymbol{\sigma}(\mu(t)) \\
& \quad + M^T \boldsymbol{\sigma}(\mu(t)) - \tilde{M}^T(t) \boldsymbol{\sigma}(\mu(t)) \\
& \quad = M^T \left( \boldsymbol{\sigma}(\bar{x}(t)) - \boldsymbol{\sigma}(\mu(t)) \right) + \tilde{M}^T \boldsymbol{\sigma}(\mu(t))
\end{align*}
\]

where $\tilde{M}(t) \triangleq M - \tilde{M}(t)$ denotes the NN weight error. Further notice that $M^T \left( \boldsymbol{\sigma}(\bar{x}(t)) - \boldsymbol{\sigma}(\mu(t)) \right)$ can be upper bounded as

\[
\left\| M^T \left( \boldsymbol{\sigma}(\bar{x}(t)) - \boldsymbol{\sigma}(\mu(t)) \right) \right\| \leq 2NM^*, \quad \forall t \geq 0
\]

where $N$ denotes the number of basis functions. In [118], it has been shown that for an unknown observable bounded process, it is possible to approximate its dynamics using a finite sample of its output history. In this case

\[
\left\| M^T \left( \boldsymbol{\sigma}(\bar{x}(t)) - \boldsymbol{\sigma}(\mu(t)) \right) \right\| \approx \mathcal{O}(d)
\]

where $d$ is introduced in (230), and hence, tends to zero as $d \to 0$. For a general situation, when there is no guarantee for the boundedness of the process, then the bound in (231) is valid. We propose the following adaptive estimator for the dynamics in (223)

\[
\begin{align*}
\dot{x}(t) & = f(\bar{x}(t)) + B_2 u(\bar{x}(t)) + K(t)(y(t) - \bar{y}(t)) + B_1 \nu_{ad}(t) \\
\dot{y}(t) & = C \bar{x}(t)
\end{align*}
\]

\[\text{(233)}\]
where $\mathbf{u}(\bar{x}(t)) = \mathbf{g}_d(\bar{x}(t))$, $\mathbf{g}_d$ is a continuous map, $\bar{x}(0) = \bar{x}_0$ and $K(t)$ is the Kalman gain history obtained through the following set of equations \[51\]:

$$
\dot{P}(t) = A(t)P(t) + P(t)A^T(t) - P(t)C^TR^{-1}CP(t) + \mathbf{Q} \quad (234)
$$

$$
K(t) = P(t)C^TR^{-1} \quad (235)
$$

where $\mathbf{Q} \geq q \ I > 0$, $R > 0$ is the measurement noise matrix and the solution $P(t)$ is bounded, symmetric, positive definite and continuously differentiable. Denote the estimation error signals $\mathbf{e}(t) \triangleq \mathbf{x}(t) - \bar{x}(t)$ and $\hat{y}(t) \triangleq y(t) - \hat{y}(t)$. Using (227), (228), the Taylor series expansion of $\mathbf{f}(\mathbf{x}(t))$, and noting that the higher order Taylor terms will be a function of the error $\mathbf{e}(t)$, we can formulate the error dynamics as follows

$$
\dot{\mathbf{e}}(t) = \bar{A}(t)\mathbf{e}(t) + \varphi(\mathbf{e}(t))
$$

$$
+ B_1 \left( M^T \sigma(\bar{x}(t)) - \hat{M}^T(t)\sigma(\mathbf{u}(t)) + \epsilon(\bar{x}(t)) \right) 
$$

$$
\hat{y}(t) = C\mathbf{e}(t)
$$

(236)

where $\bar{A}(t) = A - K(t)C$. Further using the expression in (230), we can write (236) as

$$
\dot{\mathbf{e}}(t) = \bar{A}(t)\mathbf{e}(t) + \varphi(\mathbf{e}(t))
$$

$$
+ B_1 \left( \hat{M}^T(t)\sigma(\mathbf{u}(t)) + M^T(\sigma(\bar{x}(t)) - \sigma(\mathbf{u}(t)) + \epsilon(\bar{x}(t))) \right) 
$$

(237)

We propose the following **linear observer** for the dynamics in (236)

$$
\dot{\mathbf{e}}(t) = \bar{A}(t)\mathbf{e}(t) + \bar{K}(t)(\hat{y}(t) - \hat{y}_e(t))
$$

$$
\hat{y}_e(t) = C\mathbf{e}(t)
$$

(238)

Denoting $\bar{e}(t) \triangleq \mathbf{e}(t) - \hat{e}(t)$, and $\bar{y}_e(t) \triangleq \hat{y}(t) - \hat{y}_e(t)$ we have

$$
\dot{\mathbf{e}}(t) = \bar{A}(t)\mathbf{e}(t) + \varphi(\mathbf{e}(t)) + B_1 \left( \hat{M}^T(t)\sigma(\mathbf{u}(t)) + M^T(\sigma(\bar{x}(t)) - \sigma(\mathbf{u}(t)) + \epsilon(\bar{x}(t))) \right)
$$

$$
\hat{y}_e(t) = C\mathbf{e}(t)
$$

(239)

where $\bar{A}(t) = \bar{A}(t) - \bar{K}(t)C$. The design of the error observer in (238) is based on a time varying Kalman filter design \[52\] such that $\bar{K}(t)$ is obtained through the set of equations

$$
\dot{\bar{P}}(t) = \bar{A}(t)\bar{P}(t) + \bar{P}(t)\bar{A}^T(t) - \bar{P}(t)C^T\bar{R}^{-1}C\bar{P}(t) + \bar{Q}
$$

$$
\bar{K}(t) = \bar{P}(t)C^T\bar{R}^{-1}
$$

(240)

(241)
where \( \tilde{Q} \geq \tilde{q} I > 0 \) and the solution \( \tilde{P}(t) \) is bounded, symmetric, positive definite and continuously differentiable.

**Assumption 5.** Let the equilibrium point \( \chi = 0 \) of the system

\[
\dot{\chi}(t) = f(\chi(t)) + B_2u(\chi(t))
\]

be globally exponentially stable.

Define the tracking error as \( e_d(t) = x_d(t) - x(t) \) where

\[
\dot{x}_d(t) = f(x_d(t)) + B_2u(x_d(t))
\]

\[
\dot{x}(t) = f(x(t)) + B_2u(x(t))
\]

Then the tracking error dynamics becomes

\[
\dot{e}_d(t) = f(x_d(t)) + B_2u(x_d(t)) - f(x(t)) - B_2u(x(t))
\]

The equilibrium point \( e_d = 0 \) is exponentially stable and there exists a continuously differentiable function \( V_d(t, e_d(t)) \) such that the following inequalities hold:

\[
\frac{\partial V_d}{\partial t} + \frac{\partial V_d}{\partial e_d} \dot{e}_d(t) \leq -c_3\|e_d(t)\|^2
\]

\[
\left\| \frac{\partial V_d}{\partial e_d} \right\| \leq c_4\|e_d(t)\|
\]

where \( c_1, c_2, c_3, c_4 > 0 \).

**Assumption 6.** There exists a constant \( k_u^* > 0 \) such that

\[
\|u(x(t)) - u(\hat{x}(t))\| \leq k_u^*\|e(t)\|
\]

where \( e(t) = x(t) - \hat{x}(t) \).

**Lemma 1.** Consider the following nonlinear differential equation

\[
\dot{x}(t) = f(x(t)) + B_2u(\hat{x}(t)) + B_1G_1(t)
\]
where $G_1(t) = \left( g_1(x(t), z_1(t)) - \nu_{ad}(t) \right)$ and $\dot{x}(t)$ is the solution of the differential equation (233). The tracking error dynamics

$$\dot{e}_d(t) = f(x_d(t)) + B_2u(x_d(t)) - \left( f(x(t)) + B_2u(x(t)) + B_1G_1(t) \right)$$

(251)
satisfies the following inequalities

$$\frac{\partial V_d}{\partial t} + \frac{\partial V_d}{\partial e_d} \dot{e}_d(t) \leq -c_3\|e_d(t)\|^2 + c_8\|e_d(t)\|\|e(t)\| + c_9\|e_d(t)\||G_1(t)\|$$

(253)

where $c_5$, $c_6$, $c_8$, $c_9 > 0$.

**Proof.** Differentiating the function $V_d(t, e_d(t))$ with respect to time along (251) yields

$$\dot{V}_d(t, e_d(t)) = \frac{\partial V_d}{\partial t} + \frac{\partial V_d}{\partial e_d} \dot{e}_d(t)$$

$$= \frac{\partial V_d}{\partial t} + \frac{\partial V_d}{\partial e_d} \left( f(x_d(t)) + B_2u(x_d(t)) - \left( f(x(t)) + B_2u(x(t)) + B_1G_1(t) \right) \right)$$

Further rearranging terms we obtain

$$\dot{V}_d(t, e_d(t)) = \frac{\partial V_d}{\partial t} + \frac{\partial V_d}{\partial e_d} \left( f(x_d(t)) + B_2u(x_d(t)) - \left( f(x(t)) + B_2u(x(t)) \right) \right)$$

$$+ \frac{\partial V_d}{\partial e_d} \left( u(x(t)) - u(\dot{x}(t)) \right) - \frac{\partial V_d}{\partial e_d} B_1G_1(t)$$

Using (245), (247) and (248), the Lyapunov derivative can be upper bounded as

$$\dot{V}_d(t, e_d(t)) \leq -c_3\|e_d(t)\|^2 + \left\| \frac{\partial V_d}{\partial e_d} \right\| \|B_1\| \|u(x(t)) - u(\dot{x}(t))\|$$

$$\leq -c_3\|e_d(t)\|^2 + c_4\|e_d(t)\|\|B_2\|\|u(x(t)) - u(\dot{x}(t))\|$$

$$+ c_4\|e_d(t)\|\|B_1\|\|G_1(t)\|$$

(254)

Further with Assumption 6, we have

$$\dot{V}_d(t, e_d(t)) \leq -c_3\|e_d(t)\|^2 + c_4\|B_2\|k_u^*\|e_d(t)\|\|e(t)\| + c_4\|B_1\|\|e_d(t)\|\|G_1(t)\|$$

(255)

Thus choosing $c_8 = c_4\|B_2\|k_u^*$ and $c_9 = c_4\|B_1\|$, we obtain

$$\dot{V}_d(t, e_d(t)) \leq -c_3\|e_d(t)\|^2 + c_8\|e_d(t)\|\|e(t)\| + c_9\|e_d(t)\|\|G_1(t)\|$$

(256)

**Remark 12.** Lemma 1 implies that with $e(t)$ and $G_1(t)$ as bounded inputs, the tracking error dynamics described by (251) is input-to-state stable.
5.4 Boundedness Analysis

In this section we propose a $\sigma-$ modification based adaptation law utilizing the error observer introduced in Section 5.3. Proof of boundedness of the estimation error signals $e(t)$ and $\tilde{e}(t)$ and the NN weight error $\tilde{M}(t)$ is given using Lyapunov’s direct method. For the boundedness proof we make use of the following:

Remark 13. The matrix differential Riccati equations in (234) and (240) can be represented as

\[
\dot{P}(t) = \dot{A}(t)P(t) + P(t)\dot{A}^T(t) + Q + P(t)C^T R^{-1}CP(t) \tag{257}
\]

\[
\dot{\tilde{P}}(t) = \tilde{A}(t)\tilde{P}(t) + \tilde{P}(t)\tilde{A}^T(t) + \tilde{Q} + \tilde{P}(t)C^T \tilde{R}^{-1}C\tilde{P}(t) \tag{258}
\]

Remark 14. The matrix differential Riccati equations in (257) and (258) can be represented as

\[
\dot{P}^{-1}(t) = -P^{-1}(t)\dot{A}(t) - \dot{A}^T(t)P^{-1}(t) - P^{-1}(t)Q P^{-1}(t) - C^T R^{-1}C \tag{259}
\]

\[
\dot{\tilde{P}}^{-1}(t) = -\tilde{P}^{-1}(t)\tilde{A}(t) - \tilde{A}^T(t)\tilde{P}^{-1}(t) - \tilde{P}^{-1}(t)\tilde{Q} P^{-1}(t) - C^T \tilde{R}^{-1}C \tag{260}
\]

This is seen by pre and post-multiplying (257) by $P^{-1}(t)$ and (258) by $\tilde{P}^{-1}(t)$.

Fact 2. [128] Consider the solutions of the matrix differential Riccati equations (257) and (258), with $P(0) > 0$, $R > 0$ and $\tilde{P}(0) > 0$, $\tilde{R} > 0$ respectively for (257) and (258). Then the solutions $P(t)$ and $\tilde{P}(t)$ are bounded via

\[
p_1 I \leq P(t) \leq p_2 I \tag{261}
\]

\[
p_1 I \leq \tilde{P}(t) \leq \tilde{p}_2 I \tag{262}
\]

for some $p_1$, $p_2$, $\tilde{p}_1$, $\tilde{p}_2 > 0$.

Remark 15. The bounds on the matrix differential Riccati equation solutions for $P(t)$ and $\tilde{P}(t)$ in (261) and (262) lead to the following bounds

\[
\frac{1}{p_2} I \leq P^{-1}(t) \leq \frac{1}{p_1} I \tag{263}
\]

\[
\frac{1}{\tilde{p}_2} I \leq \tilde{P}^{-1}(t) \leq \frac{1}{\tilde{p}_1} I \tag{264}
\]
Consider the composite error vector

\[
\zeta(t) = \begin{bmatrix} e^T(t) & \hat{e}^T(t) & \text{vec}(\hat{M}^T(t)) \end{bmatrix}^T
\]  

(265)

and the vector \( z(t) = \begin{bmatrix} z_1^T(t) & z_2^T(t) \end{bmatrix}^T \). Introduce the largest ball \( B_R \) defined as

\[
B_R \triangleq \left\{ \zeta \mid ||\zeta|| \leq R \right\}, \quad R > 0
\]  

(266)

that lies in

\[
\Omega_\zeta \triangleq \left\{ (e, \hat{e}, \hat{M}) \in \mathcal{R}^{n_x} \times \mathcal{R}^{n_x} \times \mathcal{R}^{n_z \times n_z} : (x, z_1, z_2, \hat{x}, \hat{e}, u) \in \mathcal{D}_x \times \mathcal{D}_{z_1} \times \mathcal{D}_{z_2} \times \mathcal{D}_\hat{x} \times \mathcal{D}_\hat{e} \times \mathcal{D}_u \right\}
\]  

(267)

such that for every \( \zeta \in B_R \), we have \( \hat{x} \in \mathcal{D}_{g_1} \). For boundedness analysis, we consider the following Lyapunov function

\[
V(\zeta, z) = e^T P^{-1} e + \hat{e}^T \hat{P}^{-1} \hat{e} + \frac{1}{2} \text{tr}
\begin{pmatrix}
\hat{M}^T \Gamma^{-1}_M \hat{M}
\end{pmatrix}
\]  

(268)

It is easily verified that

\[
\zeta^T T_1 \zeta \leq V(\zeta, z) \leq \zeta^T T_2 \zeta
\]  

(269)

where we have the following matrices:

\[
T_1 \triangleq \begin{bmatrix}
\frac{1}{p_2} & 0 & 0 \\
0 & \frac{1}{p_2} & 0 \\
0 & 0 & \frac{1}{2} \Gamma^{-1}_z
\end{bmatrix}, \quad T_2 \triangleq \begin{bmatrix}
\frac{1}{p_1} & 0 & 0 \\
0 & \frac{1}{p_1} & 0 \\
0 & 0 & \frac{1}{2} \Gamma^{-1}_z
\end{bmatrix}
\]  

(270)

Let

\[
\alpha \triangleq \min_{||\zeta|| = R} \zeta^T T_1 \zeta = R^2 T_{\min}
\]  

(271)

where \( T_{\min} \) is the minimum of the matrix \( T_1 \). Introduce the set

\[
\Omega_\alpha = \left\{ \zeta \in B_R \mid V(\zeta, z) \leq \alpha \right\}
\]  

(272)

**Assumption 7.** Let

\[
R > \gamma \sqrt{\frac{T_{\max}}{T_{\min}}} \geq \gamma
\]  

(273)
where $T_{\text{max}}$ is the maximum of the matrix $T_2$ and

$$\gamma = \max \left( \sqrt{\frac{\rho_1}{p_1^2} - 2}, \sqrt{\frac{\rho_1}{p_2^2} - 2}, \sqrt{\frac{\rho_2}{k_{M}^2 - \rho_2}} \right)$$  \hfill (274)

where

$$\begin{align*}
\rho_1 &= \kappa_1^2 \kappa_2^2 + \frac{k_M}{2} \|M\|^2_F \\
\rho_2 &= \kappa_2^2 N^2 \|B_1\|^2 \\
\kappa_1 &= \varphi^* + \|B_1\| e^* + N\|B_1\| M^* \\
\kappa_2^2 &= \left( \frac{1}{p_1^2} + \frac{1}{p_2^2} \right) \hfill (275)
\end{align*}$$

**Theorem 15.** Let the initial errors, $e(0)$, $\bar{e}(0)$ and $\bar{M}(0)$, belong to the set $\Omega_\alpha \overset{\Delta}{=} \{ \zeta \in B_R|V(\zeta, z) \leq \alpha \}$ in Fig. 26. Let the NN adaptive law be given by

$$\dot{M}(t) = -\Gamma_M \left( 2\sigma(t)e^T(t)\bar{P}^{-1}(t)B_1 - 2\sigma(t)e^T(t)\bar{P}^{-1}(t)B_1 + k_M\bar{M}(t) \right)$$  \hfill (276)

where $\Gamma_M > 0$ is the learning rate and $k_M > 0$ is the $\sigma$-modification gain. Then the estimation errors $e(t)$, $\bar{e}(t)$ and the NN weight error $\bar{M}(t)$ are uniformly ultimately bounded by $\alpha_1^{-1}(\alpha_2(\gamma))$, where $\gamma$ is given by the right hand side of (285), (286) and (287).

![Figure 26: Geometric Representation of the sets in the Error Space.](image-url)
Proof. Consider the function
\[
V(t, \zeta(t), z(t)) = e^T(t)P^{-1}(t)e(t) + e^T(t)\tilde{P}^{-1}(t)\tilde{e}(t) + \frac{1}{2}\text{tr}\left(\tilde{M}^T(t)\tilde{M}(t)\right) \tag{277}
\]
as a candidate Lyapunov function for the dynamics in (236), (239) and (276). Taking the derivative of \(V(t, \zeta(t), z(t))\) along (236), (239) and (276), and rearranging terms, we obtain
\[
\dot{V}(t, \zeta(t), z(t)) = e^T(t)\left(\bar{A}^T(t)P^{-1}(t) + P^{-1}(t)\bar{A}(t) + \dot{\tilde{P}}^{-1}(t)\right)e(t)
+ 2e^T(t)P^{-1}(t)\varphi(e(t))
+ 2e^T(t)P^{-1}(t)B_1\tilde{M}^T(t)\sigma(\mu(t)) + 2e^T(t)P^{-1}(t)B_1e(\tilde{x}(t))
+ 2e^T(t)P^{-1}(t)B_1M^T(t)(\sigma(\tilde{x}(t)) - \sigma(\mu(t)))
+ \tilde{e}^T(t)\left(\bar{A}^T(t)\tilde{P}^{-1}(t) + \tilde{P}^{-1}(t)\bar{A}(t) + \dot{\tilde{P}}^{-1}(t)\right)\tilde{e}(t)
+ 2\tilde{e}^T(t)\tilde{P}^{-1}(t)\varphi(e(t))
+ 2\tilde{e}^T(t)\tilde{P}^{-1}(t)B_1\tilde{M}^T(t)\sigma(\mu(t))
+ 2\tilde{e}^T(t)\tilde{P}^{-1}(t)B_1e(\tilde{x}(t))
+ 2\tilde{e}^T(t)\tilde{P}^{-1}(t)B_1M^T(t)(\sigma(\tilde{x}(t)) - \sigma(\mu(t)))
+ 2\tilde{e}^T(t)\tilde{P}^{-1}(t)B_1\tilde{M}^T(t)\sigma(\mu(t))
- 2\tilde{e}^T(t)P^{-1}(t)B_1\tilde{M}^T(t)\sigma(\mu(t)) + k_m\text{tr}\left(\tilde{M}^T(t)\tilde{M}(t)\right) \tag{278}
\]
Using (259) and (260) and recalling that \( \hat{e}(t) \overset{\Delta}{=} e(t) - \hat{e}(t) \), we can rewrite (278) as

\[
\dot{V}(t, \zeta(t), z(t)) = -e^T(t) \left( P^{-1}(t) Q P^{-1}(t) + C^T R^{-1} C \right) e(t) + 2e^T(t) P^{-1}(t) \varphi(e(t)) \\
+ 2(e(t) + \hat{e}(t))^T P^{-1}(t) B_1 \tilde{M}^T(t) \sigma(\mu(t)) \\
+ 2e^T(t) P^{-1}(t) B_1 \epsilon(\hat{x}(t)) \\
+ 2e^T(t) P^{-1}(t) B_1 M^T(t) (\sigma(\hat{x}(t)) - \sigma(\mu(t))) \\
- \hat{e}^T(t) \left( \tilde{P}^{-1}(t) \tilde{Q} \tilde{P}^{-1}(t) + C^T \tilde{R}^{-1} C \right) \hat{e}(t) + 2\hat{e}^T(t) \tilde{P}^{-1}(t) \varphi(e(t)) \\
+ 2(e(t) - \hat{e}(t))^T \tilde{P}^{-1}(t) B_1 \tilde{M}^T(t) \sigma(\mu(t)) \\
+ 2e^T(t) \tilde{P}^{-1}(t) B_1 \epsilon(\hat{x}(t)) \\
+ 2e^T(t) \tilde{P}^{-1}(t) B_1 M^T(t) (\sigma(\hat{x}(t)) - \sigma(\mu(t))) \\
+ 2e^T(t) \tilde{P}^{-1}(t) B_1 \tilde{M}^T(t) \sigma(\mu(t)) \\
- 2\hat{e}^T(t) P^{-1}(t) B_1 \tilde{M}^T(t) \sigma(\mu(t)) + k_M tr(\tilde{M}^T(t) \tilde{M}(t)) \tag{279}
\]

Further, cancelling out the NN terms in the adaptation law, and with \( \|\sigma(\mu(t))\| \leq N \), the Lyapunov derivative can be upper bounded as

\[
\dot{V}(t, \zeta(t), z(t)) \leq -\frac{q}{p_2} \|e(t)\|^2 + 2\|e(t)\| \|P^{-1}(t)\|\varphi^* + 2N\|\hat{e}(t)\| \|P^{-1}(t)\| \|B_1\| \|\tilde{M}(t)\|_F \\
+ 2\|e(t)\| \|P^{-1}(t)\| \|B_1\| e^* + 2N\|e(t)\| \|P^{-1}(t)\| \|B_1\| M^* \\
- \frac{q}{p_2} \|\hat{e}(t)\|^2 + 2\|\hat{e}(t)\| \|\tilde{P}^{-1}(t)\|\varphi^* + 2N\|e(t)\| \|\tilde{P}^{-1}(t)\| \|B_1\| \|\tilde{M}(t)\|_F \\
+ 2\|e(t)\| \|\tilde{P}^{-1}(t)\| \|B_1\| e^* + 2N\|\hat{e}(t)\| \|\tilde{P}^{-1}(t)\| \|B_1\| M^* \\
+ \frac{k_M}{2} \left( \|M\|_F^2 - \|\tilde{M}(t)\|_F^2 \right) \tag{280}
\]
Using (263) and (264), (280) further reduces to

\[
\dot{V}(t, \zeta(t), z(t)) \leq -\frac{q}{p_2^2} \|e(t)\|^2 + 2\frac{\|e(t)\|}{p_1} \varphi^* \\
+ 2\frac{N\|\bar{e}(t)\|}{p_1} \|B_1\| \|\bar{M}(t)\|_F + 2\frac{\|e(t)\|}{p_1} \|B_1\| \epsilon^* \\
- \frac{q}{p_2^2} \|\bar{e}(t)\|^2 + 2\frac{\|\bar{e}(t)\|}{\bar{p}_1} \varphi^* \\
+ 2\frac{N\|e(t)\|}{\bar{p}_1} \|\bar{B}_1\| \|\bar{M}(t)\|_F + 2\frac{\|\bar{e}(t)\|}{\bar{p}_1} \|\bar{B}_1\| \epsilon^* \\
+ 2N\frac{\|e(t)\|}{\bar{p}_1} \|\bar{B}_1\| M^* + 2N\frac{\|\bar{e}(t)\|}{\bar{p}_1} \|\bar{B}_1\| M^* \\
+ \frac{k_M}{2} \left(\|\bar{M}\|_F^2 - \|\bar{M}(t)\|_F^2\right) \tag{281}
\]

Rearranging the terms in (281) and using the notations in (275), we obtain

\[
\dot{V}(t, \zeta(t), z(t)) \leq -\frac{q}{p_2^2} \|e(t)\|^2 + 2\frac{\|e(t)\|}{p_1} \kappa_1 + 2\frac{N\|\bar{e}(t)\|}{p_1} \|\bar{B}_1\| \|\bar{M}(t)\|_F \\
- \frac{q}{p_2^2} \|\bar{e}(t)\|^2 + 2\frac{\|\bar{e}(t)\|}{\bar{p}_1} \kappa_1 + 2\frac{N\|e(t)\|}{\bar{p}_1} \|\bar{B}_1\| \|\bar{M}(t)\|_F \\
+ \frac{k_M}{2} \left(\|\bar{M}\|_F^2 - \|\bar{M}(t)\|_F^2\right) \tag{282}
\]

Completion of squares on the terms \((2\frac{\|e(t)\|}{p_1} \kappa_1), (2\frac{N\|\bar{e}(t)\|}{p_1} \|\bar{B}_1\| \|\bar{M}(t)\|_F), (2\frac{\|\bar{e}(t)\|}{\bar{p}_1} \kappa_1)\), \((2\frac{N\|e(t)\|}{\bar{p}_1} \|\bar{B}_1\| \|\bar{M}(t)\|_F)\) and rearranging, yields

\[
\dot{V}(t, \zeta(t), z(t)) \leq -\|e(t)\|^2 \left(\frac{q}{p_2^2} - 2\right) - \|\bar{e}(t)\|^2 \left(\frac{q}{\bar{p}_2^2} - 2\right) \\
-\|\bar{M}(t)\|^2 \left(\frac{k_M}{2} - N^2 \|\bar{B}_1\|^2 \kappa_2^2\right) + \frac{k_M}{2} \|\bar{M}\|_F^2 - \rho_2 \tag{283}
\]

and using the other notations in (275), the derivative in (283) can be further expressed as

\[
\dot{V}(t, \zeta(t), z(t)) \leq -\|e(t)\|^2 \left(\frac{q}{p_2^2} - 2\right) - \|\bar{e}(t)\|^2 \left(\frac{q}{\bar{p}_2^2} - 2\right) - \|\bar{M}(t)\|^2 \left(\frac{k_M}{2} - \rho_2\right) + \rho_1 \tag{284}
\]

Thus either of the following conditions

\[
\|e\| > \sqrt{\frac{\rho_1}{\frac{q}{p_2^2} - 2}} \tag{285}
\]

\[
\|\bar{e}\| > \sqrt{\frac{\rho_1}{\frac{q}{\bar{p}_2^2} - 2}} \tag{286}
\]

\[
\|\bar{M}\| > \sqrt{\frac{\rho_1}{\frac{k_M}{2} - \rho_2}} \tag{287}
\]

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will render $\hat{V}(t, \zeta(t), z(t)) < 0$ outside the compact set

$$B_\gamma = \left\{ \zeta \in B_R \mid \|\zeta\| \leq \gamma \right\}$$  \hspace{1cm} (288)

Notice from (273) that $B_\gamma \subset B_R$. Let $\Gamma$ be the maximum value of $V(\zeta, z)$ on the edge of $B_\gamma$

$$\Gamma \triangleq \max_{\|\zeta\| = \gamma} \zeta^T T_2 \zeta = \gamma^2 T_{\max}$$  \hspace{1cm} (289)

Introduce the level set of $V(\zeta, z)$ that touches the ball $B_\gamma$

$$\Omega_\gamma = \left\{ \zeta \mid V(\zeta, z) = \Gamma \right\}$$  \hspace{1cm} (290)

The condition in (273) ensures that $\Omega_\gamma \subset \Omega_\alpha$. Thus if the initial errors $\zeta_0 = \zeta(0)$ belong to $\Omega_\alpha$ then from Theorem 14 it follows that the error signal $\zeta$ is uniformly ultimately bounded with ultimate bound, $\alpha_1^{-1}(\alpha_2(\gamma))$, where $\gamma$ is given by (274). With the bound on $\|\tilde{M}(t)\|$, the function $G_1 = (g_1(x(t), z_1(t)) - \nu_{\text{ad}}(t))$ can be bounded as follows:

$$G_1(t) = g_1(x(t), z_1(t)) - \nu_{\text{ad}}(t)$$

$$= M^T \sigma(\bar{x}(t)) + \epsilon(\bar{x}(t)) - \hat{M}(t) \sigma(\mu(t))$$  \hspace{1cm} (291)

Now using (230) and (231) $G_1$ can be upper bounded as

$$\|G_1(t)\| \leq N\|\tilde{M}(t)\| + 2NM^* + \epsilon^*$$  \hspace{1cm} (292)

where $N$ are the number of basis functions of the NN. This implies that with $\|\tilde{M}(t)\|$ bounded, $\|G_1(t)\|$ is also bounded. Furthermore, with the bound on $\|e(t)\|$ and Remark 12, $e_{\text{ad}}(t)$ is bounded. With $x_{\text{ad}}(t)$ bounded, this implies that $x(t)$ is bounded. Together with the bound on $\|e(t)\|$ this implies that $\bar{x}(t)$ is bounded and since $u(\bar{x}(t)) = g_d(\bar{x}(t))$, where $g_d$ is a continuous map, this implies that $u$ is bounded. Thus all signals of the closed-loop system are bounded with adaptation. \hfill \Box

### 5.5 Simulation Results

In this section, we apply the theory developed in this chapter to the following applications:

1. Obstacle avoidance using a passive 2 dimensional vision system

2. Formation flight control in 2 dimension

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5.5.1 Obstacle Avoidance in 2 Dimension

We consider the problem of obstacle avoidance in two dimensions, as shown in Fig 27. The goal is for an unmanned vehicle to traverse a nominal path while avoiding obstacles in the course of travel. It is assumed that a path planning algorithm provides a path that an unmanned aerial vehicle (UAV) has to follow in order to satisfy the objectives established by a cognitive decision maker. In these simulations, this nominal path is a straight line marked at \( Y = 0 \). However, due to unforeseen obstacles in the path of the UAV, the vehicle must negotiate along the path, requiring a deviation from the planned path. Rules are developed for using camera images to detecting obstacles, estimating their relative positions to the camera, and identifying the most critical point among all the obstacles and apply the proportional navigation (PN) based guidance law to the most critical obstacle. A camera is mounted on the vehicle such that its \( X_C, Y_C \) axes lie in a horizontal plane. The vehicle has a constant speed, \( U \), and is driven by the turning rate. The camera heading is controlled to align its optical axis, \( X_C \) along the vehicle velocity vector. Simulation runs were performed for the case of stationary and moving obstacles.

**Figure 27:** Obstacle Avoidance Problem in 2 Dimension.
5.5.1.1 Case of Stationary Obstacles

The obstacles are assumed to be stationary and their positions, in an inertial frame $(X, Y)$, are denoted as $(x_{obs}, y_{obs})$.

**Camera Dynamics:** The camera dynamics is defined as

\[
\begin{bmatrix}
\dot{x}_{cam} \\
\dot{y}_{cam} \\
\dot{\psi}
\end{bmatrix} = \begin{bmatrix}
L_{LC} & U \\
0 & 0
\end{bmatrix}
\]

where $\psi$ denotes the heading angle of the vehicle upon which the camera is mounted, $\omega$ denotes the vehicle heading rate, which is the control input and $L_{LC}$ denotes the rotation matrix that takes a vector from the camera frame to the local frame and is defined as

\[
L_{LC} = \begin{bmatrix}
\cos(\psi) & -\sin(\psi) \\
\sin(\psi) & \cos(\psi)
\end{bmatrix}
\]

Substituting (294) into (293), we obtain the camera dynamics as

\[
\begin{bmatrix}
\dot{x}_{cam} \\
\dot{y}_{cam} \\
\dot{\psi}
\end{bmatrix} = \begin{bmatrix}
U \cos(\psi) \\
U \sin(\psi)
\end{bmatrix}
\]

**EKF Design:** In this section, we develop the model for the design of the EKF and explain the manner in which the EKF is designed. The relative position between the vehicle and the obstacle in the inertial frame is given as

\[
\begin{bmatrix}
x_L \\
y_L
\end{bmatrix} = \begin{bmatrix}
x_{obs} - x_{cam} \\
y_{obs} - y_{cam}
\end{bmatrix}
\]

where $x_L$, $y_L$ denote the relative position between the obstacle and the vehicle in the local frame. The relative position in the camera frame, is denoted by $x_C$, $y_C$, and is given as

\[
\begin{bmatrix}
x_C \\
y_C
\end{bmatrix} = L_{CL} \begin{bmatrix}
x_L \\
y_L
\end{bmatrix} = L_{CL} \begin{bmatrix}
x_{obs} - x_{cam} \\
y_{obs} - y_{cam}
\end{bmatrix}
\]
where \( L_{CL} = L_{CL}^T \). Differentiating (297) with respect to time we obtain

\[
\begin{bmatrix}
\dot{x}_C \\
\dot{y}_C
\end{bmatrix} = \dot{L}_{CL} \begin{bmatrix} x_{obs} - x_{cam} \\ y_{obs} - y_{cam} \end{bmatrix} - L_{CL} \begin{bmatrix}
\dot{x}_{cam} \\
\dot{y}_{cam}
\end{bmatrix} - \begin{bmatrix}
-\sin(\psi) \dot{\psi} + \cos(\psi) \dot{\psi} \\
-\cos(\psi) \dot{\psi} - \sin(\psi) \dot{\psi}
\end{bmatrix} \begin{bmatrix} x_{obs} - x_{cam} \\ y_{obs} - y_{cam} \end{bmatrix} - L_{CL} \begin{bmatrix}
\dot{x}_{cam} \\
\dot{y}_{cam}
\end{bmatrix}
\]

(298)

Substituting (294), (295) and (297) into (298) and simplifying, we get

\[
\begin{bmatrix}
\dot{x}_C \\
\dot{y}_C
\end{bmatrix} = \omega \begin{bmatrix}
-\sin(\psi) & \cos(\psi) \\
-\cos(\psi) & -\sin(\psi)
\end{bmatrix} \begin{bmatrix} x_L \\ y_L \end{bmatrix} - \begin{bmatrix} U \\ 0 \end{bmatrix}
\]

\[
= \omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_C \\ y_C \end{bmatrix} - \begin{bmatrix} U \\ 0 \end{bmatrix}
\]

(299)

Thus the model for the design of the EKF is given as

\[
\begin{bmatrix}
\dot{x}_C \\
\dot{y}_C
\end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_C \\ \dot{y}_C \end{bmatrix} - \begin{bmatrix} U \\ 0 \end{bmatrix}
\]

\[
y_m = \frac{\dot{y}_C}{x_C} + \nu
\]

(300)

where \( \dot{x}_C, \dot{y}_C \) are the estimates of the true relative position between the obstacle and the vehicle \( x_C, y_C \); \( y_m \) is the available measurement which is the position of an obstacle in an image plane and \( \nu \) is a band limited white noise process, with known standard deviation. The EKF design consists of two phases: the update phase and the prediction phase. The general equations are given as [52]:

**Update:**

\[
K_k = P_k^{-1} H_k^T \left( H_k P_k^{-1} H_k^T + R_k \right)^{-1}
\]

\[
\hat{x}_{C_k} = \hat{x}_{C_k}^- + K_k (y_m - \hat{y}_{m_k})
\]

\[
P_k = \left( P_k^- - K_k H_k P_k^- \right)
\]

(301)

where \( K_k \) is the Kalman gain, \( \hat{x}_{C_k} = \left[ \hat{x}_C \ \hat{y}_C \right]^T \) is the state estimate vector, \( P_k \) is the covariance associated with the estimation error before an update, \( P_k^- \) is the covariance associated with the estimation error after an update, \( R_k \) is the measurement noise matrix, \( H_k \) is
the measurement linearized about the current trajectory given by

$$H_k = \left. \frac{\partial y_m}{\partial \mathbf{x}_C} \right|_{x_C = \mathbf{x}_{c_k}^*} = \begin{bmatrix} -\frac{\hat{\gamma}_{y_k}}{\hat{x}_{c_k}} & \frac{1}{\hat{x}_{c_k}} \end{bmatrix}$$

and $\hat{y}_{m_k} = \frac{\hat{y}_{y_k}}{\hat{x}_{c_k}}$ is the estimate of the measurement.

**Prediction:**

$$\mathbf{x}_{c_{k+1}} = \mathbf{x}_c + \left( A(\omega_k) + BU \right) \Delta t$$

$$P_{k+1} = \phi_k P_k \phi_k^T + Q_k$$

where $\mathbf{x}_{c_{k+1}}$ is the state projected to the next time instant, $\Delta t$ is the sampling time interval and $A(\omega_k)$ and $B$ are given by

$$A(\omega_k) = \begin{bmatrix} 0 & \omega_k \\ -\omega_k & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

The state transition matrix $\phi_k = I + A(\omega_k)\Delta t$ and the covariance matrix of the discretized process noise is given by $Q_k = \int_0^{\Delta t} e^{\mathbf{F}_k s} Q e^{\mathbf{F}_k^T s} ds$, where $Q$ is the continuous process noise matrix.

**Proportional Navigation (PN) Guidance Law:** A PN guidance strategy is used for collision avoidance. An interception point for the most critical obstacle is determined as shown in Fig. 28. The details of the PN guidance law are given in [83]. The PN guidance law for obstacle avoidance is given as

$$\omega_{OA} = -\frac{3 \text{sgn}(ZEM) \left(d_{\text{min}} - |ZEM|\right)}{U_{tg}^2}$$

where $ZEM$ is the zero effort miss to the closest approach to each obstacle, $t_{go}$ is the time-to-go, which is defined as the time interval from the current time to a time when the vehicle reaches the point of closest approach to an obstacle, and $d_{\text{min}}$ defines a boundary around each obstacle that the vehicle must keep out of. Since $t_{go}$ and $ZEM$ explicitly express how closely the vehicle approaches the obstacle when no control effort is applied, it is more efficient to use estimates of $t_{go}$ and $ZEM$ than the estimates of the relative position, $\hat{\mathbf{x}}$, in a collision
avoidance strategy. The estimates for zero effort miss distance and time-to-go are constructed from the state estimates of the EKF in (300) as follows:

\[
t_{go} = \frac{\dot{x}_C}{U}, \quad ZEM = \ddot{y}_C
\]  

(306)

**Figure 28:** Obstacle Avoidance using Proportional Navigation Guidance.

### 5.5.1.2 Discussion of Results - Stationary Obstacles

Figs. 29 and 30 compare the performances of obstacle avoidance when employing the EKF design and the NN based adaptive EKF design for the case when the obstacles are stationary. As depicted in Fig. 29, the true positions of the obstacles are marked by ‘o’, while the estimated obstacle positions are marked by ‘+’. The large solid circles denote the true safe boundaries around the obstacles that the vehicle must keep out of, while the large dashed circles denote the estimated safe boundary. As seen from Fig. 29, with the estimates from only the EKF fed into the guidance law, the vehicle violates the safe boundaries around the 4th and 5th, with a major violation occurring at the safe boundary of the 5th obstacle. These violations are due to the transient in the estimation error of the obstacle positions, and are not due to the guidance law. Fig. 30 shows that with the estimates from the adaptive EKF fed into the guidance law, the vehicle is able to better maneuver around the obstacles.
Figure 29: Stationary Obstacles: guidance law uses estimates from EKF.

Figure 30: Stationary Obstacles: guidance law uses estimates from EKF+NN.
5.5.1.3 Case of Moving Obstacles

**Obstacle Dynamics:** The obstacle motion dynamics are given as

\[
\begin{align*}
\dot{x}_{\text{obs}} &= v_{x,\text{obs}} \\
\dot{y}_{\text{obs}} &= v_{y,\text{obs}} \\
\dot{v}_{x,\text{obs}} &= a_{x,\text{obs}} \\
\dot{v}_{y,\text{obs}} &= a_{y,\text{obs}} \\
\dot{a}_{x,\text{obs}} &= w_x \\
\dot{a}_{y,\text{obs}} &= w_y
\end{align*}
\]

(307)

where \( x_{\text{obs}}, y_{\text{obs}}, v_{x,\text{obs}}, v_{y,\text{obs}}, a_{x,\text{obs}}, a_{y,\text{obs}} \) denote, respectively, the \( X \) and \( Y \) components of the obstacle position, velocity and acceleration in the inertial frame, and \( a_{x,\text{obs}}, a_{y,\text{obs}} \) are modelled as random walk processes where \( w_x, w_y \) are zero mean, band limited white noise processes of known statistics.

**Camera Dynamics:** The process dynamics for the camera motion in an inertial frame are given as

\[
\begin{bmatrix}
\dot{x}_{\text{cam}} \\
\dot{y}_{\text{cam}} \\
\dot{\psi}
\end{bmatrix} = L_{\text{LC}} \begin{bmatrix} U \\ 0 \end{bmatrix}
\]

(308)

where \( \psi \) denotes the heading angle of the vehicle upon which the camera is mounted, \( \omega \) denotes the vehicle heading rate, which is the control input and \( L_{\text{LC}} \) denotes the rotation matrix to take a vector from the camera frame to the local frame and is defined in (294). Substituting (294) into (308), we obtain the camera dynamics as follows:

\[
\begin{bmatrix}
\dot{x}_{\text{cam}} \\
\dot{y}_{\text{cam}} \\
\dot{\psi}
\end{bmatrix} = \begin{bmatrix} U \cos(\psi) \\ U \sin(\psi) \end{bmatrix}
\]

(309)

**EKF Design:** To arrive at the model for the EKF design, we first consider the relative dynamics between the obstacle and the vehicle in the local frame. The relative position
between the obstacle and the vehicle is given as
\[
\begin{bmatrix}
  x_L \\
  y_L
\end{bmatrix} =
L_{\text{CL}}
\begin{bmatrix}
  x_{\text{obs}} - x_{\text{cam}} \\
  y_{\text{obs}} - y_{\text{cam}}
\end{bmatrix}
\] (310)

where \( x_L, y_L \) denote the relative position between the obstacle and the vehicle in the local frame. The relative position in the camera frame, is denoted by \( x_C, y_C \), and is given as
\[
\begin{bmatrix}
  x_C \\
  y_C
\end{bmatrix} = L_{\text{CL}}
\begin{bmatrix}
  x_L \\
  y_L
\end{bmatrix} = L_{\text{CL}}
\begin{bmatrix}
  x_{\text{obs}} - x_{\text{cam}} \\
  y_{\text{obs}} - y_{\text{cam}}
\end{bmatrix}
\] (311)

where \( L_{\text{CL}} = L_{\text{CL}}^T \). Differentiating (311) once with respect to time, we arrive at
\[
\begin{bmatrix}
  \dot{x}_C \\
  \dot{y}_C
\end{bmatrix} = L_{\text{CL}}
\begin{bmatrix}
  \dot{x}_{\text{obs}} - \dot{x}_{\text{cam}} \\
  \dot{y}_{\text{obs}} - \dot{y}_{\text{cam}}
\end{bmatrix} + \dot{L}_{\text{CL}}
\begin{bmatrix}
  x_{\text{obs}} - x_{\text{cam}} \\
  y_{\text{obs}} - y_{\text{cam}}
\end{bmatrix}
\] (312)

Substituting (307), (309) and (310), into (312) and recalling that \( L_{\text{CL}} = L_{\text{CL}}^T \), we obtain
\[
\begin{bmatrix}
  \dot{x}_C \\
  \dot{y}_C
\end{bmatrix} =
L_{\text{CL}}
\begin{bmatrix}
  x_{C} \\
  y_{C}
\end{bmatrix} + \omega
\begin{bmatrix}
  -\sin(\psi) & \cos(\psi) \\
  -\cos(\psi) & -\sin(\psi)
\end{bmatrix}
\begin{bmatrix}
  x_{\text{obs}} - x_{\text{cam}} \\
  y_{\text{obs}} - y_{\text{cam}}
\end{bmatrix}
\] (313)

Further noting that
\[
\begin{bmatrix}
  x_L \\
  y_L
\end{bmatrix} = L_{\text{LC}}
\begin{bmatrix}
  x_C \\
  y_C
\end{bmatrix} =
L_{\text{LC}}
\begin{bmatrix}
  \cos(\psi) & -\sin(\psi) \\
  \sin(\psi) & \cos(\psi)
\end{bmatrix}
\begin{bmatrix}
  x_{C} \\
  y_{C}
\end{bmatrix}
\] (314)

Substituting (314) into (313) and simplifying, we obtain
\[
\begin{bmatrix}
  \dot{x}_C \\
  \dot{y}_C
\end{bmatrix} =
\begin{bmatrix}
  0 & \omega \\
  -\omega & 0
\end{bmatrix}
\begin{bmatrix}
  x_C \\
  y_C
\end{bmatrix} + \omega
\begin{bmatrix}
  v_{x_{\text{obs}}} \cos(\psi) + v_{y_{\text{obs}}} \sin(\psi) - U \\
  -v_{x_{\text{obs}}} \sin(\psi) + v_{y_{\text{obs}}} \cos(\psi)
\end{bmatrix}
\] (315)
Therefore, the relative motion dynamics can finally be expressed as

\[
\begin{bmatrix}
\dot{x}_C \\
\dot{y}_C
\end{bmatrix} = \begin{bmatrix}
0 & \omega \\
-\omega & 0
\end{bmatrix}
\begin{bmatrix}
x_C \\
y_C
\end{bmatrix} + L_{CL}
\begin{bmatrix}
v_{x_{obs}} \\
v_{y_{obs}}
\end{bmatrix} - \begin{bmatrix}
U \\
0
\end{bmatrix}
\]  

(316)

Notice from the form of the model in (316), the EKF model is comprised of 2 states, which are \( x_C, y_C \) and the constant vector \( \begin{bmatrix} U \\ 0 \end{bmatrix} \). The states \( v_{x_{obs}}, v_{y_{obs}} \), which form a part of the unmodeled dynamics vector of the true process, as shown in (307), are considered to be unmodeled dynamics that affect the model in (316). Thus the model for the EKF design, which is the relative dynamics between the obstacle and camera in the camera frame, is given as

\[
\begin{bmatrix}
\dot{x}_C \\
\dot{y}_C
\end{bmatrix} = \begin{bmatrix}
0 & \omega \\
-\omega & 0
\end{bmatrix}
\begin{bmatrix}
x_C \\
y_C
\end{bmatrix} - \begin{bmatrix}
U \\
0
\end{bmatrix}
\]

\( y_m = \frac{\dot{y}_C}{x_C} + \nu \)  

(317)

where \( \hat{x}_C, \hat{y}_C \) denote the estimates of the relative positions between the camera and the obstacle in the camera frame along the camera \( X_C \) and \( Y_C \) axes respectively, \( y_m \) is the measurement of the obstacle position in an image plane, as shown in Fig 27, and \( \nu \) is a zero mean, band limited white noise process of standard deviation 0.01. The manner in which the EKF is designed for state estimation is summarized in (301)-(304).

**NN Design:** We wish to augment the EKF with a linearly parameterized NN to improve the estimation performance of the EKF in the presence of the unmodeled dynamics. Comparing the model in (317) with (223), we see that

\[
z_1 = \begin{bmatrix} v_{x_{obs}} \\ v_{y_{obs}} \end{bmatrix}^T
\]  

(318)

is a part of

\[
z = \begin{bmatrix} v_{x_{obs}} \\ v_{y_{obs}} \\ a_{x_{obs}} \\ a_{y_{obs}} \end{bmatrix}^T
\]  

(319)

The NN based augmented EKF model is given as

\[
\begin{bmatrix}
\dot{x}_C \\
\dot{y}_C
\end{bmatrix} = \begin{bmatrix}
0 & \omega \\
-\omega & 0
\end{bmatrix}
\begin{bmatrix}
\hat{x}_C \\
\hat{y}_C
\end{bmatrix} - \begin{bmatrix}
U \\
0
\end{bmatrix} + L_{CL}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\nu_{ad}(t)
\]

\( y_m = \frac{\dot{y}_C}{x_C} + \nu \)  

(320)
where \( \mathbf{v}_{ad}(t) = \begin{bmatrix} \mathbf{v}_{ad_1}(t) & \mathbf{v}_{ad_2}(t) \end{bmatrix}^T \) is the output of a linearly parameterized NN which is designed as
\[
\mathbf{v}_{ad}(t) = \hat{M}^T(t) \sigma(\mathbf{\mu}(t))
\]
where \( \hat{M}(t) \) is the estimate of the NN weights adapted online according to (276) and \( \mathbf{\mu}(t) \) is the vector of delayed values as described by (229).

**Proportional Navigation Guidance Law**: The PN guidance strategy used for collision avoidance is detailed in [83] and is summarized in the section which discusses the case of stationary obstacles. In this section we explain the manner in which the estimates for \( t_{go} \) and \( ZEM \) are calculated for the case of moving obstacles. The estimates of \( t_{go} \) and \( ZEM \) can be obtained by solving the dynamics of (316), setting the control input \( \omega = 0 \), and assuming constant obstacle velocity. Thus from (307), we have
\[
\begin{bmatrix}
\dot{v}_{x,obs} \\
\dot{v}_{y,obs}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\Rightarrow v_{x,obs}(t_k) = v_{x,obs}(t), \quad v_{y,obs}(t_k) = v_{y,obs}(t)
\]
and with \( \omega = 0 \), the \( \dot{\psi} \) equation of (309) reduces to
\[
\dot{\psi} = \omega = 0
\Rightarrow \psi(t_k) = \psi(t)
\]
Recall that the relative motion dynamics between the vehicle and the obstacle expressed in the camera frame is
\[
\begin{bmatrix}
\dot{x}_C \\
\dot{y}_C
\end{bmatrix} =
\begin{bmatrix}
0 & \omega \\
-\omega & 0
\end{bmatrix}
\begin{bmatrix}
x_C \\
y_C
\end{bmatrix} + L_{CL}
\begin{bmatrix}
v_{x,obs} \\
v_{y,obs}
\end{bmatrix} -
\begin{bmatrix}
U \\
0
\end{bmatrix}
= 
\begin{bmatrix}
\omega y_C + v_{x,obs} \cos(\psi) + v_{y,obs} \sin(\psi) - U \\
-\omega x_C - v_{x,obs} \sin(\psi) + v_{y,obs} \cos(\psi)
\end{bmatrix}
\]
Setting \( \omega = 0 \), and using (322), (323) we have
\[
\begin{bmatrix}
\dot{x}_C \\
\dot{y}_C
\end{bmatrix} =
\begin{bmatrix}
v_{x,obs} \cos(\psi) + v_{y,obs} \sin(\psi) - U \\
-v_{x,obs} \sin(\psi) + v_{y,obs} \cos(\psi)
\end{bmatrix}
\]
Solving the first equation of (325) for $x_C(t)$, we obtain

$$x_C(t_k) = x_C(t) + \left( v_{x_{obs}} \cos(\psi) + v_{y_{obs}} \sin(\psi) - U \right)(t_k - t)$$

$$= x_C(t) + \left( v_{x_{obs}} \cos(\psi) + v_{y_{obs}} \sin(\psi) - U \right)t_{go} \tag{326}$$

To obtain an expression for $t_{go}$, we set $x_C(t_k) = 0$ in (326) and use the estimate $\hat{x}_C$ of $x_C$ to construct an estimate for $t_{go}$ as

$$t_{go} = \frac{\hat{x}_C}{v_{x_{obs}} \cos(\psi) + v_{y_{obs}} \sin(\psi) - U} \tag{327}$$

Solving the second equation of (325) for $y_C(t)$ we obtain

$$y_C(t_k) = y_C(t) - \left( v_{x_{obs}} \sin(\psi) - v_{y_{obs}} \cos(\psi) \right)(t_k - t)$$

$$= y_C(t) - \left( v_{x_{obs}} \sin(\psi) - v_{y_{obs}} \cos(\psi) \right)t_{go} \tag{328}$$

To obtain an expression for $ZEM$, we solve the second equation of (328) for $y_C(t_k)$ and use the estimate $\hat{y}_C$ of $y_C$. Thus we obtain

$$ZEM = \hat{y}_C(t_k)$$

$$= \hat{y}_C - \left( v_{x_{obs}} \sin(\psi) - v_{y_{obs}} \cos(\psi) \right)t_{go} \tag{329}$$

where $t_{go}$ is the expression in (327), and $\hat{x}_C(t_k)$, $\hat{y}_C(t_k)$ are the estimates of the adaptive EKF in (320).

**Remark 16.** When the obstacles are stationary, the expressions for time-to-go and zero effort miss distance in (327) and (329) reduce to the expression in (306).

**Remark 17.** For the case of moving obstacles the purpose of the NN based adaptive element is to approximate the obstacle velocity which form the unmodeled dynamics vector, $g_1(x(t), z_1(t))$. Notice that in the absence of the NN the expressions for $t_{go}$ and $ZEM$ are simply given by (306), which are the expressions of $t_{go}$ and $ZEM$ for the case of stationary obstacles. This is a potential source of modelling error when only the EKF is used, for the case of moving obstacles. In the presence of the NN the expressions for $t_{go}$ and $ZEM$ can be
expressed as

\[
t_{go} = -\frac{\hat{x}_C}{\left( \begin{bmatrix} \cos(\psi) & \sin(\psi) \\ \nu_{ad_1} \\ \nu_{ad_2} \end{bmatrix} \right) - U}
\]

\[ZEM = \hat{y}_C - \left( \begin{bmatrix} \sin(\psi) & -\cos(\psi) \\ \nu_{ad_1} \\ \nu_{ad_2} \end{bmatrix} \right) t_{go}\]

where \(\nu_{ad_1}\) and \(\nu_{ad_2}\) are designed as described in (321).

5.5.1.4 Discussion of Results - Moving Obstacles

Figs. 31 and 32 compare the performances of obstacle avoidance when employing the EKF design and the NN based adaptive EKF design for the case when the obstacles are moving. The true positions of the obstacles are marked by 'o', while the estimated obstacle positions are marked by '+'.

The large solid circle centered around each 'o' denotes the true safe boundary of the obstacles while the large dashed circles centered around each '+' denotes the estimated safe boundary. For situations in which an obstacle boundary is not violated, the estimated obstacle positions are shown for the instant in time when the obstacle first leaves the field of view of the camera. For situations in which an obstacle boundary is violated, the estimated obstacle positions are shown for the instant in time that the obstacle boundary is violated.

As seen from Fig. 31, the vehicle, with estimates from the EKF is unable to avoid the safe boundaries of the 4th and 5th obstacles, with a major violation occurring at the boundary of the 5th obstacle.

Fig. 32 shows that with the adaptive estimates the vehicle is able to better maneuver around the boundaries of the obstacles. Figs. 33 and 34 show a significant improvement in the estimates of the position of the 6th obstacle with adaptation in the EKF.

This can also be seen from Fig. 32 where the estimation of the safe boundaries of the 5th and 6th obstacles are close to the true safe boundaries respectively, and all the obstacles are avoided.
**Figure 31:** Moving Obstacles: guidance law uses estimates from EKF.

**Figure 32:** Moving Obstacles: guidance law uses estimates from EKF+NN.
Figure 33: Moving Obstacles: position estimation errors for the 5th obstacle.

Figure 34: Moving Obstacles: position estimation errors for the 6th obstacle.
5.5.2 Formation Flight Control in 2 Dimension

In this section, we consider the problem of target-follower tracking in two dimensions, wherein the range between the target and the follower aircrafts is being regulated to 2 wing spans by feeding back estimates of the target velocity obtained by processing camera images. The goal is to accomplish a task in an unmanned system that is commonly performed in a manned system that relies primarily on visual information. The tracking scenario is shown in Fig 35, wherein $T$ and $F$ represent the target/leader and follower respectively, $(V_{x_T}, V_{y_T})$ denotes the components of follower velocity and $(V_{x_T}, V_{y_T})$ denotes the components of target velocity, $\beta$ is the bearing angle, $r$ represents the range between the target aircraft and the follower aircraft, $b$ represents the size of the target and $\alpha$ the angle subtended by the target in an image plane.

Figure 35: Target Follower Tracking Scenario in 2 Dimension.

This problem is approached from the perspective of using:

1. bearings-only measurement

2. bearing angle and the angle subtended by the target in the image plane as measurements, which will be referred to as a "two-angles problem".
Remark 18. An advantage of using two angles as measurements is that observability is preserved even when the follower aircraft is not maneuvering.

5.5.2.1 Bearings only measurement

Remark 19. The equations of motion for the true dynamics and the equations of motion for the model of the design of the EKF are derived in detail and shown in section 4.5 of chapter 4. Hence we only re-write the model for the design of the EKF in this section.

Remark 20. For the purpose of simulations, all the dimensional variables except time are nondimensionalized with respect to gravity, \( g = 32.2 \text{ ft/s}^2 \).

The process dynamics can be described in polar coordinates by the following set of nonlinear differential equations [90]:

\[
\frac{d}{dt} \begin{bmatrix} \dot{\beta} \\ \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -2 \dot{\beta}^2 + \frac{1}{r}(a_y \cos \beta - a_x \sin \beta) \\ \dot{\beta}^2 - (\dot{\beta})^2 + \frac{1}{r}(a_y \sin \beta + a_x \cos \beta) \\ \dot{\theta} \end{bmatrix} \tag{332}
\]

and the measurements are given by the following equations:

\[
\beta_m(t) = \beta(t) + \nu_\beta(t) \tag{333}
\]

where \( a_x \) and \( a_y \) are the horizontal relative acceleration components in a Cartesian frame. The measurement noise \( \nu_\beta \) is a band limited zero mean white noise processes with a standard deviation of 0.01. The initial covariance matrix was chosen as

\[
P_0 = \begin{bmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix} \tag{334}
\]

where the individual elements on the diagonal represent the variances of the initial errors in the state variables. The measurement noise matrix, \( R \) is a scalar and corresponds to the variance of the measurement noise

\[
R = 0.01^2 \tag{335}
\]
The adaptive law in (165) was implemented for the simulation runs with the learning rate \( \Gamma_M = 1 \) and the gain \( k_o = 1.4 \) sampled delayed values of the measurement were used. The following sigmoidal basis function was used

\[
\sigma(x) = \frac{1}{1 + e^{-\alpha x}}
\]

with the activation potential \( \alpha = 10 \).

5.5.2.2 Discussion of Results - Bearings only Measurement

The simulation run was performed for a target maneuvering in a sinusoidal manner with an acceleration of \( 0.3g \). Figs. 36, 37 and 38 show the performance of the EKF and Figs. 39, 40 and 41 show the performance of the EKF + NN. In Figs. 37 and 40, the top sub-figure shows the regulated range between the target and follower aircrafts on full scale, while the bottom sub-figure is a magnified version of the top sub-figure. It is clear from the range plot of Fig. 37 that the performance of the EKF is severely degraded. Fig. 40 shows that there is a remarkable improvement when the EKF is augmented with the NN based adaptive element.

5.5.2.3 Discussion of figure 36

From Fig. 36, we see that even though the target trajectory estimates are clearly unbounded, without NN augmentation, one would expect that with the state estimates being used to form the feedback controller, the performance of the follower in tracking the target would be severely degraded. However, this does not seem to be the case. One possible explanation for this is that the follower velocity commands along the line-of-sight are bounded via saturation limits. Hence even though the velocity estimates go unbounded, the saturation command limits these estimates.
Figure 36: Sinusoidal Target Maneuver - Performance of EKF

Figure 37: True and Estimated LOS Range - Performance of EKF
**Figure 38:** State Estimation Errors - Performance of EKF

**Figure 39:** Sinusoidal Target Maneuver - Performance of EKF + NN
Figure 40: True and Estimated LOS Range - Performance of EKF + NN

Figure 41: State Estimation Errors - Performance of EKF + NN
5.5.2.4 Two angles measurement

The process dynamics can be described in polar coordinates by the following set of non-linear differential equations [90]:

\[
\frac{d}{dt} \begin{bmatrix} \dot{\beta} \\ \frac{\dot{r}}{r} \\ \frac{1}{r} \\ b \end{bmatrix} = \begin{bmatrix} -2\dot{\beta} \frac{\dot{x}}{r} + \frac{1}{r}(a_y \cos \beta - a_x \sin \beta) \\ \dot{\beta}^2 - \left(\frac{\dot{x}}{r}\right)^2 + \frac{1}{r}(a_y \sin \beta + a_x \cos \beta) \\ \frac{\dot{r}}{r} \\ 0 \end{bmatrix}
\]  

(337)

and the measurements are given by the following equations:

\[
\begin{align*}
\beta_m(t) &= \beta(t) + \nu_\beta(t) \\
\alpha_m(t) &= 2 \tan^{-1}\left(\frac{b}{2r}\right) + \nu_\alpha(t)
\end{align*}
\]  

(338)

where \(b\), which denotes the size of the target, is assumed to be constant and \(a_x\) and \(a_y\) are the horizontal relative acceleration components in a Cartesian frame. The measurement noise \(\nu_\beta\) and \(\nu_\alpha\) are band limited zero mean white noise processes with a standard deviation of 0.01. The initial covariance matrix was chosen as

\[
P_0 = \begin{bmatrix}
0.01 & 0 & 0 & 0 & 0 \\
0 & 0.3 & 0 & 0 & 0 \\
0 & 0 & 0.01 & 0 & 0 \\
0 & 0 & 0 & 0.1 & 0 \\
0 & 0 & 0 & 0 & 0.001
\end{bmatrix}
\]  

(339)

The measurement noise matrix, \(R\), corresponds to the variance of the measurement noise channels, \(\beta_m(t)\) and \(\alpha_m(t)\) and is given by \(R = \text{diag} \begin{bmatrix} 0.01^2 & 0.01^2 \end{bmatrix}\). The adaptive law in (165) was implemented for the simulation runs with the learning rate \(\Gamma_M = 3.2\), the gain \(k_\sigma = 8\) and 5 sampled delayed values of each measurement were used. The following sigmoidal basis function was used \(\sigma(x) = \frac{1}{1 + e^{-\alpha x}}\), with the activation potential \(\alpha = 10\).

5.5.2.5 Discussion of Results - Two Angles Measurement

We considered the situation in which a follower aircraft regulates its range from the leader (target) aircraft by feeding back estimates of the target velocity. The line-of-sight angle,
\( \beta_m \), and the angle subtended by the target in the image plane, \( \alpha_m \), are the two available measurements. The goal was to maintain a commanded range of 2 wing spans between the target and the follower, with the target performing the maneuvers described below.

**Circular Trajectory Target Maneuver with a heading command change of \( \frac{\pi}{30} \) rad/s:**

Figs. 42 and 43 respectively show the trajectories of the leader and follower aircrafts, for the EKF and EKF + NN. Figs. 44 and 45 respectively show the performance of the EKF and the EKF + NN in estimating the range between the leader and the follower aircrafts. From Fig. 44 we see that the bias in the range estimates between the follower and leader aircrafts is removed with NN augmentation, as shown in Fig. 45. The estimation errors of the LOS range histories between the leader and the follower aircrafts with and without NN augmentation is shown in Fig. 46. The lower sub figure of Fig. 46 shows the range estimation errors on a magnified scale, and we see that the NN augmented EKF provides nearly an unbiased estimate. Fig. 47 shows that augmenting the EKF with an NN based adaptive element greatly reduces the bias in the estimation error of the target size, \( b \).

**Box Trajectory Target Maneuver:**

Figs. 48 and 49 respectively show the trajectories of the leader and follower aircrafts, for the EKF and EKF + NN. Figs. 50 and 51 respectively show the performance of the EKF and the EKF + NN in estimating the range between the leader and the follower aircrafts. From Fig. 50 we see that the bias in the range estimates between the follower and leader aircrafts is removed with NN augmentation, as shown in Fig. 51. The estimation errors of the LOS range histories between the leader and the follower aircrafts with and without NN augmentation is shown in Fig. 52. We see that the NN augmented EKF produces a nearly unbiased estimate as seen from the lower window of Fig. 52. Fig. 53 shows the target size estimation error for the case of EKF and EKF + NN. We observe here that augmenting the EKF with the NN helps to reduce the bias in the error of the target size.

**Sinusoidal Target Maneuver with an acceleration of 0.3g:**

Figs. 54 and 55 respectively show the trajectories of the leader and follower aircrafts, for the EKF and EKF + NN. Figs. 56 and 57 respectively show the performance of the EKF and
the EKF + NN in estimating the range between the leader and the follower aircrafts. Once again from Fig. 56 we see that the bias in the range estimates between the follower and leader aircrafts is removed with NN augmentation, as shown in Fig. 57. The estimation errors of the LOS range histories between the leader and the follower aircrafts with and without NN augmentation is shown in Fig. 58. We see that the NN augmented EKF produces a nearly unbiased estimate as seen from the lower window of Fig. 58. Fig. 59 shows that augmenting the EKF with an NN based adaptive element greatly reduces the bias in the estimation error of the target size.

![Diagram](image)

**Figure 42:** Formation Trajectory and Target Trajectory Estimates - Performance of EKF

### 5.6 Conclusions

In this paper we address the problem of augmenting an EKF with an adaptive element. The estimated states from the adaptive EKF are used for feedback control. An application to obstacle avoidance for the cases of stationary and moving obstacles has been treated. In the case of moving obstacles the main source of uncertainty/unmodeled dynamics arises by designing the EKF assuming that the obstacles are fixed. The velocity of the obstacles are
treated as the source of unmodeled dynamics which are approximated by the output of a linearly parameterized NN. In the case of fixed obstacles, the main source of uncertainty is in the approximation of the EKF to treat only the first order terms and neglect the higher order terms. In the case of the formation flight control example, simulations show that augmenting the EKF with an NN helps in removing the bias in the range estimates and also improves the estimate of the target size. In the bearings-only case the NN was able to correct for the divergent estimates that were produced by the EKF. The approach is applicable to uncertain multivariable nonlinear systems with uncertain parameters and unmodeled dynamics coupled to the process. Boundedness of error signals is shown through Lyapunov’s direct method. Simulations show that the estimation accuracy is improved by augmenting the EKF with an NN based adaptive element that compensates for estimation errors due to the nonlinearity or unmodeled dynamics, consequently helping the UAV to avoid the safe boundaries of the obstacles.
Figure 44: True and Estimated LOS Range - Performance of EKF

Figure 45: True and Estimated LOS Range - Performance of EKF + NN
Figure 46: Range estimate error - two angles measurement case

Figure 47: Target size estimate error - two angles measurement case
**Figure 48:** Formation Trajectory and Target Trajectory Estimates - Performance of EKF

**Figure 49:** Formation Trajectory and Target Trajectory Estimates - Performance of EKF + NN
Figure 50: True and Estimated LOS Range - Performance of EKF

Figure 51: True and Estimated LOS Range - Performance of EKF + NN
Figure 52: Range estimate error - two angles measurement case

Figure 53: Target size estimate error - two angles measurement case
Figure 54: Formation Trajectory and Target Trajectory Estimates - Performance of EKF

Figure 55: Formation Trajectory and Target Trajectory Estimates - Performance of EKF + NN
Figure 56: True and Estimated LOS Range - Performance of EKF

Figure 57: True and Estimated LOS Range - Performance of EKF + NN
Figure 58: Range estimate error - two angles measurement case

Figure 59: Target size estimate error - two angles measurement case
CHAPTER 6

MISSILE TARGET INTERCEPTION

In chapter 5, we developed a theory for adaptive estimation and control of uncertain systems. The adaptive estimator takes the form of augmenting an extended Kalman filter (EKF) which is designed for the system without unmodeled dynamics. The design of the adaptive element employs a linearly parameterized neural network (NN). The states of the adaptive estimator are used to form the feedback controller. The approach is applicable to uncertain multivariable nonlinear systems with uncertain parameters and unmodeled dynamics coupled to the process. We illustrated the theory developed on the following examples:

1. obstacle avoidance in two dimension for the cases of stationary and moving obstacles
2. formation flight control in two dimension

In this chapter we illustrate the theory developed in chapter 5 to the problem of missile-target interception in two dimension. We consider the cases of low speed maneuvering target and high speed maneuvering target.

6.1 Low Speed Maneuvering Target

In this section, we show the true dynamics for the missile-target problem formulation, the model for the design of the EKF, the augmented proportional navigation (APN) guidance law designed to guide the missile towards the target and some simulation results are shown depicting the performance of the EKF and the EKF augmented with the NN.

6.1.1 Problem Formulation

Consider the two dimensional point mass missile-target engagement scenario as shown in Fig. 60 where M and T denote missile and target respectively, $a_T$, $V_T$, $\theta_T$ denote the acceleration, velocity and flight path angle of the target respectively, $a_M$, $V_M$, $\theta_M$ are the
acceleration, velocity and flight path angle of the missile respectively, $\lambda$ denotes the line-of-sight (LOS) angle and $R$ denotes the LOS range.

![Diagram](image)

**Figure 60**: 2D Missile-Target Engagement Geometry.

The true dynamics of the interception scenario are given by the following differential equations:

$$
\dot{\lambda} = \frac{V_T \sin(\theta_T - \lambda) - V_M \sin(\theta_M - \lambda)}{R}
$$

$$
\dot{R} = V_T \cos(\theta_T - \lambda) - V_M \cos(\theta_M - \lambda)
$$

$$
\dot{\theta}_T = \frac{a_T}{V_T}
$$

$$
\dot{\theta}_M = \frac{a_M}{V_M}
$$

(340)

The measurements for the process are $\lambda$ and $R$ and are given by the following equations:

$$
y_R = R + \nu_R
$$

(341)

$$
y_\lambda = \lambda + \nu_\lambda
$$

(342)

where $\nu_R$ and $\nu_\lambda$ are zero mean, band limited white noise processes. Before proceeding further, we impose the following assumptions:
**Assumption 8.** The target acceleration is normal to the target velocity and the missile acceleration is normal to the missile velocity, i.e., \( a_T \perp V_T \) and \( a_M \perp V_M \).

**Assumption 9.** The missile velocity, \( V_M \) and target velocity, \( V_T \), are assumed to be constant.

### 6.1.2 Extended Kalman Filter (EKF) Design

In this section, we first obtain the model for the design of the EKF and then summarize the manner in which the EKF is designed to estimate the states. The EKF is a nonlinear filter that relinearizes the trajectory about the current state estimate \( \hat{\mathbf{x}}(t) \). This improves the filtering accuracy since \( \hat{\mathbf{x}}(t) \) is usually closer to the actual state than the assumed nominal trajectory [51]. To obtain the equations of motion for the EKF model, we differentiate the \( \dot{\lambda} \) and \( \dot{R} \) equations of (340). Taking the derivative of the \( \dot{R} \) equation in (340) once with respect to time and substituting for \( \dot{\lambda}, a_M \) and \( a_T \), we arrive at the following

\[
\dot{\dot{R}} = -V_T \sin(\theta_T - \lambda)(\dot{\theta}_T - \dot{\lambda}) + V_M \sin(\theta_M - \lambda)(\dot{\theta}_M - \dot{\lambda}) \\
= \dot{\lambda} \left( V_T \sin(\theta_T - \lambda) - V_M \sin(\theta_M - \lambda) \right) + \dot{\theta}_M V_M \sin(\theta_M - \lambda) - \dot{\theta}_T V_T \sin(\theta_T - \lambda) \\
= \dot{\lambda}^2 R + \dot{\theta}_M V_M \sin(\theta_M - \lambda) - \dot{\theta}_T V_T \sin(\theta_T - \lambda) \\
\Rightarrow \dot{\dot{R}} = \dot{\lambda}^2 R + a_M \sin(\theta_M - \lambda) - a_T \sin(\theta_T - \lambda)
\]  

(343)

Similarly, differentiating the \( \dot{\lambda} \) equation of (340) once with respect to time and substituting for \( \dot{\dot{R}}, a_M \) and \( a_T \), we arrive at the following

\[
\ddot{\dot{\lambda}} = \frac{R \left( V_T \cos(\theta_T - \lambda)(\dot{\theta}_T - \dot{\lambda}) - V_M \cos(\theta_M - \lambda)(\dot{\theta}_M - \dot{\lambda}) \right)}{R^2} \\
- \frac{\dot{\dot{R}} \left( V_T \sin(\theta_T - \lambda) - V_M \sin(\theta_M - \lambda) \right)}{R^2} \\
= \frac{-\dot{\lambda} \left( V_T \cos(\theta_T - \lambda) - V_M \cos(\theta_M - \lambda) \right)}{R} \\
+ \frac{\left( \dot{\theta}_T V_T \cos(\theta_T - \lambda) - \dot{\theta}_M V_M \cos(\theta_M - \lambda) \right)}{R} \\
- \frac{\dot{\dot{R}} \left( V_T \sin(\theta_T - \lambda) - V_M \sin(\theta_M - \lambda) \right)}{R^2} \\
\Rightarrow \ddot{\dot{\lambda}} = -2 \frac{\dot{\lambda} \dot{\dot{R}}}{R} + \frac{a_T \cos(\theta_T - \lambda) - a_M \cos(\theta_M - \lambda)}{R}
\]  

(344)
To arrive at the model for the design of the EKF we impose the following assumption:

**Assumption 10.** The model for the design of the EKF assumes that the target and missile are accelerating normal to the LOS. Thus the component of $a_T$ and $a_M$ along the LOS are neglected. This contributes to a potential source of modelling error, between the true dynamics and the model for the design of the EKF.

Thus, with Assumption 10, the equations of motion for the EKF model design are

\[
\begin{align*}
\ddot{R} &= \dot{\lambda}^2 \dot{R}, \\
\ddot{\lambda} &= -2\dot{\lambda} \frac{\dot{R}}{R} + \frac{\dot{a}_T - a_M}{R}, \\
\dot{a}_T &= -\frac{\dot{a}_T}{\tau_T} + \frac{w_T}{\tau_T},
\end{align*}
\]  

where $w_T$ is a zero mean, band limited white noise process described by $N(0, q_T)$, where $q_T$ is the magnitude of the power spectral density and is given by

\[q_T = 2\sigma_T^2 \tau_T\]  

where $\sigma_T$ is the standard deviation of the first order Markov process described by the 3$^{rd}$ equation of (345) and $\tau_T$ is the time constant of the process. The measurement equations are given by (341) and (342). Defining $\hat{\dot{R}} = x_1$, $\hat{\dot{\lambda}} = x_3$ and $\hat{a}_T = x_5$, we obtain the following state space representation of system described by (345) as

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_1 x_4^2, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= -2x_2 x_4 + \frac{x_5 - a_M}{x_1}, \\
\dot{x}_5 &= -\frac{x_5}{\tau_T} + \frac{w_T}{\tau_T}, \\
z_{r_k} &= x_1 + \nu_{r_k}, \\
z_{\lambda_k} &= x_3 + \nu_{\lambda_k}
\end{align*}
\]  

Since the system equations described by (345) are nonlinear, a first order approximation for the systems dynamic matrix is obtained by defining

\[F_k = \left. \frac{\partial f(x)}{\partial x} \right|_{x = \dot{x}_k}\]  

121
Thus $F_k$ turns out to be

$$F_k = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
x_4^2 & 0 & 0 & 2x_4x_1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\frac{2x_4}{x_1} & \frac{-2x_4}{x_1} & \frac{-2x_4}{x_1} & \frac{1}{x_1} & \frac{-1}{\tau_r} \\
0 & 0 & 0 & 0 & \frac{1}{\tau_r}
\end{bmatrix}_{x=x_k^e}$$

The Ricatti equations required for the computation of the Kalman gains are given as:

$$M_k = \Phi_k P_{k-1} \Phi_k^T + Q_k$$  \hspace{0.5cm} (349)

$$K_k = M_k H_k^T (H_k M_k H_k^T + R_k)^{-1}$$  \hspace{0.5cm} (350)

$$P_k = (I - K_k H_k) M_k$$  \hspace{0.5cm} (351)

where $P_k$ is the covariance matrix representing errors in state estimates before an update, $M_k$ is the covariance matrix representing errors in state estimates after an update, $K_k$ is the Kalman gain matrix, $\Phi_k$ is the state transition matrix which is usually approximated by the first two terms of the Taylor series expansion of $e^{F_k \Delta t} \approx (I + F_k \Delta t)$, and the discrete measurement noise matrix is

$$R_k = E[\nu_k \nu_k^T]$$  \hspace{0.5cm} (352)

where $\nu_k = \begin{bmatrix} \nu_{r_k} & \nu_{\lambda_k} \end{bmatrix}^T$. The discrete process noise matrix $Q_k$ can be found from the continuous process noise matrix $Q$ according to

$$Q_k = \int_0^{\Delta t} e^{F_k s} Q e^{F_k^T s} ds$$  \hspace{0.5cm} (353)

where $Q$ is the strength of the autocorrelation function associated with the process noise

$$E[w(t)w^T(t+\tau)] = Q\delta(\tau)$$  \hspace{0.5cm} (354)

From (347) we see that

$$w = \begin{bmatrix} 0 & 0 & 0 & 0 & w_T \end{bmatrix}^T$$  \hspace{0.5cm} (355)

Thus $Q$ is a matrix of zeros except for the last diagonal element

$$Q(5,5) = q_T$$  \hspace{0.5cm} (356)

where $q_T$ is given by (346).
6.1.3 Neural Network (NN) Design

One of the potential application areas this thesis investigates is missile-target tracking problems when we do not have a good to characterize the random behavior of the target. The differences between the true target dynamics and the nominal target model, used in the design of the EKF, leads to a potential source for modelling errors in the problem. In this problem the EKF is augmented with an NN based adaptive element, to correct for target acceleration. We assume that the target acceleration is formed as follows:

\[ a_T^* = a_T + a_{TN} \]  (357)

where \( a_T^* \) is the true target acceleration, \( a_T \) is the part of the target acceleration that can be modelled as shown in (345), and \( a_{TN} \) is the correction in the target acceleration that accounted for by the NN. The adaptive EKF model can be written as

\[
\begin{bmatrix}
\dot{\hat{R}} \\
\dot{\hat{\lambda}} \\
\dot{\hat{a}}_T
\end{bmatrix} = \begin{bmatrix}
\hat{\lambda}^2 \dot{\hat{R}} \\
-2\hat{\lambda} \frac{\dot{\hat{R}}}{R} + \frac{\ddot{a}_T - a_{MN}}{R} \\
-\frac{\ddot{a}_T}{\tau_T} + \frac{w_T}{\tau_T}
\end{bmatrix} + \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} \nu_{ad}(t) \quad (358)
\]

where \( \nu_{ad}(t) \) is the output of a linearly parameterized NN which is designed as follows:

\[ \nu_{ad}(t) = \hat{M}^T(t) \sigma(\mu(t)) \]  (359)

where \( \hat{M}(t) \) is the estimate of the NN weights adapted on and \( \mu(t) \) is the vector of delayed values of the measurement and control variables.

6.1.4 Guidance Law

In this section, the type of guidance law used for guiding the missile towards the target is described. Proportional navigation (PN) is extensively used in the tactical missile world as an interceptor guidance law because it is easy to implement and is very effective. Theoretically, the PN guidance law issues acceleration commands, perpendicular to the instantaneous missile-target LOS, which are proportional to the LOS rate and the closing velocity. In tactical radar homing missiles using PN guidance, the seeker provides an effective measurement of LOS rate, and a Doppler radar provides closing velocity information. The closing velocity
is defined as the negative rate of change of distance from the missile to the target. From a
guidance point of view, it is desired to make the range between the missile and the target at
the expected intercept time as small as possible. It is widely known that PN guidance law
produces better performance for a non-maneuvering target [129]. However, in the case of a
highly maneuvering target, the performance of the PN guidance law is much worse [130-134],
hence there is an increasing need to develop more complex advanced guidance laws that can
replace PN guidance laws. The price paid for more advanced guidance laws is that more
information is required for their successful implementation. However these advanced guidance
laws relax the interceptor acceleration requirements and yield smaller miss distances,
the point of closest approach of the missile and the target. In this thesis, an augmented
proportional navigation (APN) guidance law is implemented to account for the randomly
maneuvering targets [52]. Mathematically, the APN law is written as

\[ a_M = N'\dot{V}_c\lambda + N'\dot{\hat{a}}_T \]  \hspace{1cm} (360)

where \( N' \) is a unitless gain, usually in the range of \((3 - 5)\), \( \dot{V}_c \) is the closing velocity given by
\( \dot{V}_c = -\dot{R} \), and \( \dot{R}, \dot{\lambda} \) and \( \dot{\hat{a}}_T \) are the estimates from the adaptive EKF described by (358).

6.1.5 Simulation Results

In this section we present simulation results in order to compare the performances of
the EKF with and without adaptation. The simulations performed here were for constant
velocity missile and target. The case considered here is a low speed maneuvering target,
where the target behavior is modelled as a first order Markov process as shown in (345). The
total time for the simulation run was 100 s and a time step of \( \Delta t = 0.05 \) s was used.

6.1.5.1 True Process Parameters

The velocity of the missile and target were set to

\[ V_M = 115 \text{ ft/s}, \quad V_T = 100 \text{ ft/s} \]  \hspace{1cm} (361)

The true process described by (340) is initialized as

\[ \lambda(0) = 0, \quad \dot{\lambda}(0) = 0, \quad R(0) = 300 \text{ ft}, \quad V_c(0) = 15 \text{ ft/s}, \quad \theta_T(0) = 0, \quad \theta_M(0) = 0 \]  \hspace{1cm} (362)
The true target acceleration, $a_T$, is implemented as the first order Markov process driven by a random square wave, $d(t)$, and is given as

$$ \dot{a}_T = -\frac{a_T}{\tau} + d(t) \quad (363) $$

where $\tau = 0.5$ s is the time constant for the true target model and $d(t)$ is defined such that

$$ d(t) = \begin{cases} 
  a_T & \text{if } w(t) > 0 \\
  -a_T & \text{if } w(t) \leq 0 
\end{cases} \quad (364) $$

where $w(t)$ is a band limited white noise process with known power spectral density. The signal $w(t)$ is fed into a switch that toggles between $\pm 2 \ g$, depending on the sign of $w(t)$.

6.1.5.2 EKF Design Parameters

The initial conditions chosen for the model in (345) are

$$ \hat{R}(0) = 100 \ ft, \ \dot{\hat{R}}(0) = 0, \ \hat{\lambda}(0) = 0, \ \dot{\hat{\lambda}}(0) = 0, \ \hat{a}_T(0) = 0 \quad (365) $$

The standard deviations of the state variables in (345) are

$$ \sigma_R = 100 \ ft, \ \sigma_{\dot{R}} = 1000 \ ft/s, \ \sigma_\lambda = \pi \ rad, \ \sigma_{\dot{\lambda}} = 1000 \ rad/s, \ \sigma_{a_T} = g \ ft/s^2 \quad (366) $$

Thus the initial covariance matrix is

$$ P(0) = \begin{bmatrix}
100^2 & 0 & 0 & 0 & 0 \\
0 & 1000^2 & 0 & 0 & 0 \\
0 & 0 & \pi^2 & 0 & 0 \\
0 & 0 & 0 & 1000^2 & 0 \\
0 & 0 & 0 & 0 & g^2
\end{bmatrix} \quad (367) $$

The standard deviation for the range measurement noise was chosen to be $10 \ ft$ and the standard deviation for the $\lambda$ measurement noise was chosen to be $0.1 \ rad$. The target acceleration for the design of the EKF is modelled as a first order Gauss-Markov process with a standard deviation of $g$ and a time constant of $\tau_T = 1 \ s$, where $\tau_T$ is the filter estimate of the true target acceleration time constant, $\tau$. 

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6.1.5.3 NN Design Parameters

The design of the linearly parameterised NN is carried out in a continuous time domain as explained in chapter 5, and the implementation of the NN signal is done in discrete time domain. The learning rate, $\Gamma_M$ and the $\sigma-$ modification gain for the NN described by (276) were each chosen to be 0.1. We used the sigmoidal function of the form $\sigma(x) = \frac{1}{1+e^{-x}}$, where $\alpha$ is the activation potential and was set to 1. The inputs to the NN include the two measurements described by (341) and (342) and the control signal described by (360). The unitless gain $N'$ of the APN guidance law was chosen to be 3. The missile is assumed to generate a maximum acceleration of $a_{M_{\text{max}}} = 3.5g$. We implemented the following $2^{\text{nd}}$ order autopilot model:

$$a_L(s) = \left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)a_M(s)$$

(368)

where the damping was set to $\zeta = 0.8$ and the natural frequency was $\omega_n = \frac{1}{\tau_c} = 2.5 \text{ rad/s}$.  

![Figure 61: Implementation of missile-target interception in SIMULINK.](image)

6.1.6 Discussion of Results

Remark 21. The true target acceleration vector is normal to the target velocity vector, i.e. $a_T \perp V_T$. However, in deriving the model (345) for the design of the EKF, it is assumed that
the target acceleration is normal to the LOS vector as stated in Assumption 10. Thus while comparing the true and the estimated target accelerations, we make a comparison between 
\[(a_T \cos(\theta_T - \lambda))\] and \(\ddot{a}_T\).

6.1.6.1 True Initial Missile-Target Range, \(R(0) = 300\ ft\)

Figs. 62, 63, 64 and 65 show the estimation performance of the EKF described by (345). Figs. 66 and 67 show the missile acceleration profiles and the missile-target trajectories, wherein the state estimates, \(\dot{\hat{R}}, \dot{\lambda}\) and \(\ddot{a}_T\), from the EKF in (345), are used as inputs to the APN guidance law which is described by (360). Figs. 68, 69, 70 and 71 show the estimation performance of the EKF + NN described by (358). Figs. 72 and 73 show the missile acceleration profiles and the missile-target trajectories, wherein the state estimates, \(\dot{\hat{R}}, \dot{\lambda}\) and \(\ddot{a}_T\), from the NN based adaptive EKF in (358), are used as inputs to the APN guidance law which is described by (360). Fig 74 shows the range profile between the missile and target for the case of the EKF only while Fig 75 shows the range profile between the missile and target for the case of NN based adaptive EKF. An immediate conclusion that can be drawn by looking at the range plot of Fig. 74 is that with the EKF the closest the missile gets is to within 23 ft of the target. With the NN based adaptive EKF, as seen in Fig. 75, the missile gets to within 12 ft of the target at 48 s and 3.7 ft of the target at 68 s. The benefit of augmenting the EKF with an NN seems to be in aiding the missile to make closer passing attacks at the target.

6.1.6.2 Hitting the target

We performed some simulation runs for different true initial missile-target range, \(R(0)\), for \(R(0) = 550\ ft\), \(R(0) = 800\ ft\) and \(R(0) = 1000\ ft\). Figs. 76, 80 and 84 show the missile-target trajectories when the state estimates from the EKF are fed into the guidance law, while Figs. 77, 81 and 85 show the missile-target trajectories when the state estimates from the NN based adaptive EKF are fed into the guidance law. As seen from Figs. 77, 81 and 85, in the presence of NN augmentation, the missile was able to successfully intercept the target. We notice from Figs. 76, 80 and 84, that when the estimates from only the EKF are fed into the APN guidance law, even though the range between the missile and
target becomes quite small, the missile still does not intercept the target. This becomes of paramount importance when the goal is to hit the target.

6.1.6.3 Varying the true process initial conditions

In this section, we discuss the sensitivity of the EKF and the NN based adaptive EKF to changes in the true process initial conditions. In particular, we varied $R(0)$ in the range from [10, 1000] ft and $H_E(0)$ in the range from $[-180^\circ, 180^\circ]$. It should be noted that we do not vary the initial conditions at the same time, i.e., $H_E(0)$ is fixed and $R(0)$ is varied in the interval specified to observe the sensitivity of the EKF and EKF + NN respectively. Similarly, $R(0)$ is fixed and $H_E(0)$ is varied in the specified interval.

1. Varying true initial missile-target range, $R(0)$: Fig. 88 compares the minimum ranges achieved between the missile and target with and without augmenting the EKF with an NN based adaptive element over a simulation of 100 seconds for different initial ranges, $R(0)$. The black solid line in the figure is marked at a minimum range of 1 ft. This indicates that when the minimum range between the missile and target gets to a foot, the simulation automatically stops since it is assumed that at 1 feet, interception takes place. It should be noted from Fig. 88 that with adaptation in the EKF missile-target interception takes place 5 times. It is also evident from the figure that with adaptation in the EKF the minimum range achieved is lower than without adaptation and in some cases is significantly lower.

2. Varying initial heading error, $H_E(0)$: Fig. 89 is a plot of the minimum range achieved between the missile and target over a simulation of 100 seconds for different initial missile heading errors, $H_E(0)$. Once again, the black solid line in the figure is marked at a minimum range of 1 ft. This indicates that when the minimum range between the missile and target gets to a foot, the simulation automatically stops since it is assumed that at 1 feet, interception takes place. It should be noted from Fig. 89 that with adaptation in the EKF missile-target interception takes place 10 times. It is also evident from the figure that with adaptation in the EKF the minimum range achieved is lower than without adaptation and in some cases is significantly lower.
**Figure 62:** Estimation performance of Range, \((\vec{R})\) and Range-rate, \((\vec{\dot{R}})\) - EKF.

**Figure 63:** Estimation performance of LOS, \((\lambda)\) and LOS-rate, \((\lambda_{\dot{\cdot}})\) - EKF.
Figure 64: Estimation performance of target acceleration, $a_T$ - EKF.

Figure 65: Estimation performance of closing velocity, $V_c$ - EKF.
**Figure 66:** Missile acceleration profile ($g$'s): commanded (dashed), achieved (solid).

**Figure 67:** Missile-Target trajectories (ft) - guidance law uses estimates from EKF.
**Figure 68:** Estimation performance of Range, ($R$) and Range-rate, ($\dot{R}$) - EKF + NN.

**Figure 69:** Estimation performance of LOS, ($\lambda$) and LOS-rate, ($\dot{\lambda}$) - EKF + NN.
**Figure 70:** Estimation performance of target acceleration, $a_T$ - EKF + NN.

**Figure 71:** Estimation performance of closing velocity, $V_c$ - EKF + NN.
Figure 72: Missile acceleration profile: commanded (dashed), achieved (solid).

Figure 73: Missile-Target trajectories (ft) - guidance law uses estimates from EKF + NN.
Figure 74: Missile-Target range history ($R(0) = 300 \text{ ft}$) - EKF.

Figure 75: Missile-Target range history ($R(0) = 300 \text{ ft}$) - EKF + NN.
Figure 76: Missile-Target trajectories - guidance law uses estimates from EKF.

Figure 77: Missile-Target trajectories - guidance law uses estimates from EKF + NN.
**Figure 78:** Missile-Target range history \((R(0) = 550 \text{ ft})\) - EKF.

**Figure 79:** Missile-Target range history \((R(0) = 550 \text{ ft})\) - EKF + NN.
Figure 80: Missile-Target trajectories (ft) - guidance law uses estimates from EKF.

Figure 81: Missile-Target trajectories (ft) - guidance law uses estimates from EKF + NN.
Figure 82: Missile-Target range history \((R(0) = 800 \text{ ft})\) - EKF.

Figure 83: Missile-Target range history \((R(0) = 800 \text{ ft})\) - EKF + NN.
Figure 84: Missile-Target trajectories (ft) - guidance law uses estimates from EKF.

Figure 85: Missile-Target trajectories (ft) - guidance law uses estimates from EKF + NN.
Figure 86: Missile-Target range history ($R(0) = 1000 \text{ ft}$) - EKF.

Figure 87: Missile-Target range history ($R(0) = 1000 \text{ ft}$) - EKF + NN.
Figure 88: Initial Range (ft) vs. Minimum Range Achieved (ft)

Figure 89: Initial heading (deg) vs. Minimum Range Achieved (ft)
6.2 High Speed Maneuvering Target

In this section, we show the true dynamics for the missile-target problem formulation, the model for the design of the EKF, the augmented proportional navigation (APN) guidance law designed to guide the missile towards the target and some simulation results are shown depicting the performance of the EKF and the EKF augmented with the NN. One of the differences between the implementation of the high speed target as compared to the low speed target is that here we only have one available measurement, LOS \( \lambda \), while in the low speed target case discussed in section 6.1 the available measurements were LOS range, \( R \) and \( \lambda \).

6.2.1 Problem Formulation

Consider the two dimensional point mass missile-target engagement scenario as shown in Fig. 90 where M and T denote missile and target respectively, \( a_T, V_T, \theta_T \) denote the acceleration, velocity and flight path angle of the target respectively, \( a_M, V_M, \theta_M \) are the acceleration, velocity and flight path angle of the missile respectively, \( \lambda \) denotes the line-of-sight (LOS) angle and \( R \) denotes the LOS range.

![Figure 90: 2D Missile-Target Engagement Geometry.](image-url)
The true dynamics of the interception scenario are given by the following equations

\[
\begin{align*}
\dot{\lambda} &= \frac{V_T \sin(\theta_T - \lambda) - V_M \sin(\theta_M - \lambda)}{R} \\
\dot{\hat{R}} &= V_T \cos(\theta_T - \lambda) - V_M \cos(\theta_M - \lambda) \\
\dot{\theta}_T &= \frac{a_T}{V_T} \cos(\theta_T - \lambda) \\
\dot{\theta}_M &= \frac{a_M}{V_M} \cos(\theta_M - \lambda) \\
\dot{V}_T &= a_T \sin(\theta_T - \lambda) \\
\dot{V}_M &= a_M \sin(\theta_M - \lambda)
\end{align*}
\] (369)

The measurement for the process is the LOS angle, \( \lambda \) and is given as

\[
y_\lambda = \lambda + \nu_\lambda
\] (370)

where \( \nu_\lambda \) is a zero mean, band limited white noise process.

6.2.2 Extended Kalman Filter (EKF) Design

To obtain the equations of motion for the EKF model, we differentiate the \( \dot{\lambda} \) and \( \dot{\hat{R}} \) equations of (369). Taking the derivative of the \( \dot{\hat{R}} \) equation in (369) once with respect to time and substituting for \( \dot{\lambda} \), \( a_M \) and \( a_T \) from (369), we arrive at the following:

\[
\begin{align*}
\ddot{R} &= \ddot{V}_T \cos(\theta_T - \lambda) - V_T \sin(\theta_T - \lambda)(\dot{\theta}_T - \dot{\lambda}) - V_M \cos(\theta_M - \lambda) \\
&\quad + V_M \sin(\theta_M - \lambda)(\dot{\theta}_M - \dot{\lambda}) \\
&= a_T \sin(\theta_T - \lambda) \cos(\theta_T - \lambda) - a_T \sin(\theta_T - \lambda) \cos(\theta_T - \lambda) + \dot{\lambda} V_T \sin(\theta_T - \lambda) \\
&\quad - a_M \sin(\theta_M - \lambda) \cos(\theta_M - \lambda) + a_M \sin(\theta_M - \lambda) \cos(\theta_M - \lambda) - \dot{\lambda} V_M \sin(\theta_M - \lambda) \\
\Rightarrow \ddot{R} &= \dot{\lambda}^2 R
\end{align*}
\] (371)

Similarly, differentiating the \( \dot{\lambda} \) equation of (369) once with respect to time and substituting for \( \dot{\hat{R}}, \dot{V}_T \) and \( \dot{V}_M \) from (369), we arrive at the following

\[
\ddot{\lambda} = -2 \frac{\dot{\lambda} \dot{R}}{R} + \frac{a_T - a_M}{R}
\] (372)
Thus the equations of motion for the EKF model design are

\[
\begin{align*}
\ddot{\lambda} &= \dot{\lambda}^2 \dot{R}, \\
\dot{\lambda} &= -2\lambda \frac{\dot{\lambda}}{\dot{R}} + \frac{\dot{\alpha}_T - \dot{a}_M}{\dot{R}}, \\
\dot{\alpha}_T &= -\frac{\dot{a}_T}{\tau_T} + \frac{w_T}{\tau_T} 
\end{align*}
\]

(373)

where \( w_T \) is a zero mean, band limited white noise process described by \( N(0, q_T) \), where \( q_T \) is the magnitude of the power spectral density and is given by (346). The measurement equation is given by (370). The manner in which the EKF is designed to estimate the states of the model described in (373), is summarized in (348)-(356).

6.2.3 Neural Network (NN) Design

As discussed in section 6.1.3, the EKF is augmented with an NN based adaptive element to correct for the target acceleration, which is a source of unmodeled dynamics which gets coupled to the model through the term \( g_1(x(t), z(t)) \) as described by (223). The target acceleration of the true process is a sinusoidal weave of frequency \( \omega_T \). The model for the design of the EKF assumes that the target acceleration is modelled as a first order Markov process. This anomaly in modelling contributes to a potential source of unmodeled dynamics.

6.2.4 Guidance Law

The APN guidance law is implemented as \( a_M = N' \dot{V}_c \dot{\lambda} + N'' \frac{\dot{a}_T}{\dot{R}} \), where \( N' \) is a unitless gain, \( \dot{V}_c \) is the closing velocity and is given by \( \dot{V}_c = -\dot{R} \) and \( \dot{R}, \dot{\lambda} \) and \( \dot{a}_T \) are the estimates coming from (373).

6.2.5 Simulation Results

In this section we present simulation results in order to compare the performances of the EKF with and without adaptation. The case considered here is a high speed maneuvering target, where the true target behavior is modelled as a sinusoidal process.

6.2.5.1 True Model Parameters

The true process is described by (369) and is initialized as follows:

\[
\begin{align*}
\lambda(0) &= 0, \quad R(0) = 1000,000 \text{ ft}, \quad V_T(0) = 25,000 \text{ ft/s}, \quad V_M(0) = 25,000 \text{ ft/s} 
\end{align*}
\]

(374)
The initial flight path angle of the target, \( \theta_T(0) \), and the missile, \( \theta_M(0) \), are computed as follows:

\[
\theta_T(0) = (180 - \theta_{T_a}) \frac{\pi}{180}, \quad \theta_M(0) = \theta^*_M + H_E
\]

(375)

where \( \theta_{T_a} \) is the target aspect angle taken to be 10\(^\circ\), \( H_E \) is the heading error which is taken to be 2\(^\circ\) and \( \theta^*_M \) is the ideal missile heading that results in \( \dot{\lambda}(0) = 0 \) given the values of \( V_T(0), V_M(0), \theta_T(0) \) and \( \lambda(0) \). Thus we obtain

\[
\theta^*_M = \lambda(0) + \sin^{-1}\left( \frac{V_T(0)}{V_M(0)} \sin \left( \theta_T(0) - \lambda(0) \right) \right)
\]

(376)

The true target acceleration, \( a_T \), is implemented as the following sinusoidal process

\[
a_T = a_{T_{amp}} \sin(2\pi f_{ar} t)
\]

(377)

where \( a_{T_{amp}} = 15g \) and \( f_{ar} = 1.5Hz \).

6.2.5.2 EKF Design Parameters

Model Initial Conditions:

The initial conditions chosen for the model in (373) are

\[
\dot{R}(0) = 1001,000 \text{ ft}, \quad \dot{\dot{R}}(0) = -\dot{V}_c(0), \quad \dot{\lambda}(0) = 0, \quad \dot{\dot{\lambda}}(0) = 0, \quad \mathring{\dot{a}}_T(0) = 0
\]

(378)

where \( \dot{V}_c(0) \) is computed by taking into account \( \theta_M \) in (375) and substituting it into the \( \dot{R} \) equation of (369). Thus we obtain

\[
\dot{V}_c(0) = V_M \cos(\theta_M(0)) - V_T \cos(\theta_T(0))
\]

(379)

Initial covariance matrix, \( P(0) \):

To start the Riccati equations of the EKF, we need an initial covariance matrix, \( P_0 = P(0) \). The \( P_0(2,2), P_0(4,4) \) and \( P_0(2,4) \) elements are obtained by perturbing the \( \dot{\lambda} \) and \( \dot{R} \) equations of (369) with respect to \( \theta_M \). Accordingly we have

\[
P_0(2,2) = V_M^2 \sin^2(\theta_M(0)) H_E^2
\]

(380)

\[
P_0(4,4) = V_M^2 \cos^2(\theta_M(0)) H_E^2
\]

(381)

\[
P_0(2,4) = P_0(4,2) = -V_M^2 \frac{\sin(\theta_M(0)) \cos(\theta_M(0))}{R(0)} H_E^2
\]

(382)
Thus the initial covariance matrix is given as
\[
P(0) = P_0 = \begin{bmatrix}
1000^2 & 0 & 0 & 0 & 0 \\
0 & \rho_0(2, 2) & 0 & \rho_0(2, 4) & 0 \\
0 & 0 & \frac{1000^2}{\rho_0(0)} & 0 & 0 \\
0 & \rho_0(4, 2) & 0 & \rho_0(4, 4) & 0 \\
0 & 0 & 0 & 0 & g^2
\end{bmatrix}
\]  \hspace{1cm} (383)

*Measurement noise parameters:*

The standard deviation of the $\lambda$ measurement is chosen to be range independent and is $\sigma_{\lambda} = 150 \times 10^{-6} \text{ rad}$. The time constant for the target acceleration in (373) is $\tau_T = 0.5 \text{ s}$.

### 6.2.5.3 Other Simulation Parameters

The missile is assumed to generate a maximum acceleration of $a_{M_{\text{max}}} = 20g$ and the value of the gain $N'$ in (??) is set to 3. We implemented the 2nd order autopilot model as described in (368). The time step for the simulations was taken to be $\Delta t = 0.05 \text{ s}$. To compare the true and estimated values of the target flight path angle, $\theta_T$, we compare the true $\dot{\theta}_T$ as generated by the third equation of (369) with the estimate of $\dot{\theta}_T$. To compute $\dot{\hat{\theta}}_T$ we solve the first and second equations of (369) and obtain the following relation for $\dot{\hat{\theta}}_T$:
\[
\dot{\hat{\theta}}_T = \theta_M + \cos^{-1} \left( \frac{\hat{V}_T^2 + V_M^2 - \lambda^2 \hat{R}^2 - \hat{R}'^2}{2\hat{V}_TV_M} \right)
\]  \hspace{1cm} (384)

where $\hat{R}$, $\dot{\hat{R}}$ and $\ddot{\hat{R}}$ are the estimates from the EKF. To compare the true and estimated values of the target velocity, $V_T$, we compare the true $V_T$ as generated by the 5th equation of (369) with the estimate of $V_T$. To compute $\hat{V}_T$ we solve the 1st and 2nd equations of (369) and obtain the following relation for $\hat{V}_T$:
\[
\hat{V}_T = \sqrt{\left( \frac{\hat{R}\ddot{\lambda} + V_M \sin (\theta_M - \lambda)}{\dot{\theta}_M} \right)^2 + \left( \frac{\hat{R} + V_M \cos (\theta_M - \lambda)}{\dot{\theta}_M} \right)^2}
\]  \hspace{1cm} (385)

### 6.2.6 Discussion of Results

Figs. 91, 92, 93 and 94 show the estimation performance of an EKF. Fig. 95 shows the true and estimated flight path angle and velocity of the target, while Fig. 96 shows the missile acceleration profile. Fig. 97 shows the missile-target trajectories, wherein the state
estimates, \( \dot{R}, \lambda \) and \( \ddot{a}_T \), from the EKF are supplied to the APN guidance law given by (??) and fig. 107 shows the missile-target range histories.

Figs. 98, 99, 100 and 101 show the estimation performance of the EKF + NN. Fig. 102 shows the true and estimated flight path angle and velocity of the target for the NN based adaptive EKF, while Fig. 103 shows the missile acceleration profile. Fig. 104 shows the missile-target trajectories, wherein the state estimates, \( \dot{R}, \lambda \) and \( \ddot{a}_T \), from the NN based EKF are supplied to the APN guidance law given by (??) and fig. 108 shows the missile-target range histories.

As seen from Fig. 93, the EKF has difficulty in estimating the acceleration of the target, producing an estimate of the target acceleration which is out of phase and that starts to diverge. This phase shift is removed when the EKF is augmented with an NN based adaptive element as shown in Fig. 100. The other states that are reconstructed from the estimates of the EKF in (373) are target flight path angle, \( \theta_T \) and target velocity, \( V_T \), the estimates of which are constructed as shown in (384) and (385). The performance of the EKF in estimating \( \dot{\theta}_T \) and \( \dot{V}_T \) is shown in Fig. 95, while the performance of the adaptive NN based EKF in estimating \( \dot{\theta}_T \) and \( \dot{V}_T \) is shown in Fig. 102. Fig. 105 shows the estimation errors of \( V_T \) and \( \theta_T \) when the estimates of the EKF are used in (384) and (385), while Fig. 106 shows the estimation errors of \( V_T \) and \( \theta_T \) when the estimates of the NN based adaptive EKF are used in (384) and (385). We see that augmenting the EKF with an NN based adaptive element helps to significantly remove the bias in the errors of the state estimates that characterize the behavior of the maneuvering target.

Another interesting point to be noted is that without NN augmentation of the EKF, the commanded missile acceleration, Fig. 96, almost always saturates at the time instant when the amplitude of the target maneuver reaches a maximum or a minimum. When the estimates from the adaptive EKF are used, instead, as inputs to the APN guidance law, the commanded missile acceleration rarely saturates as seen in Fig. 103.
Figure 91: Estimation performance of Range, \((\bar{R})\) and Range-rate, \((\bar{\dot{R}})\) - EKF.

Figure 92: Estimation performance of LOS, \((\lambda)\) and LOS-rate, \((\dot{\lambda})\) - EKF.
Figure 93: Estimation performance of target acceleration, $a_T$ - EKF.

Figure 94: Estimation performance of closing velocity, $V_c$ - EKF.
Figure 95: Target flight path angle ($\theta_T$), Target velocity ($V_T$): true and estimated - EKF.

Figure 96: Missile acceleration profile: commanded (dashed), achieved (solid).
Figure 97: Missile-Target trajectories - guidance law uses estimates from EKF.

Figure 98: Estimation performance of Range, \( R \) and Range-rate, \( \dot{R} \) - EKF + NN.
Figure 99: Estimation performance of LOS, ($\lambda$) and LOS-rate, ($\dot{\lambda}$) - EKF + NN.

Figure 100: Estimation performance of target acceleration, $a_T$ - EKF + NN.
Figure 101: Estimation performance of closing velocity, $V_c$ - EKF + NN.

Figure 102: Target flight path angle ($\theta_T$), Target velocity ($V_T$) : true and estimated - EKF + NN.
Figure 103: Missile acceleration profile: commanded (dashed), achieved (solid).

Figure 104: Missile-Target trajectories - guidance law uses estimates from EKF + NN.
Figure 105: Estimation errors of $V_T$ and $\theta_T$ - EKF.

Figure 106: Estimation errors of $V_T$ and $\theta_T$ - EKF + NN.
Figure 107: Missile-Target range history ($R$) - EKF.

Figure 108: Missile-Target range history ($R$) - EKF + NN.
CHAPTER 7

CONCLUDING REMARKS

7.1 Contributions of this Research Work

This thesis has investigated the following areas:

7.1.1 Adaptive Estimation for Time Invariant Uncertain Nonlinear Systems

In chapter 3 an estimation approach for general multivariable nonlinear systems that augments an existing linear time invariant observer with an NN based adaptive element is proposed. The approach developed here relaxes the SPR condition in earlier work, by employing a linear observer called the error observer. The error observer is used to train the NNS on line. The resulting estimator is robust with respect to both parametric uncertainty and unmodeled dynamics. Ultimate boundedness of the error signals is proven with adaptation being based on a modified gradient algorithm with sigma-modification [124]. Ultimate boundedness of the error signals has also been proven with projection based adaptation [119]. The theory developed here is applied to a system of unknown dimension with unmodelled dynamics coupled to both the process dynamics and the measurement. Simulation results show that the estimation performance of a steady state Kalman filter (SSKF) for this class of systems goes through severe degradation, and that the estimation performance of the SSKF augmented with NN based adaptive elements is greatly improved.

7.1.2 Adaptive Estimation for Time Varying Uncertain Nonlinear Systems

In chapter 4 an approach that adaptively augments a time varying observer, such as an extended Kalman filter (EKF), is also developed. Training of the NN does not rely on an SPR filter or an error observer. Instead, the NNs are trained directly by the residuals (difference between the actual measurement and its estimate) of the EKF. Ultimate boundedness of the error signals is proven using an \( \epsilon \)-modification based adaptation [135]. A typical application of adaptive EKFs developed in chapter 4 is in the area of tracking randomly maneuvering
targets, wherein the greatest source of modelling error in the problem is in the behavior of the target. Rather than come up with a better model for the target behavior, which is not an easy task, we treat this as a potential source of modelling error in the problem. The NN based adaptive elements compensate for this modelling error and improve the overall performance of the EKF based adaptive observer. The adaptive element designed here is parameterized in a linear manner.

7.1.3 Adaptive Estimation for Control of Uncertain Nonlinear Systems

In chapter 4, the problem of augmenting a time varying observer with an adaptive NN element was addressed. The theory was developed from the perspective of adaptive estimation, without treating control input in the analysis. However we are interested in developing approaches for addressing practical applications such as missile-target tracking, formation flight control and obstacle avoidance, to name a few. These applications require the implementation of a control/guidance law that relies on the estimates of the states of the process that are not directly available. The theory developed in chapter 5 uses the proposed adaptive observer to obtain estimates of the state which are used to form the control law.

7.1.4 Practical Application Areas for the Theory Developed in Chapter 5

The proposed approach in chapter 5 is validated for the following practical applications:

1. **Obstacle avoidance using a passive 2D vision system**: We consider the problem of obstacle avoidance in which the sensor is a passive 2D vision system. Simulation results illustrate the benefit of augmenting the EKF with an NN based adaptive element. We considered the separate cases when the obstacles were stationary and when the obstacles were moving wherein the obstacle acceleration is modelled as a random walk process.

2. **Formation flight control in 2 dimension**: We consider the problem of target-follower tracking in two dimensions, wherein the range between the target and the follower aircrafts is being regulated to 2 wing spans by feeding back estimates of the target velocity obtained by processing camera images. The goal is to accomplish a task in an unmanned system that is commonly performed in a manned system that relies primarily
on visual information. This problem is approached from the perspective of using, (i) bearings-only measurement, and (ii) bearing angle and the angle subtended by the target in the image plane as measurements.

3. **Missile-Target Interception:** We consider the problem of missile-target interception, wherein the missile guidance/control law is based on the state estimates from the adaptive NN based EKF. This example is known to lead to instability when applying the theory, in its current form, developed in chapter 4. We treat the missile-target interception problem, with the cases of high speed and low speed maneuvering targets.

### 7.2 Recommendations for Future Work

Future research work is recommended in the following areas:

1. **Controlling the Ultimate Bounds on the Error Signals:**

   One of the most important issues in developing a theory for adaptive estimation/control is in controlling the size of the ultimate bounds within which reside the error signals such as the tracking errors, the estimation errors and the NN weight errors. In this thesis, conditions are developed that establish ultimate bounds of the error signals, in a purely estimation setting and also for the problem of adaptive estimation with control. However, the stability proofs in this thesis do not provide a means for achieving an arbitrarily small ultimate bound for the error signals. In [135], it is shown that in the setting of state feedback control, one can control the size of the ultimate bounds of the error signals to be arbitrarily small by increasing a controller design gain. A key feature of the approach proposed in [135] is that the domain in which the initial errors lie is independent of the NN initial weights. In the setting of output feedback control, most of the existing approaches in the literature have addressed the problem of the "size" of the ultimate bounds by designing a high gain observer [29,30] as reported in [136,137]. In the setting of adaptive estimation theory using the error observer approach, this issue has not yet been fully explored and hence future research work in this direction is required.
2. **Adaptive Estimation and Adaptive Control:**

In this thesis, we have addressed the issue of adaptive estimation and control of uncertain nonlinear systems by augmenting a time varying estimator such as an EKF with an NN based adaptive element to improve the performance of the EKF. The adaptive state estimates are then used as inputs to the control/guidance law. In other words, this thesis has addressed the problem of adaptation in only the estimator design. The design of the control law, while relying on the states from the adaptive estimator, is not in itself adaptive. A potential area for future research work involves addressing the issue of adaptation in both the estimator and the controller design. This is a challenging research problem and could be approached from the perspective of:

(a) using 2 NNs, one for the estimator design and the other for the controller design
(b) using 1 NN for the adaptive estimator design and using the techniques of classical adaptive control.

3. **Discrete Time Analysis:** In this thesis, the theory for NN based adaptation is developed for continuous-time systems, while in the simulations the theory is implemented in a discrete manner by sampling the data at a fixed sampling rate. Developing a discrete-time counterpart for the theory developed in this thesis is a potential direction for future research work. In [138], NN based adaptive laws for discrete-time systems are developed. One of the major difficulties in dealing with discrete-time systems is in the stability analysis. Unlike in the continuous-time case, wherein an adaptive law can be found during stability analysis by choosing it such that the time derivative of the candidate Lyapunov function is forced to be negative definite outside a compact domain, the adaptive laws for the discrete-time case must be selected before hand.

4. **Novel Approaches to Adaptation:** In [139], a novel modification term, called Q-modification, is suggested for use in adaptive control for systems where the system uncertainty is parameterized linearly. In this approach the NN weight error signal is shown to be uniformly bounded, while the tracking error signal is shown to asymptotically converge to zero as \( t \to \infty \). Addressing the problem of adaptive estimation for control of
uncertain nonlinear systems, from the perspective of using Q-modification adaptation in the estimator design, is a potential future research area. This can be particularly felt in applications such as missile-target interception, formation flight control, aerial refuelling problem and obstacle avoidance to name a few, wherein the efficiency of the control/guidance law relies primarily on the estimates of the unknown states.

In the recent years, considerable research has been carried out in the area of support vector machines (SVMs) as reported in [140–143]. SVM is another category of universal feedforward networks and, like multilayer perceptrons and radial-basis function networks, have been used for pattern classification and nonlinear regression analysis (functional approximation). In the context of pattern classification, the main idea of SVM is to construct a hyperplane as the decision surface in such a way that the margin of separation between positive and negative samples is maximized. The SVM learning algorithm can be used to construct different types of networks such as radial-basis networks, polynomial networks or single hidden layer networks. In contrast to the back-propagation algorithm which is devised to specifically train a single hidden layer network with a fixed number of hidden units, SVMs can automatically determine the required number of hidden units, given a set of training data [110]. There is no structural difference between NNs and SVMs in terms of their implementation. The main difference between the two arises from the learning mechanism.

5. More Future Work:

In this thesis, the NN based adaptive approaches proposed in chapters 4 and 5 are from the perspective of augmenting a linearly parameterized NN. As a potential future direction of research, the use of nonlinearly parameterized NNs should be explored.

Also, in this thesis, NN based adaptation is addressed from the perspective of using single hidden layer NNs which are static in nature. In [47], the authors use recurrent NNs to approximate the unknown dynamics of the system. Recurrent NNs are those which have at least one feedback loop as shown in Fig. 2. The presence of feedback loops has a profound impact on the learning capability of the network and on its performance and will subsequently reduce the modelling efforts.
REFERENCES


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VITA

Venky Madyastha, born in Pune, India, was awarded a Bachelor of Engineering degree in Mechanical Engineering from Bangalore University in 1996. He worked at the Indian Institute of Science, Bangalore, India, as a research scientist from 1997 to 1998 where he assisted in the design and development of a novel 2 stroke petrol engine. He joined the department of mechanical engineering at the University of Houston, Houston, Texas, in 1998, where he received a master of science degree in mechanical engineering in 2000 and began a doctorate degree at the school of aerospace engineering, Georgia Institute of Technology, soon after. His research is mainly focused on adaptive nonlinear state estimation for control of uncertain systems with applications to such problems as missile-target tracking, target rendezvous and interception, formation flight control and obstacle avoidance.