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OFFICE OF NAVAL RESEARCH
END-OF-THE-YEAR REPORT
PUBLICATIONS/PATENTS/PRESENTATIONS/HONORS/STUDENTS REPORT

for

GRANT: N00014-96-1-0045
PR Number 96PR00715

NONLINEAR DYNAMICS OF COUPLED LASER SYSTEMS

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June 1996

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OFFICE OF NAVAL RESEARCH
PUBLICATIONS/PATENTS/PRESENTATIONS/HONORS/STUDENTS REPORT

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a. Number of papers submitted to refereed journals, but not published: 4 (one accepted for publication in Optics Letters)

b. Number of papers published in refereed journals: 2


c. Number of books or chapters submitted, but not yet published: None
d. Number of book or chapters published: None
e. Number of printed technical reports/non-refereed papers: None
f. Number of patents filed: None
g. Number of patents granted: None
h. Number of invited presentations: 8

Imperial College, London (November 1995)
University of Strathclyde, Glasgow (November 1995)
IEEE/LEOS Annual Meeting, San Fransisco (November 1995)
University of California, Davis (November 1995)
Cornell University (April 1996)
Pennsylvania State University (April 1996)
ARO Workshop, Durham, North Carolina (June 1996)

i. Number of submitted presentations: 5, for the Optical Society of America Annual Meeting, Rochester, N.Y., 1996.
j. Honors/Awards/Prizes for grant employees:

Editorial Board of Journal of Semiclassical and Quantum Optics (IOP)
Editorial Board of Nonlinear Science Today (Springer)
Fellow of the Optical Society of America

k. Total number of Full Time Equivalent Graduate Students and Post-Doctoral Associates supported during this period, under this R&T:

Graduate Students: 3
Post-Doctoral Associates:
Including number of
Female Graduate Students: Darlene Hart (currently post-doctoral fellow at NRL)
Female Post-Doctoral Associates:
Minority Graduate Students: Quinton Williams (currently Member of Technical Staff, Lucent Technologies)
Minority Post-Doctoral Associates:
Asian Graduate Students:
Asian Post-Doctoral Associates:

l. Other funding received this year:


This work examined the fluctuations in laser light fields that are stochastic in origin, and the impact of these fluctuations on interaction of light with atoms and molecules. The grant ended this year. The ONR work focuses on instabilities and nonlinear dynamics of coupled lasers in arrays, and on techniques to analyse and control such instabilities.
Nonlinear Dynamics of Coupled Laser Systems

Rajarsi Roy, Georgia Institute of Technology

Technology Issues:
- Design and Fabrication of Laser Arrays
- Information Storage and Processing in Fiber Optic Communication Systems

Objectives:
- Investigate Amplitude Stability and Phase Coherence of Laser Array Elements fabricated with controllable spacing
- Coupled Polarization Mode Dynamics of Erbium Doped Fiber Ring Laser

Approach:
- Build two laser system, observe amplitude dynamics and phase coherence
- Build Nd:YAG laser array with controlled spacing using Dammann phase grating
- Develop model for laser array

Accomplishments:
- Observed amplitude instability
- Predict array decay of spatial correlations observed experimentally
- Unusual clock-like square waves observed. Model developed.

Impact:
- New understanding of design requirement
- Important guidelines for array fabrication
- Predict effect of parameter distribution on coherence properties of array

Could use as storage loop for optical communications
FIGURES

FIG. 1. Experimental system for generating two laterally coupled lasers in a Nd:YAG crystal and observing the amplitude instability. RP is a rectangular prism; translating this device changes the pump beam separation, and thus the infrared beam separation. The Nd:YAG crystal is coated for high reflectivity (HR) on one side and anti-reflection coated (AR) on the other. The output coupler (OC) is 2% transmissive; both mirrors are flat. FPI is a scanning Fabry-Pérot interferometer, and is used to measure the mode spectrum of both lasers.
Experimental Data

Numerical Results
Numerically calculated parameter space plot of the amplitude instability of two lasers as a function of both the separation $d$ and detuning $\Delta \omega$. Here $p_1 = 0.053$, $p_2 = 0.051$, and $\alpha_1 = \alpha_2 = 0.04$. The height of the graph indicates the largest intensity value recorded at a given value of separation and detuning, while the color of the graph denotes the degree of phase synchronization between the two lasers, as indicated by the fringe visibility. Blue colors indicate low visibilities, while red colors indicate visibilities approaching unity, as shown in the legend. The amplitude instability around $\Delta \omega = 5 \times 10^5$ s$^{-1}$ is clearly evident.

Experimental time trace, measured at a pump separation of $d = 1.03$ mm, which illustrates the bursting nature of the amplitude instability. The average interspike interval (ISI) is 1.9 ms, the normalized standard deviation $\sigma_I / I = 0.10$, and the standard deviation of the detuning $\sigma_{\Delta \omega} = 2 \pi \cdot 5$ MHz.

Numerically simulated time trace of the intensity of laser 1 with an exponentially correlated, stochastic detuning term of strength $D = 5 \cdot 10^9$ s$^{-1}$ and correlation time $\lambda^{-1} = 3$ ms. The mean detuning $\Delta \omega = 5 \cdot 10^5$ s$^{-1}$, and the standard deviation of the detuning $\sigma_{\Delta \omega} = 1.4 \cdot 10^6$ s$^{-1}$. The average ISI was 1.7 ms, and the normalized standard deviation $= 0.12$. The separation $d$ is 1.13 mm, $p_1 = 0.0533$, $p_2 = 0.0531$, and $\alpha_1 = \alpha_2 = 0.04$. 
A fundamental instability of coupled nonlinear oscillators is illustrated by a system of two coupled lasers. It is shown, both experimentally and theoretically, that two detuned lasers exhibit an amplitude instability as the coupling strength is varied across the phase-locking threshold. Stochastic detuning fluctuations are included to obtain quantitative agreement between numerical simulations and experimental measurements.

Coupled nonlinear oscillators can display a fascinating array of dynamical behavior. Pairs of neurons [1], pacemaker cells [2], chemical oscillators [3], Josephson junctions [4] and lasers [5,6] provide diverse examples of such systems. Many investigations of coupled oscillator systems consider only phase effects; they treat limit cycle oscillations with a stable amplitude, and study the phase difference of the two oscillators. More recently, it has been theoretically recognized that the amplitudes of the coupled oscillators can display a rich variety of unstable behaviors for certain regimes of coupling strength [7]. There are very few experiments on physical systems, to the best of our knowledge, that have quantitatively probed the amplitude instability of coupled nonlinear oscillators. It is often difficult to specify and vary the coupling strength of such systems, and a direct comparison of theory and experiment appears to be missing.

In the context of laser systems, many studies of coupled lasers have been motivated by the need for high power coherent sources. It is desirable to achieve both phase synchronization and amplitude stability of the elements of laser arrays. Coupled semiconductor, solid state, and CO₂ lasers have been studied [5,6,8], but it is the spatial properties of the output radiation that have received the most attention, rather than the dynamical characteristics of the emitted light [9]. An earlier work [6] contains mention of the amplitude instability we study here – but no comparison between theory and experiment was made, and no attempt was made at quantitative analysis of dynamical behavior as coupling strength and detuning were varied.

In this Letter we report the observation of an amplitude instability near the phase-locking threshold of two single-mode, linearly polarized, Nd:YAG (neodymium doped yttrium aluminum garnet) lasers excited by pump beams in the same crystal, and coupled to each other through the overlap of the infrared light. The detuning between the lasers is very small, roughly 1 part in 10⁸ of the oscillation frequency.

The following equations describe the time evolution of the complex electric field $E$ and gain $G$ of a pair of spatially coupled, single transverse and longitudinal mode class B lasers [10]:

\[
\frac{dE_1}{dt} = \tau_c^{-1} \left( (G_1 - \alpha_1) E_1 - \kappa E_2 \right) + i\omega_1 E_1, \quad (1a)
\]

\[
\frac{dG_1}{dt} = \tau_f^{-1} \left( p_1 - G_1 - G_1 |E_1|^2 \right), \quad (1b)
\]

\[
\frac{dE_2}{dt} = \tau_c^{-1} \left( (G_2 - \alpha_2) E_2 - \kappa E_1 \right) + i\omega_2 E_2, \quad (1c)
\]

\[
\frac{dG_2}{dt} = \tau_f^{-1} \left( p_2 - G_2 - G_2 |E_2|^2 \right). \quad (1d)
\]

In these equations, $\tau_c$ is the cavity round trip time ($\approx 400$ ps for a cavity of length 6 cm), $\tau_f$ is the fluorescence time of the upper lasing level of the Nd³⁺ ion (240μs for the 1064 nm transition), $p_1$ and $p_2$ are the pump coefficients, $\alpha_1$ and $\alpha_2$ are the cavity loss coefficients, and $\omega_1$ and $\omega_2$ (angular frequencies) are the detunings of the lasers from a common cavity mode, respectively. The lasers are coupled linearly to each other with strength $\kappa$, assumed to be small, and the sign of the coupling terms is chosen to account for the observed stable phase-locked state in which the lasers have a phase difference of 180 degrees. For laser beams of Gaussian intensity profile and $1/e^2$ beam radius $r \approx 200\mu$m, the coupling strength, as determined from the overlap integral of the two fields, is defined as $\kappa = \exp(-d^2/2r^2)$. Control parameters are the frequency detuning of the lasers ($\Delta \omega = \omega_2 - \omega_1$) and the coupling coefficient $\kappa$.

By integrating equations (1) using different values of $\kappa$ and $\Delta \omega$, the dependence of the system on parameters can be numerically investigated. Fig. 1 displays the predicted amplitude instability of the two lasers as...
a function of both the laser separation $d$ and the detuning $\Delta \omega$. We use slightly different pump parameters in the simulation to account for the fact that the two lasers may be non-identical in the experiment. The height of the graph shows the largest intensity value of laser 1 recorded during the 5 ms integration time. The color coding shows the degree of phase synchronization between the two lasers, as measured by the fringe visibility. The visibility $V$ of the fringe pattern formed by the small angle interference of the laser beams is defined as $V = (I_{\text{max}} - I_{\text{min}})/(I_{\text{max}} + I_{\text{min}})$, where $I_{\text{max}}$ and $I_{\text{min}}$ are adjacent maxima and minima in the fringe profile. Low visibilities, shown as blue colors in this figure, indicate states of low phase synchronization, while reds indicate visibilities approaching one and therefore high degrees of phase synchronization. One can clearly see from Fig. 1 that the area where intensity instabilities exist occur just before the onset of phase-locking and that significant intensity oscillations appear only around a rather narrow band of detuning values between $10^5$ and $10^6$ s$^{-1}$. Also evident is the increased value of the visibility, relative to surrounding regions, that exists in the area of the amplitude instability.

Insight into the amplitude instability can be obtained by considering the special case of identical laser parameters and by assuming that the two laser amplitudes and gains are identical. Equations (1) then reduce to

$$\frac{dE}{dt} = \tau_c^{-1}(G - \alpha)E - \tau_c^{-1}\kappa E \cos(\Phi)$$  
(2a)

$$\frac{dG}{dt} = \tau_f(p - G - GE^2)$$  
(2b)

$$\frac{d\Phi}{dt} = 2\tau_c^{-1}\kappa \sin(\Phi) + \Delta \omega$$  
(2c)

for the laser amplitudes $|E_1| = |E_2| = E$, gains $G_1 = G_2 = G$ and phase difference $\Phi = \phi_2 - \phi_1$.

Equations 2(a)-(c) are the rate equations describing a single mode class B laser with variable losses. The phase equation can be integrated exactly, and $\Phi(t)$ is an unbounded function of time if

$$|\Delta \omega| > \Delta \omega_c \equiv 2\kappa \tau_c^{-1}.$$  
(3)

This is the critical condition for an amplitude instability. As a result, the laser equations 2(a)-(b) are periodically modulated by the $\cos(\Phi(t))$ term. The frequency $\omega_M$ of these modulations is given by

$$\omega_M \equiv \alpha \sqrt{\frac{\tau_f}{2\tau_c} \left(\Delta \omega^2 - \Delta \omega_c^2\right)}.$$  
(4)

On the other hand, it is known that the laser relaxation oscillation frequency $\omega_R = 2\pi \nu_R$ for small $\tau_c/\tau_f$ and $\kappa = 0$ is given by

$$\omega_R \equiv \sqrt{\frac{2(p - \alpha)}{\tau_c \tau_f}}.$$  
(5)

This implies the possibility of subharmonic resonance if the ratio of $\omega_R$ to $\omega_M$ is close to an integer. These resonances then produce branches of subharmonic solutions which destabilize the laser system [11,12].

We have tested the prediction of the amplitude instability with the experimental system of Fig. 2, which consists of two parallel, laterally separated lasers created by pumping a single Nd:YAG rod of 5 mm length and diameter in a plane parallel cavity. The pump beams are generated from the argon ion laser output ($\lambda = 514.5$ nm) by a system of beam-splitters and prisms that ensure parallel propagation at an adjustable separation symmetric with respect to the YAG rod axis. The optical cavity consists of one high reflection coated end face of the rod and of an external planar output coupler with 2% transmittance. A Brewster plate and thick etalon within the cavity ensure linear polarization and single longitudinal mode operation. The lasers were operated at approximately 33% above threshold pump power. For these parameters, the relaxation oscillation frequency, $\nu_R$, is of
the order of 100 kHz. The frequency detuning between the two lasers, can be adjusted by tilting the output coupler slightly, thereby introducing a minute difference in cavity lengths.

Thermal lensing induced in the YAG crystal by the pump beams of waist radius $\approx 20\mu m$ is responsible for generating two stable, separate cavities [10]. The TEM$_{00}$ infrared laser beams have radii (at $1/e^2$ of the maximum intensity of the Gaussian profile) of $r \approx 200\mu m$ and their overlap may be continuously changed by varying the lateral separation $d$ of the pump beams over a range of 0.5 - 3 mm. The pump beam separation and profiles are measured directly by a rotating slit technique. In this range, there is no appreciable overlap of the pump beams and coupling is entirely due to the spatial overlap of the infrared laser fields.

The individual temporal dynamics of the output intensities is recorded with fast photodetectors and a two channel digital oscilloscope. The optical frequency difference of the lasers is measured with a radio frequency spectrum analyzer after combining the two beams on a photodetector.

The change of dynamical behavior of the detuned, coupled system can be seen as the separation of the pump beams is varied. For a large separation ($d \geq 1.20$ mm) the lasers were stable and incoherent. The visibility of the fringes was low ($V \approx 0$), and the heterodyne signal was measured to be between 30 and 40 MHz. For a small separation ($d \leq 0.8$ mm), the lasers are stable and phase-locked. The fringe visibility was high ($V \approx 1$), and the heterodyne signal was absent since the lasers were frequency locked. Fig. 3(a) shows a typical intensity time trace characteristic of the unstable regime. Large bursts of the intensity occur, separated by quiescent periods. Here the lasers were separated by 1.03 mm, which implies $\kappa \approx 2.0 \cdot 10^{-5}$. Using (3), we find that the condition for an amplitude instability requires $|\Delta\omega| > 10^8 s^{-1}$, which is verified in our experiments ($\Delta\omega \approx 5 \cdot 10^8 s^{-1}$). The intermediate visibility of $V = 0.20$ signifies the onset of phase-locking.

In the experiment, a substantial amount of fluctuation in the detuning between the two lasers was observed; the beat signal was between 0 and 5 MHz. In order to obtain quantitative comparison between simulations and experiments, we numerically investigated the behavior of equations (1) with a stochastic detuning term, such that $\Delta\omega(t) = \omega_2 - \omega_1$, where $\omega_i = \omega_{0,i} + \delta\omega_i(t)$. Here $\delta\omega_i(t)$ is a colored noise term of strength $D$ and correlation time $\lambda^{-1}$, with the properties $\langle \delta\omega_i(t) \rangle = 0$ and
\( \langle \delta \omega_i(t) \delta \omega_j(t + \Delta t) \rangle = \delta_{ij} D \lambda \exp (-\lambda |\Delta t|) \) [13].

We used three different statistical measures to compare the numerically simulated and experimental traces—the normalized standard deviation of the intensity \( \sigma_I / \bar{I} \), the average interspike interval (ISI), and the standard deviation of the detuning \( \sigma_{\Delta \omega} \). The average ISI is determined by measuring the average time between adjacent bursts whose intensities are greater than some threshold, here defined to be 1.2 times the average intensity. To avoid counting the same burst twice, a "quiescence time" \( \tau_q \) of 0.8 ms was used such that a new spike would be detected no sooner than \( \tau_q \). The standard deviation of the detuning in the experiment was measured to be on the order of 10 MHz or less; numerically, \( \sigma_{\Delta \omega} = \sqrt{D \lambda} \). Using these statistical measures, the parameters \( D \) and \( \lambda \) were adjusted to give quantitative agreement between the observed experimental results and the numerical simulations. The range of parameters \( D \) and \( \lambda \) that gave quantitative agreement with experiment is very limited; \( D \sim O(10^3) \) s\(^{-1}\) and \( \lambda^{-1} \sim O(10^{-3}) \) s. Fig. 3(b) shows a good match with the experimental data.

In conclusion, we have demonstrated a fundamental amplitude instability of coupled nonlinear oscillators as illustrated by a system of two coupled lasers. The amplitude instability occurs over a range of coupling strengths close to the phase-locking threshold for the lasers. Theoretical and numerical predictions, using a dynamical model, of the range of coupling strengths where the instability is expected to occur agree very well with experimental observations. For large separations, both the model and experiment reveal stable intensities, no appreciable phase correlation, and finite frequency detunings. As the separation is decreased to just below the phase locking point, both the model and experiment see large fluctuations, small amounts of phase correlation, and finite frequency detunings. Finally, for separations below the phase locking threshold, stable intensities, large fringe visibilities, and no frequency difference are seen. Stochastic detuning fluctuations were found to be necessary to achieve quantitative agreement between experiment and simulation. These studies reveal a rich and complex dynamical scenario which should be systematically explored in the future for a variety of different oscillator systems.

We acknowledge support from the Division of Chemical Sciences, Office of Basic Energy Sciences, Office of Energy Research, U.S. Department of Energy, and the Office of Naval Research. MM acknowledges support from the Deutsche Forschungsgemeinschaft (DFG, Germany). RR thanks Neal Abraham, Edgar Knobloch, and Steve Strogatz for helpful discussions. TE acknowledges support from the US Air Force Office of Scientific Research grant AFOSR-93-1-0084, the National Science Foundation grant DMS-9308009, the Fonds National de la Recherche Scientifique (Belgium) and the InterUniversity Attraction Pole of the Belgian government.
Fast Polarization Dynamics of an Erbium-doped Fiber Ring Laser

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June 10, 1996

Abstract

The polarization dynamics of a unidirectional erbium-doped fiber ring laser has been observed for individual round trips in the cavity. A rich variety of dynamical states, including square wave pulses and irregular temporal patterns were observed as operating parameters were changed. A model with coupled delay and differential equations is used to interpret the dynamics.
Rare-earth doped silica fiber lasers have recently received much attention in the context of long time scale polarization resolved dynamics. Phenomena such as antiphase dynamics in orthogonal polarization states, self-pulsing, and polarization switching induced by optical feedback have been reported. Experimental evidence of the quasiperiodic route to chaos in an erbium-doped fiber laser has been published. The previous reports have been studies that were done on the millisecond time scale which corresponds to the relaxation oscillation frequency of the fiber laser.

In this Letter, we will present some measurements of the fast temporal (on the nanosecond time scale) dynamics of the Er$^{3+}$-doped fiber ring laser (EDFRL). The fiber laser output beam contains two linearly polarized components. It is within the two groups of orthogonal polarization eigenmodes that the various dynamical states are observed and investigated. Computational results from a model based on coupled delay and differential equations of the Ikeda type provide an explanation of the experimental observations. A similar model has been developed earlier by Loh and Tang for polarization dynamics of an external cavity semiconductor laser.

The EDFRL presents a very unique opportunity in the study of laser nonlinear dynamics. The small longitudinal mode spacing and large gain bandwidth make the EDFRL a practical experimental system in which the collective behavior of a large number of globally coupled nonlinear oscillators can be observed. Such models have been studied in the context of physical and biological systems by Strogatz et al. and by many others recently.

A diagram of the experimental configuration is shown in Figure 1. The coherent pump source was the 514.5 nm wavelength line from an argon-ion laser. A 6 meter length of erbium-doped fiber with an ion concentration of ~240 ppm was taken as the gain medium. A Faraday optical isolator was included in the laser cavity to ensure unidirectional operation. An output coupler removed 3% of the intracavity power. The polarization controller functioned as a discrete birefringence inducing element. Overall, the laser cavity was 20 meters long, 14 meters being passive optical fiber.
ends of the couplers were placed in index matching fluid to suppress the small, but parasitic, Fresnel reflections. The output at $\lambda = 1.561 \, \mu m$ was sent through a $\lambda/2$ waveplate and a polarization beam splitter cube where the orthogonal polarization eigenmodes could be observed simultaneously with high speed photodetectors. Data was recorded by a fast digital oscilloscope with a 1 GHz sample rate. The roundtrip time for the cavity was $\sim 100$ ns and it was possible to store 100 data points per roundtrip.

The EDFRL lases on a very broad 3 dB optical gain bandwidth that is $\sim 10^{10}$ Hz. The longitudinal mode separation is 9.8 MHz; the number of active oscillating modes is well over 2000. An optical spectrum analyzer reveals that the modes oscillate within orthogonally polarized mode groups which have been modelled as two supermodes.¹

While pumping the EDFRL well above threshold (the threshold pump power was about 175 mW), self-pulsing was observed on the nanosecond time scale. Figures 2(a)-2(b) are resolved polarization components of the total output intensity. In Figure 2(a), the distinct sharp pulses are separated by the fundamental cavity roundtrip time of $\sim 100$ ns. Fig. 2(b) shows a highly complex time series which is quasiperiodic or nearly perfectly repeating with a period of $\sim 7$ cavity roundtrips. Inspection of the irregular waveforms in Fig. 2(a) shows that these patterns repeat for several hundred cavity roundtrips before eventually evolving to other irregular waveforms. At the higher pump levels (three to four times above threshold), antiphased square pulses were formed in the orthogonal polarization intensities for a narrow range of adjustment of the polarization controller. Figure 2(c) shows pulses that are 30 ns in duration. This corresponds to the 6 m length of the gain medium. Figure 2(d) shows 70 ns pulses that correspond to the 14 m length of the passive fiber within the laser cavity. Another detail to note is the highly structured intensity fluctuations that ride on top of the square pulses and repeat over many roundtrips.
A laser model based upon an Ikeda-type set of delay-differential equations has been used to investigate the dynamical behavior of the EDFRL. Loh and Tang derived a set of difference-differential equations to study ultrahigh frequency polarization self-modulation in semiconductor lasers.\(^6\) It is in the same spirit that we derive our set of equations from the Maxwell-Bloch equations; they take the form

\[
S_1(t) = \frac{R^2}{2} \left[ S_1(t - \tau_R) e^{2A_1(W(t))} (1 + \cos \phi) + S_2(t - \tau_R) e^{2A_2(W(t))} (1 - \cos \phi) \right.
\]

\[
-2\sqrt{S_1(t - \tau_R)S_2(t - \tau_R)} e^{A_1(W(t)) + A_2(W(t))} \sin(\kappa(W(t))) \sin \phi \]

\[
S_2(t) = \frac{R^2}{2} \left[ S_1(t - \tau_R) e^{2A_1(W(t))} (1 - \cos \phi) + S_2(t - \tau_R) e^{2A_2(W(t))} (1 + \cos \phi) \right.
\]

\[
+2\sqrt{S_1(t - \tau_R)S_2(t - \tau_R)} e^{A_1(W(t)) + A_2(W(t))} \sin(\kappa(W(t))) \sin \phi \]

\[
\frac{dW(t)}{dt} = P - \gamma_1(W(t) + W_\tau) - S_1(t - \tau_R) \left[ e^{a_1[W(t) - N_0 L]} - 1 \right] - S_2(t - \tau_R) \left[ e^{a_2[W(t) - N_0 L]} - 1 \right]
\]

where \( W(t) = \frac{1}{2} \int_0^L N(z, t + \tau_R/\gamma_1) \, dz \), \( \kappa(W(t)) = q_1(W(t)) - q_2(W(t)) - \beta \),

\[
S_{1,2}(t - \tau_R) = \left| E_{1,2}(t - \tau_R) \right|^2 / 4 \sigma_{1,2}, \quad \sigma_{1,2}(W(t)) = \frac{a_{1,2}}{2} (W(t) - N_0 L),
\]

\[
q_{1,2}(W(t)) = \frac{a_{1,2}}{2} (W(t) - W(t = 0)), \quad \text{and} \ \phi \ \text{is the relative phase difference between the polarized fields}. \ \text{The mode detuning factor is defined as} \ \alpha_m = \Delta_m/\gamma_1 \ \text{where} \ \Delta_m = \omega_m - \omega_0 \ \text{and} \ m = 1, 2. \ \text{Other parameters are defined in Table 1.}
\]

In this model, the gain is taken to be a linear function of the population inversion. \( S_1 \) and \( S_2 \) are the photon number densities for the \( x \)- and \( y \)-polarization modes, and \( W \) represents the inversion. The differential equation was integrated with a fourth-order Runge-Kutta routine with a 1 ns integration timestep, corresponding to the experimental sampling time for observation of laser dynamics in a single cavity round trip.
In the equations, the lumped parameter $\beta$ is due to the phase shift associated with fiber birefringence over the entire cavity. Changing the polarization controller, one introduces a local birefringence by applying stress to the fiber which appears as a discrete phase shift in the section of passive fiber. We take the phase term $\phi$ to be $\phi_A$ in the active region and $\phi_P$ in the passive region which contains the polarization controller. The value of $\phi_A$ was taken to be small, but nonzero. The small phase shift in the active fiber could be a result of the active fiber being wound on a spool. This feature of separate phases in the active and passive fiber portions is necessary to reproduce the observed asymmetric nature of the square wave pulsations. $\phi_P$ was taken to be approximately $\pi$ radians because the polarization controller functions roughly as a $\lambda/2$ waveplate. The birefringence causes the two mode groups to travel at different speeds, ultimately resulting in a mode group detuning $\Delta \lambda = (\lambda_2 - \lambda_1)$.

Typical results from the numerical model showing output intensities in orthogonal polarization directions are displayed in Figure 3. Table 1 gives values for the physical parameters of the system. These parameters yield a good match between theory and experiment, as seen from Figs. 2 and 3. However, these computations are merely representative of the enormous variety of waveforms that emerge for different parameter values - they are not meant to provide a detailed reproduction of the experimental waveforms. The sharp pulses are seen to be distinctly separated by the fundamental cavity roundtrip time of 100 ns in Fig. 3(a) and Fig. 3(b). One sees that the irregular waveforms actually repeat over cavity single cavity roundtrips for the parameters chosen. Figs. 3(c)-3(d) show antiphased square wave pulses that form when the parameter value settings of $\phi_A$, $\phi_P$, $\beta$, and $\Delta \lambda$ are in the proper regime.

Essential experimental features captured by the model are (1) the dynamics occur on the nanosecond time scale, (2) self-pulsing at the cavity roundtrip time or multiples with repeating irregular waveforms is present, (3) antiphased square wave pulses form when parameter values are conducive, (4) the dynamics of the system take place with a dc background, (5) highly structured
fluctuations are present on the tops of the square pulses, and (6) the time duration of the square pulses correspond to the lengths of active and passive fiber in the ring.

In conclusion, measurements of the fast temporal dynamics during a single cavity roundtrip have been made for an Er-doped fiber ring laser. Square wave pulsing and irregular dynamics which repeat at roundtrip times have been observed in the two orthogonal polarization eigenstates. The experimentally observed properties are described by a unified model based on an Ikeda-type delay-differential equation model of the laser. We have shown that fiber birefringence, polarization controller adjustment, and the frequency difference between the orthogonal mode groups influence the nature of the dynamics. In a future publication we will include stochastic noise sources in the laser model to explain the slow changes of temporal patterns that occur continually in the system.

We acknowledge support from the Division of Chemical Sciences, Office of Basic Energy Sciences, Office of Energy Research, U.S. Department of Energy and from the Office of Naval Research. It is a pleasure to thank J. Garcia-Ojalvo, R. Hilborn, K. McCoy and C. Verber for many discussions and help with the experiments. We also thank S. Strogatz for many discussions of coupled nonlinear oscillator models.
References


Figure Captions

Fig. 1. Experimental arrangement. Pump, Ar+-ion laser (λp = 514.5 nm); WDM, 514.5 nm/1550 nm wavelength division multiplexer optical coupler; FI, Faraday optical isolator; OC; output coupler (97/3 coupling ratio); MO1, 5x microscope objective; MO2, 20x microscope objective; ND, neutral density filter with 10% transmission at 1.55 μm; λ/2; half-wave plate at 1.55 μm; GP, Glan polarizer; Det; fast response InGaAs/PIN photodetectors.

Fig. 2 Experimentally measured polarization resolved traces of (a) self-pulsing at the cavity roundtrip time in the x-polarization direction from EDFRL with 10% output coupling. (b) irregular trace in the y-polarization direction. The EDFRL was pumped 4 times threshold. (c) and (d) antiphased square pulses in the x- and y-polarization directions from EDFRL with 3% output coupling, respectively. The EDFRL was pumped at 3.3 times threshold.

Fig. 3 Numerical simulations of (a) time trace showing self-pulsing at the cavity roundtrip time in the x-polarization direction. (b) time trace in the y-polarization direction. (c - d) antiphased square-wave pulses in the x- and y-polarization directions corresponding to those of Fig. 2 (c - d). Parameters for Figs. (a) and (b): φA = 0.027, φp = π - 0.175, β = 1.5×10^{-3}, and Δλ = 0.125 fm. Parameters for Figs. (c) and (d): φA = 0.027, φp = π - 0.015, β = 10^{-2}, and Δλ = 4.09 fm. The pump rate is 3.2 times threshold.
Table Captions

Table 1. Parameter values used in the numerical simulations.
<table>
<thead>
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<th>Parameter</th>
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<th>Units</th>
<th>Definition</th>
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<td>m</td>
<td>total cavity length</td>
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<td>$L$</td>
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<td>m</td>
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<td>$\tau_R$</td>
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<td>1.44x10$^{28}$</td>
<td>m$^{-2}$ s$^{-1}$</td>
<td>pump term</td>
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</table>
Experimental Data

Figure 2
Numerical Results

Figure 3
NOISE AMPLIFICATION IN A STOCHASTIC IKEDA MODEL

J. García-Ojalvo* and R. Roy

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Abstract

The effect of spontaneous emission noise on the light circulating in a ring cavity with a nonlinear absorbing medium is studied by means of a set of stochastic delay-differential equations based on the deterministic Ikeda model. Noise fluctuations are found to be amplified as the first bifurcation from the steady state of the system is approached.

*On leave from: Dept. de Física i Enginyeria Nuclear, E.T.S. d'Enginyers Industrials de Terrassa, Univ. Politècnica de Catalunya, Colom 11, E-08222 Terrassa, Spain
spatio-temporal white stochastic process accounting for spontaneous emission processes. This noise term is chosen to have zero mean and correlation equal to

\[ \langle \mu(\tau, z) \mu^*(\tau', z') \rangle = 2D \delta(\tau - \tau') \delta(z - z') \]  

(3)

These equations can be obtained in a straightforward way from the standard Maxwell-Bloch equations which describe the propagation of the electric field inside the absorber by adiabatically eliminating the polarization of the medium, whose relaxation rate is much larger than those of \( N \) and \( E \). One need also assume that the nonlinear term in the population inversion equation does not depend on \( N \) itself. The space variable \( z \) corresponds to the direction of light propagation (transverse effects which might appear in the directions perpendicular to propagation [13,14] are not considered here). The time variable \( \tau \) is written in a reference frame which moves with the velocity \( v_g \) of light in the medium, \( \tau = t - z/v_g \). \( \alpha (> 0) \) is the absorption coefficient of the medium, \( \beta \) is a parameter depending on the detuning between the cavity and the transition resonance frequencies, and \( \gamma \) is the population decay rate. The coefficient \( \Omega \) of the nonlinear term in (2) depends on both the dipole moment of the transition and the asymptotic value of \( N \).

Let \( L \) denote the length of the absorbing medium, \( \mathcal{L} \) that of the whole cavity and \( l = \mathcal{L} - L \). Then, the relation between the incident field \( E_I \) and the field propagating inside the cavity is given by the following boundary condition:

\[ E(t, 0) = \sqrt{T} E_I + R \exp(ikL) E \left( t - \frac{l}{v_g}, L \right) \]  

(4)

where \( T \) is the transmission coefficient of the input mirror M1 and \( R = 1 - T \) is the reflexion coefficient of both the input and output mirrors M1 and M2 (see Fig. 1). Mirrors M3 and M4 are assumed to be perfectly reflecting. \( k \) is the light wavenumber.

The space dependence of the previous equations can be removed by using this boundary condition. First, we formally integrate Eq. (1) with respect to \( z \) and introduce the result
\[ \psi(t) \equiv E(t, 0) \exp(\alpha W_0) \sqrt{\frac{\Omega L \beta}{\gamma}} \]
\[ \phi(t) \equiv \beta W(t - \tau_R, L) \]  

where \( W_0 \) is the asymptotic value of the population inversion integral.

The noise sources \( \eta(t) \) and \( \xi(t) \) are dimensionless and space-independent versions of \( \Gamma \) and \( \chi \). It can easily be seen that its variances \( D_\eta \) and \( D_\xi \) are related to the original physical parameters by:

\[ D_\eta = D R^2 e^{2\alpha W_0} \Omega L^2 \beta \]  
\[ D_\xi = D e^{-2\alpha W_0} \Omega L^2 \beta \]

Besides these two noise strengths, this model has four other independent parameters: the dimensionless incident field \( A = \sqrt{T E_I} \exp(\alpha W_0) \sqrt{\Omega L \beta/\gamma} \), the dissipation \( B = R \exp(\alpha W_0) \), the phase shift due to propagation \( \phi_0 = k L \) and the dimensionless cavity round-trip time \( \tau_R = \gamma L/v_g \).

Eqs. (10) and (11) define the stochastic version of the standard Ikeda model, which includes the existence of spontaneous emission processes of the two-level atoms forming the absorber. It is worth noting that what is initially an additive noise in the original partial-differential equation scheme has become multiplicative in the difference-differential equation model. This may be considered as an indication of the non-trivial influence of the spontaneous emission process.

III. INFLUENCE OF NOISE ON DYNAMICS

As stated above, the dynamical properties of even the deterministic version of the Ikeda model lead to a highly complex behavior of the model. In particular, the steady state solution of the model, which can be seen to obey the following transcendental equation
plot in each of Figs. 4(a) and 4(b)) shows a radical change. A distinct peak in the power spectrum can be observed for a non-zero finite frequency in the stochastic case, in contrast to the delta function of the deterministic case. This frequency is seen to be the same as that of the periodic attractor which appears after the bifurcation. Figure 5 demonstrates this fact, by means of a comparison between the light intensity time series and its power spectrum for the deterministic (Fig. 5(a)) and noisy (Fig. 5(b)) cases. The main peak in both spectra coincide, as seen in Fig. 5(b). The oscillation amplitudes are however very different. The fact that the oscillations are much smaller in the first case \( A = 9.80 \) than in the second \( A = 9.90 \) proves that this is not a mere advance of the bifurcation caused by the noise. However, the amplitude in the pre-bifurcation case is much larger than the noise source variance would have us expect. We are hence observing an amplification of noise fluctuations, which takes place at the natural frequency selected by the dynamics of the system.

A clear picture of the amplification of noise fluctuations can be obtained by computing the standard deviation of the intensity time series as the first bifurcation is approached. This is shown in Fig. 6, where a horizontal dashed line indicates the value that is to be expected from the real noise intensity which is being handled. The amplification effect is plainly revealed.

**IV. CONCLUSION**

The main objective of this paper was to systematically derive the equations for the stochastic Ikeda model of a ring cavity with a nonlinear absorber. Spontaneous emission noise has been found to significantly influence the dynamical behavior of the system. We observe substantial amplification of noise fluctuations before the steady state loses stability; this amplification occurs at a natural frequency of the system.
REFERENCES


FIGURE 3
FIGURE 4(b)
FIGURE 5(b)
OFFICE OF NAVAL RESEARCH

END-OF-THE-YEAR REPORT

PUBLICATIONS/PATENTS/PRESENTATIONS/HONORS/STUDENTS REPORT

for

GRANT: N00014-96-1-0045

PR Number 96PR00715

NONLINEAR DYNAMICS OF COUPLED LASER SYSTEMS

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September 1997

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OFFICE OF NAVAL RESEARCH
PUBLICATIONS/PATENTS/PRESENTATIONS/HONORS/STUDENTS REPORT

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Principal Investigator: Rajarshi Roy
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a. Number of papers submitted to refereed journals, but not published: 2
b. Number of papers published in refereed journals: 6


c. Number of books or chapters submitted, but not yet published: None
d. Number of book or chapters published: None
e. Number of printed technical reports/non-refereed papers: None
f. Number of patents filed: None
g. Number of patents granted: None
h. Number of invited presentations: 13

Heraeus Foundation Physics and Dynamics Conference, Berlin (1996)
Fluctuations, Disorder and Nonlinearity, Crete (1996)
University of Houston (1996)
University of Maryland (1996)
International Conference on Dynamical Systems, Bangalore (1997)
University of Alabama, Birmingham (1997)
Suzhou University, China (1997)
Beijing Normal University (1997)
University of Nice, France (1997)
Control of Chaos Conference, Montecatini (1997)
SIAM Dynamical Systems Conference (1997)
College of William and Mary (1997)
Rensselaer Polytechnic (1997)

i. Number of submitted presentations: 0

j. Honors/Awards/Prizes for grant employees:

   Editorial Board of Journal of Semiclassical and Quantum Optics (IOP)
   Editorial Board of Nonlinear Science Today (Springer)
   Editorial Board of International Journal of Bifurcations and Chaos, Jan. 1998 -
   Fellow of the Optical Society of America

k. Total number of Full Time Equivalent Graduate Students and Post-Doctoral Associates
   supported during this period, under this R&T:

   Graduate Students: 2
   Post-Doctoral Associates: 1
   Including number of
      Female Graduate Students: 0
      Female Post-Doctoral Associates: 0
      Minority Graduate Students: 0
      Minority Post-Doctoral Associates: 0
      Asian Graduate Students: 0
      Asian Post-Doctoral Associates: 0

l. Other funding received this year:

   National Science Foundation, "Communication With Synchronized Chaotic Lasers", (1996 -
   1999), joint grant of $1,069,000 with Profs. Henry Abarbanel (UCSD) and Steve Strogatz
   (Cornell). The Georgia Tech part of the contract is $326,337.
Program Objective

To design and build coupled solid state lasers and arrays and develop models to understand and predict coherence and synchronization properties of the light emitted. Also the study of coupled waves in optical fibers, their nonlinear interactions, transport of polarized light.

Significant Results During Last Year

I. Dynamics of Coupled Lasers

Our experimental, numerical and analytic results on the dynamics of two coupled lasers were published in Physical Review E in April [1]. A novel result demonstrated was the possibility of phase synchronization of the lasers even though the amplitudes of the laser fields are unstable and chaotic. Stochastic fluctuations of the detuning between the lasers were accounted for in simulations to reproduce the characteristics of the intensity time traces measured.

A new set of experiments on a linear array of three lasers was initiated. We find the remarkable result that the outer two lasers may be beautifully synchronized with each other though the middle laser is not. We are investigating this phenomenon in the light of recent work on generalized synchronization of nonlinear oscillators [2]. An illustration of the phenomenon is being sent by mail.

Extensive simulations of the correlations of the intensity fluctuation of a nine laser array have been done this past year, and a paper is in preparation, to be submitted to Physical Review E [3]. The conclusion from experimental observations and numerical computations is that the spatial correlations of intensity correlations for the elements of the array can decay sharply or very slowly depending on the coupling strength of the lasers.

II. Nonlinear Dynamics in Optical Fibers

We are about to submit a paper to Physical Review E on an extensive study of nonlinear wave propagation in a single mode optical fiber [4]. It is shown that the evolution of new sidebands in the fiber due to four wave mixing can be significantly affected by phase fluctuations along the fiber length, as well as by fine spectral structure of the pump waves.

Two papers were published in Physics Letters A [5,6], that developed a stochastic version of the Ikeda model. Two papers, one in Optics Letters [7] and one in Physical Review A [8], reported the results of extensive measurements of the polarization dynamics of the laser intensity on nanosecond time scales. A new laser model based on the Ikeda equations was developed and used to explain the formation of sharp pulses, irregular chaotic dynamics, as well as the formation of square waves in this system. These experiments and the corresponding models open a new regime for the investigation of fiber laser dynamics and future applications.

References:


Chaos and coherence in coupled lasers

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(Received 5 August 1996)

A fundamental chaotic instability in a system of two coupled lasers is investigated both experimentally and theoretically. The amplitude instability and mutual coherence of the light emitted by the lasers is investigated as a function of the detuning and coupling parameters. A quantitative comparison of the intensity fluctuations is made with numerical simulations that include noise in the laser detuning. [S1063-651X(97)03904-4]

PACS number(s): 05.45.+b, 42.50.Lc, 42.55.Rz

Haken's seminal analogy between fluid dynamics and laser instabilities initiated extensive studies of the Lorenz-like chaotic dynamics of the single mode far-infrared ammonia laser over the last two decades [1,2]. While this is conceptually the simplest chaotic laser system, it is also of great fundamental interest that two single-mode lasers that are stable individually can exhibit a chaotic instability when coupled [3,4]. Such a system provides a beautiful illustration of the rich and complex dynamical behavior of two coupled nonlinear oscillators. Pairs of neurons [5], pacemaker cells [6], chemical oscillators [7], and Josephson junctions [8] provide other examples of coupled nonlinear oscillator systems. It has been theoretically recognized that the amplitudes of the coupled oscillators can display a rich variety of unstable behaviors for certain regimes of coupling strength [9]. However, there are no experiments on physical systems that have quantitatively probed the relationship between the chaotic amplitude instability and phase coherence of coupled nonlinear oscillators. In this paper we report the results of precise measurements of the amplitude dynamics and phase coherence of coupled lasers and make quantitative comparisons with numerical models.

Many studies of coupled lasers have been motivated by the need for high power coherent sources. Coupled semiconductor, solid state, and CO2 lasers have been studied [4,10–12], but it is the spatial properties of the output radiation that have received the most attention, rather than the dynamical characteristics of the emitted light [13]. Here, we study the chaotic dynamics and mutual coherence [14] of two coupled single-mode Nd:YAG (neodymium doped yttrium aluminum garnet) lasers that are detuned from each other by a very small amount (roughly 1 part in 108 of the oscillator frequency) and for which we can vary the coupling strength over many orders of magnitude.

The following equations describe the time evolution of the complex, slowly varying electric field $E$ and gain $G$ of a pair of spatially coupled, single transverse and longitudinal mode class B lasers [15,16]

$$\frac{dE_1}{dt} = \tau_c^{-1}[(G_1 - \alpha)E_1 - \kappa E_2] + i \omega_1 E_1,$$

$$\frac{dG_1}{dt} = \tau_f^{-1}(p_1 - G_1 - G_1|E_1|^2),$$

$$\frac{dE_2}{dt} = \tau_c^{-1}[(G_2 - \alpha_2)E_2 - \kappa E_1] + i \omega_2 E_2,$$

$$\frac{dG_2}{dt} = \tau_f^{-1}(p_2 - G_2 - G_2|E_2|^2).$$

In these equations, $\tau_c$ is the cavity round trip time ($\approx 450$ ps for a cavity of length 6 cm), $\tau_f$ is the fluorescence time of the upper lasing level of the Nd$^{3+}$ ion (1 $\mu$s for the 1064 nm transition), $p_1$ and $p_2$ are the pump coefficients, $\alpha_1$ and $\alpha_2$ are the cavity loss coefficients, and $\omega_1$ and $\omega_2$ (angular frequencies) are the detunings of the lasers from a common cavity mode, respectively. The lasers are coupled linearly to each other with strength $\kappa$, assumed to be small, and the sign of the coupling terms is chosen to account for the observed stable phase-locked state in which the lasers have a phase difference of 180°. For laser beams of Gaussian intensity profile and $1/e^2$ beam radius $r$ the coupling strength, as determined from the overlap integral of the two fields, is defined as $\kappa = \exp(-d^2/2r^2)$. Control parameters are the frequency detuning of the lasers ($\Delta \omega = \omega_2 - \omega_1$) and the coupling coefficient $\kappa$.

The dependence of the system dynamics on parameters can be numerically investigated by integrating Eqs. (1) using different values of $\kappa$ and $\Delta \omega$. Figure 1 displays the predicted amplitude instability of the two lasers and its relationship to the coherence of the laser light as a function of both the laser separation $d$ and the detuning $\Delta \omega$. The height of the graph shows the largest intensity value of laser 1 recorded during the 5 ms integration time. The color coding shows the degree of mutual coherence between the two lasers, as measured by...
the fringe visibility. The visibility $V$ of the fringe pattern formed by the small angle interference of the laser beams is defined as $V = (I_{\text{max}} - I_{\text{min}})/(I_{\text{max}} + I_{\text{min}})$ where $I_{\text{max}}$ and $I_{\text{min}}$ are adjacent maxima and minima in the fringe profile. The fringe visibility is directly proportional to the absolute value of the complex degree of mutual coherence [14]. Low visibilities, shown as blue colors in this figure, indicate states of low mutual coherence, while reds indicate visibilities approaching one and therefore high degrees of mutual coherence. One can clearly see from Fig. 1 that the area where the intensity instabilities exist occurs just before the onset of phase locking and that significant intensity oscillations appear only around a rather narrow band of detuning values between $10^5$ and $10^6$ s$^{-1}$. A single positive Lyapunov exponent was computed in this regime with a typical value of $-10^4$ s$^{-1}$, demonstrating the chaotic nature of the instability.

Insight into the amplitude instability can be obtained by considering the special case of identical laser parameters and by assuming that the two laser amplitudes and gains are identical. Equations (1) then reduce to

$$\frac{dE}{dt} = \tau_e^{-1}[G - \alpha - \kappa \cos(\Phi)]E,$$  \hspace{1cm} (2a)

$$\frac{dG}{dt} = \tau_f^{-1}(p - G - GE^2).$$  \hspace{1cm} (2b)
FIG. 2. Experimental system for generating two laterally coupled lasers in a Nd:YAG crystal and observing the amplitude instability. RP is a rectangular prism; translating this device changes the pump beam separation, and thus the infrared beam separation. The Nd:YAG crystal is coated for high reflectivity (HR) on one side and antireflection coated (AR) on the other. The output coupler (OC) is 2% transmissive; both mirrors at flat. FPI is a scanning Fabry-Perot interferometer, and as used to measure the mode spectrum of both lasers.

\[
\frac{d\Phi}{dt} = 2\tau_c^{-1}\kappa\sin(\Phi) + \Delta \omega
\]  

(2c)

for the laser amplitudes \( |E_1| = |E_2| = \mathcal{E} \), gains \( G_1 = G_2 = G \) and phase difference \( \Phi = \phi_2 - \phi_1 \), where \( \phi_1 \) is the phase of the field \( E_1 \).

Equations (2a)-(2c) are the rate equations describing a single mode class B laser with variable losses. The phase equation can be integrated exactly, and \( \Phi(t) \) is an unbounded function of time if the detuning \( |\Delta \omega| \) exceeds a critical detuning \( \Delta \omega_c \), where

\[
\Delta \omega_c = 2\kappa \tau_c^{-1}. 
\]  

(3)

This is the critical condition for an amplitude instability [4]; we also note that the lasers are phase locked for detunings smaller than \( \Delta \omega_c \) [16]. If condition (3) is obeyed, then the laser equations (2a) and (2b) are periodically modulated by the \( \cos[\Phi(t)] \) term. The frequency of these modulations is given by

\[
\omega_M = \sqrt{\Delta \omega^2 - \Delta \omega_c^2}. 
\]  

(4)

On the other hand, it is known that the laser relaxation oscillation frequency \( \omega_R = 2\pi \nu_R \) for small \( \tau_c / \tau_f \) and \( \kappa = 0 \) is given by

\[
\omega_R = \left( \frac{2(\rho - \alpha)}{\tau_c \tau_f} \right)^{1/2}. 
\]  

(5)

This implies the possibility of subharmonic resonance if the ratio of \( \omega_M \) to \( \omega_R \) is close to an integer. These resonances then produce branches of subharmonic solutions which explain the destabilization of the laser system [17,18].

We have tested the prediction of the amplitude instability with the experimental system of Fig. 2, which consists of two parallel, laterally separated lasers created by pumping a single Nd:YAG rod of 5 mm length and diameter in a plane parallel cavity. The pump beams are generated from the argon ion laser output (\( \lambda = 514.5 \text{ nm} \)) by a system of beam splitters and prisms that ensure parallel propagation at an adjustable separation symmetric with respect to the YAG rod axis. The optical cavity consists of one high reflection coated end face of the rod and of an external planar output coupler with 2% transmittance. A Brewster plate and thick etalon within the cavity ensure linear polarization and single longitudinal mode operation. The lasers were operated at approximately 33% above threshold pump power. For these parameters, the relaxation oscillation frequency, \( \nu_R \), is of the order of 100 kHz. The frequency detuning between the two lasers can be adjusted by tilting the output coupler slightly, thereby introducing a minute difference in cavity lengths.

Thermal lensing induced in the YAG crystal by the pump beams of waist radius \( \approx 20 \mu \text{m} \) is responsible for generating two stable, separate cavities [16]. The TEM\(_{00}\) infrared laser beams have radii of (at 1/e\(^2\) of the maximum intensity of the Gaussian profile) of \( r \approx 200 \mu \text{m} \) and their overlap may be continuously changed by varying the lateral separation \( d \) of the pump beams over a range of 0.5 mm to 3 mm. The pump beam separation and profiles are measured directly by a rotating slit technique. In this range, there is no appreciable overlap of the pump beams and coupling is entirely due to the spatial overlap of the infrared laser fields.

The individual output intensity time series are recorded with fast photodetectors and a two channel digital oscilloscope. The optical frequency difference of the lasers is measured with a radio frequency spectrum analyzer after combining the two beams on a photodetector. A scanning Fabry-Perot interferometer was used to ensure that both lasers oscillated only on a single longitudinal mode.

The change of dynamical behavior of the detuned, coupled system can be seen as the separation of the pump beams is varied. For a large separation (\( d \approx 1.20 \text{ mm} \)) the lasers were stable and incoherent. The visibility of the fringes was low (\( V \approx 0 \)), and the heterodyne single was measured to be between 30 and 40 MHz. For a small separation (\( d \approx 0.8 \text{ mm} \)), the lasers are stable and phase locked. The fringe visibility was high (\( V \approx 1 \)), and the heterodyne signal was absent since the lasers were frequency locked. Figure 3(a) shows a typical intensity time trace characteristic of the unstable regime. Large bursts of the intensity occur, separated by quiescent periods. Here the lasers were separated by 1.03 mm, which implies \( \kappa = 2.0 \times 10^{-5} \). Using Eq. (3), we find that the condition for an amplitude instability requires \( |\Delta \omega| > 10^5 \text{ s}^{-1} \), which is verified in our experiments (\( \Delta \omega \approx 1 \text{ MHz} \)). The intermediate visibility of \( V \approx 0.20 \) signifies the onset of phase locking. The experimentally measured visibilities are in excellent agreement with the numerically computed values represented in Fig. 1.

In the experiment, a substantial amount of fluctuation in the detuning between the two lasers was observed; the beat signal frequency in the unstable regime fluctuated between 0 and 10 MHz. In order to obtain quantitative comparison between measured intensity time series and simulations, we numerically investigated the behavior of Eqs. (1) with a stochastic detuning term, such that \( \Delta \omega(t) = \omega_2 - \omega_1 \), where \( \omega_1 = \omega_0 + \delta \omega_1(t) \). Here \( \delta \omega_1(t) \) is a colored noise term of
FIG. 3. Intensity time traces of (a) experiment and (b) numerical simulation. The time trace in (a) was measured at a pump separation of $d = 1.03 \text{ mm}$, and illustrates the bursting nature of the amplitude instability. The average interspike interval (ISI) is 1.9 ms, the normalized standard deviation $\sigma_{1}/\bar{T} = 0.10$, and the standard deviation of the detuning $\sigma_{\Delta \omega} = 10^8 \text{ s}^{-1}$. (b) The numerically computed time trace of the intensity of laser 1 with an exponentially correlated, stochastic detuning of strength $D = 5 \times 10^9 \text{ s}^{-1}$ and correlation time $\lambda^{-1} = 3 \text{ ms}$. The mean detuning $\Delta \omega_0 = 5 \times 10^7 \text{ s}^{-1}$, and the standard deviation of the detuning $\sigma_{\Delta \omega} = 1.4 \times 10^8 \text{ s}^{-1}$. The average ISI was 1.7 ms, and $\sigma_{1}/\bar{T} = 0.12$. The cavity losses were taken to be 4% and the lasers were pumped one-third above threshold, with a 0.5% asymmetry.

The average time between adjacent bursts whose intensities are greater than some threshold, here defined to be 1.2 times the average intensity. To avoid counting the same burst twice, a “quiescence time” $\tau_q$ of 0.8 ms was used such that a new spike would be detected no sooner than $\tau_q$. The standard deviation of the detuning in the experiments was measured to be on the order of 10 MHz or less; numerically, $\sigma_{\Delta \omega} = \sqrt{D\lambda}$. Using these statistical measures, the parameters $D$ and $\lambda$ were adjusted to give quantitative agreement between the observed experimental results and the numerical simulations. The range of parameters $D$ and $\lambda$ that gave quantitative agreement with experiment is very limited: $D - O(10^9 \text{ s}^{-1})$ and $\lambda^{-1} - O(10^{-3})$. Figure 3(b) shows a good match with the experimental data.

In conclusion, we have demonstrated a fundamental amplitude instability of two coupled lasers and its relationship to the mutual coherence of the total field. Theoretical and numerical predictions, using a dynamical model, of the range of coupling strengths where the instability is expected to occur agree very well with experimental observations. For large separations, both the model and experiment reveal stable intensities and no appreciable coherence. As the separation is decreased to just above the phase-locking point, large amplitude fluctuations are observed, in agreement with numerical predictions. The laser fields exhibit a low degree of mutual coherence for this range of coupling strength. It was necessary to include stochastic detuning fluctuations to achieve quantitative agreement between experimental and simulation in the unstable regime. Finally, for even smaller separations, phase locking is achieved. The lasers are now found to be stable, mutually coherent, and frequency locked. These studies are directly relevant to the design of laser arrays; they also reveal a rich and complex dynamical scenario which should be systematically explored in the future for a variety of different oscillator systems.

We acknowledge support from the Division of Chemical Sciences, Office of Basic Energy Sciences, Office of Energy Research, U.S. Department of Energy, and the Office of Naval Research. M.M. acknowledges support from the Deutsche Forschungsgemeinschaft (DFG, Germany). R.R. thanks Neal Abraham, Edgar Knobloch, and Steve Strogatz for helpful discussions. T.E. acknowledges support from U.S. Air Force Office of Scientific Research Grant AFOSR-93-1-0084, National Science Foundation Grant DMS-9308009, the Fonds National de la Recherche Scientifique (Belgium), and the InterUniversity Attraction Pole of the Belgian government.

Fast polarization dynamics of an erbium-doped fiber ring laser

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The polarization dynamics of a unidirectional erbium-doped fiber ring laser has been observed for individual
round trips in the cavity. A rich variety of dynamic states, including square-wave pulses and irregular
temporal patterns, was observed as operating parameters were changed. A model with coupled delay and
differential equations is used to interpret the dynamics. © 1996 Optical Society of America

Rare-earth-doped silica fiber lasers have recently re-
ceived much attention in the context of long-time-scale
polarization resolved dynamics. Phenomena such as
antiphase dynamics in orthogonal polarization states,1
self-pulsing,2 and polarization switching induced by
optical feedback3 have been reported. Experimental
evidence of the quasi-periodic route to chaos in an
erbium-doped fiber laser has been published.4 The
previous reports were done on the millisecond time
scale, which corresponds to the relaxation oscillation
frequency of the fiber laser.

We present some measurements of the fast temporal
(on the nanosecond time scale) dynamics of the Er3+-
doped fiber ring laser (EDFRL). The fiber laser out-
put beam contains two linearly polarized components.
It is within the two groups of orthogonal polarization
eigenmodes that the various dynamic states are ob-
erved and investigated. Computational results from
a model based on coupled delay and differential equa-
tions of the Ikeda type5 provide an explanation of the
experimental observations. A similar model was de-
developed by Loh and Tang6 for polarization dynamics
of an external-cavity semiconductor laser.

The EDFRL presents a unique opportunity for the
study of laser nonlinear dynamics. The small longi-
tudinal mode spacing and large gain bandwidth make
the EDFRL a practical experimental system in which
the collective behavior of a large number of globally
coupled nonlinear oscillators can be observed. Such
models have been studied in the context of physical
and biological systems by Strogatz and co-workers7 and by
many others recently.

A schematic of the experimental configuration is
shown in Fig. 1. The coherent pump source was the
514.5-nm wavelength line from an argon-ion laser. A
6-m length of erbium-doped fiber with an ion con-
centration of ~240 parts in 10^6 was taken as the
gain medium. A Faraday optical isolator was included
in the laser cavity to ensure unidirectional opera-
tion. An output coupler removed 3% of the intra-
cavity power. The polarization controller functioned
as a discrete birefringence-inducing element. Overall,
the laser cavity was 20 m long, 14 m being passive
optical fiber. Free ends of the couplers were placed in
index-matching fluid to suppress the small, but para-
sitic, Fresnel reflections. The output at \( \lambda = 1.561 \mu m \)
was sent through a \( \lambda/2 \) wave plate and a polarization
beam splitter cube, where the orthogonal polari-

Experimental Data

![Graphical data](image)

Fig. 2. Experimentally measured polarization resolved traces of (a) self-pulsing at the cavity round-trip time in the x-polarization direction from an EDFRL with 10% output coupling, (b) irregular trace in the y-polarization direction. The EDFRL was pumped four times threshold. (c), (d) Antiphase square pulses in the x- and y-polarization directions, respectively, from an EDFRL with 3% output coupling. The EDFRL was pumped at 3.3 times threshold.

levels (three to four times above threshold), antiphase square pulses were formed in the orthogonal polarization intensities for a narrow range of adjustment of the polarization controller. Figure 2(c) shows 30-ns pulses. This corresponds to the 6-m length of the gain medium. Figure 2(d) shows 70-ns pulses that correspond to the 14-m length of the passive fiber within the laser cavity. Another detail to note is the highly structured intensity fluctuations that ride on top of the square pulses and repeat over many round trips.

A laser model based on an Ikeda-type set of delay-differential equations was used to investigate the dynamical behavior of the EDFRL. Loh and Tang derived a set of difference-differential equations to study ultrahigh-frequency polarization self-modulation in semiconductor lasers. It is in the same spirit that we derive our set of equations from the Maxwell–Bloch equations; they take the form

\[
S_1(t) = \frac{R_1}{2} \left( S_1(t - \tau_R) \exp\{2A_1[W(t)]\}(1 + \cos \phi) + S_2(t - \tau_R) \exp\{2A_2[W(t)]\}(1 - \cos \phi) - 2[S_1(t - \tau_R)S_2(t - \tau_R)]^{1/2} \times \exp[A_1[W(t)] + A_2[W(t)] + \sin(\kappa[W(t)]\sin \phi)\right),
\]

\[
S_2(t) = \frac{R_2}{2} \left( S_1(t - \tau_R) \exp\{2A_1[W(t)]\}(1 - \cos \phi) + S_2(t - \tau_R) \exp\{2A_2[W(t)]\}(1 + \cos \phi) + 2[S_1(t - \tau_R)S_2(t - \tau_R)]^{1/2} \times \exp[A_1[W(t)] + A_2[W(t)] \times \sin(\kappa[W(t)])\sin \phi\right),
\]

\[
\frac{dW(t)}{dt} = P - \gamma_W(W_T + W(t)) - S_1(t - \tau_R) \times (\exp[a_1[W(t) - N_0L]] - 1) - S_2(t - \tau_R)(\exp[a_2[W(t) - N_0L]] - 1),
\]

where \(W(t) = \int_0^Z N(z,t) + z/\nu_g)dz\), \(\kappa[W(t)] = q_1[W(t)] - q_2[W(t)] - \beta, S_{1,2} = [E_{1,2}(t - \tau_R)]^2/\hbar\omega_{1,2}, A_{1,2}[W(t)] = (a_{1,2}/2)[W(t) - N_0L], q_{1,2}[W(t)] = a_{1,2}(a_{1,2}/2)[W(t) - W(t = 0)], \) and \(\phi\) is the relative phase difference between the polarized fields. The mode detuning factor is defined as \(\alpha_m = -\Delta_m/
u_\phi\), where \(\Delta_m = \omega_m - \omega_0\) and \(m = 1, 2\). Other parameters are defined in Table 1.

In this model the gain is taken to be a linear function of the population inversion. \(S_1\) and \(S_2\) are the photon number densities for the x- and y-polarization modes, respectively, and \(W\) represents the inversion. The differential equation was integrated with a fourth-order Runge–Kutta routine with a 1-ns integration step, corresponding to the experimental sampling time for observation of laser dynamics in a single cavity round trip.

In Eqs. (1)–(3) the lumped parameter \(\beta\) is due to the phase shift associated with fiber birefringence over the

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
<th>Definition</th>
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<td>(R_{1,2})</td>
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<td>(\tau_R)</td>
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<td>Cavity round-trip time</td>
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<td>(P)</td>
<td>(1.44 \times 10^{23})</td>
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Fig. 3. Numerical simulations of time traces showing self-pulsing at the cavity round-trip time in (a) the x-polarization direction, (b) the y-polarization direction, (c), (d) Antiphase square-wave pulses in the x- and y-polarization directions corresponding to those of Figs. 2(c)–2(d). Parameters for (a) and (b): $\phi_A = 0.027$, $\phi_P = \pi - 0.175$, $\lambda = 1.5 \times 10^{-3}$, and $\Delta \lambda = 0.125$ fm. Parameters for (c) and (d): $\phi_A = 0.027$, $\phi_P = \pi - 0.015$, $\lambda = 10^{-2}$, and $\Delta \lambda = 4.09$ fm. The pump rate is 3.2 times threshold.

entire cavity. By changing the polarization controller, one introduces a local birefringence by applying stress to the fiber, which appears as a discrete phase shift in the section of passive fiber. We take the phase term $\phi$ to be $\phi_A$ in the active region and $\phi_P$ in the passive region that contains the polarization controller. The value of $\phi_A$ was taken to be small but nonzero. The small phase shift in the active fiber could be a result of the active fiber's being wound onto a spool. This feature of separate phases in the active and passive fiber portions is necessary for reproducing the observed asymmetric nature of the square-wave pulsations. $\phi_P$ was taken to be approximately $\pi$ rad because the polarization controller functions roughly as a $\lambda/2$ wave plate. The birefringence causes the two mode groups to travel at different speeds, ultimately resulting in a mode group detuning $\Delta \lambda = (\lambda_2 - \lambda_1)$.

Typical results from the numerical model showing output intensities in orthogonal polarization directions are displayed in Fig. 3. Table 1 gives values for the physical parameters of the system. These parameters yield a good match between theory and experiment, as seen from Figs. 2 and 3. However, these computations are merely representative of the large variety of waveforms that emerge for different parameter values; they are not meant to provide a detailed reproduction of the experimental waveforms. The sharp pulses are seen to be distinctly separated by the fundamental cavity round-trip time of 100 ns in Figs. 3(a) and 3(b). One sees that the irregular waveforms actually repeat over single cavity round trips for the parameters chosen. Figures 3(c) and 3(d) show antiphase square-wave pulses that form when the parameter value settings of $\phi_A$, $\phi_P$, $\beta$, and $\Delta \lambda$ are in the proper regime.

Essential experimental features captured by the model are the following: (1) the dynamics occur on the nanosecond time scale, (2) self-pulsing at the cavity round-trip time or multiples with repeating irregular waveforms is present, (3) antiphase square-wave pulses form when parameter values are favorable, (4) the dynamics of the system take place with a dc background, (5) highly structured fluctuations are present on the tops of the square pulses, and (6) the time durations of the square pulses correspond to the lengths of active and passive fiber in the ring.

In conclusion, measurements of the fast temporal dynamics during a single cavity round trip have been made for an erbium-doped fiber ring laser. Square-wave pulsing and irregular dynamics that repeat at round-trip times have been observed in the two orthogonal polarization eigenstates. The experimentally observed properties were described by a unified model based on an Ikeda-type delay-differential equation model of the laser. We have shown that fiber birefringence, polarization controller adjustment, and the frequency difference between the orthogonal mode groups influence the nature of the dynamics.

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References

Fast intracavity polarization dynamics of an erbium-doped fiber ring laser: Inclusion of stochastic effects

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The dynamics of a unidirectional erbium-doped fiber laser is investigated on a time scale short enough to observe, with good resolution, its behavior for individual round-trips in the laser cavity. With an intracavity polarization controller, a rich variety of nonlinear phenomena, ranging from self-pulsing to square-wave antiphase patterns in two orthogonal states of polarization, are observed. These patterns evolve continuously in time. A stochastic delay-differential equation model is proposed to describe this system. Numerical simulations show that this model satisfactorily accounts for all types of qualitative behavior and reveal that the inclusion of spontaneous-emission noise is necessary to reproduce the observed continuous pattern evolution. Two different, typical types of nonlinear dynamical states are found both numerically and experimentally: a deterministic, low-dimensional regime and a noise-driven high-dimensional motion. [S1050-2947(97)01403-0]

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I. INTRODUCTION

The idea of doping glass to obtain amplifying optical fibers is very attractive from both a technological and a fundamental point of view. Technologically, fiber amplifiers are very promising useful devices in all-optical telecommunication schemes, through their use to replace repeaters in fiber-optic transmission lines. For instance, when these materials are complemented with a cavity resonator and a pumping scheme, laser emission can be obtained. Such systems are used for the generation of ultrashort pulses and solitons.

Besides their evident practical applications, fiber lasers are very interesting from a basic physics perspective. The conjunction of the inherent nonlinear character of both the optical fiber and the light amplification process makes this type of laser specially suited for investigations of nonlinear dynamics in optical systems. Furthermore, because of the amorphous character of the glass host, fiber lasers are the ideal counterpart of the more extended and well-known doped-crystal solid-state lasers.

Due to the optical-guiding characteristics of their amplifying medium, fiber lasers can have cavity lengths of the order of tens of meters, orders of magnitude higher than in most other lasers. This fact, along with the broad gain profile of doped fibers, ensures that a large number of longitudinal modes experience gain and coexist inside the cavity, coupled through gain sharing. Hence fiber lasers usually operate in a strongly multimode regime. The dynamics of multimode lasers is very rich, including antiphase behavior and self-organized collective oscillations [1]. Previous experiments in fiber lasers [2,3] have shown this kind of phenomenon in the dynamics of two orthogonal states of polarization, which suggest a description of this system in terms of two supermodes associated with two different polarization eigenstates of the field. Another dynamical feature that is usually observed in experiments is self-pulsing [4], which has recently been related to an absorption effect due to interaction between dopant ions [5]. All the previous experiments have been done in the millisecond to microsecond time scale, which corresponds to the relaxation-oscillation frequency of the laser. But this system, due to its large cavity length and thus long round-trip time, gives us a unique chance to observe its dynamics for individual round-trips inside the cavity. This work aims at the characterization of this fast polarization dynamics in the regimes previously mentioned.

We report experimental observations of the intracavity dynamics of an erbium-doped fiber laser. A polarization-controlling device has been included in the cavity and, as a result, a fast polarization-switching effect, on a time scale of the order of nanoseconds, has been observed. This kind of effect is known to occur in semiconductor lasers [6] when a wave plate is inserted in the laser cavity. Recently, optical feedback has been found to induce this effect also in Nd-doped fiber lasers [7], but on a much slower time scale (on the order of microseconds). A model is proposed to explain the behavior observed. Most models used so far in doped-fiber lasers to account for antiphase [2], self-pulsing [3], and polarization-switching [7] behavior are based on semiclassical rate equations for each of the two polarization supermodes, which are coupled to one another through cross saturation and gain sharing. In some cases, the need of explicitly taking into account the dependence of the system variables on the propagation direction has been stressed [3]. This consideration, which is, in general, advisable in this system due to its long cavity, is in our case unavoidable given the time scale in which the observations are made. Following Loh and Tang [8,9] in their modeling of fast polarization self-modulation in semiconductor lasers, we develop a delay-differential equation model that accounts for all kinds of features observed. The inclusion of spontaneous-emission noise is seen to be necessary to obtain a more complete agreement. Indeed, the importance of spontaneous emission in the dynamics of guided lasers is a well-established fact [2]. Finally,
in order to simplify the modeling of the system, a ring-cavity configuration is used. Preliminary results of this investigation have been reported elsewhere [10]. The outline of the paper is the following. Section II contains a description of the experimental apparatus and a report of the behavior observed. Section III establishes a theoretical model that reproduces this behavior, as shown by numerical simulations. Finally, some conclusions and comments are made in Sec. IV.

II. EXPERIMENTAL FEATURES

A. Experimental setup

Several wavelengths can be used to optically pump an erbium-doped fiber amplifier in order to obtain laser emission. In our case the pump wavelength is fixed at 514.5 nm and is provided by an Ar" ion laser. Under these conditions, the lasing frequency lies in the near infrared, at 1.561 μm. The experimental setup is shown in Fig. 1. The amplifying medium is a 6-m-long erbium-doped fiber, with an ion concentration of approximately 240 ppm (corresponding to 4.98 × 10²⁵ ions/m³). The total cavity length is made to be 20 m long with the addition of 14 m of passive optical fiber. The fiber is closed on itself in order to form a ring cavity. To ensure unidirectional operation, an optical isolator is placed inside the cavity. The optical isolator is based on the Faraday effect and is polarization insensitive. The pump light coming from the argon laser is launched into the ring fiber through a wavelength division multiplexer (WDM), while an output coupler removes part of the light that circulates inside the cavity. In both cases, fiber ends were placed in an index-matching fluid to prevent possible parasitic Fresnel reflections, as shown in Fig. 1. Two different output couplers have been used, with coupling ratios 90/10 (10%) and 97/3 (3%), respectively. A 5 × microscope objective is used to optimize the coupling of the pump light into the input port of the WDM. The output emission is passed through a 10% transmission neutral density (ND) filter and a half-wave plate to a polarization beam splitter, which separates the light into its two orthogonal polarization components. These components are measured with two high-speed photodetectors connected to the two input channels of a fast digital oscilloscope with a 1-GHz sampling rate. This setup allows us to measure the intensity with 100 data points per cavity round-trip.

In order to modify the polarization state of the light traveling inside the fiber, a polarization controller is used. Polarization controllers produce a phase shift by introducing a local birefringence into a portion of the fiber. This is accomplished by winding the fiber around mandrels of the proper diameter. It is very important to correctly choose both the diameter of the mandrels and the number of turns of the fiber around them: if the diameter is too small, the bending loss of the device becomes too high; too few turns would undesirably reduce the phase shift. We found that, for wavelengths of the circulating light, a diameter of 38 mm and three turns of fiber around each mandrel was a good choice to produce a small loss and a retardation effect similar to that of a half-wave plate.

B. Characterization of the system

A measurement of the total output power as a function of pumping is the first standard procedure used to characterize this laser system. Such a procedure shows that the lasing threshold is ~150 mW. When the the output light is separated into its two orthogonal polarization components, one can see that the two states have slightly different thresholds and very different output vs pump slopes in the lasing regime. This is a first indication of the well-known two-mode-like behavior of doped-fiber lasers [2,3]. By suitably modifying the state of the polarization controller, it is possible to separate the two main groups of modes that are amplified inside the cavity. The optical spectra in two orthogonal polarization directions, as obtained from an optical spectrum analyzer, show that the two mode groups are indeed orthogonal and linearly polarized, with spectral peaks centered around ~1.560 52 μm and ~1.561 05 μm, respectively.

The behavior of output vs pump power in the lasing regime is observed to be linear, which is a characteristic of most lasers. Nevertheless, at high pump powers, an increase of output power fluctuations occurs while making the measurements. In order to quantify this effect, one can measure the standard deviation of these fluctuations as a function of the mean light intensity and pump power. The results are shown in Fig. 2 for the case of 10% output coupling. For the sake of clarity, we should remark at this point that the pump power that appears in this figure is just the recorded output of the pump laser; it does not correspond exactly to the actual power that is being injected into the fiber laser, due to the imperfect launching of pump light into the cavity through the WDM. In any case, an analysis of this figure reveals a steady increase of the fluctuations as both pump level and output power are raised. This phenomenon is rather unexpected: in most single mode lasers, fluctuations produced by spontaneous emission are independent of pump level once lasing has been achieved. This is so because the spontaneous-emission rate is proportional to the population inversion in the amplifying medium, and this is constant beyond threshold, as can be seen from any rate-equation model [11]. In multimode lasers, these fluctuations may be deterministic and originate in the nonlinear dynamics of modes coupled through sharing of the population inversion.

We can calculate the number of modes inside the cavity by measuring the optical spectrum of the output light. The ratio of its full width at half maximum to the free spectral
FIG. 2. Standard deviation of the output intensity fluctuations vs the mean output level and pump power. Two different sample times (shown in the legend) have been used. The pump power shown in the lower figure does not correspond to the power that is actually injected into the fiber.

range of the cavity (longitudinal mode spacing) gives us an estimate of this quantity. We observe a pronounced spectral narrowing and a corresponding sharp decrease in the number of modes (from $\sim 3 \times 10^5$ to $\sim 2 \times 10^3$) as the lasing threshold is crossed. Note, however, that even in the lasing regime the number of amplified modes is very large. This fact shows the strongly multimode character of fiber lasers.

C. Dynamical behavior

1. Self-pulsing

A characteristic time trace of the total output intensity extracted by the output coupler in the higher loss case (10% coupling) is shown in Fig. 3 for a pump rate well above threshold. Self-pulsing is observed with a periodicity of $\sim 100$ ns. This corresponds to the cavity round-trip time of our system, which is estimated as $L/v$, where $L = 20$ m is the cavity length and $v = c/n$ is the speed of light in the fiber. The index of refraction of erbium-doped fiber is $n = 1.46$.

FIG. 3. Total output intensity time trace showing self-pulsing with 10% output coupler. The pump power is 400 mW.

FIG. 4. Two different polarization-resolved time traces. Both polarization directions are shown in each case.

One can now resolve the output in terms of its orthogonal linear polarization components. These components, although coupled, may exhibit very different dynamics. Figure 4(a) shows quasiperiodic behavior in one polarization direction and random evolution in the other, also for 10% output coupling. In other experimental situations, one can observe different quasiperiodic evolution in the two modes. Figure 4(b) corresponds to a case with period-1 behavior in one direction and period-7 in the other. The 3% output coupler has been used in this case.

2. Influence of the polarization controller

Another way of establishing the distinct character of the polarization-resolved intensity time traces compared to those of the total output intensity is through their power spectrum. In the latter case, a typical spectrum shows peaks separated by the fundamental cavity frequency of 9.8 MHz. However, when the signal used comes from a single polarization direction, sideband peaks appear between the main ones. These sidebands can be tuned by manipulating the polarization controller and eventually can be made to overlap. When this happens and losses are small enough (i.e., the light intensity inside the cavity is high enough), the pulsed behavior disappears and square pulses develop in the output intensity of the
FIG. 5. Antiphase square pulsing in the two orthogonal polarization components of the output light for a given setting of the polarization controller and a 3% output coupling. The pump power is 600 mW.

orthogonal polarization states. This behavior is antiphase in the two states and is periodic at the cavity round-trip time, as shown in Fig. 5. The requirement that losses have to be low for this effect to occur is reflected in the fact that square pulses are observed when the 3% output coupler is used, but not in the 10% case. It is also worth noting that the time durations of the plateaus correspond to the lengths of the active and passive part of the fiber. In other words, the 70-ns upper part of the pulse in the y-polarization trace of Fig. 5 corresponds to the 14 m of passive fiber, whereas the 30-ns lower part is related to the 6 m of active erbium-doped fiber. A threshold pump power is typically observed for the onset of square pulsing. For the measurements shown, square pulses formed at a pump power ~2.2 times above threshold. In addition to square pulses, other antiphase pulse patterns have been observed. One of them is shown in Fig. 6. A final remark on this behavior is that the irregular intensity patterns superimposed on the plateaus of the square waves evolve continuously and slowly in time.

FIG. 6. Another antiphase pattern observed. Here the pump power is here 700 mW.

D. Nonlinear analysis

We have observed so far that this system has a wide variety of nonlinear attractors: regular or irregular temporal patterns in which the dynamics are trapped. Which specific attractor occurs depends on many factors: the state of the polarization controller, the pump power, the output losses, etc. We are now interested in dynamical characteristics of these attractors.

The state space of a dynamical system can be reconstructed from the information obtained through a scalar measurement (the intensity of one of the polarization eigenstates, for instance) by means of time-delay vectors [12]. If we denote as \( x(n) = x(t_0 + n \Delta) \) a scalar set of measurements sampled at equally spaced time intervals \( \Delta \), one can construct \( d \)-dimensional vectors

\[
y(n) = (x(n), x(n+T), \ldots, x(n+[d-1]T)).
\]

The evolution of these time-delayed vectors in state space describes an attractor. There exists a minimum value of \( d \) for this attractor to properly represent the dynamical behavior of the system. This value is called the embedding dimension of the system. Both the embedding dimension and the time lag \( T \) have to be chosen carefully if one wants this state space reconstruction to be really useful.

To obtain a reasonable value of the time delay \( T \), one has to reach a compromise between the high correlation between vector components that would arise if \( T \) is chosen too small \([x(n+iT) \text{ and } x(n+(i-1)T)]\) would be nearly identical\) and their statistical independence if \( T \) is too large. All these features are reflected in the so-called average mutual information function, which can be interpreted as a nonlinear correlation function between the time series \( x(n) \) and \( x(n+\Delta t) \) as a function of the time lag \( \Delta t \). Its definition is

\[
M(\Delta t) = \sum_{n=1}^{N} P(s(n), s(n+\Delta t)) \times \log_2 \left[ \frac{P(s(n), s(n+\Delta t))}{P(s(n))P(s(n+\Delta t))} \right].
\]

where \( P(s(n)) \) is the probability density of the process \( s(n) \) and \( P(s(n), s(n+\Delta t)) \) is the joint probability of the two time-shifted series. A high value of this function represents a high correlation between the series and a low value corresponds to a high degree of independence. A suitable value of \( T \) will be intermediate between these two regimes. A reasonable prescription that is frequently used [13] is to choose \( T \) as the first minimum of \( M(\Delta t) \).

We can compute the average mutual information function (2) for the time series measurements obtained from our experiment. A typical result is shown in Fig. 7, corresponding to the intensity for a single polarization direction. The behavior is roughly the same for the other polarization direction and for the total output and also for the other different dynamical regimes investigated. The results suggest that an adequate value for the time delay is \( T = 3 \).

Once the time delay has been chosen, one needs to determine the embedding dimension. To do so, we use a method proposed in Ref. [14]. This procedure determines the minimum useful embedding dimension as that for which the per-
FIG. 7. Self-pulsing time series for the intensity output in one of the polarization directions and its corresponding average mutual information function. The pulse separation is $2\tau_e$ and the pump power is 1000 mW.

The percentage of false nearest neighbors (FNNs) in the attractor drops to zero. Two points of the attractor are said to be FNNs when they seem to be close only because the attractor is embedded in a dimension that is too low, but they are actually separated from one another. They can be identified by measuring the distance between them in two consecutive dimensions. When this distance is very small in the lowest-dimensional space and much larger in the highest-dimensional space, the two points are FNNs. The procedure consists of computing the percentage of FNNs for increasing dimensions. The embedding dimension is then determined as the dimension for which this percentage drops to a very small number. Figure 8 presents the result of this method for two different time series exhibiting very different dynamical behaviors. Figure 8(a) corresponds to a quasiperiodic low-dimensional regime with an embedding dimension $d_e = 4$. Figure 8(b), on the other hand, shows a nonperiodic time series whose percentage of FNNs does not go to zero as the embedding dimension increases. This indicates that the dynamics in this case is high dimensional and hence noise driven. A similar coexistence of deterministic and stochastic behavior in the same dynamical system has recently been observed in a Nd:YAG laser (where YAG denotes yttrium aluminum garnet) exhibiting deterministic chaos [15].

III. MODELING

A. A delay-differential equation model

To develop a theoretical model that reproduces the observations made so far, several important characteristics of this system have to be taken into account. This requirement is completely unavoidable in our case since we are observing the dynamics within individual cavity round-trips.

(iii) The polarization controller is acting as an important half-wave plate that almost completely switches polarizations once every cavity round-trip.

(iv) Only part of the total cavity length is active medium. Hence this system has two different characteristic lengths which are reflected in the experimental results (see Sec. II C 2) and must appear also in the theoretical model.

(v) Spontaneous-emission noise is known to have an important influence on the behavior of guided lasers such as the one we are dealing with in this experiment [2]. It thus seems necessary to include it in any realistic model of fiber lasers.

The first three points in the previous list have already been faced by Loh and Tang [8,9] in their description of ultrafast polarization self-modulation in semiconductor lasers. In this study, they developed a delay-differential equation model similar in approach to that used by Ikeda and co-workers [16,17] to analyze instabilities in the absorption of light by a passive medium placed inside a ring cavity and
by Otsuka and Iwamura [18] to model the dynamics of semiconductor laser amplifiers. We will follow the spirit of Loh and Tang’s study to derive a model for our system.

Let $E_1(t,z)$ and $E_2(t,z)$ be the complex field envelopes of the two polarization modes of the amplified radiation. The following set of equations can be derived for their time evolution after adiabatically eliminating the polarization of the medium from the corresponding Maxwell-Bloch equations [8].

$$\frac{\partial E_1(t,z)}{\partial t} + \frac{1}{v_g} \frac{\partial E_1(t,z)}{\partial z} = \frac{a_1}{2}(1-i\alpha_1)[N-N_0]E_1(t,z)$$
$$+ \mu_1(t,z). \tag{3}$$

$$\frac{\partial E_2(t,z)}{\partial t} + \frac{1}{v_g} \frac{\partial E_2(t,z)}{\partial z} = \frac{a_2}{2}(1-i\alpha_2)[N-N_0]E_2(t,z)$$
$$+ \mu_2(t,z). \tag{4}$$

$$\frac{\partial N(t,z)}{\partial t} = P - \gamma_1[N(t,z) + N_i] - a_1[N(t,z) - N_0] |E_1|^2 \frac{1}{\hbar \omega_1}$$
$$- a_2[N(t,z) - N_0] |E_2|^2 \frac{1}{\hbar \omega_2}. \tag{5}$$

It can be seen that the two modes are coupled through $N(t,z)$, the population inversion of the medium. $N_0$ is the population inversion necessary for transparency (i.e., for zero gain) and $N_i$ the density of erbium ions in the fiber. The quantity $N_i$ has to be taken into account because erbium-doped fiber, when pumped at 514.5 nm, behaves as a threel level medium with incoherent pumping [11]. $a_1$ and $a_2$ are gain coefficients. $\alpha_1$ and $\alpha_2$ represent the detuning between the corresponding mode frequency $\omega$ and the resonance frequency of the cavity $\omega_0$ [$\alpha_0 = (\omega_0 - \omega)/\gamma_1$, where $\gamma_1$ is the decay rate of the polarization of the medium], $v_g$ is the velocity of light in the medium (assumed equal for the two modes), and $P$ is the pump rate. These parameters have been defined in Ref. [10]. $z$ is the direction of propagation of the light inside the cavity. $\mu_1(t,z)$ and $\mu_2(t,z)$ are spatiotemporal Gaussian and white stochastic processes that account for spontaneous emission. They have zero mean and correlation given by

$$\langle \mu_1(t,z)\mu_1^*(t',z') \rangle = 2D_1 \delta_{ij} \delta(t-t') \delta(z-z'). \tag{6}$$

We are now going to map the spatial dependence of the system into time by making use of the boundary conditions that have to be fulfilled by the fields $E_1$ and $E_2$ inside the cavity. These boundary conditions are

$$E_1(t,0) = \frac{1}{2} R_1 e^{ik_1L} \left[ E_1(t-l_p/v_g, l_A) (e^{i\phi} + 1) \right.$$  
$$+ E_2(t-l_p/v_g, l_A) (e^{i\phi} - 1) e^{-i\beta} \right]. \tag{7}$$

$$E_2(t,0) = \frac{1}{2} R_2 e^{ik_2L} \left[ E_1(t-l_p/v_g, l_A) (e^{i\phi} - 1) \right.$$  
$$+ E_2(t-l_p/v_g, l_A) (e^{i\phi} + 1) e^{-i\beta} \right]. \tag{8}$$

Here $l_A$ and $l_p$ are the lengths of the active and passive parts of the fiber, respectively, and $L = l_A + l_p$ is the total fiber length. The reference frame is chosen in such a way that $z = 0$ corresponds to one end of the active fiber. $R_1$ and $R_2$ are the return coefficients of the output coupler for each one of the modes. The parameter $\varphi$ represents the phase shift caused by the polarization controller. In the perfect half-wave case ($\varphi = \pi$) it can be seen that the previous boundary conditions merely represent an exchange of polarizations every round-trip. We shall consider $\varphi$ near, but not equal to, its perfect half-wave value. Finally, the parameter $\beta$ represents the birefringence of the fiber, which causes different phase shifts in the two polarization modes. These different phase shifts are produced by the different velocities that the two modes actually have when traveling through the fiber, which can be modeled satisfactorily by including the parameter $\beta$ while keeping $v_g$ equal in both modes [9].

The previous boundary conditions can be used in combination with an integration of Eqs. (3)–(5) with respect to $z$ to obtain the difference-differential model [9,19].

$$\psi_1(t) = \frac{1}{2} R_1 A_1 \left[ \psi_1(t - \tau_R) \exp \left( \frac{\Gamma_1}{2} (1 - i\alpha_1) [\phi(t) - 1] \right) + \eta_1(t) \right] (e^{i\phi} + 1)$$
$$\times \left[ \psi_2(t - \tau_R) \exp \left( \frac{\Gamma_1}{2} (1 - i\alpha_2) [\phi(t) - 1] \right) + \eta_2(t) \right] (e^{i\phi} - 1) e^{-i\beta}, \tag{9}$$

$$\psi_2(t) = \frac{1}{2} R_2 A_2 \left[ \psi_1(t - \tau_R) \exp \left( \frac{\Gamma_2}{2} (1 - i\alpha_1) [\phi(t) - 1] \right) + \eta_1(t) \right] (e^{i\phi} - 1)$$
$$\times \left[ \psi_2(t - \tau_R) \exp \left( \frac{\Gamma_2}{2} (1 - i\alpha_2) [\phi(t) - 1] \right) + \eta_2(t) \right] (e^{i\phi} + 1) e^{-i\beta}. \tag{10}$$

$$\frac{\partial \phi}{\partial t} = q - \phi(t) - |\psi_1(t - \tau_R)|^2 \left( \exp[\Gamma_1 (\phi(t) - 1)] - 1 \right) - \text{Re} \left[ \psi_1(t - \tau_R) \xi_1(t) \right] - |\psi_2(t - \tau_R)|^2 \left( \exp[\Gamma_2 (\phi(t) - 1)] - 1 \right)$$
$$- \text{Re} [\psi_2(t - \tau_R) \xi_2(t)], \tag{11}$$

where $A_1$ and $A_2$ represent the amplitudes of the two polarization modes, $\tau_R$ is the round-trip time of the light in the cavity, $\phi(t)$ is the relative phase shift between the two polarization modes, $\Gamma_1$ and $\Gamma_2$ are the decay rates of the two modes (assumed equal for both modes), $\alpha_1$ and $\alpha_2$ represent the detuning between the corresponding mode frequency $\omega$ and the resonance frequency of the cavity $\omega_0$.
where new dimensionless variables have been defined. \( \psi_1 \) and \( \psi_2 \) are related to the electric field envelopes at \( z = 0 \),

\[
\psi_i(t) = \frac{E_{i}(t,0)}{\sqrt{\hbar \omega_i / \gamma_i} N_0}
\]

(12)

and \( \phi \) is the dimensionless total population inversion in the active fiber,

\[
\phi(t) = \frac{W(t-\tau_R, l_A)}{l_A N_0} = \frac{1}{l_A N_0} \int_{0}^{l_A} N(t-\tau_R, z) dz.
\]

(13)

Time is now measured in units of \( \gamma_i^{-1} \). \( \tau_R \) is the cavity round-trip time, also measured in units of \( \gamma_i^{-1} \) (\( \tau_R = L / v_g / \gamma_i \)). We have defined a dimensionless gain parameter and an effective pump rate as

\[
\Gamma_i = a_i l_A N_0, \quad q = \frac{P}{\gamma_i N_0} - \frac{N_i}{N_0}.
\]

(14)

\( \Lambda_1 \) and \( \Lambda_2 \) are phase-shift coefficients that can be evaluated as [9]

\[
\Lambda_i = e^{i\alpha_i} = \exp \left[ i \alpha_i \frac{\Gamma_i}{2} (\phi(0) - 1) \right].
\]

(15)

An inspection of the delay-differential model (9)–(11) shows that the original spontaneous emission noise sources \( \mu_i(t, z), i = 1, 2, \) have given rise to new noise terms \( \eta_i(t) \) and \( \xi_i(t) \) in all three equations for the electric fields and the population inversion. The stochastic processes \( \eta_i(t) \) come from the formal integration of the spontaneous emission noise sources \( \mu_i(t, z) \) over the space variable \( z \), whereas \( \xi_i(t) \) appear through the introduction of the result of this integration into Eq. (5). It is worth noticing that these new stochastic processes are no longer space dependent: this is true of all the other quantities of the model as well. Note also that in the population inversion equation the noise terms are multiplicative [19]. They are all Gaussian distributed with zero mean, and we will denote their variances by \( D_i^\eta \) and \( D_i^\xi \). We will treat these noise strengths as adjustable parameters for our studies; they can be related to the physical properties of the system as [19]

\[
D_i^\eta = \frac{D_i}{\hbar \omega_i N_0},
\]

\[
D_i^\xi = 4 a_i^2 (N_{SS} - N_0)^2 l_i^2 / \hbar \omega_i N_0.
\]

(16)

where \( N_{SS} \) is the steady-state value of the population inversion.

In summary, we have obtained a delay-differential equation model that translates the space dependence of the population direction \( z \) (and hence its infinite-dimensional character) into a dependence on time-delayed quantities. The model also includes the influence of intrinsic noise sources. We have performed extensive numerical computations with this model, and the results obtained will be described in the following subsections.

The simulations are performed as follows. Each cavity round-trip time is divided into equal-size time intervals (or, equivalently, the cavity is discretized in a number of equal cells). The evolution of the fields \( \psi_1 \) and \( \psi_2 \) depends on their values one round-trip earlier [Eqs. (9) and (10)] and the total inversion \( \phi \) evolves according to the differential equation (11), which is discretized in the equally spaced time intervals defined above (or in the one-dimensional spatial lattice in which the cavity has been divided). Since in our case the cavity round-trip time is much smaller than the population inversion decay time, the integration time steps resulting from a not very dense cavity subdivision are small enough to ensure numerical stability in the algorithm that integrates the differential equation. We will usually choose a subdivision of the cavity in 100 parts and use a Heun algorithm (a stochastic version of a second-order Runge-Kutta scheme) [20] to simulate that equation. The multiplicative noise terms are treated according to a Stratonovich interpretation.

A distinction has to be made at this point between the active and the passive fiber. Since the polarization controller is located in the passive part of the cavity and the delay differential model maps time into space, the value of the phase shift \( \phi \) will be close to \( \pi \) only in the time instants corresponding to the passive part of the fiber. The rest of the time \( \phi \) will be near zero (not exactly zero because any small winding in the active fiber may also have a small phase-shifting effect). Hence we will take \( \phi \) to be equal to \( \varphi_A \) (small) in the active region and to \( \varphi_P \) (close to \( \pi \)) in the passive part.

Several of the parameters of the model will be fixed by physical requirements of the active medium and the experimental setup, whereas others will be used as adjustable parameters. Among the former, we have the gain coefficients \( a_1 \) and \( a_2 \), which will be taken to be coincident and equal to \( 2.03 \times 10^{-23} \) m\(^3\) s\(^{-1}\). The detuning factors \( \alpha_1 \) and \( \alpha_2 \) will also be taken to be the same and equal to \( 3.52 \times 10^{-2} \). The population decay rate \( \gamma_i \) is \( 10^2 \) s\(^{-1}\) and its inverse is the time unit we use throughout this section. The lengths of the active and passive fibers are the ones used in the experiment (6 m and 14 m, respectively), with a total cavity length of \( L = 20 \) m, which gives a round-trip time equal to \( \tau_R = nL/c \gamma_i = 10^{-5} \) dimensionless time units. The dopant ion concentration is \( N_i = 4.98 \times 10^{25} \) m\(^{-3}\) and the transparency inversion is \( N_{0} = 4 \times 10^{20} \) m\(^{-3}\). The return coefficient of the output coupler will be taken to be, according to the experimental setup, \( R_1 = 0.97 \), equal for both polarization modes. The noise strengths are chosen to be \( D_i^\eta = D_1^\eta = D_2^\eta = D_i^\xi = 10^{-5} \). To put these noise source variances in perspective, we should remark at this point that the magnitude of the light intensity in the lasing regime is, in our dimensionless units, of the order of \( 10^6 \). The pump rate will take several values for the different regimes. The phase shifts \( \varphi_A \) and \( \varphi_P \) and the birefringence coefficient \( \beta \) will be adjustable parameters. A summary of the previous values is shown in Table I.

B. Characterization of the model

A first comparison between the numerical model that has just been derived and the experimental observations shown previously is made by computing how the laser output
TABLE I. Parameters used in the delay-differential model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>α₁₂</td>
<td>2.03 × 10⁻²</td>
<td>m²</td>
<td>gain coefficients</td>
</tr>
<tr>
<td>α₂₁</td>
<td>3.52 × 10⁻²</td>
<td>s⁻¹</td>
<td>detuning factors</td>
</tr>
<tr>
<td>γₜ</td>
<td>10²</td>
<td></td>
<td>population decay rate</td>
</tr>
<tr>
<td>lₜ</td>
<td>6</td>
<td>m</td>
<td>length of active fiber</td>
</tr>
<tr>
<td>L</td>
<td>20</td>
<td>m</td>
<td>total cavity length</td>
</tr>
<tr>
<td>τₚ</td>
<td>10⁻⁷</td>
<td>s</td>
<td>cavity round-trip time</td>
</tr>
<tr>
<td>N₀</td>
<td>10²⁰</td>
<td>m⁻³</td>
<td>transparency inversion</td>
</tr>
<tr>
<td>Nₙ</td>
<td>4.98 × 10²⁵</td>
<td>m⁻³</td>
<td>dopant ion concentration</td>
</tr>
<tr>
<td>R₁₂</td>
<td>0.97</td>
<td></td>
<td>output coupler return</td>
</tr>
<tr>
<td>Dₜ₀</td>
<td>10⁻⁵</td>
<td></td>
<td>coefficients</td>
</tr>
<tr>
<td>Dₜ₁</td>
<td>10⁻⁵</td>
<td></td>
<td>electric-field noise strength</td>
</tr>
<tr>
<td>Dₜ₂</td>
<td>10⁻⁵</td>
<td></td>
<td>population inversion noise strength</td>
</tr>
</tbody>
</table>

changes with increasing pump power. A sudden jump in photon number (over nine orders of magnitude) is observed and represents the transition from a spontaneous emission (no lasing) to a stimulated emission regime (lasing). Linear behavior is observed in the lasing regime. The estimated value of the lasing threshold (∼5 × 10⁵ in dimensionless units) is in qualitative agreement with the experimental result.

It has also been found that this model reproduces the striking experimental observation of an increase in intensity fluctuations for higher pumping and output power. The results are shown in Fig. 9 and should be compared with their experimental counterpart presented in Fig. 4. The fact that this behavior persists even in the absence of stochastic terms in the simulations indicates that these intensity fluctuations are of deterministic origin. They are related to spiking and pulsing phenomena occurring in the time evolution of the light intensity and may be caused by the coupling dynamics between the many modes that are undergoing amplification, as mentioned in Sec. III A.

FIG. 9. Standard deviation of the total intensity output vs total output power I₁ + I₂ and pump rate q.

FIG. 10. Polarization-resolved quasiperiodic self-pulsing time traces. A periodicity equal to one cavity round-trip time is observed. The pump rate is q = 2 × 10⁶ dimensionless units, roughly 5 times above threshold.

C. Dynamical behavior

1. Self-pulsing

Typical time traces of the output intensity Iᵢ = |ψᵢ|², as obtained from our delay-differential model, are shown in Figs. 10 and 11. Self-pulsing behavior is clearly observed, with different overall characteristics depending on the values of the parameters. Figure 10 presents antiphase quasiperiodic self-pulses at a periodicity of one cavity round-trip time. Figure 11 shows period-2 behavior. The difference between both cases lies only in the value of the birefringence factor, equal to 0.0015 in the first case and taken to be exactly zero in the second. The values chosen for the phase shifts are 0.027 in the active fiber and π - 0.175 in the passive fiber. All the other parameters are those of Table I. It is worth noting that in all cases we obtain antiphase motion for the two polarization modes. The structures immersed in this self-pulsing behavior are observed to drift slowly as time.

FIG. 11. Polarization-resolved quasiperiodic self-pulsing time traces with a period equal to 2τᵢ. The pump rate is the same as in Fig. 10.
FIG. 12. Three snapshots of the self-pulsing behavior of one polarization mode, showing the slow drift of temporal patterns due to the effect of stochastic noise sources. These snapshots are separated in time by several hundred round-trips. (a) Numerical simulation (parameters are the same as in the previous figures) and (b) experimental behavior (in this case the cavity round-trip time is ~130 ns).

evolves, as observed in the experiments (see Fig. 12). This pattern evolution does not occur if the noise sources are neglected in the model, which indicates the importance of spontaneous emission in this system.

2. Influence of the phase shifts

The value of $\varphi_p$ used in the previous simulations corresponds to an imperfect half-wave plate. By taking a value of this phase shift closer to $\pi$ (which amounts to properly tuning the polarization controller mandrels in the experiment), we can reproduce the square-wave behavior observed in the real system. Figure 13 is the result of making $\varphi_p = \pi - 0.015$ and $\beta = 0.020$. As in the experimental output, these square waves are antiphase in both polarization components, with a period equal to the cavity round-trip time, and a relation between the lengths of the upper and lower plateau equal to that between the lengths of the active and passive part of the cavity. Also, as in the experiment, the patterns on top of the square pulses change continuously and slowly with time, as shown in Fig. 14, where three series of ten cavity round-trip times occurring at different instants of the same dynamical evolution are compared. Again, this behavior is not obtained if the spontaneous-emission noise is not taken into account.

D. Nonlinear analysis

To complete our comparison between the results given by the delay-differential model that has been derived in this section and the results obtained from the experimental system, we will analyze the numerical time traces from a nonlinear dynamics point of view. We can compute the average mutual information function of a polarization-resolved output time trace. Figure 15 shows the typical behavior of this function [which in this case corresponds specifically to the time trace shown in Fig. 16(a)]. We conclude that a reasonable value
FIG. 15. Typical example of the average mutual information function obtained numerically. The actual time trace from which this function has been derived is shown in Fig. 16(a).

for the time lag to be used in phase-space reconstruction is \( \Delta t = 10^{-7} \) dimensionless units, which corresponds to one time interval in the cavity subdivision we have chosen throughout this work. We now compute the percentage of false nearest neighbors for different dimensions in two different regimes. Figure 16(a) shows a time trace exhibiting a high degree of periodicity and its corresponding false-nearest-neighbor percentage vs embedding dimension. This result shows that the behavior of the system in this regime is low-dimensional and deterministic, with an embedding dimension \( d_e = 4 \). Figure 16(b), on the other hand, shows a nonperiodic time trace and a false-nearest-neighbor percentage that does not go to zero for increasing dimension, implying that the behavior in this case is high dimensional and noise driven. We remind the reader that these two different regimes have also been obtained experimentally (Fig. 8). We regard this agreement as a significant indication of the success of our model in capturing the dynamical behavior of the laser system.

IV. CONCLUSION

We have analyzed the fast, intracavity dynamics of an erbium-doped fiber laser in a ring cavity. Since it is well known that this kind of system presents interesting polarization dynamics, we have introduced a polarization controller inside the laser cavity. Self-pulsing has been observed in a very broad range of system configurations, both in the total output intensity of the laser and in the polarization-resolved dynamics, in periods of the order of the cavity round-trip time. In this regime the two different polarization modes can behave independently, i.e., one may show quasiperiodic dynamics and the other chaotic behavior, for instance. Due to the long cavity and fast detection devices, we have been able to sample the behavior inside a cavity round-trip. By carefully tuning the polarization controller, the self-pulsing behavior can be transformed into square-wave dynamics. In this case, the behavior of the two polarization modes is usually antiphase, as predicted for lasers with a strong multimode character. All these features can be reproduced by a stochastic delay-differential equation model, which takes into account the fact that a mean-field approximation in the propagation direction is misleading in this kind of long-cavity laser. Spontaneous emission is introduced via a noise term in the original Maxwell-Bloch equations and leads to a nontrivial stochastic contribution to the delay-differential model. This model is able to reproduce both the self-pulsing and the square-wave behavior. Spontaneous-emission noise is necessary to obtain the observed slow time drift of the patterns underlying the square-pulse structure. However, even though spontaneous emission (and hence the noise sources in the model) is always present in the laser operation, we observe, numerically and experimentally, both a deterministic and a noise-driven regime for slightly different values of the system parameters. The first situation corresponds to a quasiperiodic, low-dimensional motion and the second to a random, high-dimensional behavior. The coexistence of these two types of behavior in the same nonlinear dynamical system is a remarkable feature that deserves further study.

FIG. 16. Quasiperiodic time trace and its percentage of false nearest neighbors vs dimension. Full circles represent the numerical result, which corresponds to a pump rate of \( q = 6 \times 10^5 \) dimensionless units, ~1.1 times above threshold. Empty circles are the experimental result of Fig. 8(a). (b) Nonperiodic time trace and its percentage of false nearest neighbors vs dimension, which displays a residual percentage of FNNs, implying random dynamics. Full circles represent the numerical result, which corresponds to a pump rate of \( q = 1 \times 10^6 \) dimensionless units, ~2 times above threshold. In this case, the noise source strengths have been increased to a value of \( 1.2 \times 10^{-4} \) dimensionless units to obtain better agreement with the experiments, which are represented by empty circles (from Fig. 8(b)).
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Noise amplification in a stochastic Ikeda model

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Abstract

The effect of spontaneous emission noise on the light circulating in a ring cavity with a nonlinear absorbing medium is studied by means of a set of stochastic delay-differential equations based on the deterministic Ikeda model. Noise fluctuations are found to be amplified as the first bifurcation from the steady state of the system is approached.

1. Introduction

Delay-differential equations are frequently used to model nonlinear dynamical systems. Among them, the so-called Ikeda model is particularly well known in the analysis of the dynamical behavior of nonlinear optical media. Since its introduction by Ikeda and coworkers [1,2] in the investigation of the light transmission process by a nonlinear absorber contained in a ring cavity, it has increasingly been applied to the study of the interaction of light with either passive [3–5] or active [6,7] media. Also, due to its highly complex multistable behavior, the model eventually leads to chaos through a rich variety of routes [8]. Hence, its simplified map version has become a paradigm in the analysis of chaotic systems [9–12]. It is therefore of interest to investigate the influence of spontaneous emission noise on the dynamics of this system. It is particularly important to consider the physical origin of the noise source; here we begin with the Maxwell–Bloch equations and outline the inclusion of spontaneous emission noise in a physically meaningful way, leading to a stochastic version of the deterministic Ikeda model.

2. Derivation of the stochastic Ikeda model

Let us consider the simple situation, originally analysed by Ikeda in his seminal paper [1], of a nonlinear absorbing medium placed in a ring cavity. This medium shall be assumed to be a set of homogeneously broadened two-level atoms, whose interaction with an incident light beam can be described by the following equations,

\[
\frac{\partial E}{\partial z} = (\alpha + i\beta)(N - N_0)E + \mu,
\]

(1)

\[
\frac{\partial N}{\partial \tau} = -\gamma N - \Omega(N - N_0)|E|^2,
\]

(2)

where \(E(\tau, z)\) is the complex envelope of the electric field which propagates in the absorber, \(N(\tau, z)\) is the population inversion \((N < 0\) for an absorber) and \(\mu(\tau, z)\) is a Gaussian and spatio-temporal white stochastic process accounting for spontaneous emission.

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sion processes. This noise term is chosen to have zero mean and correlation equal to

$$\langle \mu(\tau, z) \mu^*(\tau', z') \rangle = 2D\delta(\tau - \tau')\delta(z - z'). \quad (3)$$

It should be noted that the white character of this noise source is a mathematical idealization of the spontaneous emission process, which actually has correlations in time and space that are very small in comparison to all other time and length scales of the system.

Eqs. (1), (2) can be obtained in a straightforward way from the standard Maxwell–Bloch equations which describe the propagation of the electric field inside the absorber by adiabatically eliminating the polarization of the medium, whose relaxation rate is much larger than those of $N$ and $E$. The space variable $z$ corresponds to the direction of light propagation (transverse effects which might appear in the directions perpendicular to propagation [13,14] are not considered here). The time variable $\tau$ is written in a reference frame which moves with the velocity $v_g$ of light in the medium, $\tau = t - z/v_g$. $\alpha$ ($> 0$) is the absorption coefficient of the medium, $\beta$ is a parameter depending on the detuning between the cavity and the transition resonance frequencies, and $\gamma$ is the population decay rate. The coefficient $\Omega$ of the nonlinear term in (2) depends on the dipole moment of the transition. $N_0$ is the value of $N$ corresponding to transparency.

Let $L$ denote the length of the absorbing medium, $L$ that of the whole cavity and $l = L - L$. Then, the relation between the incident field $E_i$ and the field propagating inside the cavity is given by the following boundary condition,

$$E(t, 0) = \sqrt{T} E_i + R \exp(ikL) E \left( t - \frac{l}{v_g}, L \right) \quad (4)$$

where $T$ is the transmission coefficient of the input mirror $M1$ and $R = 1 - T$ is the reflection coefficient of both the input and output mirrors $M1$ and $M2$ (see Fig. 1). Mirrors $M3$ and $M4$ are assumed to be perfectly reflecting. $k$ is the light wavenumber.

The space dependence of the previous equations can be removed by using this boundary condition. First, we formally integrate Eq. (1) with respect to $z$ and introduce the result into Eq. (2). As a result, the originally additive noise $\mu$ generates a multiplicative noise term in the equation for the population inversion,

$$E(t, z) = \frac{z}{v_g} \frac{\partial W(\tau, z)}{\partial \tau} = -\gamma W$$

$$+ \frac{\Omega}{2\alpha} \left\{ \exp \left[ 2\alpha(W - N_0z) \right] - 1 \right\} |E(t, 0)|^2$$

$$+ 2\Omega(N_1 - N_0)\text{Re}(E(t, 0)\chi(\tau, z)) \quad (5)$$

where $W(\tau, z)$ is defined as

$$W(\tau, z) = \int_0^z \frac{dz'}{v_g} \left( \tau - \frac{z'}{v_g} \right)$$

To obtain the evolution equation for $W(\tau, z)$ (1 (6)), the variations of the population inversion $N_1$ (5) have been assumed to be negligible. Numerical simulations show that the variations in this quantity are a very small fraction of its average value $N_1$ (Fig. 2). Also, two new stochastic processes have been defined, which are also Gaussian with zero mean and zero correlations.

$$\langle \Gamma(\tau, z)\Gamma^*(\tau', z) \rangle = 2D\delta(\tau - \tau') \quad (6)$$

$$\langle \chi(\tau, z)\chi^*(\tau', z) \rangle = 2Dz^2\delta(\tau - \tau') \quad (7)$$

$$\langle \chi(\tau, z)\Gamma(\tau', z) \rangle = 2Dz^3\delta(\tau - \tau') \quad (8)$$

These noise sources arise from the application of the integral operators to the original spontaneous emission noise. In order to obtain the simple expressions above for the variances, the population inversion has been assumed to be constant. Notice also that the cross-correlation between $\chi(\tau, z)$ and $\Gamma(\tau', z)$
\[ z(t, z) + \Gamma(t, z). \]  
(5)

\[ |E(t,0)|^2 \]  
(6)

non-zero. Nevertheless, the influence of this cross-correlation in the dynamics of the system was found negligible in numerical simulations, and it has not been considered in what follows. It should also be noted that, in deriving Eq. (6), a contribution proportional to \( \Gamma \) has been discarded, due to the small strength of this noise source.

Introduction of Eq. (5) into boundary condition (4) and use of the population inversion Eq. (6) at point \( z = L \) leads to the following difference-differential equations.

\[ \psi(t) = A + B \psi(t - \tau_R) \exp \left\{ \left[ \phi(t) + \phi_0 \right] \right\} + \eta(t), \quad (11) \]

\[ \frac{d\phi}{dt} = -\phi + (\psi(t - \tau_R))^2 + 2 \text{Re} \left\{ \psi(t - \tau_R) \xi(t) \right\}. \quad (12) \]

Time \( t \) is now measured in units of \( \gamma^{-1} \) and the following dimensionless variables have been defined.

\[ \psi(t) \equiv E(t,0) \exp(\alpha W_0) \sqrt{\frac{\Omega \beta}{2 \alpha \gamma}}, \quad (13) \]

\[ \phi(t) \equiv \beta W(t - \tau_R, L), \quad (14) \]

where \( W_0 = W_0 L \).

The noise sources \( \eta(t) \) and \( \xi(t) \) are dimensionless and space-independent versions of \( \Gamma \) and \( \chi \). It can easily be seen that its variances \( D_\eta \) and \( D_\xi \) are related to the original physical parameters by

\[ D_\eta = D_\xi = \alpha \Omega \beta / 2 \alpha. \quad (15) \]

\[ D_\xi = D_\eta = 2 \alpha \Omega \beta L^2 \beta (N_e - N_0). \quad (16) \]

A remark should be made at this point in relation to the difference equation (11). This equation makes no sense mathematically if the stochastic process \( \eta(t) \) is taken to be white. This interpretation problem can be avoided by recalling that the original spontaneous emission noise has a very small, but non-negligible, correlation time. In this case the parameter \( D_\eta \) corresponds to the (finite) value of the correlation function of the noise at equal times.

Besides these two noise strengths, this model has four other independent parameters: the dimensionless incident field \( A = \sqrt{\Gamma} \exp(\alpha W_0) \sqrt{\Omega \beta / 2 \alpha \gamma} \), the dissipation \( B = R \exp(\alpha W_0) \), the phase shift due to propagation \( \phi_0 = kLC - \beta N_0 L \) and the dimensionless cavity round-trip time \( \tau_R = \gamma LC / \beta \).

Eqs. (11) and (12) define the stochastic version of the standard Ikeda model, which includes the existence of spontaneous emission processes of the two-level atoms forming the absorber. It is worth noting that what is initially an additive noise in the original partial-differential equation scheme has become multiplicative in the difference-differential equation model. This may be considered as an indication of the non-trivial influence of the spontaneous emission process.

3. Influence of noise on dynamics

As stated above, the dynamical properties of even the deterministic version of the Ikeda model lead to a highly complex behavior of the model. In particular, the steady state solution of the model, which can be seen to obey the following transcendental equation,

\[ |\psi(t)|^2 \left[ 1 + B^2 - 2B \cos \left( |\psi(t)|^2 + \phi_0 \right) \right] = A^2. \quad (17) \]

is a multivalued function of the input parameter \( A \) (see Fig. 3). This means that even in the simplest case in which the system evolves towards a fixed-point attractor, it faces a high degree of multistability. The effects of this fact can immediately be seen by looking at the bifurcation diagram of the light intensity extrema \( \psi(t) \) versus the input parameter \( A \) (Fig. 4). The step-like appearance of this diagram is a clear indication of the multistable character of the attractor structure of the system, each step corresponding to a jump between two equally stable states. The position of the jumps
is slightly affected by the choice of initial conditions, suggesting that the role of spontaneous emission noise might be relevant as well.

In order to analyse the influence of noise in the behavior of the system, we will first compare how the transition to chaos is produced in the deterministic and the stochastic cases. The algorithm used to integrate the differential equation appearing in the stochastic model is a standard Heun algorithm [15], where the integration time step is the one imposed by the discrete equation for $\psi$ and the number of subdivisions made within one cavity round-trip time (100 in our calculations). The Stratonovich interpretation is used to derive the integration algorithm including the influence of noise sources.

Fig. 5 shows the power spectral density of $\psi$ for different values of $A$ in deterministic (a) and stochastic (b) cases. Parameters are the same as in Fig. 4. The vertical scale is the same for all graphs except the steady-state case (first graph in Fig. 5a), where the zero level is explicitly shown. In this last case, an arrow in the vertical indicates the existence of a Dirac delta function at $\omega = 0$.
The existence of a small noise source (Fig. 5b) does not substantially modify the period-doubling scenario after the first bifurcation has taken place ($A > A_c$, $A_c \approx 9.85$ for the parameters chosen, corresponding to the last three plots in each of Figs. 5a and 5b). A noisy background superimposed on the deterministic spectral density appears, as expected. On the other hand, the situation before the first bifurcation is reached (first plot in each of Figs. 5a and 5b) shows a radical change. A distinct peak in the power spectrum can be observed for a non-zero finite frequency in the stochastic case, in contrast to the delta function of the deterministic case. This frequency is seen to be the same as that of the periodic attractor which appears after the bifurcation. Fig. 6 demonstrates this fact, by means of a comparison between the light intensity time series and its power spectrum for the deterministic (Fig. 6a) and noisy (Fig. 6b) cases. The main peak in both spectra coincide, as seen in Fig. 6b. The oscillation amplitudes are however very different. The fact that the oscillations are much smaller in the first case ($A = 9.80$) than in the second ($A = 9.90$) proves that this is not a mere advance of the bifurcation caused by the noise. However, the amplitude in the pre-bifurcation case is much larger than the noise source variance would have us expect. We are hence observing an amplification of noise fluctuations, which takes place at the natural frequency selected by the dynamics of the system. We note that the fluctuation-enhanced peak observed here is of the same shape and occurs at the same frequency as that which appears after the bifurcation; this behavior seems different from that of the "noisy precursors" studied by Wiesenfeld and others (see Ref. [16], and references therein).

A clear picture of the amplification of noise fluctuations can be obtained by computing the standard deviation of the intensity time series as the first bifurcation is approached. This is shown in Fig. 7, where a horizontal dashed line indicates the value that is to be expected from the real noise intensity which is being handled. The amplification effect is plainly revealed.
4. Conclusion

The main objective of this paper was to systematically derive the equations for the stochastic Ikeda model of a ring cavity with a nonlinear absorber. Spontaneous emission noise has been found to significantly influence the dynamical behavior of the system. We observe substantial amplification of noise fluctuations before the steady state loses stability; this amplification occurs at a natural frequency of the system.

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References

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Intracavity chaotic dynamics in ring lasers with an injected signal

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Abstract

The intracavity dynamics of optically injected ring lasers is studied by means of an extended delay-differential Ikeda model. The behavior of this kind of lasers is, in some aspects, strikingly different from that of a nonlinear absorber placed in a ring cavity, for which the Ikeda model was originally derived. In particular, chaotic behavior in the laser case is seen to occur on much faster time scales than for the absorber. The scenario in which the transition to chaos occurs is also different.

Injection of coherent light into laser systems has been a common practice since the early years of the laser era. The reasons for using such a technique are diverse. At high injection levels, the laser locks its frequency and phase to those of the injected signal; this is called the injection-locking regime, and is very useful for obtaining a stable and narrow-band laser output at a desired frequency. On the other hand, if the injected signal is not strong enough, locking is not possible and a competition arises between the two coherent signals which coexist inside the laser resonator, giving rise to a wide and interesting variety of dynamical behavior (see Ref. [1] for a general review on the subject).

In the present work, we are interested in the chaotic regimes that frequently appear in a laser with an injected signal, and in the transitions and instabilities leading to them. Much attention has been paid to this problem in the past years [2–6], and evidence of chaos has been obtained from both an experimental and a theoretical point of view. Nevertheless, similarly to almost all investigations of laser dynamics, these studies were done on time scales longer than the round-trip time of the laser. For “typical” laser systems, such as semiconductor, gas, or Nd:YAG solid-state lasers, this quantity usually takes values in the range ~ 10 ps–1 ns, which places the analysis of intracavity phenomena beyond the reach of standard measurement devices. The recent development of optical fiber lasers, mainly for communication purposes [7], has changed this situation. In such lasers, the amplifying medium is an optical fiber that has been doped with rare-earth ions.

The waveguiding properties of optical fibers enable the construction of lasers with very large (even of the order of km) cavities, and hence with round-trip times long enough (of the order of µs) to be able to observe their behavior inside the cavity [8]. The question of analysing the intracavity dynamics of

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lasers with an injected signal when operating in a chaotic regime naturally arises. It is interesting to investigate, for instance, if chaos occurs in time scales faster than the cavity round-trip time, and if this is the case, whether these time scales can be fast enough to be interesting for encoding purposes [9,10] in all-optical communication systems.

The intracavity dynamics of long-cavity erbium-doped fiber lasers has been recently analysed by means of a system of delay-differential equations of the Ikeda type [8,11]. This model was first introduced by Ikeda to study the absorption of light by a nonlinear medium placed in a ring cavity [12], and predicted the existence of chaotic behavior in this system. This prediction was first tested in a passive optical fiber ring [13], and became one of the first examples of optical chaos. The Ikeda model has also been occasionally used in a laser context by several authors. In the few years following its appearance, Otsuka and coworkers used it to describe the nonlinear dynamics of a semiconductor laser amplifier with delayed feedback [14,15]. Several years later, Loh and Tang derived a delay-differential model, following Ikeda, to analyse ultrafast polarization modulation in semiconductor lasers [16,17]. Again, these studies were done on time scales larger than the cavity round-trip time of the system. It is our aim here to compare the dynamical behavior of these two versions of the Ikeda model, namely that corresponding to a nonlinear absorber and the one used to analyse laser systems. As we will show in what follows, the time scales in which the second model evolves correspond to nontrivial intracavity dynamics.

The Ikeda delay-differential model can be written in dimensionless form as follows.

\[ E(t) = \sqrt{T} E(t) + R E(t - \tau_R) \]
\[ \times \exp \left[ a (1 + i \alpha) (\phi - 1) \right] . \]

(1)

\[ \frac{d\phi(t)}{dt} = q - \phi - |E(t - \tau_R)|^2 \]
\[ \times \{ \exp[2a(\phi - 1)] - 1 \} . \]

(2)

These equations describe the interaction between light and a nonlinear medium (absorbing or amplifying) placed in a ring cavity. \( E(t) \) is the complex envelope of the electric field, measured at a given reference point inside the cavity. \( \phi \) is the total population inversion between the two energy levels of the nonlinear medium which interact with the propagating light. Time is measured in units of \( \tau_f^{-1} \), where \( \tau_f \) is the decay time of the atomic transition. The delay \( \tau_R \) is the dimensionless cavity round-trip time, i.e. the time the light takes to travel once around the cavity, in units of the inverse of the transition decay time \( \tau_f \). \( E \) is the amplitude of the injected field, assumed constant. \( R \) is the return coefficient of the ring (fraction of light that remains in the cavity after one round-trip), and \( T = 1 - R \). The parameter \( \alpha \) is the dimensionless detuning between the atomic transition frequency and the light frequency. The coefficient \( a \) represents either absorption or gain, depending on whether we are studying a nonlinear absorber or a laser, respectively.

In this last case, the amplifying medium has to be pumped, which is represented by the dimensionless pump rate \( q \).

In the absorbing case \( (q = 0) \), this model has been extensively studied both numerically and analytically \([18-21]\). In particular, Kaiser and coworkers \([19,21]\) numerically obtained the bifurcation structure of the system for several sets of parameters, displaying different routes to chaos for round-trip times of the order of the transition decay time \( (i.e. for \tau_R \text{ of order 1}) \). Otsuka used similar time scales in his study of semiconductor lasers with optical feedback \([15]\).

We, on the other hand, are interested in another region of parameter space, where the round-trip time is several orders of magnitude smaller than the transition decay time \( (\text{in fiber lasers, the difference can be of 5 orders of magnitude [11]}) \). Throughout this paper, we will consider \( \tau_R = 0.01 \).

We have numerically integrated Eqs. (1), (2) to obtain the bifurcation structure of the Ikeda model with and without pumping in the \( (a, \alpha) \) space. The result is shown in Fig. 1, for \( R = 0.95, \tau_R = 0.01 \) and \( E_t = 5.0 \). The pump rate is \( q = 0 \) in Fig. 1a and \( q = 5 \) in Fig. 1b. The influence of pumping is evident. When \( q = 0 \), the situation is qualitatively similar to that reported in Ref. \([21]\), with islands of

\[ 1 \text{ We choose } a \text{ to be always positive, so that it is the sign of the population inversion } \phi \text{ what makes the medium absorbing or amplifying.} \]
periodic behavior embedded in a chaotic background. When \( q \neq 0 \), on the other side, this scenario changes drastically. First, the fixed point loses stability at smaller values of the detuning (compare the x-axis scales in Figs. 1a and 1b). Second, the instability threshold is virtually independent of the gain, and leads almost immediately to the chaotic regime, so that the transition to chaos is much sharper now. Third, there are no islands of periodic behavior in the chaotic regime. A comparison between typical routes to chaos in both cases is presented in Fig. 2. The system parameters are the same as in the previous figure. Fig. 2a corresponds to a slice of Fig. 1a at \( \alpha = 3.5 \) for increasing values of \( \alpha \), starting shortly after the fixed point loses stability, and clearly shows a period-doubling sequence leading to a chaotic attractor of annular shape. The situation is again very

![Fig. 1. Phase diagram of the delay-differential model presented in Eqs. (1), (2). The parameters are: \( E_r = 5.0, R = 0.95 \) and \( t_R = 0.01 \). (a) \( q = 0 \), (b) \( q = 5.0 \). White regions correspond to fixed-point dynamics, black regions to chaotic behavior, and the different grey areas represent periodic motions with different repetition rates. The period of the motion is represented by the numbers shown in the figure. Different superscripts correspond to qualitatively different periodic orbits.](image)

![Fig. 2. Time series (left) and attractor in complex-E space (right) for different values of \( \alpha \) as the system goes from a fixed-point to chaos without (a) and with (b) pumping. Parameters are those on the previous figure and \( \alpha = 3.5 \). The values of \( \alpha/\pi \) are, from top to bottom: (a) 6.25, 7.00, 8.00, 8.05, and 8.90; (b) 2.29, 2.31, 2.34, 3.00, and 6.00.](image)

different in the presence of pumping (Fig. 2b), with periodic behavior in a very narrow band of \( \alpha \) values separating regions of fixed-point and chaotic behavior. In the periodic regime, the attractor is similar in shape to that of the absorbing case, whereas in the chaotic region, the attractor fills all the space inside its boundaries, and covers a much larger region of phase space. It should also be noted that the time scale for the chaotic dynamics is much faster in the laser case than it is in the absorber case. Taking into account that the round-trip time is 0.01 in both cases, it can be seen from Fig. 2 that the system exhibits intracavity chaos for \( q \neq 0 \), but not for \( q = 0 \), where variations in the intensity, although chaotic, occur in a time interval larger than \( t_R \).
difference corresponds to several orders of magnitude (cf. the scales of the frequency axis in both cases).

Finally, we now address the question of whether the faster dynamics observed in the laser case corresponds to a higher dimensional motion. This seems to be suggested by the differences in the chaotic attractors shown in Fig. 2. We can estimate the dynamical dimension of the two systems by using a phase-space reconstruction method [22] and computing the percentage of false nearest neighbors as we increase the dimension of the space in which the intensity time series is embedded. To perform this calculation we make use of a method developed in Ref. [23]. False nearest neighbors are points in phase space which seem to be nearby only because the dynamics has been embedded in a space of too low dimension. They can be revealed by increasing the dimension of the space in which the dynamics is trying to be reconstructed. In this way, when the percentage of false nearest neighbors (with respect to all points in the attractor) drops to zero beyond a given dimension, we can expect that the phase space has been correctly reconstructed. The minimum dimension for which this happens constitutes a measure of the dynamical dimension of the system. Fig. 4 shows the percentage of false nearest neighbors vs. increasing dimension of the embedding space for the Ikeda model, both with and without pumping. As can be observed in the inset, the embedding dimension in the pumped case (circles) is equal to 4, and coincides with that of the "classical" Ikeda model for an absorber, i.e. for no pumping (diamonds).

In order to further investigate and corroborate the existence of different time scales in the two different chaotic dynamics observed so far, it is also useful to analyze their respective power spectral densities (PSD). A comparison between these functions for typical time traces in the two chaotic regimes (with and without pumping) is shown in Fig. 3. The existence of much higher frequencies in the laser case (Fig. 3b) as compared with the absorber case (Fig. 3a), can be easily observed in the time-domain representations (upper plots), and is quantitatively described in the frequency domain (lower plots). The PSD is seen to be much broader for $q = 0$. This

Fig. 3. Two time series (top) and their corresponding power spectral densities (bottom) with and without pumping. Parameters are those of the previous figure, and $\alpha/\pi = 8.60$. (a) $q = 0.0$; (b) $q = 5.0$.

Fig. 4. Percentage of false nearest neighbors versus embedding dimension for the time traces shown in Fig. 3.
In summary, we have analysed and compared the dynamics of the delay-differential Ikeda model for an absorber and a laser with long cavity. The phase diagrams of the two systems display fixed-point, periodic and chaotic behaviors, and in the periodic regime the time scale seems to be similar in both systems. This is not the case in the chaotic regime; here, the time scale for intensity fluctuations is still long for the absorber, but much faster for the laser. In this last situation, the dynamics is chaotic within a single cavity round trip. In spite of the different frequencies involved in the two cases, both systems seem to have equal dimensionality, as shown by a false nearest neighbor analysis. The transition between the regions of "slow" and "fast" dynamics can be seen not to be discontinuous at \( q = 0 \). On the contrary, the standard, "slow" behavior can be observed for a finite range of \( q \) values up to a given threshold, beyond which the fast regime appears. This bifurcation might correspond to a jump towards a higher branch of the multistable system. Further research, both numerical and analytical, is needed in order to clarify this point. The existence of intracavity chaotic dynamics for lasers with a long cavity, such as optical fiber lasers, might be important for chaotic encoding of information at frequencies in the GHz–THz range, in all-optical communication systems. In this sense, it would be of interest to analyze how a time variation of the injected signal (the message to be encoded, for instance) would affect the scenario presented here.

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References

Influence of noise on chaotic laser dynamics

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The Nd:YAG laser with an intracavity second harmonic generating crystal is a versatile test bed for concepts of nonlinear time series analysis as well as for techniques that have been developed for control of chaotic systems. Quantitative comparisons of experimentally measured time series of the infrared light intensity are made with numerically computed time series from a model derived here from basic principles. These comparisons utilize measures that help to distinguish between low and high dimensional dynamics and thus enhance our understanding of the influence of noise sources on the emitted laser light.

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I. INTRODUCTION

The Nd:YAG (neodymium doped yttrium aluminum garnet) laser with an intracavity KTP (potassium titanyl phosphate) crystal is a chaotic dynamical system for which it is possible to directly compare statistical aspects of measured time series with predictions from a numerical model that has been derived from basic theory. When operated with three or more longitudinal cavity modes, this laser is known to display chaos, and attempts have previously been made to write dynamical equations that could capture certain aspects of observed behavior [1–3]. These models have successfully predicted the existence of antiphase dynamical states, energy sharing of chaotic polarization modes of the laser, and also the possibility of obtaining stable operation through rotational orientation of the KTP and YAG crystals. The laser system has also served as an example of which algorithms for the control of chaotic lasers have been successfully applied, both experimentally and in numerical simulations [4–7].

It was, however, the observation that simple control algorithms failed in certain operating regimes that motivated us in a previous paper to apply methods of nonlinear time series to experimentally recorded intensity time series with the goal of discovering qualitative and quantitative differences in the operating regimes. The laser was thus operated specifically in three longitudinal modes in two polarization configurations by careful adjustment of crystal orientations in the cavity. In the first configuration, all three longitudinal modes were polarized parallel to each other. In the second, one mode was polarized orthogonal to the other two. All other parameters of the laser system such as the cavity loss, pump level, etc. were maintained constant, and the instrumentation for the measurements was operated with exactly the same sampling times and other settings.

The dynamics observed in these two polarization configurations were labeled type I and type II. Nonlinear time series analysis allowed us to determine the dimensionality of the chaotic attractors for the two cases and estimate the Lyapunov exponents in the two cases. A major conclusion of our previous study was that while the type I behavior was established to be low dimensional, there was clear evidence that the type II behavior was significantly influenced by noise, indicating the presence of high dimensional dynamics as well. At the end of that paper we sketched the outline of a theoretical approach to the derivation of a model that would allow us to simulate intensity time series and apply the nonlinear analysis techniques to make a direct comparison with the experimental results.

In this paper we present the derivation outlined in [8], and obtain the equations that describe the dynamics of a three mode laser with an intracavity KTP crystal. Previous models [1–3] were found not to reproduce type I dynamical behavior after conducting extensive searches in parameter space. It is shown here that the inclusion of nondegenerate four wave mixing, which leads to a model that includes the phase dynamics of the electric fields, overcomes this difficulty. Type II behavior of the infrared light has very different characteristics, and is accompanied by emission of substantial amounts of green light, in contrast to type I dynamics. Degenerate four wave mixing is the dominant process in this case. A major purpose of the research reported here is to include noise sources appropriately in the numerical equations and to explore their influence on type I and type II deterministic chaotic dynamics.

The next section reviews the main aspects of type I and type II chaotic dynamics of the laser. The experimentally observed differences (time series behavior, controllability, mode structure, and green output power) are summarized. We describe a noise measurement method called false nearest neighbors, an algorithm normally used to find the embedding dimension of a chaotic time series. We demonstrate that the two types of dynamics differ significantly in the amount of high dimensional (noisy) dynamics of the laser. Section II provides the basis for comparison with numerical computations that are the focus of this paper.

Section III contains a derivation of the model equations of motion from a Hamiltonian. Three infrared cavity modes are modeled as harmonic oscillators coupled to heat baths. A mode that represents green light generated by the KTP crys...
tal is also included. It is nonlinearly coupled to the infrared modes so as to model the interaction in the KTP crystal. The cavity loss for the green light is very high compared to that for the infrared modes, hence it is sufficient to just consider a single mode of green light and to eliminate its dynamics from the final set of equations that describe the evolution of the field amplitudes of the infrared modes and of the population inversion of the two level atoms that drive them.

In Sec. IV we describe the results from numerically integrating the equations of motion derived in Sec. III. There is a qualitative match between the wave forms of the model and experimental data in both chaos regimes. We also present the false neighbors results when noise is added to the system and find that the resulting noise in the output intensity differs in the two chaotic regimes for the same input noise, leading us to conclude that the susceptibility of the dynamics to noise differs for the two chaotic behaviors.

Section V attempts to locate the source of noise that is seen in the laser time series. Four intrinsic quantum fluctuation sources (cavity loss of infrared light, cavity loss of green light, intrinsic conversion noise, and spontaneous emission) are analyzed for their expected noise levels. These noise sources are all too weak by many orders of magnitude to contribute the amount of noise evidenced in the laser dynamics. We also consider and eliminate extrinsic pumping fluctuations as the noise source.

II. TYPE I AND TYPE II BEHAVIOR

The basic elements of the laser system are a diode laser pumped Nd:YAG crystal and an intracavity KTP crystal with an output mirror that is highly reflecting at the 1.064 μm line of the Nd:YAG crystal but highly transmitting for the green light [1]. It has been shown that this laser can be configured so that few modes (≈ 3–10) are present in the cavity; each mode can have one of two polarizations.

Using the methods of nonlinear time series analysis [8] we are able to distinguish between chaotic behavior where the noise level is very low and situations where the output is still chaotic but substantial noise is also present. The former we call type I chaos; it is observed when all three modes are polarized parallel to each other. The latter we label type II chaos; it is observed when one of the three modes is polarized perpendicular to the other two. Very little green light is generated for type I behavior, which is demonstrably low dimensional chaos, and is controllable by the method of occasional proportional feedback (OPF) [4,5]. Type II chaos is accompanied by the generation of a substantial amount of green light and a clear signature of noise is evident in its chaotic dynamics. It is typically not controlled by OPF.

The laser system displays chaotic intensity output when operated with three or more longitudinal modes. In the present experiments the system parameters were adjusted to obtain three mode operation in the two distinct polarization configurations. An appropriate orientation of the crystal axes allowed us to select these configurations. The pump level, set to about twice the threshold pump power, was similar for the two configurations. The total intensity (the sum of the intensities of each individual mode) was observed with a photodiode having a rise time of less than 1 ns and was sampled using a 100 MHz eight bit digital oscilloscope capable of storing 10⁶ samples. In Fig. 1(a) we show the total intensity when all three modes are polarized parallel to each other (type I chaos). In Fig. 1(b) we show the total intensity with one mode polarized perpendicular to the other two (type II chaos).

In the time traces we can see the distinction between these two operating regimes. Type I consists of long “bursts” of relaxation oscillations, while type II appears far more irregular. During type I operation very little green light, less than 1 μW, was observed, while more than 25 μW of power in green light accompanied type II activity.

We use the total laser intensity \( I(n) = I(t_0 + n \tau_s) \), with the sampling time \( \tau_s = 100 \) ns, and its time delayed values to reconstruct the system phase space [9–12] by forming vectors

\[
y(n) = \{I(n), I(n+T), \ldots, I(n+(d_y-1)T)\}.
\]
where $d_E$ is the integer embedding dimension of the reconstructed phase space and $T$ is the integer time lag in units of $\tau_1$. Our ability to use this phase space reconstruction for extracting physical properties from the observations rests on a proper choice of the time delay $T$ and the embedding dimension $d_E$. For $T$ we use the first minimum of the average mutual information $[9,10,13]$ between $I(n)$ and $I(n+T)$ evaluated as a function of $T$.

$d_E$ is chosen by using the false nearest neighbors algorithm $[14,9,10]$. This relies on the property of autonomous dynamical systems that their trajectories in phase space do not cross each other unless the system is observed in a space with too low a dimension. To determine the $d_E$ necessary to unfold the trajectories using time delay coordinates we observe each point along the trajectory $y(n)$ and its nearest neighbor as the dimension of the space is increased from $d_E$ to $d_E+1$. If the point and its nearest neighbor move sufficiently far from each other as the dimension is increased, we conclude they were falsely seen to be nearest neighbors because of projection from a higher dimensional object, the attractor. When the percentage of false nearest neighbors drops to zero, we have established the value of $d_E$. Here, we use the property of the algorithm that in the presence of noise $[9,10]$, a residual percentage of false nearest neighbors is observed. The amount of residual is a measure of the noise level.

The original data sets of $10^6$ points were oversampled. These were down sampled by a factor of 8, resulting in 125 000 data points. Using the time delay suggested by the average mutual information, we evaluated the percentage of false nearest neighbors for types I and II chaos. This percentage averaged over five type I data traces is shown in Fig. 2(a) (solid line) and enlarged in Fig. 2(b). We see that $d_E = 5$ where the percentage of false nearest neighbors drops well below 0.5%. The dotted lines in Figs. 2(a) and 2(b) represent the corresponding average over four type II data sets. In these data it is clear that there is a residual number of false neighbors that is not eliminated by going to higher embedding dimensions. We have consistently observed this much larger fraction of residual false nearest neighbors for type II dynamics compared to type I dynamics in the many time series of total intensity from our laser system. In fact, the mean type II residual is 40 times the mean type I residual at $d_E = 6$.

Table I contains a summary of the differences between type I and type II chaos as found from experimental measurements and from the nonlinear analysis of the data.

**III. MODEL OF THE PROCESS**

The laser is modeled using three interacting components: the infrared cavity modes, a green cavity mode, and a two level active medium. We write the whole Hamiltonian as

$$H = H_{IR} + H_{green} + H_{conv} + H_{level} + H_{driving}$$

where $H_{IR}$ represents the Hamiltonian of the infrared cavity modes, $H_{green}$ is the Hamiltonian of the green cavity mode, $H_{conv}$ is the Hamiltonian of the conversion process, $H_{level}$ is the Hamiltonian of the two level active medium, and $H_{driving}$ is the Hamiltonian of the driving field.

**TABLE I. Type I and type II chaos summary.**

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Type I</th>
<th>Type II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time series</td>
<td>Bursting</td>
<td>Irregular</td>
</tr>
<tr>
<td>Green output</td>
<td>&lt; 1 $\mu$W</td>
<td>$\geq 25$ $\mu$W</td>
</tr>
<tr>
<td>Mode configuration</td>
<td>3-0</td>
<td>2-1</td>
</tr>
<tr>
<td>OPF controllable</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Embedding dimension</td>
<td>$\approx 5$</td>
<td>$\approx 5$</td>
</tr>
<tr>
<td>False neighbors</td>
<td>$&lt; 1%$</td>
<td>$\approx 5%$</td>
</tr>
</tbody>
</table>
\[ [a_n, a_m^\dagger] = \delta_{nm}, \quad n, m = 1, 2, \ldots, M. \] (4)

For us, \( M = 3. \)

Each mode is coupled to independent heat baths or reservoirs which are represented by boson operators \( b_i \) for the \( k \)th reservoir mode of infrared mode \( i \). This harmonic oscillator has a frequency of \( \Omega_{ik} \). We assume that all of the reservoir modes are independent of each other and the infrared modes (except through the coupling), that is

\[ [b_{qn}, b_{pm}^\dagger] = \delta_{mn} \delta_{pq} \] (5)

and

\[ [b_{pn}, a_{m}^\dagger] = 0. \] (6)

The reservoir modes are bilinearly coupled to the infrared modes with real coupling constants \( \Gamma_{ik} \), which leads to

\[ H_{1R} = \sum_{k=1}^{M} \hbar \omega_i b_{ik}^\dagger b_{ik} \]

\[ + i\hbar \Gamma_{ik} (b_{ik} a_{i}^\dagger - a_{i} b_{ik}^\dagger). \] (7)

There is a single green mode represented by annihilation and creation operators \( g \) and \( g^\dagger \) that satisfies

\[ [g, g^\dagger] = 1 \] (8)

and

\[ [g, a_{m}^\dagger] = 0. \] (9)

It is bilinearly coupled (via real coupling constants \( \Gamma_{ik} \)) to a reservoir that is independent of the infrared mode reservoirs. The \( k \)th reservoir mode of the green mode is represented as \( b_{g} \) and has a frequency of \( \Omega_{g} \). The green mode Hamiltonian is

\[ H_{green} = \hbar \omega_g g^\dagger g + \sum_{k} [\hbar \Omega_{g} b_{g}^\dagger b_{g} + i\hbar \Gamma_{g} (b_{g} g^\dagger - g b_{g}^\dagger)]. \] (10)

In the KTP frequency conversion process, modeled by \( H_{conv} \), conversion occurs when two infrared photons are destroyed to create a green photon and when one green photon is destroyed to create two infrared photons. We assume the coupling tensor \( \kappa_{ij} \) is real and symmetric:

\[ H_{conv} = i\hbar \sum_{i,j=1}^{M} \kappa_{ij} (a_{j}^\dagger a_{j}^\dagger g - g^\dagger a_{j} a_{j}). \] (11)

The laser driving system is represented by a distribution of spin-1/2 systems along the axis over the length of the laser cavity. The Pauli spin operators \( S_3(z,t) \) and \( S_\pm(z,t) \) are used to represent the two level systems and satisfy

\[ [S_3(z), S_\pm(z')] = \pm 2 S_\pm(z) \delta(z-z') \] (12)

and

\[ [S_\pm(z), S_\pm(z')] = S_3(z) \delta(z-z'). \] (13)

In addition, it can be shown that

\[ S_\pm(z) S_\pm(z') = \mp S_\pm(z) \delta(z-z'). \]

\[ S_\pm(z) S_\pm(z') = \mp \frac{1}{2} (I + S_3(z)) \delta(z-z'). \] (14)

The two level system is damped by a cavity mode reservoir represented by boson operators \( b_{c} \) and \( b_{c}^\dagger \). The Hamiltonian is

\[ H_{2level} = \int_{0}^{L} \left\{ \frac{\hbar}{2} S_3(z) + \sum_{k} \left[ i\hbar \Gamma_{ck} (S_+(z)b_{ck} + S_-(z)b_{ck}^\dagger) \right] dz \right\} \]

\[ + i\hbar \Gamma_{g} (s_{g}^\dagger S_+(z) b_{g} + S_-(z) b_{g}^\dagger). \] (15)

The coupling between the medium and the cavity modes is bilinear and the driving efficiency \( \sigma_i \) is assumed to be real:

\[ H_{driving} = \int_{0}^{L} i\hbar \sum_{i=1}^{M} \sigma_i [S_+(z)a_{i} \sin(K_i z) - a_{i}^\dagger S_-(z) \sin(K_i z)] dz. \] (16)

A derivation of the equations of motion for this system can be found in the Appendix. Here we give an overview of the physics of the model and the approximations that are made in the derivation.

First we use the Hamiltonian to determine the standard Heisenberg equations of motion for the system. The reservoir model allows us to apply the Wigner-Weisskopf approximation (see Appendix and Chap. 19.2 of [15]) to write a Langevin equation for the green mode:

\[ \frac{dg}{dt} = - (\gamma_{g} + i\omega_{g}) g - \sum_{i=1}^{M} \kappa_{im} a_{m}^\dagger a_{m}, \]

\[ - \sum_{k} \Gamma_{gk} b_{gk}(0) e^{-i\Omega_{gk} t}, \] (17)

where \( \gamma_{g} \) represents the damping rate and the last term is a fluctuation or noise term. Integrating this equation and taking advantage of the fact that the decay rate \( \gamma_{g} \approx 10^{10} \text{Hz} \) is much faster than the characteristic rate at which \( g \) fluctuates \( (10^{5} \text{Hz}) \), we can find an equation for the green mode:

\[ g = \frac{1}{\gamma_{g} + i(\omega_{g} - \Omega_{g})} \sum_{i=1}^{M} \kappa_{im} a_{m}^\dagger a_{m} + \eta_{g}, \] (18)

where \( \eta_{g} \) is a dimensionless fluctuation term

\[ \eta_{g} = \frac{1}{\gamma_{g} + i(\omega_{g} - \Omega_{g})} \sum_{k} \Gamma_{gk} b_{gk}(0) e^{-i\Omega_{gk} t}. \] (19)

The green mode is seen here to be "slaved" to the infrared dynamics, namely, \( g(t) \) is determined solely in terms of the infrared modes and fluctuations associated with its coupling to the external world. The use of a single green mode operator is justified as the green light escapes from the laser cavity.
and its dynamics is not observed. In what follows, we shall see it acts as a damping factor, and the detailed mode structure is not important.

We do the same with the infrared reservoir and infrared equations of motion and substitute in the green evolution equation to get

$$
\frac{dA_i}{dt} = -\gamma_p A_i - \frac{2}{\gamma_p} \sum_{j=1}^{M} \kappa_{ij} \kappa_{jm} A_j^* A_m
$$

$$
+ 2 \sum_{j=1}^{M} A_j^T \eta_e e^{i(\omega_s + \omega_j)t + \eta_i e^{i\omega_s t}}
$$

$$
- \int_0^L \sigma_i^T e^{i\omega_s t} S_-(z) \sin(Kz) \, dz.
$$

(20)

The noise ($\eta_e$ and $\eta_i$) and damping ($\gamma_p$ and $\gamma_s$) can be related through a fluctuation-dissipation relation, which we derive in a later section.

Now we turn to the two level system equations of motion. Although the Nd:YAG laser is actually a four level system, this model works well for determining the equations of motion. It fails when computing the spontaneous emission noise power, so we compute this power in another way. In the meanwhile we will ignore all noise contributions from the two level system.

The equations of motion are found again, and we formally integrate the reservoir operators, substitute them into the $S_-(z)$ equation of motion, and make the Langevin approximation to get

$$
\frac{dS_+(z)}{dt} = (\gamma_s + i \omega_s) S_+(z) + \eta_s(z) S_3(z)
$$

$$
- \sum_{i=1}^{M} \sigma_i^T a_i^T S_3(z) \sin(Kz).
$$

(21)

At this point, we note that the Nd:YAG laser is a class B laser and its polarization decay rate is much higher than $\gamma_s$ because the polarization of the active medium is affected by the surrounding crystal lattice. For Nd:YAG, $\gamma_s^{-1}$ is approximately 240 $\mu$s. The actual polarization decay time $\gamma_p^{-1}$ is on the order of $10^{-11}$ s.

So we substitute the faster decay rate $\gamma_p$ for $\gamma_s$ and ignore the associated fluctuations.

In the interaction frame moving at the driving frequency $\omega_d$ we find that the driving terms are slaved to the population inversion $S_3(z)$ due to the high polarization decay rate. In a way similar to the method used to determine the green mode equation of motion we determine the driving terms to be

$$
S_+(z) = -\frac{1}{\gamma_p} \sum_{i=1}^{M} \sigma_i a_i^* e^{-i\omega_d t} \sin(Kz) S_3(z),
$$

$$
S_-(z) = -\frac{S_3(z)}{\gamma_p} \sum_{i=1}^{M} \sigma_i a_i e^{i\omega_d t} \sin(Kz).
$$

(22)

We now take the $S_3(z)$ equation, substitute the reservoir solutions, and perform the Langevin approximations.

$$
\frac{dS_3(z)}{dt} = 2\Lambda - 2\gamma_p [I + S_3(z)] - 2[S_+(z) \eta_s(z)]
$$

$$
- \frac{1}{\tau_f} \sum_{i=1}^{M} \sigma_i^T [S_+(z) a_i
$$

$$
+ a_i^T S_-(z)] \sin(Kz).
$$

(23)

A constant population inversion $2\Lambda$ has been added to account for optical pumping. Further manipulations and associating the operator $S_3(z)$ with the population inversion $n(z)$, we find and equation for the population inversion of the laser,

$$
\frac{dn(z)}{dt} = -\frac{1}{\tau_f} [n(z) - \bar{n}(z)] - n(z) \sum_{i=1}^{M} 4 \frac{\sigma_i^2}{\gamma_p} [A_i^T A_i \sin^2(Kz)].
$$

(25)

where $\tau_f$ is the fluorescence decay time of the Nd:YAG medium (240 $\mu$s) and $\bar{n}$ is the mean population inversion.

After substituting the driving terms into the field equation we get

$$
\frac{dA_i}{dt} = -\gamma_p A_i - \frac{2}{\gamma_p} \sum_{j=1}^{M} \kappa_{ij} \kappa_{jm} A_j^* A_m
$$

$$
+ 2 \sum_{j=1}^{M} A_j^T \eta_e e^{i(\omega_s + \omega_j)t + \eta_i e^{i\omega_s t}}
$$

$$
+ \frac{\sigma_i^2}{N \gamma_p} \int_0^L \sin^2(Kz) n(z) \, dz A_i.
$$

(26)

We have identified $n(z)$ here. At this point we recall that the number of photons in the cavity is large ($10^9$) and treat the quantum mechanical operators $A_i$ and $A_i^*$ as if they are real numbers.

Since we now have a partial differential equation for $n(z)$, we break this equation into the component normal modes as described in detail in [3]. To do this, we define a mode gain $G_i$ as

$$
G_i = \frac{2 |\sigma_i|^2 \tau_c}{N \gamma_p} \int_0^L n(z) \sin^2(Kz) \, dz,
$$

(27)

where $\tau_c$ is the round trip cavity time of the laser (0.2 ns). Assuming that $n(z,t)$ separate into time and space components we can write down equations for the mode gains instead of the population inversion. After rescaling the equations so that the electric field has measurable units we obtain

$$
\frac{dE_i}{dt} = -2 \tau_c \left[ (G_i - a_i) E_i - e \sum_{j=1}^{M} \xi_{ij} \xi_{jm} E_j E_m \right]
$$

$$
+ 2 \kappa \sum_{j=1}^{M} \xi_{ij} E_j^* \eta_e e^{i(\omega_s + \omega_j)t + \sqrt{\frac{\hbar \omega_d}{\tau_c}} \eta_i e^{i\omega_s t}}.
$$

(28)

$$
\frac{dG_i}{dt} = \frac{1}{\tau_f} \left[ \rho_i - G_i \left( 1 + \sum_{j=1}^{M} \beta_{ij} |E_j|^2 \right) \right].
$$

(29)
At this point we make use of an earlier model of the laser [3]:

$$2 \tau_c \frac{dE_i}{dt} = (G_i - \alpha_i)E_i - \epsilon g_i^2 E_i^2 - 2\epsilon \sum_{j \neq i} \mu_{ij} |E_j|^2 E_i,$$

(30)

$$\tau_f \frac{dG_i}{dt} = \rho_i - G_i \left(1 + \sum_{j=1}^{M} \beta_{ij} |E_j|^2 \right).$$

(31)

where $\mu_{ij} = g_i^2$ if the modes are parallel polarized and $\mu_{ij} = (1 - g_i^2)$ if the modes are orthogonally polarized. These values of $\mu_{ij}$ have been determined in [3] after consideration of the phase-matching conditions for the intracavity KTP crystal in the presence of the polarized modes of the laser field. Notice that Eq. (30) is a special case of Eq. (28) having the terms where $i = k$ and $j = l$ or $i = l$ and $j = k$. This is called degenerate four wave mixing. Matching coefficients in the degenerate case, we find that $\zeta_{ij} = \sqrt{g_i}$ when modes $i$ and $j$ are parallel polarized and $\zeta_{ij} = \sqrt{1 - g_i}$ when they are perpendicularly polarized.

We expect that the degenerate and nondegenerate four wave mixing rates differ in the different laser configurations. Type I chaos exhibits nondegenerate four wave mixing with little, if any, degenerate four wave mixing. This implies that the green photons never have a chance to leave the cavity before being downconverted to infrared again. The opposite is true for type II chaos where the green photons immediately leave the cavity. In order to separate these two cases, it is necessary to define a four wave mixing tensor $\epsilon_{ijkl}$ where

$$\epsilon_{ijkl} = \begin{cases} \epsilon_d \zeta_{ij} \zeta_{kl} & \text{if } i = k \text{ and } j = l \\ \epsilon_{dij} & \text{if } i = l \text{ and } j = k \\ \epsilon_{sk} \zeta_{kj} & \text{otherwise}. \end{cases}$$

(32)

Here, $\epsilon_d$ is the degenerate four wave mixing rate and $\epsilon_n$ is the nondegenerate four wave mixing rate. We see that Eq. (28) is a special case where the two rates are identical while Eq. (30) is the case when there is only degenerate four wave mixing and no nondegenerate four wave mixing.

The equations we numerically integrate are

$$\frac{dE_i}{dt} = \frac{1}{2 \tau_c} \left[ (G_i - \alpha_i)E_i - \sum_{j,k=1}^{M} \epsilon_{ijkl} E_j^* E_k E_i \right] + \eta_i',$$

(33)

$$\frac{dG_i}{dt} = \frac{1}{\tau_f} \left[ \rho_i - G_i \left(1 + \sum_{j=1}^{M} \beta_{ij} |E_j|^2 \right) \right].$$

(34)

In these equations $i = 1, 2, \ldots, M$. We have lumped all of the noise terms into the single additive noise term $\eta_i'$. This is possible because the multiplicative noise in Eq. (28) is much smaller than the additive noise (see below).

We use the parameters shown in Table II. $\epsilon_{ijkl}$ is the four wave mixing efficiency in inverse watts and has a magnitude on the order of $10^{-3} \text{ W}^{-1}$. It depends on the mode configuration and the relative orientations of the Nd:YAG and KTP crystals. $\beta_{ij}$ is the cross saturation parameter between modes.

### TABLE II. Model parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_c$</td>
<td>0.2 ns</td>
<td>Round trip cavity time</td>
</tr>
<tr>
<td>$\tau_f$</td>
<td>240 $\mu$s</td>
<td>Fluorescence decay time of Nd:YAG</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.01</td>
<td>Cavity loss factor</td>
</tr>
<tr>
<td>$\epsilon_{ijkl}$</td>
<td>See Tables III and IV</td>
<td>Four wave mixing efficiency</td>
</tr>
<tr>
<td>$\rho_i$</td>
<td>0.02</td>
<td>Pumping power</td>
</tr>
<tr>
<td>$\beta_{ij}$</td>
<td>See Tables III and IV</td>
<td>Cross saturation parameter</td>
</tr>
</tbody>
</table>

$i$ and $j$ in units of inverse watts. These values are different for type I and type II chaos and are discussed below.

### IV. NUMERICAL INTEGRATION RESULTS

These model equations were numerically integrated using a standard stiff integrator from the Los Alamos CLAMS library with a time step of 100 ns. The reservoir noise $\eta_i$ was simulated by adding a complex Gaussian distributed with a variance of $10^{-4} \text{ W}$ to the electric field of each mode between integration steps.

Type I behavior is obtained in numerical integration when all modes are polarized in the same direction and no nondegenerate four wave mixing is present, as shown in Table III.

The absence of degenerate four wave mixing is consistent with the experimental absence of measurable green output. Figure 3(a) shows a type I time trace obtained by numerical integration of the equations. The bursting behavior and the relaxation oscillation period echo the experimental type I data in Fig. 1(a).

An approximation to type II behavior is obtained when degenerate four-wave mixing dominates over nondegenerate four-wave mixing as shown in Table IV.

Note that the factors $\xi_{ij}^2$ in (32) are all equal regardless of whether mode $i$ and mode $j$ are parallel or perpendicular. The predominance of degenerate four wave mixing is consistent with experiment; with type II behavior we observe a high amount of green output. An example of a type II time trace obtained from numerical integration is shown in Fig. 3(b).

### A. Data preparation

In our previous paper [8] we discussed the digital signal processing methods we used to extract more resolution from

### TABLE III. Type I model parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Type I chaos Condition</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon_{ijkl}$</td>
<td>$i = k \text{ and } j = l$</td>
<td>0 W$^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$i = l \text{ and } j = k$</td>
<td>0 W$^{-1}$</td>
</tr>
<tr>
<td></td>
<td>Otherwise</td>
<td>$2.1 \times 10^{-6} \text{ W}^{-1}$</td>
</tr>
<tr>
<td>$\beta_{ij}$</td>
<td>$i = j$</td>
<td>1.0 W$^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$i \neq j$</td>
<td>0.6 W$^{-1}$</td>
</tr>
</tbody>
</table>
our data acquired using an eight bit sampling oscilloscope. The resolution affects the local false neighbors and the Lyapunov exponent calculations so in order to use these tools to compare the experimental data and the numerical model, it was necessary to perform the same manipulations. In summary, the numerical model was integrated for $10^5$ points with a time step of 100 ns, matching the maximum storage capacity and the sampling time of the oscilloscope. The data were then quantized to eight bits. For the false nearest neighbors test and the average mutual information calculation, the data were down sampled by a factor of eight, that is, seven out of every eight samples were thrown out. This leaves 125 000 points at a sampling rate of 1.25 MHz ($\tau_e=800$ ns). The down sampling preserves the broadband noise level.

For the local false nearest neighbors test and the Lyapunov exponents, the quantized data were interpolated using a digital linear filter. This filter is designed to remove frequencies from 500 kHz to the Nyquist frequency $f_s/2=5$ MHz and pass all frequencies below 500 kHz. This was needed to get higher resolution from the experimental data traces. In order to match our results, we did this with the numerical traces as well. After performing the interpolation, the data were also down sampled by a factor of 8, leaving 125 000 points at a sampling rate of 1.25 MHz (800 ns).

B. Power spectrum

When we compare the power spectra of the numerical results and the experimental data, we find similarities. Figure 4 shows the power spectra for the experimental data [Fig. 4(a)] and the numerical data [Fig. 4(b)] for type I chaos. The peaks and their structure are very similar. Figure 5 shows the same information for type II chaos. Here, it is not clear from the spectra whether the type II chaos is well modeled.

C. Average mutual information

The average mutual information of the model is strikingly similar to the experimental data. Figure 6(b) is the average mutual information as a function of time lag for the numerically integrated model for type I chaos, and has essentially the same shape as the average mutual information function of the experimental data [Fig. 6(a)]. Note that the relaxation oscillation time is slightly different between the model and the data, however, this can be adjusted with a small change in the pump power.

The average mutual information function for type II chaos is also very similar between model and experiment as shown in Fig. 7. Again, the relaxation oscillation time can be refined by changing the pumping power.

D. False nearest neighbors

When we examine how the model dynamics respond to noise using the false nearest neighbors algorithm, we find that the type I dynamics tend to suppress noise while the type II dynamics do not. Figure 8 shows the false nearest neighbors results for the numerically integrated time traces (125 000 points) for both types of dynamics, with and without reservoir noise. It is clear, especially in Fig. 8(b) that when no noise is present, both type I and type II dynamics exhibit low-dimensional behavior with almost no residual.

When Gaussian noise ($\sigma=0.01|\mathcal{E}_{\text{nominal}}|$) is added to the electric field for every integration time step of 100 ns, we find that type I dynamics have no residual, or in other words, the reservoir noise has been suppressed by the dynamics. However, in the type II dynamics, the residual is around 5%, which indicates that the dynamics have been significantly
affected by the reservoir noise. These findings are numerically consistent with our observations. When we normalize the noise levels using the maximum amplitude of the type I and type II time series, we find that type II is three times more susceptible to noise than type I.

E. Local false nearest neighbors

We also performed a test called local false nearest neighbors on the numerical data [8]. This is used to find the local dimension, or number of equations of motion of the system that generated the data. The results for type I chaos are shown in Fig. 9. For the experimental data [Fig. 9(a)] the predictability of the data has become independent of the number of neighbors and the embedding dimension. We find that numerical results [Fig. 9(b)] match well; both sets have a local dimension $d_L \approx 6$ and the same fraction of poor predictions. For type II chaos (Fig. 10) the match is not so good—the fraction of poor predictions is different by a factor of $2$ and the local dimension appears substantially smaller for the model than for the experiment.

F. Average local Lyapunov exponents

The average local Lyapunov exponents matched well between the model and experimental types I and II. These are computed using the methods described in [8]. Figure 11 shows the average local Lyapunov exponents for the experimental type I data [Fig. 11(a)] and numerical model type I data [Fig. 11(b)] using $d_L = 7$ and $d_L = 7$. Figure 12 shows a closeup of these graphs. Note that in both cases, there are two positive Lyapunov exponents and a zero exponent. The negative Lyapunov exponents are slightly larger for the model dynamics. It is likely that a small parameter change can improve the match.

For the type II data, the match is not as good. Figure 13 shows the average local Lyapunov exponents for the experimental type II data [Fig. 13(a)] and the numerical model type II data [Fig. 13(b)] using $d_L = 7$ and $d_L = 7$. Figure 14 is a closeup of these graphs. The experimental data have three positive Lyapunov exponents while the numerical model has 2. The largest Lyapunov exponent from the experimental data exceeds that of the model by a factor of two. We conclude that the model of type II dynamics does not match the experiment well.
FIG. 6. (a) The average mutual information as a function of time lag for the experimental time series shown in Fig. 1(a) (type I chaos). The time lag is given in units of 8/100 MHz or 800 ns. (b) The average mutual information as a function of time lag for the numerically integrated time series shown in Fig. 3(a) (type I chaos).

Table V gives the average Lyapunov exponent values for \( L = 2048 \), which is a good approximation of the global Lyapunov exponents for the experimental data and the model data. From these numbers, it is clear that type I chaos is modeled well, while type II chaos is not.

V. NOISE SOURCES

In an attempt to determine the source of the noise in the equations, we discuss four sources of intrinsic quantum fluctuations: fluctuations due to cavity damping of the infrared, fluctuations due to the green light leaving the cavity, fluctuations due to spontaneous emission, and fluctuations inherent in the conversion process. We also examined the possibility of fluctuations in the pumping power, and concluded that these could not cause the noise in the output intensity.

We choose to compute the noise levels in photons/s, so we abandon our current units and go back to using the c numbers associated with the creation and annihilation operators \( A_i^\dagger \) and \( A_i \). \( A_i^\dagger A_i \) is simply the number of photons in mode \( i \) and we call this quantity \( N_{IR} \). We repeat the differential equation governing \( A_i \) using a generic source of noise \( \eta(t) \):

\[
\frac{dA_i}{dt} = -\gamma_i A_i - \frac{2}{\gamma_i} \sum_{j,i,m=1}^{M} \kappa_{ij} \kappa_{lm} A_j^\dagger A_j A_m + \sqrt{D} \eta. \tag{35}
\]

where \( \eta(t) \) satisfies

\[
\langle \eta(t') \eta(t) \rangle = \delta(t-t') \tag{36}
\]

and \( D \) is the noise variance or strength in units of s\(^{-1}\).
FIG. 8. (a) The percentage of false nearest neighbors (FNN) vs the embedding dimension $d_E$ for the numerically integrated model. The graphs depict type I with no noise (circles), type II with no noise (squares), type I with reservoir noise ($\sigma^2 = 10^{-4}$, diamonds) and type II with the same reservoir noise (triangle). (b) An enlargement of (a) showing that the percentage of FNN drops to 0.1% and stays there as $d_E$ increases for both types of dynamics with no noise added, and type I dynamics with noise. However, the percentage of type II FNN when noise is added is much higher, around 3%.

The noise power in units of photons/s that is added to each mode can be computed using the number equation:

$$\frac{dA_i^n A_i}{dt} = -2 \gamma_i A_i^n A_i - \frac{2}{\gamma_i} \sum_{j,m=1}^M \kappa_{ij} \kappa_{im} A_j^n A_m^n + \frac{\sqrt{D}}{D} \eta A_i^n + \frac{1}{\sqrt{D}} \eta^* A_i^n.$$  \hspace{1cm} (37)

The amount of noise added to the numerical integration in these units can be determined by converting the noise term in the above equation to real units $E$ where $|E|^2$ is in watts.

FIG. 9. (a) Local false nearest neighbors for the experimental type I time series shown in Fig. 1(a). (b) Local false nearest neighbors for the numerically integrated type I time series.

$$\frac{dE_i}{dt} = -\gamma_i E_i - \frac{2\tau_c}{\gamma_i \hbar \omega_{d}} \sum_{j,m=1}^M \kappa_{ij} \kappa_{im} E_j E_m + \frac{\sqrt{D \eta}}{\tau_c}.$$  \hspace{1cm} (38)

The noise strength in the simulation is $10^3$ W/s. Thus, $D = 10^{21}$ s$^{-1}$. Using Eq. (37) we find that the noise in photons/s is

$$N_{num} \approx 2 \sqrt{N}$ \sqrt{D} \frac{1}{s^{1/2}}.$$  \hspace{1cm} (39)

where $N_{num}$ is the number of IR photons in mode $i$. The strange units in Eq. (39) occur because the units of $\eta$ are the units of a square root of a $\delta$ function in time.

From the experiment we find that about 1 mW of infrared light is output from the laser. Given a transmission loss of $\sim 0.1\%$, this means that there is approximately 1 W of infrared power inside the cavity. Since each photon has an energy of $\hbar \omega_d = 2 \times 10^{-19}$ J and the round trip cavity time is
\[ \tau_c = 2 \times 10^{-10} \text{ s}, \text{ we find that } N_r = 10^9. \text{ This puts the numerical integration noise at } 2 \times 10^{15} \text{ photons/s.} \]

Similarly, the output green power of 100 \( \mu \)W with a fully transmitting cavity implies that the number of green photons in the cavity \( N_g \) is about \( 10^5 \).

### A. Damping fluctuations

First, we wish to find the noise power due to damping of infrared light. We compute the noise strength \( D_{ii} \).

\[ \langle \eta_i(t) \eta_i(t') \rangle = D_{ii} \delta(t-t'). \]  \hspace{1cm} (40)

Based on Chap. 19-2 in [15] we find that

\[ N_i^{IR} = \gamma_i \langle n(\omega_i) \rangle = \frac{\alpha}{\tau_c} \langle n(\omega_i) \rangle. \]  \hspace{1cm} (41)

where \( \langle n(\omega_i) \rangle \) is the mean occupation number of bosons and

\[ \langle n(\omega_i) \rangle = \frac{1}{e^{\hbar \omega_i/kT} - 1}. \]  \hspace{1cm} (42)

Here, \( k \) is Boltzmann's constant and \( T \) is the temperature of the reservoir, which we take to be the cold cavity temperature of 300 K. The energy of an infrared photon is 1.2 eV while \( kT \) at room temperature is about 0.026 eV. Thus \( \langle n(\omega) \rangle \) is \( 10^{-20} \). The noise strength is \( 10^{-13} \text{ s}^{-1} \). The noise added to each mode due to IR damping is approximately

\[ N_{IR} = 2 \sqrt{N_r D_{ii}} = 0.02 \text{ photons/s.} \]  \hspace{1cm} (43)

Similarly, the noise power due to IR damping is

\[ \langle \eta_i(t) \eta_i(t') \rangle = D_{ii} \delta(t-t') = \frac{\langle n(\omega_i) \rangle}{\gamma_i} \delta(t-t'). \]  \hspace{1cm} (44)

The noise power in infrared mode \( i \) due to green cavity damping fluctuations from Eq. (A47) is

\[ N_i^{green} = 2 \sum_{j=1}^{M} \kappa_j A_j A_j \sqrt{D_{ii} \eta_j} + 2 \sum_{j=1}^{M} \kappa_j A_j A_j \sqrt{D_{i} \eta_j}. \]  \hspace{1cm} (45)

which we approximate as

\[ N_i^{green} = 4 \kappa_i \sqrt{\frac{\langle n(2\omega_d) \rangle}{\gamma_i}}. \]  \hspace{1cm} (46)
Since we know the value of $\epsilon$,

$$\epsilon = \frac{4 \tau_p^2 \kappa^2}{\hbar \omega_d \gamma_\delta} \approx 10^{-5} \text{ W}^{-1},$$

(47)

we find $\kappa \approx 500 \text{ s}^{-1}$. We assume that the decay time of the green is one cavity round trip time, or $1/\tau_c$, which leads to a noise power of $2 \times 10^{-12} \text{ photons/s}$, which is so tiny that it can be ignored.

B. Spontaneous emission noise

The spontaneous emission power can be determined in a similar way to the infrared and green contributions shown above. A simpler method following [16] is used instead.

The Nd:YAG medium has a spontaneous emission spectrum with a Lorentzian shape of width $\gamma_p$ or 6 cm$^{-1}$ (180 GHz). Knowing the density of photon modes in a cavity with volume $V$,

$$\frac{dN}{df} = \frac{8 \pi V f^2}{c^3},$$

(48)

and assuming a cavity volume of 0.25 cm$^3$ we find that the number of modes in the spontaneous emission width $df$ is $p = 3 \times 10^9$.

The total spontaneous emission power in photons/s into a single mode is simply

$$N_i^{\text{spont}} = \frac{N_2}{p \tau_f}.$$

(49)

where $N_2$ is the population of the second level. We can determine this population at threshold especially easily for a Nd:YAG laser because it is a four level laser where $N_2 > N_1$. According to [16], just below threshold,

$$(N_2 - N_1)_{\text{threshold}} = \frac{p \tau_f}{\tau_p} N_2,$$

(50)

where $\tau_p$ is the cavity decay time or, using our constants, $\tau_p = \tau_c / \alpha$. What this says is that no net stimulated emission occurs, and the entire population inversion fluoresces at the same rate as the resulting photons leak away. In our laser the population inversion is about $3 \times 10^{13}$ at threshold.

Substituting the expression for population inversion into the power equation, we find that at threshold,

$$N_i^{\text{spont}} = \frac{\alpha}{\tau_c} = 5 \times 10^7 \text{ photons/s}.$$

(51)

This is still 7 orders of magnitude lower than the levels we expect from numerical integration.
equation much like Eq. (A47). In the process, a new noise appears, which is related to the diffusion of probability that occurs with nonlinear terms in the Hamiltonian. Since our derivation of this noise term follows [17] almost exactly, we will simply present the results.

Starting with the perturbation related to the KTP conversion process,

$$V = i\hbar \sum_{i,j=1}^{M} \kappa_{ij}(a_i^\dagger a_i^\dagger g - g^\dagger a_i a_i),$$  \hspace{1cm} (52)

we find that the terms due to this perturbation in the differential equations are

$$\frac{d}{dt} \begin{bmatrix} A_1 \\ A_1^\dagger \\ A_2 \\ A_2^\dagger \\ A_3 \\ A_3^\dagger \end{bmatrix} = \begin{bmatrix} -2 \sum_{j=1}^{3} \kappa_{1,j} A_j^\dagger G \\
-2 \sum_{j=1}^{3} \kappa_{1,j} A_j G^\dagger \\
-2 \sum_{j=1}^{3} \kappa_{2,j} A_j^\dagger G \\
-2 \sum_{j=1}^{3} \kappa_{2,j} A_j G^\dagger \\
-2 \sum_{j=1}^{3} \kappa_{3,j} A_j^\dagger G \\
-2 \sum_{j=1}^{3} \kappa_{3,j} A_j G^\dagger \end{bmatrix} + B \begin{bmatrix} \eta_1 \\ \eta_1^\dagger \\ \eta_2 \\ \eta_2^\dagger \\ \eta_3 \\ \eta_3^\dagger \end{bmatrix},$$  \hspace{1cm} (53)

$$\frac{d}{dt} [G] = \sum_{i,j=1}^{M} \kappa_{ij} [A_i A_j^\dagger].$$  \hspace{1cm} (54)

Here, $A_i$ is a c number similar to (and can be considered to be equivalent to) $a_i$ used earlier. The noise matrix $B$ is defined by

$$BB^\dagger = \begin{bmatrix}
\kappa_{11} G & 0 & \kappa_{12} G & 0 & \kappa_{13} G & 0 \\
0 & \kappa_{11} G^\dagger & 0 & \kappa_{12} G^\dagger & 0 & \kappa_{13} G^\dagger \\
\kappa_{12} G & 0 & \kappa_{22} G & 0 & \kappa_{23} G & 0 \\
0 & \kappa_{12} G^\dagger & 0 & \kappa_{22} G^\dagger & 0 & \kappa_{23} G^\dagger \\
\kappa_{13} G & 0 & \kappa_{23} G & 0 & \kappa_{33} G & 0 \\
0 & \kappa_{13} G^\dagger & 0 & \kappa_{23} G^\dagger & 0 & \kappa_{33} G^\dagger
\end{bmatrix}.$$  \hspace{1cm} (55)

and the $\eta$ terms are zero-mean fluctuation terms satisfying

$$\langle \eta_i(t) \eta_j(t') \rangle = \delta_{ij} \delta(t-t').$$  \hspace{1cm} (56)

Other than the noise term, the four wave mixing perturbations are the same as what were derived earlier. The multiplicative noise term is much larger than the one derived previously. A rough estimate of the number of noise photons added to the IR mode every second is

---

**TABLE V. Lyapunov exponents of experimental data and model.**

<table>
<thead>
<tr>
<th>Average Lyapunov exponents</th>
<th>Type 1 Chaos</th>
<th>Type 2 Chaos</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>L = 2048; d_E = 7; d_L = 7</strong></td>
<td>Experiment</td>
<td>Model</td>
</tr>
<tr>
<td>0.080</td>
<td>0.080</td>
<td>0.244</td>
</tr>
<tr>
<td>0.041</td>
<td>0.038</td>
<td>0.172</td>
</tr>
<tr>
<td>0.008</td>
<td>0.009</td>
<td>0.091</td>
</tr>
<tr>
<td>-0.033</td>
<td>-0.044</td>
<td>0.007</td>
</tr>
<tr>
<td>-0.102</td>
<td>-0.152</td>
<td>-0.104</td>
</tr>
<tr>
<td>-0.278</td>
<td>-0.338</td>
<td>-0.298</td>
</tr>
<tr>
<td>-1.017</td>
<td>-1.266</td>
<td>-0.788</td>
</tr>
</tbody>
</table>
or 10⁷ photons/s. This can be arrived at by other means. According to Eq. (37) the green production rate is approximately 4κ²N₁²/γ₂, which is 2 × 10¹⁴ photons/s. This is the mean value of a process whose standard deviation we would expect to be the noise added to the infrared mode. Since the green mode is a coherent state and therefore has a Poisson distribution in the number states, we would expect the standard deviation to be the square root of the mean. Thus, the noise added to the infrared mode should be around 10⁷ photons/s. Table VI summarizes the power estimates in photons/s for the noise sources described above.

It appears that there must be another source of noise in our system that contributes much more than these quantum mechanical sources. Pumping fluctuations were considered. To determine if these were the noise source, we substituted

\[ \rho_e = \rho_e [1 + \sigma \eta(t)] \]  

into Eq. (34) where \( \eta(t) \) is a zero mean unit variance random number. We found that in order to reproduce the noise levels of the integrated time series with the experimental time series using the false nearest neighbors algorithm, we had to set \( \sigma = 0.05 \) or 5% fluctuation in \( \rho_e \). This level is unrealistically high for pumping fluctuations. In addition, the fluctuations seen in the experimental data have a characteristic frequency that is much higher than the relaxation oscillation rate, which is impossible to attain through pumping fluctuations because of the slow time constant in Eq. (34).

### VI. SUMMARY AND CONCLUSIONS

In summary, we have developed a model that captures key features of the intensity dynamics of the three-mode Nd:YAG laser with an intracavity KTP crystal. This model consists of three equations for each infrared mode: two describe the complex electric field and one describes the gain.

The inclusion of both degenerate and nondegenerate four wave mixing features not found in previous models of the laser. Both qualitative and quantitative behaviors found in the experimental system are captured by this model, which is especially successful in its description of the type I case. The distance between type I and type II chaos is seen as a difference in structure of the four wave mixing tensor, which also leads to a difference in the noise susceptibility of the equations of motion.

Type I chaos occurs when all modes are parallel polarized and is controllable by the OPF chaos control algorithm. The model captures the bursting behavior found in the time traces. Extremely low levels of green light are measured in type I output, which is described by the model as a predominance of nondegenerate four wave mixing in the laser cavity. Low noise levels are measured in the intensity dynamics, which agrees with the suppression of noise in the type I model dynamics. The local false nearest neighbors test and the Lyapunov exponents match between model and experiment.

Type II chaos occurs when one mode is polarized perpendicular to the other two and is not controllable by the OPF scheme. The bursting, highly irregular time series behavior is captured by the model. The large amount of green light produced by the laser is due to a large amount of degenerate four wave mixing in the laser cavity. The high noise levels found in the intensity dynamics agree with the model's tendency to suppress reservoir noise but to amplify it instead. However, the local false nearest neighbors test and the Lyapunov exponents do not match well between the model and the experiment, leading us to believe that type II chaos is not fully modeled. We have found that the parameter space of the model is quite complex, especially when degenerate four wave mixing is present. It is possible that additional noise sources remain to be identified and included in the model.

Nonlinear time series analysis has aided this investigation by revealing the link between the high noise levels in the data and the large green light output. A more sophisticated model that reproduces type I behavior almost perfectly and approximates type II behavior is the major result of this paper. Time series analysis allows us to make a quantitative comparison of the model with the experiment. This is the first case we know of where chaotic time series analysis has significantly aided the development of a more complete physical model of the dynamics. This system and model provide a means to study the influence of noise on chaotic systems.

### ACKNOWLEDGMENTS

Z.G. and R.R. thank R. Fox, B. Kennedy, and K. Wiesenfeld for many helpful discussions on the material discussed here. Z.G. was financially supported by AT&T. R.R. acknowledges support from NSF Grant No. ECS-9114232. C.L. and C.I.A. thank the members of INLS for numerous discussions on this subject. Their work was supported in part by the Department of Energy, Office of Basic Energy Sciences, Division of Engineering and Geosciences, under Contract DE-FG03-90ER14138, and in part by the Office of Research (Contract No. N00014-91-C-0125).
We reiterate the Hamiltonian:
\[
H = \sum_{i=1}^{M} \hbar \omega_i a_i^\dagger a_i + \hbar \omega g a_i^\dagger a_i + i \hbar \sum_{i,j=1}^{M} \kappa_{ij} (a_i^\dagger a_j^\dagger g - g^* a_i a_j) + \sum_{i=1}^{M} \sum_k \left[ \hbar \Omega_{ik} b_i^\dagger b_k + i \hbar \Gamma_{ik} (b_i^\dagger a_k - a_k b_i^\dagger) \right] + \sum_k \left[ \hbar \Omega_{ek} b_k^\dagger b_k + i \hbar \Gamma_{ek} (b_k b_k^\dagger - a_k^\dagger a_k) \right] + \sum_k \left[ \frac{\hbar \omega}{2} b_k^\dagger b_k \right] + \sum_k \hbar \Omega_{ek} b_k^\dagger b_k.
\]

Using this Hamiltonian and the standard Heisenberg equations of motion
\[
i \hbar \frac{d \cdot}{dt} = [-, H],
\]
we arrive at the equations of motion governing the system.

\[
\frac{da_i}{dt} = -i \omega_i a_i + 2 \sum_{j=1}^{M} \kappa_{ij} a_j^\dagger g + \sum_k \Gamma_{ik} b_k,
\]
\[
- \int_0^L \sigma_+^*(z) \sin(Kz) dz,
\]
\[
\frac{dg}{dt} = -i \omega g - \sum_{i,m=1}^{M} \kappa_{im} a_i a_m + \sum_k \Gamma_{ek} b_k,
\]
\[
\frac{db_k}{dt} = -\Gamma_{ik} a_i - i \Omega_{ik} b_k,
\]
\[
\frac{db_k}{dt} = -\Gamma_{ek} g - i \Omega_{ek} b_k.
\]

The green reservoir equation (A6) is linear in \( b_k(t) \) so we can integrate it:
\[
b_k = -b_k(0) e^{-i \Omega_{ek} t} - \Gamma_{ek} \int_0^t g(t') e^{-i \Omega_{ek} (t' - t')} dt'.
\]

Substituting this into (A4) we arrive at
\[
\frac{dg}{dt} = -i \omega g - \sum_{i,m=1}^{M} \kappa_{im} a_i a_m + \sum_k \Gamma_{ek} \left[ -b_k(0) e^{-i \Omega_{ek} t} - \Gamma_{ek} \int_0^t g(t') e^{-i \Omega_{ek} (t' - t')} dt' \right].
\]

The fourth term can be approximated by a damping term using the Wigner-Weisskopf approximation where the modes are assumed to form a continuous spectrum and the interference time of sum on \( g(t) \) is assumed to be much smaller than the characteristic time scale of the equation. This approximation is discussed in detail in Sec. 19-2 of [15] and will not be elaborated further here. This leads us to

\[
\frac{dg}{dt} = -(\gamma_g + i \omega_g) g - \sum_{i,m=1}^{M} \kappa_{im} a_i a_m - \sum_k \Gamma_{ek} b_k(0) e^{-i \Omega_{ek} t},
\]

where \( \gamma_g \) represents the damping rate.

Since this equation is linear in \( g(t) \), we can integrate it to find
\[
g = -g(0) e^{-(\gamma_g + i \omega_g) t} - \int_0^t \sum_{i,m=1}^{M} \kappa_{im} a_i(t') a_m(t') e^{-(\gamma_g + i \omega_g) (t' - t)} dt'.
\]
\[
- \sum_k \frac{\Gamma_{ek} b_k(0)}{\gamma_g + i (\omega_g - \Omega_{ek})} \left( e^{(\gamma_g + i \omega_g) t} - 1 \right).
\]

In the integral, we replace the rapidly varying infrared operator \( a_i(t) \) with the more slowly varying interaction representation forms \( A_i(t) \) in the rotating coordinate system where
\[
a_i(t) = e^{-i \omega_i t} A_i(t).
\]

then we perform the integrations by removing the slowly varying operators from under the integral. This method assumes that the damping rate \( \gamma_g \approx 10^{10} \) Hz is much higher than the characteristic time scale of the evolution of the slowly varying interaction form of the green operator. We find through experimental observation that the green intensity varies at the same 100 kHz rate as the infrared operator. For times large compared to \( \gamma_g^{-1} \) we can ignore the decaying transients. Thus we find
\[
g(t) = - \sum_{i,m=1}^{M} \frac{\kappa_{im} a_i a_m}{\gamma_g + i (\omega_g - \omega_i - \omega_m)} - \sum_k \frac{\Gamma_{ek} b_k(0)}{\gamma_g + i (\omega_g - \Omega_{ek})} e^{-i \Omega_{ek} t}.
\]

This expression is further simplified if we assume that in order for significant infrared-green conversion to occur, \( \omega_g = \omega_i + \omega_m \).
\[
g = - \frac{1}{\gamma_g} \sum_{i,m=1}^{M} \kappa_{im} a_i a_m + \eta_g.
\]
where \( \eta_s \) is a dimensionless fluctuation term

\[ \eta_s = -\sum_k \frac{\Gamma_{sk} b_k(0)}{\gamma + i(\omega - \Omega_{sk})} e^{-i\Omega_{sk}t}. \quad (A14) \]

The green mode is seen here to be "slaved" to the infrared dynamics; namely, \( g(t) \) is determined solely in terms of the infrared modes and fluctuations associated with its coupling to the external world. The use of a single green mode operator is justified as the green light escapes from the laser cavity and its dynamics is not observed. In what follows, we shall see it acts as a damping factor, and the detailed mode structure is not important.

Performing the same operations on the infrared equations (without the final integration) we arrive at the equations of motion for the \( M \) infrared modes.

\[ \frac{d a_i}{d t} = - (\gamma_i + i\omega_i) a_i + \eta_i - \sum_{j=1}^M 2\kappa_{ij} a_j g \]

\[ - \int_0^L \sigma_i^* S_-(z) \sin(K_i z) dz. \quad (A15) \]

where

\[ \gamma_i = \pi |\Gamma_i(\omega_i)|^2 D(\omega_i) \]

\[ \eta_i = -\sum_k \Gamma_{ik} b_k(0) e^{-i\Omega_{ik} t}. \quad (A17) \]

\( M = 3 \) in our problem.

Next we substitute \( g \) into this equation, move to a coordinate system rotating with frequency \( \omega_i \) by substituting

\[ A_i = e^{i\omega_i t} a_i, \quad (A18) \]

and assume that in order for significant four wave mixing to occur, \( \omega_1 + \omega_2 = \omega_3 + \omega_m \):

\[ \frac{d A_i}{d t} = -\gamma_i A_i - \frac{2}{\gamma_i} \sum_{j,l,m=1}^M \kappa_{ijl} A_j A_l A_m \]

\[ + 2\sum_{j=1}^M A_j^* \eta_j e^{i(\omega_j + \omega_m) t} + \eta_i e^{i\omega_i t} \]

\[ - \int_0^L \sigma_i^* S_-(z) \sin(K_i z) dz. \quad (A19) \]

Now we turn to the two level system equations of motion. Though the Nd:YAG laser is actually a four level system, this model works well for determining the equations of motion. It fails when computing the spontaneous emission noise power, so we compute this power in another way. In the meanwhile we will ignore all noise contributions from the two level system.

The pertinent equations of motion are

\[ \frac{d S_+(z)}{d t} = 2\sum_{i=1}^M \sigma_i [S_-(z) a_i + a_i^* S_-(z)] \sin(K_i z) \]

\[ + 2\sum_k [\Gamma_{zk} S_+(z) a_k + \Gamma_{zk}^* S_-(z) a_k^*] \]

\[ + \Gamma_{zk} S_+(z) a_k^* + 2\Lambda. \quad (A20) \]

\[ \frac{d S_-(z)}{d t} = i\omega S_+(z) - \sum_k \Gamma_{zk}^* S_-(z) S_+(z) \]

\[ - \sum_{i=1}^M \sigma_i a_i^* S_-(z) \sin(K_i z). \quad (A21) \]

\[ \frac{d S_-(z)}{d t} = -i\omega S_-(z) - \sum_k \Gamma_{zk} S_+(z) S_-(z) \]

\[ - \sum_{i=1}^M \sigma_i S_+(z) a_i \sin(K_i z). \quad (A22) \]

\[ \frac{d b_{ik}}{d t} = -i\Omega_{ik} b_{ik} - \int_0^L \Gamma_{ik}^* S_-(z) dz, \]

\[ \frac{d b_{ik}^*}{d t} = i\Omega_{ik} b_{ik}^* - \int_0^L \Gamma_{ik} S_+(z) dz. \quad (A23) \]

Note that we have added a constant \( 2\Lambda \) to the \( S_3 \) equation to account for the steady-state population inversion due to optical pumping. \( \Lambda \) is a pumping rate density and has units of \( 1/(\text{level} \times \text{time}) \).

Formally integrating the reservoir operators, substituting these into the \( S_+ \) equation of motion, and making the Langevin approximation we get

\[ \frac{d S_+(z)}{d t} = (-\gamma_p + i\omega) S_+(z) + \eta_3 S_3(z) \]

\[ - \sum_{i=1}^M \sigma_i a_i^* S_-(z) \sin(K_i z). \quad (A25) \]

At this point, we need to note that the Nd:YAG laser is a class \( S \) laser and its polarization decay rate is much higher than \( \gamma_\perp \) because the polarization of the active medium is affected by the surrounding crystal lattice. For Nd:YAG,

\[ \gamma_\perp \approx \text{approximately } 240 \text{ ms}. \]  

The actual polarization decay time \( T_1 \) is on the order of \( 10^{-11} \) s. So we substitute the faster decay rate \( \gamma_p \) for \( \gamma_\perp \) and ignore the associated fluctuations.

Now we transform the \( S_+ \) equation to a rotating frame with the driving term frequency \( \omega_d \) by substituting

\[ S_+(z) = e^{-i\omega_d t} S_+(z), \quad (A26) \]

\[ \frac{d S_+(z)}{d t} = [-\gamma_p + i(\omega_d - \omega_d)] S_+(z) \]

\[ - \sum_{i=1}^M \sigma_i a_i^* e^{-i\omega_d} S_-(z) \sin(K_i z). \quad (A27) \]

Since the polarization decay rate is so high, the \( S_-(z) \) equation is slaved to the population \( S_3(z) \) and the field \( a_i e^{i\omega_d t} \). So we can dynamically eliminate this equation by setting
\[
\frac{dS_+ (z)}{dt} = 0. \tag{A28}
\]

We also assume \( \gamma_p \gg \omega_i - \omega_d \), which is equivalent to saying that the modes that the laser are very near the peak of the Lorentzian line shape of the transition. The \( S_+ (z) \) equation is similar.

\[
S_+ (z) = - \sum_{p=1}^{M} \sigma_i a_i^\dagger e^{i\omega_d z} \sin(K_iz) S_3 (z), \tag{A29}
\]

\[
S_- (z) = - \sum_{p=1}^{M} \sigma_i a_i e^{i\omega_d z} \sin(K_iz). \tag{A30}
\]

We now take the \( S_3 (z) \) equation, substitute the reservoir solutions, and perform the Langevin approximations:

\[
\frac{dS_3 (z)}{dt} = 2 \Lambda - 2 \gamma_p [I + S_3 (z)] - 2 S_3 (z) \eta_i (z)
- \eta_i (S_+ (z)) + 2 \sum_{i=1}^{M} \sigma_i [S_+(z) a_i]
+ a_i^\dagger S_- (z) \sin(K_iz). \tag{A31}
\]

For simplicity, we ignore the noise contribution term \( \eta_i (z) \). Substituting \( S_+(z) e^{-i\omega_d t} \) for \( S_+(z) \), assuming \( S_3 (z) \) commutes with \( a_i \), and ignoring cross terms we have

\[
\frac{dS_3 (z)}{dt} = 2 \Lambda - 2 \gamma_p [I + S_3 (\cdot \cdot \cdot)] - \frac{1}{\gamma_p} S_3 (z)
- \sigma_i^2 a_i^\dagger a_i \sin^2(K_iz). \tag{A32}
\]

We substitute in \( A_i \), and multiply the entire equation by \( N \), the total number of atoms.

\[
\frac{dNS_3 (z)}{dt} = 2N \Lambda - 2 \gamma_p [NI + NS_3 (\cdot \cdot \cdot)] - 4 \frac{1}{\gamma_p} NS_3 (z)
- \sum_{i=1}^{M} \sigma_i^2 A_i^\dagger A_i \sin^2(K_iz). \tag{A33}
\]

Associating the operator \( NS_3 (z) \) with the population inversion \( n(z) \), we find that \( NI \) must be the density of two level systems in the medium \( N/L \). We also define

\[
\tau_f = (2 \gamma_f)^{-1}, \tag{A34}
\]

\[
\bar{n} = \left[ \frac{N \Lambda}{\gamma_s} - \frac{N}{L} \right], \tag{A35}
\]

\[
\frac{dn (z)}{dt} = - \frac{1}{\tau_f} (n(z) - \bar{n}) - n(z) \sum_{i=1}^{M} \frac{4 \sigma_i^2}{\gamma_p} A_i^\dagger A_i \sin^2 (K_iz). \tag{A36}
\]

where \( \tau_f \) is the fluorescence decay time of the Nd:YAG material.

Now we return to the field equation and substitute \( S_-(z) e^{-i\omega_d t} \) for \( S_+(z) \) and take advantage of the orthogonality condition of the normal modes

\[
\int_0^L \sin(K_iz) \sin(K_jz) dz = \delta_{ij} \tag{A37}
\]

to get

\[
\frac{dA_i}{dt} = - \gamma_p A_i - 2 \sum_{j=1}^{M} \kappa_{ij} \kappa_{im} A_j^\dagger A_m
+ 2 \sum_{j=1}^{M} \kappa_{ij} \kappa_{im} A_j^\dagger A_m
+ \frac{\sigma_i^2}{N \gamma_p} \int_0^L \sin^2 (K_iz) n(z) dz A_i. \tag{A38}
\]

We have identified \( n(z) \) here. At this point we recall that the number of photons in the cavity is large (10^9) and treat the quantum mechanical operators \( A_i \) and \( A_i^\dagger \) as if they are c numbers.

We break the population equation into the component normal modes as described in detail in [3]. To do this, we define a mode gain \( G_i \) as

\[
G_i = \frac{2 |\sigma_i|^2 \tau_c}{N \gamma_p} \int_0^L n(z) \sin^2 (K_iz) dz. \tag{A39}
\]

where \( \tau_c \) is the round trip cavity time of the laser (0.2 ns). Then

\[
\frac{dG_i}{dt} = \frac{2 |\sigma_i|^2 \tau_c}{N \gamma_p} \int_0^L \frac{1}{\tau_f} [n(z) - \bar{n}]
- n(z) \sum_{j=1}^{M} \frac{4 |\sigma_j|^2}{\gamma_p} [A_j^\dagger A_j^\dagger] \sin^2 (K_iz) \int_0^L \sin^2 (K_iz) dz. \tag{A40}
\]

We integrate the first two terms on the right hand side and substitute the pumping power

\[
\rho_i = \frac{\sigma_i^2 \bar{n} \tau_c}{N \gamma_p}, \tag{A41}
\]

\[
\frac{dG_i}{dt} = \frac{1}{\tau_f} (\rho_i - G_i) - \frac{2 |\sigma_i|^2 \tau_c}{N \gamma_p} \sum_{j=1}^{M} \frac{2 |\sigma_j|^2}{\gamma_p} [A_j^\dagger A_j^\dagger]
\times \int_0^L n(z) \sin^2 (K_iz) dz + \frac{2 |\sigma_i|^2 \tau_c}{N \gamma_p} \sum_{j=1}^{M} \frac{2 |\sigma_j|^2}{\gamma_p} [A_j^\dagger A_j^\dagger]
\times \int_0^L n(z) \cos(2K_iz) \left( \frac{1 - \cos(2K_iz)}{2} \right) dz. \tag{A42}
\]

We define the mode coupling constant \( \xi_{ij} \) :

\[
\xi_{ij} = \frac{\int_0^L n(z) \cos(2K_iz) [1 - \cos(2K_iz)] dz}{\bar{n}}. \tag{A43}
\]
\[ |E_i|^2 = |A_i|^2 \frac{\hbar \omega_d}{\tau_c} \]  

(A48)

This coefficient is truly a constant if \( n(z, t) \) factors into separate time and space dependent components; this is probably a good assumption for a standing wave cavity. Thus

\[
\frac{dG_i}{dt} = \frac{1}{\tau_f} (\rho_i - G_i) - G_i \sum_{j=1}^{M} \frac{2|\sigma_j|^2}{\gamma_p} |A_j|^2 \\
+ G_i \sum_{j=1}^{M} \frac{2|\sigma_j|^2}{\gamma_p} |A_j|^2 \xi_{ij} \\
= \frac{1}{\tau_f} \left[ \rho_i - G_i \left( 1 + \sum_{j=1}^{M} \beta_{ij} |A_j|^2 \right) \right],
\]

(A44)

where

\[
\beta_{ij} = \frac{2\sigma_j^2}{\gamma_p} \tau_f (1 - \xi_{ij}).
\]

(A46)

The field equation is simplified.

\[
\frac{dA_j}{dt} = \left( \frac{G_i - \gamma_j}{\tau_c} \right) A_j - \frac{2}{\gamma_p} \sum_{j=1}^{M} \kappa_{ij} \kappa_{im} A_j A_m \\
+ \frac{2}{\gamma_p} \sum_{j=1}^{M} \kappa_{ij} A_j \eta_j e^{i\omega_j t} + \eta_j e^{i\omega_j t},
\]

(A47)

We rescale the field equation so that it has measurable units. We define the electric field \( E \) so that \( I = |E|^2 \) has units of watts. That is,

\[ |E_i|^2 = |A_i|^2 \frac{\hbar \omega_d}{\tau_c} \]

(A48)

since \( |A_i|^2 \) is simply the number of photons in mode \( i \). We also substitute \( \alpha_i = 2\gamma \tau_c \) and assume the \( \kappa_{ij} \) are real:

\[
\frac{dE_i}{dt} = \frac{1}{2\tau_c} \left[ (G_i - \alpha_i)E_i - \frac{4\tau_c}{\hbar \omega_d} \sum_{j=1}^{M} \kappa_{ij} \kappa_{im} E_j E_m \right] \\
+ 2 \sum_{j=1}^{M} \kappa_{ij} E_j^* \eta_j e^{i(\omega_j + \omega_i) t} + \sqrt{\frac{\hbar \omega_d}{\tau_c}} \eta_i e^{i\omega_i t}.
\]

(A49)

Now we define \( \kappa \) and \( \xi_{ij} \) so that \( \kappa \xi_{ij} = \kappa_{ij} \) and \( \xi_{ij} \) is unitless and of order unity and define

\[
\epsilon = \frac{4\tau_c^2 \kappa^2}{\hbar \omega_d \gamma_p},
\]

(A50)

which has units of inverse watts. We also define

\[
\beta_{ij} \gamma = \frac{2|\sigma_j|^2\hbar \omega_d \tau_f}{\gamma_p \tau_c} (1 - \xi_{ij})
\]

(A51)

which has units of inverse watts. The resulting equations are

\[
\frac{dE_i}{dt} = \frac{1}{2\tau_c} \left[ (G_i - \alpha_i)E_i - \epsilon \sum_{j=1}^{M} \xi_{ij} E_j^* \right] E_m \right] \\
+ 2 \kappa \sum_{j=1}^{M} \xi_{ij} E_j^* \eta_j e^{i(\omega_j + \omega_i) t} + \sqrt{\frac{\hbar \omega_d}{\tau_c}} \eta_i e^{i\omega_i t}.
\]

(A52)

\[
\frac{dG_i}{dt} = \frac{1}{\tau_f} \left[ \rho_i - G_i \left( 1 + \sum_{j=1}^{M} \beta_{ij} \right) |E_j|^2 \right].
\]

(A53)

Encoding and decoding messages with chaotic lasers

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We investigate the structure of the strange attractor of a chaotic loss-modulated solid-state laser utilizing return maps based on a combination of intensity maxima and interspike intervals, as opposed to those utilizing Poincaré sections defined by the intensity maxima of the laser ($\phi=0,\phi<0$) alone. We find both experimentally and numerically that a simple, intrinsic relationship exists between an intensity maximum and the pair of preceding and succeeding interspike intervals. In addition, we numerically investigate encoding messages on the output of a chaotic transmitting laser and its subsequent decoding by a similar receiver laser. By exploiting the relationship between the intensity maxima and the interspike intervals, we demonstrate that the method utilizes to encode the message is vital to the system's ability to hide the signal from unwanted deciphering. In this work alternative methods are studied in order to encode messages by modulating the magnitude of pumping of the transmitter laser and also by driving its loss modulation with more than one frequency. [S1063-651X(97)01803-4]

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I. INTRODUCTION

There has been great interest in the use of chaotic signals as the carriers of analog and digital information over the last few years, initiated by the work of Pecora and Carroll [1], who suggested that synchronized chaotic systems could be employed to encode and decode messages in real time. Recent experiments have demonstrated that using chaos to communicate is practically feasible with electronic circuits [2,3]. The characteristic property of the chaotic carrier wave is in these circuits is of the order of 10 kHz, and even with the prospects of speeding up these circuits by several orders of magnitude, it is still of interest to consider communication with chaotic optical signals which have the potential for even higher information transmission and reception rates.

Two groups have experimentally demonstrated that chaotic lasers can be synchronized. Roy and Thornburg [4] showed that synchronization could be achieved in a pair of pump-modulated Nd:YAG (yttrium gallium garnet) lasers by altering the mutual evanescent coupling between the lasers. Sugawara et al. [5] demonstrated synchronization of two CO$_2$ lasers by injecting the output of a master laser into a receiver laser with a saturable absorber. Colet and Roy [6] have suggested a scheme involving the synchronization of a chaotic Nd:YAG carrier laser to a receiver laser, and the subsequent decoding of the hidden message in real time by subtraction of the receiver input from its output. The sharp pulses generated by loss modulation of the laser serve as a natural background for encoding and camouflaging digital information. The authors demonstrated the validity of their proposed scheme in numerical simulations. This idea has been extended to a model of synchronized chaotic semiconductor lasers by Mirasso et al., who have included the effects of a fiber optic channel on the information processing [7].

In this paper we explore issues related to the communication scheme as proposed in [6]. In Sec. II the loss-modulated Nd:YAG laser model is introduced along with techniques for encoding and decoding of messages. Next, in Sec. III we present an analysis of numerical and experimental time series via return maps based upon interspike intervals. We find that a simple relationship exists between an intensity peak and interspike intervals that precede and follow the peak. The consequences of this relationship on the issue of deciphering the message encrypted in the chaotic carrier is explored in Sec. IV. Two schemes for encoding information are then introduced that make it more difficult to decipher the message. These consist of laser parameter modulation of the transmitter to encode the message and the use of quasiperiodic parameter modulation in both transmitter and receiver, so that the interspike interval return maps become ineffectual as deciphering tools, while the receiver's ability to decode the message is retained. The main results of the paper are summarized in Sec. V, and conclusions are drawn.

II. SCHEME FOR COMMUNICATING WITH SYNCHRONIZED CHAOTIC LASERS

The scheme proposed by Colet and Roy [6] for communicating signals via chaotic synchronized lasers is composed of a pair of loss-modulated Nd:YAG lasers operated in the chaotic regime. The hidden signal is decoded by subtraction.
such as Fig. 1(b), which are recorded at slightly different parameter values. Although the numerical simulation and the experimental data appear similar with respect to the irregularity of the intensity maxima, the temporal sequence of intensity maxima appears more regularly spaced in the experimental data than in the numerical simulation. We will return to this point, and to a more detailed description of the output intensity, in Sec. III.

The equations describing the receiving laser in which the encoded signal from the transmitter laser has been injected are given by

$$\frac{dE_R}{dt} = \gamma_c (G_R - \alpha_0 T - \alpha_1 \cos \Omega t) E_R + i \omega_R E_R$$

$$+ \sqrt{\epsilon_R} \eta_R + \kappa A_b \gamma_c (E_R - \frac{A_S}{A_b} E_T), \quad (2.2a)$$

$$\frac{dG_R}{dt} = \gamma_T [P_R - G_R (1 + |E_R|^2)]. \quad (2.2b)$$

In the above equations all variables have the same meaning as for the transmitter. In addition, the modulated loss coefficient of the receiver, \(\alpha_R(t) = \alpha_0 R + \alpha_1 \cos(\Omega t)\), is operated at the condition for synchronization (i.e., \(E_R = E_T\)), \(\alpha_0 R = \alpha_0 T + \kappa\). The quantity \(\kappa \ll \alpha_0 T\) is the coupling coefficient between the transmitting and receiving laser which also accounts for any losses in the transmission process [9].

The transmission coefficients \(A\) in Eqs. (2.2) describe the encoding of the signal on the external output of the transmitting laser. The output intensity of the transmitting laser is slightly attenuated by an external filter by a fixed bias factor \(A_0\), so that the intensity at the receiver is given by \(E_T = \kappa A_b E_T\). This implies that synchronization is achieved when no signal is encoded at a setting of \(\alpha_0 R = \alpha_0 T + \kappa A_b\). To encode a "1" bit, the transmission is increased a few percent to \(A_S > A_b\), while to send a "-1" bit the transmission is decreased a few percent to \(A_S < A_b\). Thus the message is encoded as small amplitude modulations of the high-intensity output of the transmitting laser. To avoid encoding signals on the low intensity pulses, the pulses are monitored before attenuation and only those pulses whose intensity exceeds some predetermined, fixed threshold intensity are used for encoding.

In Eq. (2.2a) the signal difference term can be written as

$$-\kappa A_b \gamma_c (E_R - \frac{A_S}{A_b} E_T) = -\kappa A_b \gamma_c (E_R - E_T)$$

$$+ \kappa \gamma_c (A_S - A_b) E_T. \quad (2.3)$$

The first term \(-\kappa A_b \gamma_c (E_R - E_T)\) is responsible for the synchronization of the transmitter laser to the receiver laser. For values of \(\kappa\) above some threshold, the damping is sufficient to drive the signal difference to zero, thereby synchronizing the receiver to the chaotic transmitter carrier wave. If \(\gamma_c (A_S - A_b) E_T\) is small, the transmitter carrier wave plus signal is then entrained by the receiver laser. The signal can be deciphered precisely because it is a small perturbation of the carrier signal. As long as the \(\kappa\) is above some threshold value (which usually needs to be found empirically, see Fig. 1(a) and in similar experimental data of the receiver and transmitter intensities.

The model for the transmitting laser is described by [6,9]

$$\frac{dE_T}{dt} = \gamma_c (G_T - \alpha_0 T - \alpha_1 \cos \Omega t) E_T + i \omega_T E_T + \sqrt{\epsilon_T} \eta_T,$$

$$\frac{dG_T}{dt} = \gamma_T [P_T - G_T (1 + |E_T|^2)], \quad (2.1b)$$

where \(E_T\) is the complex, slowly varying amplitude of the electric field, \(G_T\) is the gain of the active medium, \(\tau_c = 1/\gamma_c = 450 \text{ ps}\) is the cavity round-trip time, \(\tau_f = 1/\gamma_f = 240 \mu s\) is the decay time of the upper lasing level, \(\omega_T\) is the detuning of the laser frequency from the nearest empty cavity mode, \(P_T\) is the pump parameter, \(\epsilon_T\) is the spontaneous emission noise strength, and \(\eta_T\) is a complex Gaussian white noise term of zero mean and correlation \(\langle \eta_T(t) \eta_T^{*}(t') \rangle = 2 \delta(t-t')\). The loss modulation is given by \(\alpha_T(t) = \alpha_0 + \alpha_1 \cos(\Omega t)\) where \(\alpha_1 / \alpha_0 \ll 1\). The modulation frequency \(\Omega\) is chosen to be close to a submultiple of the relaxation frequency \(\omega = \sqrt{2 \gamma_c \gamma_T (P_T - \alpha_0 T)}\). The output intensity of the chaotic laser is a series of irregularly spaced pulses having a spiky appearance, as evidenced in the numerical simulation of a loss-modulated solid-state laser in Fig. 1(a) and in similar experimental data.
rather on a combination of the intensity maxima with the
time intervals between intensity maxima, which we call the
interspike intervals (ISI). In a recent paper, Sauer [10]
proposed the use of interspike intervals as a means for attractor
reconstruction from time series, in analogy with Takens’
thorom [11]. In this work we use the interspike intervals to
find a useful relationship between the ISI and the intensity
maxima of a chaotic loss-modulated solid-state laser.

Figure 2(b) is a plot of numerically generated transmitter
laser intensity maxima \( I_T(n) \) of Fig. 1(a) occurring at time
\( t(n) \) versus the pair of interspike intervals \( \Delta t_T(n+1) = I_T(n+1) - I_T(n) \),
where \( \Delta t_T(n+1) = I_T(n+1) - I_T(n) \). Here \( \Delta t_T(n+1) \) is the time between the
nth intensity maxima \( I_T(n) \) at time \( t_T(n) \) and the occurrence of the
next intensity maxima at time \( t_T(n+1) \). Similarly, \( \Delta t_T(n) \) is the time
between the nth intensity maxima \( I_T(n) \) at time \( t_T(n) \) and the
previous intensity maxima at time \( t_T(n-1) \). A reconstruction
of the attractor solely using interspike intervals, i.e.,
\( \Delta t_T(n+1) \) versus \( \Delta t_T(n) \) and \( \Delta t_T(n-1) \) reveals no added
information over a reconstruction solely using intensity
maxima \( I_T(n+1) \) versus \( I(n) \) and \( I(n-1) \). It is the
combination of intensity maxima and interspike intervals as shown
in Fig. 2(b) which uncovers structure, and a relationship between
physical quantities.

Figure 2(b) shows results of the numerical simulation
with noise ("++") and without the inclusion of noise (diamonds).
The level of noise was chosen to be typical of that experienced in
laboratory experiments (see Fig. 1 in [6]).
Both the noise-free and noisy maxima fall on a nearly two-
dimensional surface that is essentially planar. Figure 2(a)
shows the intensity maxima of the noise-free simulation,
while Fig. 3(b) shows the same plot from an edge-on view,
that is \( I_T(n) \) versus

\[
\cos \theta = \frac{-\Delta t_T(n+1) + \Delta t_T(n)}{\sqrt{2}}
\]

\[
= \frac{\Delta t_T(n+1) - \Delta t_T(n-1)}{\sqrt{2}}
\]

where the angle \( \theta = \pi/4 \) gave the optimal view. Here we
clearly see an almost one dimensional structure of the return
map. Figure 3(c) shows the noisy simulation with the similar
nearly one-dimensional structure in the return map, viewed
edge on in Fig. 3(d). Note that the intensity maxima of the
noisy simulation Fig. 3(c), fall on the same nearly two-
dimensional surface of Fig. 3(a), but in that portion of the
surface corresponding to lower intensity maxima. We return
to this point shortly.

Figure 3(e) is an edge-on plot of the experimental data
(diamonds) of Fig. 1(b). Because the experimental data was
taken at parameter values slightly different from that of the
numerical simulation, the temporal variation of the interspike
intervals in Fig. 1(b) is on a finer scale. In fact, the data in
Fig. 3(e) corresponds to the upper-right-hand, high intensity
maxima corner of the data of the numerical simulation, Fig. 3(c).
However, even for this more uniform variation of the
interspike intervals, a plot of the intensity maxima- ISI return
map reveals structure and a relationship between physical
variables. The experimental data is again essentially planar
as evidenced by a global least squares fit of the experimental
intensity maxima to the experimentally derived interspike

\[
M = \int_{\text{pulse}} (|A_5 E_T|^2 - |A_6 E_R|^2) dt.
\]

(2.4)

The quantity \( M \) will equal zero when no signal is sent,
\( A_5 = A_6 \), and will have a strong positive (negative) value
when a "1" ("-1") bit, \( A_5 > A_6 \) (\( A_5 < A_6 \)), is being sent.
Figure 2(a) shows a first return map of the numerically
generated receive intensity \( I_R(n+1) \) vs \( I_R(n) \), evaluated at the
intensity maxima, when a signal has been encoded on the
transmitter. The carrier wave maxima, and the "1" and
"-1" bit are depicted by the diamonds, pluses, and squares,
respectively. One sees that the encoded signal is seemingly
inextricably mixed with the carrier wave. Higher
dimensional intensity peak return maps, \( I(n+1) \) versus
\( \{I(n), I(n-1), I(n-2), \ldots \} \), offer no additional help
towards unraveling the signal from the carrier [8].

III. ANALYSIS VIA INTERSPIKE INTERVALS

A useful representation of the data occurs when one
considers return maps based not solely on intensity maxima, but
FIG. 3. Intensity-interspike intervals return maps of the transmitter laser: (a) $I_T(n) \ vs \ \Delta I_T(n+1) = I_T(n+1) - I_T(n)$ and $\Delta I_T(n) = I_T(n) - I_T(n-1)$ corresponding to Fig. 1(a); (b) edge-on view of (a); (c) part (a), with noise, $\varepsilon_T = 8.33 \times 10^{-9}$ s; (d) edge-on view of part (c); (e) edge-on view of experimental data (diamonds), Fig. 1(b), with global least square fit ("+" s) of intensity maxima $I_T(n)$ to $\Delta I_T(n+1)$ and $\Delta I_T(n)$. Note that this data corresponds to the upper right-hand corner (15–20 $\mu$s) of (c).

Intervals $\Delta I_T(n+1)$ and $\Delta I_T(n)$. The plane of this least square fit of intensity maxima to ISI is shown in an edge-on view as the overlaid heavy straight line (" + ") in the center of Fig. 3(e). One should note that it is not important in plotting these return maps to utilize the precise maxima of the intensity, which may be hard to resolve in an actual experiment. Any convenient threshold value on the intensity spike could be used to replace $I_T(n)$, and $\Delta I_T(n)$ would then be measured as the time between successive crossings of this threshold.

If one were to plot a three-dimensional return map of the interspike intervals alone, i.e., $\Delta I_T(n+1) \ vs \ \Delta I_T(n)$ and $\Delta I_T(n-1)$, the result would be an unfolding of the attractor, topologically equivalent to an unfolding utilizing only the intensity maxima, $I(n+1)$ vs $I(n)$ and $I(n-1)$. The significance of the nearly planar (linear) structure of the intensity-ISI return maps in Fig. 3(a) [Figs. 3(b) and 3(e)] is that it implies that there exists a nearly linear relationship between the intensity maxima $I_T(n)$ and the interspike intervals $\Delta I_T(n+1)$ and $\Delta I_T(n)$.

Schwartz and Erneux [12] explored this loss-modulated laser system and found explicit representations for the Poincaré mapping between the (dimensionless) gain and the ISI applicable to the period 1 and 2 orbits. Though it is not the goal of this paper, Figs. 3 suggest that such a map might be found also in the chaotic regime. They analyzed the periodi-
FIG. 4. Plot of scaled transmitter intensity $y(t) = \frac{(I(t) - I_{ss})}{I_{ss}}$ vs the normalized gain $x(t)$ from Eq. (2.1): (a) no noise, (b) with noise. $I_{ss}$ is the steady-state intensity of the conservative system, which to lowest order approximates the system of equations in Eq. (2.1). The large intensity maxima occur when the previous intensity minima reach very low values.

FIG. 5. Decoding of the signal hidden in the chaotic carrier of the transmitter laser by the return map of $I_S(n)$ vs $I_{SS}(n+1) = \frac{I_S(n+1) - I_S(n-1)}{\sqrt{2}}$. The transmitter laser has been modulated with $A_S = 1.0 + 0.15A_b$, with $A_b = 0.85$: Upper branch $\rightarrow$ "1" bit, middle branch $\rightarrow$ no signal, lower branch $\rightarrow$ "-1" bit. (a) no noise, (b) with noise.

IV. CONSEQUENCES FOR CHAOTIC COMMUNICATION

The regular structure of the intensity versus ISI return maps has important implications for communicating signals via a chaotic transmitter laser. As proposed by Colet and Roy [6], the transmitter laser encodes the signal by an amplitude modulation external to the laser. Since the transmitter is not intrinsically perturbed, Figs. 3 suggests that the intensity maxima- ISI return maps applied to the transmitter signal alone could be used to decode the signal. Figure 5(a) is a plot of the simulated transmitter intensity output maxima $I_{SS}(n+1)$ vs the ISI combination $I_S(n+1) - I_S(n-1)/\sqrt{2}$, when the laser has been modulated with $A_S = 1.0 + 0.15A_b$ with $A_b = 0.85$, without the inclusion of noise. Signals were encoded only on intensity maxima with values approximately greater than 10. The middle branch is the no-signal maxima $A_S = A_b$, while the upper branch corresponds to a "1" bit $A_S = 1.15A_b$, and the lower branch corresponds to a "-1" bit $A_S = 0.85A_b$. Figure 5(b) is the corresponding simulation when noise has been included using a value of $\epsilon_T = \epsilon_S = 8.33 \times 10^{-9}$ s$^{-1}$, typical of actual experiments [6.9]. Even though the noise smear the branches out some-
FIG. 6. The return map of the transmitter laser, $r_i(n)$ vs $[r_i(n+1) - r_i(n-1)]/\sqrt{2}$, when the transmitter laser pump has been modulated with $P_s = 1.0 \pm 0.50 P_r$ (no noise). Compared with Fig. 5(a), the signal branches corresponding to the encoded bit $(1, -1)$ have been merged with the no-signal branch.

what, they are still clearly distinguishable. Plots at 5% encoding modulation for $A_b = 0.9$ show similar behavior of clearly distinguishable signal branches.

The signals embedded in the chaotic output of the transmitting laser were decipherable because of the inherent relationship between intensity maxima and interspike intervals exhibited in the intensity-ISI return maps. This encoding scheme, in which the transmitter laser intensity was modulated outside of the laser cavity, did not dynamically alter the relationship between intensity maxima and interspike intervals. Therefore, to encode and hide signals on the chaotic transmitter carrier, we suggest that it would be more advantageous to have the encoding method fundamentally perturb the ISI. This can be achieved by modulating the transmitter pump across an intensity peak. The actual beginning and ending of the modulation could occur in the intensity troughs, as long as the pump change persists over the intensity peak. In Figure 6 we have simulated the intensity output of the transmitter laser when its pump has been modulated with $P_s = 1.0 \pm 0.50 P_r$ and without the inclusion of noise. The resulting edge-on view of intensity-ISI return map for the transmitter shows that the attractor surfaces have been essentially merged together onto the no-signal surface. Even if we were to look at the logarithmic signal differences, our success rate of distinguishing a "+1" bit from no-signal via the transmitter return maps is greatly diminished due to the severe overlapping of the surfaces. For smaller modulations or with the inclusion of noise, things only become more difficult. However, the signal can be decoded when the transmitting laser output is synchronized to the receiver and the integrated signal difference of Eq. (2.4) is utilized.

We point out an interesting feature of this encoding scheme that differs from the original encoding scheme in which the laser intensity is externally modulated. Figure 7(a) shows the decoded message bits (solid line) when the external intensity is used to encode the signal. The dashed line in this figure is the value of the discrete bits "+1," encoded on the transmitter laser. Although it is drawn as a continuous line, the value of the encoded bit only has meaning across the intensity peak. In these figures we plot $M(t) = |E_r|^2 - |A_b E_r|^2$ vs $t$, where $|E_r|^2$ is the intensity of the modulated transmitter laser at the receiver laser. Positive values of $M(t)$ can be associated with a transmitted "1" bit and negative values of $M(t)$ can be associated with a "0" bit. Figure 7(b) shows a decoded message bit when the pump is modulated by $\pm 50\%$ to encode the signal. The signal is again decoded by the integrated signal difference $M(t)$ and the dashed line in this figure is the value of the discrete bits "+1" encoded on the transmitter laser. The first two "blips" ($r < 490$) in Fig. 7(b) represent no signal encoded. The coding of signals consisting of random values of "+1" beginning at $r > 490$. Note that $M(t)$ in Fig. 7(b) is neither all positive nor all negative as is essentially the case when the signal is encoded by modulating $|E_r|^2$ outside the laser, such as in Fig. 7(a). Figure 7(c) shows a magnified view of the region $525 < r < 535$ of Fig. 7(b) where a "1" (left pulse) and "1" bit were decoded from the signal. For the "1" bit the positive area is slightly larger than the negative area leading to an overall positive integrated area, while
FIG. 6. The return map of the transmitter laser, $I_r(n)$ vs $\left[ I_r(n+1) - I_r(n-1) \right] / \sqrt{2}$, when the transmitter laser pump has been modulated with $P_p = 1.0 \pm 0.5P_r$, (no noise). Compared with Fig. 5(a), the signal branches corresponding to the encoded bits $(1, -1)$ have been merged with the no-signal branch.

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the reverse is true for the "+1" bit. Again, positive values of \( M(t) \) can be associated with a "1" bit encoded on the transmitter laser and negative values of \( M(t) \) can be associated with a "−1" bit. Net magnitudes of \( M(t) \) clustered around zero can be interpreted as no signal sent. For the case of encoding the signal by pump modulation, the receiver is synchronized to the modulated transmitter laser in all regions outside of the area of the receiver intensity peak. Under the peak the transmitter and receiver intensities are slightly out of synchronization, yet still mutually entrained, with the receiver lagging the transmitter, leading to the double humped decoded signals in Figs. 7(b) and 7(c).

We note that when this pump modulation scheme is used to encode the signal onto the transmitting laser, an intensity return map \( I_T(n+1) \) vs \( I_T(n) \), is again useless in deciphering the hidden signal, having an appearance similar to that of Fig. 2(a). In addition, we also explored signal encoding with pump modulations of 10% and 90%. For both these modulation values, the intensity vs ISI return maps are similar in structure to Fig. 6, i.e., the encoded bit surfaces nearly coincide with the no-signal surface and they are all intertwined.

The signals again could be decoded by an integrated signal difference at the receiver. However, for weak modulations values (e.g., 10%) decoded bit values could occasionally be misinterpreted because the difference between the positive and negative areas in Fig. 7(b) was small enough that a signal could be interpreted as a nonsignal.

On the other hand, for stronger modulation values (e.g., 90%) the decoding would occasionally fail, and decoded bits would be interpreted incorrectly. These instances would occur when perturbations to the transmitter carrier were enough to make it sufficiently dissimilar to the receiver that entrainment was momentarily lost for that signal pulse. As discussed in Sec. II, the first term of Eq. (2.3) \(-\kappa A_T \gamma_1(E_R - E_T)\) is responsible for the synchronization of the transmitter laser to the receiver laser. If the second term \(\kappa \gamma_2(A_R - A_T)E_T\) is small with respect to the first term, the transmitter carrier wave plus signal can still be entrained by the receiver laser. However, for large pulse modulations this second term may, on occasion, not be small, and across this spiky peak entrainment is lost. In general, it appeared that intermediate values of the modulation (around 50%) produced the best results for reliably decoding the message at the receiver.

In a final numerical experiment, we explored the consequences of quasiperiodically modulating the loss coefficient of both the transmitter and receiver laser. The form of the modulation was modified to

\[
\alpha(t) = \alpha_0 + \alpha_1 \left[ \cos(\Omega t) + \alpha_2 \cos(f_2 \Omega t) + \alpha_3 \cos(f_3 \Omega t) \right],
\]

where the amplitudes \(\{\alpha_2, \alpha_3\}\) and frequencies multipliers \(\{f_2, f_3\}\) are fixed, but arbitrarily chosen constants. Again the receiver was operated at conditions for optimal synchronization \(\alpha_{0R} = \alpha_{0T} + \kappa A_R\), and noise (typical for these lasers) was included in the calculations.

When a single additional frequency was used, \(\alpha_2 \neq 0, \alpha_3 = 0\) and \(f_2 \neq 0, f_3 = 0\), the branches of the two-dimensional intensity-ISI return maps [as in Fig. 5(b)] thickened and merged as the amplitude \(\alpha_2\) approached unity. This thickening and merging effect was pronounced when two additional frequencies were utilized. Figure 8(a) shows the intensity vs ISI return map for the transmitter laser for the case \(\alpha_2 = \alpha_3 = 1\), with a choice of incommensurate relative frequencies \(f_2 = \sqrt{2}\) and \(f_3 = (\sqrt{2} - 1)/2\). Quasiperiodic driving led to an increase in the dimensionality of the attractor, exhibited by the thickening and merging of the intensity vs ISI map in Fig. 8(a). This renders the intensity vs ISI map ineffective for deciphering the hidden message from the transmitter laser alone. The effect was qualitatively the same when additional incommensurate frequencies were added to the driving. However, when utilizing incommensurate frequencies, the remnants of the separate attractor surfaces for the encoded bits [as in Fig. 5(b)] could be inferred, if barely. However, the surfaces were thickened and merged enough, as in Fig. 8(a), to render the intensity vs ISI map ineffective as a deciphering tool.

With no signal encoded, the transmitter synchronized effortlessly to the receiver laser. When the signal was encoded by amplitude modulation external to the transmitter laser [as in (6)], the signal could be decoded at the receiver laser by...
means of an integrated signal difference, as evidenced in Fig. 8(b). Occasionally there were misinterpretations of the decoded bits for reasons similar to those discussed above for the case of encoding with pump modulations. We purposely increased the dimension of the attractor by adding more driving frequencies of arbitrary amplitude. Therefore, when a signal is impressed upon the transmitter carrier wave, it is occasionally different enough from the receiver signal so that the second term on the right-hand side of Eq. (4.1) perturbs the system enough so that entrainment is lost for this signal peak. The details of the modification of the local Lyapunov spectrum in the presence of multiple driving frequencies was not investigated, but would make for an interesting topic of exploration.

V. SUMMARY AND CONCLUSIONS

We have investigated the chaotic loss-modulated Nd:YAG laser and have found, both numerically and experimentally, that a return map utilizing intensity maxima and interspike intervals (ISI) reveals a regular, almost planar structure. This observation indicates that a simple relationship exists between the intensity maxima and the interspike intervals centered about that maxima, i.e.,

\[ I(n) = g[I(n+1), I(n-1)] \]

In fact, by plotting the \( n \)th intensity maxima \( I(n) \) versus the difference between the preceding and preceding interspike intervals, i.e.,

\[ (n+1) - t(n) - (n-1) - t(n) \]

we observe a nearly one-dimensional, one-to-one relationship between these variables, even in the presence of noise. This relationship was observed in numerical simulations as well as in experimental data taken at slightly different parameter values, leading to a variation of the interspike intervals on a much finer scale. However, even in this latter case, a plot of the intensity maxima-ISI return map reveals an almost planar structure and therefore a relationship between physical variables. Such a result would be useful, for example, in time series prediction of the future intensity maxima. In constructing the intensity-ISI return map it was not essential that the peak of the intensity be utilized. The ISI could have been defined relative to some arbitrary threshold value under the region of the peak and the return map then reconstructed.

The relationship between the intensity maxima of the laser and the interspike intervals has consequences for the use of a transmitter-receiver pair of chaotic loss-modulated Nd:YAG lasers as a system to transmit encoded messages privately. By plotting the intensity-ISI return map of the transmitter laser alone, the message of "\( \pm 1 \)" bits, encoded by means of external cavity modulation, appears on surfaces above and below the no-signal surface. Even in the presence of moderate noise, the message can be deciphered.

As an alternative encoding scheme, we suggest encoding the signal by modulating the pump across the intensity maxima. This intrinsically perturbs the ISI return maps of the transmitter laser alone. The subsequent attempt to decode the embedded message by means of intensity-ISI return maps of the transmitter laser alone is unsuccessful because the signal attractor surfaces are merged onto the no-signal attractor surface. However, the message can still be decoded by means of driving the receiver laser with the output of the transmitter laser and extracting the message from an integrated intensity difference.

In addition, quasiperiodic driving of the loss coefficient of both the transmitter and receiver laser produced an increase in the dimensionality of the system. This led to a thickening of the intensity-ISI return maps with the merging of the individual surfaces corresponding to the \( 1, 0, -1 \) encoded bits. This rendered the intensity-ISI return maps ineffectual as means to decipher the signal from the transmitter laser alone. However, the signal could once again be extracted by means of an integrated signal difference at a receiver laser synchronized to the transmitter carrier wave.

Finally, the lessons learned in this study are twofold.

First, an intensity-ISI or purely ISI return map can be a useful tool in the study of a pair of loss-modulated Nd:YAG lasers because of the implicit relationship between the intensity peak to the interspike intervals centered about that peak. Second, as applied to chaotic communications, the intensity-ISI return maps can be used to decipher the hidden message from the transmitter carrier wave alone. Care must be taken to intrinsically perturb the system or to increase the dimensionality of the system (though not high enough to void synchronization) so that the signal is safe from undesired deciphering by means of mapping techniques.

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Dynamical Evolution of Multiple Four-Wave-Mixing Processes
in an Optical Fiber

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Abstract

We present unique results of detailed experimental and theoretical investigations of the dynamical evolution of four wave mixing spectra in an optical fiber. The experimental measurements probe the evolution of sidebands generated through four wave mixing as they co-propagate with the pumps along the fiber. We find that standard theoretical models are inadequate to predict the experimental results and that it is necessary to modify the approach to modeling the dynamics in two ways. The first modification is to include a pump laser input with multiple longitudinal modes. This reflects the fact that the pump laser fields may actually have internal structure that is not resolved by the spectrometer used and which is very small compared to the spacing of the central frequencies of the pump fields. Yet the evolution of the fields is dramatically altered for the sidebands generated by nonlinear processes in the fiber medium. The second is the inclusion of phase noise added along the propagation length; this causes damping of the sideband oscillations. These two modifications lead to excellent agreement of the measurements with numerical predictions of the sideband evolution.
I. Introduction

The study of wave propagation in a nonlinear dispersive medium, such as an optical fiber, is of interest in many areas of science and engineering. The past few decades have seen enormous growth in the use of optical fibers in communications systems. With this growth, engineers and researchers have been challenged with a wide range of physical phenomena associated with high intensity light waves propagating in optical fibers. Specifically, some of the interesting characteristics of silica glass, of which fibers are made, are low loss, dispersion, and especially nonlinearity. Since optical fibers have a relatively small cross section, a comparatively small amount of power is required to generate high intensities; thus, many nonlinear optical processes are easily observed in the medium [1].

Some of the earliest work in nonlinear fiber optics consisted of both experimental and theoretical investigations of such effects as stimulated Brillouin and Raman scattering [2]. This work stimulated the expansion of research to other nonlinear phenomena, such as four-wave-mixing [3], optically induced birefringence [4], self-phase modulation [5], and cross-phase modulation [6]. Advances in communications technology came when researchers realized that the nonlinearity in optical fibers could be exploited. In 1973, Hasegawa suggested that optical fibers would support soliton pulses in which the nonlinear effects balance the effects of dispersion [7]. Shortly thereafter, optical solitons were experimentally observed [8]. Technologies using solitons are promising for high bit rate optical communication systems [9]. Nonlinear fiber optics has found many uses beyond communications systems; for example, pulse compression [10] and sensor devices [11].

Until recently, communications systems using optical fibers supported one communication channel per fiber. To increase the information capacity of communications systems, engineers
have turned to wavelength division multiplexed (WDM) systems in which each communication channel is represented by a unique wavelength. The dominant nonlinear process which limits the information capacity of a WDM system is four-wave-mixing. The parameters that set this limit are the power coupled in the fiber and the frequency spacing between adjacent channels.

Nonlinear fiber optics is not only relevant to telecommunications; it is also of great interest in mathematics and physics. The equation which governs wave propagation in a single-mode optical fiber is a nonlinear second-order partial differential equation (the nonlinear Schrodinger equation). This particular equation has been studied extensively for its mathematical properties, for example, its analytic solutions give rise to the possibility of soliton propagation [12]. The nonlinear dynamics accessible in optical fibers is rich and varied and makes an excellent experimental system for the study of many nonlinear phenomena.

In this paper, the nonlinear dynamics of four-wave-mixing processes resulting from two waves copropagating in an optical fiber is investigated. Multiple waves at different frequencies copropagating in an optical fiber can interact through the nonlinear susceptibility of the fiber medium to generate new frequencies, sidebands, through four-wave-mixing (FWM). Two pump waves at $\omega_1$ and $\omega_2$ input to an optical fiber can generate first order sidebands at frequencies $\omega_3=2\omega_1 - \omega_2$ and $\omega_4=2\omega_2 - \omega_1$. Second order sidebands are found at $\omega_5=2\omega_3 - \omega_4$ and $\omega_6=2\omega_4 - \omega_3$. The number of sidebands generated is determined by the input power and frequency separation between the pumps, e.g. higher order sidebands may easily be generated by either increasing the pump power or decreasing the pump detuning.

We present detailed studies of the dynamical evolution of sidebands, generated from two input pump waves at $\omega_1$ and $\omega_2$, as they propagate along an optical fiber. Previous numerical studies
have shown that two critical parameters, the pump power and the frequency separation (detuning) between the pump waves, determine the dynamical evolution of power in the sidebands and the number of sidebands generated in a particular length [13]. Previous theoretical studies have shown interesting and sometimes complex dynamical evolution of the sidebands with length in the fiber [14,15]. Section II reviews the nonlinear dynamical equations used to study the evolution of FWM processes in the optical fiber. There were two sets of equations used throughout this research to model the system; the nonlinear Schrodinger equation (NLSE) [1] and a set of coupled amplitude equations derived from the NLSE [13]. Numerical simulations based on these models that show the sensitivity of the sideband dynamics on the input pump power and frequency detuning are presented. These simulations motivated the initial choice of parameter regimes to investigate in this research.

A unique set of experimental measurements of multiple FWM processes along an optical fiber were performed for this research. The experimental apparatus used to conduct the measurements is presented in Section III. The key elements of the system were two tunable dye lasers which were pumped by a frequency doubled Nd:YAG (neodymium doped yittrium aluminum garnet) laser, polarization maintaining optical fiber supplied by AT&T Bell Labs, a spectrometer, and a high resolution, low noise CCD (charge coupled device) camera supplied by Georgia Tech Research Institute (GTRI). The GTRI CCD camera was a critical instrument in the experiments. Standard CCD cameras would have been inadequate to detect very weak sidebands; the regime most of the experiments probed. The experimental results presented here are unique in two ways; first, the GTRI CCD camera allowed for detection of weak (<1% of the pump waves) sidebands and second, these are the only detailed measurements tracing the dynamical evolution of the sidebands along a fiber in existence at this time.
Section IV presents the experimental investigation of the dynamical evolution of multiple FWM processes in an optical fiber. Measurements tracing the power in the sidebands along a length of 50 meters of fiber are presented. These measurements were done at two input pump powers which yield different sideband dynamics. The power in the sidebands was observed to evolve periodically with fiber length. However, the periodic evolution appears to damp to a constant value of power for each sideband. Furthermore, each of the sidebands evolves along the fiber with different dynamics. Other studies in which the pump power was varied for a fixed length of fiber are presented as well. The initial growth of the sidebands in the first 5 meters of fiber was found to be fairly well predicted by the standard theoretical models. However, for longer lengths, the inadequacy of the models to predict the experimental observations is apparent.

Section V discusses the interpretations of the experimental results. To understand the measurements, the theoretical models are modified by including two effects previously not considered; a pump input with multiple longitudinal modes and phase fluctuations added to the waves as they propagate along the fiber length. The impact of a multimode input is examined and found to dramatically alter the dynamical evolution of the individual sidebands when compared with the standard theory using a single mode pump input. Weak stochastic phase perturbations, added to the copropagating waves are also included in the modeling and found to damp the periodic evolution of the power in the sidebands. Neither the relatively straightforward multimode input nor the phase fluctuations which were not so obvious have ever been considered when modeling multiple wave propagation in an optical fiber. Both effects are found critical to understanding and predicting the dynamics of the experimentally observed sideband evolution. This research has probed a very specific regime of a complex nonlinear system. The experimental research pointed to several inadequacies of the standard theoretical models to predict the experimental results. Section VI summarizes the conclusions of this research.
II. Theoretical Considerations

Propagation of optical pulses in single mode optical fibers is described by the well-known nonlinear Schrödinger equation [1]:

$$\frac{\partial U}{\partial Z} + \frac{\beta^{(2)}}{T_0^2} \frac{\partial^2 U}{\partial \tau^2} = i\gamma |U|^2 U \quad (1)$$

where $U$ is the complex electric field envelope normalized to the absolute amplitude of the field $\sqrt{P_0}$, $P_0$ is the total power in the fiber, $\tau$ is time normalized to the pulse width and measured in a reference frame moving with the group velocity of the pulse ($\tau = (t-z/v_g)/T_0$), $T_0$ is the pulse width, $\beta^{(2)}$ is the group velocity dispersion (GVD) and is given by the second order derivative of $\beta$, the axial wavevector, with respect to the angular frequency $\omega_{\text{ave}}$. The nonlinearity coefficient $\gamma$ is given by the relationship,

$$\gamma = \frac{\omega_{\text{ave}} n_2}{c A_{\text{eff}}} \quad (2)$$

where $A_{\text{eff}}$ is the effective core area of the fiber determined by the size of the fundamental mode, $n_2$ is the Kerr coefficient for the intensity dependent refractive index and $\omega_{\text{ave}}$ is the average angular frequency of the wave envelope [1].

In order to obtain the nonlinear Schrödinger equation (eqn. (1)), several assumptions are made. One assumption is an instantaneous nonlinear response of the medium. This is valid for pulses longer than 100 femtoseconds since the third order susceptibility of the medium, $\chi^{(3)}$, has electronic contributions on the 1 to 10 femtosecond timescale [1]. The experimental research used relatively long pulses ~5 ns. The slowly varying envelope approximation is also used where
the second order derivative of the field with respect to the length is neglected. This assumes that the change in slope of the field envelope over a distance of one wavelength is small compared with the slope of the field envelope itself. The optical field is assumed to maintain its polarization along the fiber, thus the scalar form of the NLSE (eqn. (1)). This is justified for the experiments presented here, since linearly polarized light from the lasers was propagated with the polarization aligned along one of the principal axes of a polarization preserving fiber (Section III). The axial wavevector, $\beta(\omega)$, is approximated by a Taylor series expansion. For wavelengths near the zero dispersion regime ($\lambda \sim 1.3$ microns), where $\beta^{(2)}$ approaches zero, higher order terms from the Taylor series need to be included. The experiments in this research were performed in the visible regime ($\lambda \sim 633$ nm) thus only terms up to $\beta^{(2)}$ were retained. The linear fiber loss is also assumed negligible. This is justified for the wavelength regime and fiber lengths ($L < 50$ m) investigated, since the loss is approximately $6 \text{ dB/km}$ at ($\lambda \sim 633$ nm) which amounts to $< 1\%$ loss over 50 meters.

There are two wavelength regimes of interest in optical fibers; the anomalous dispersion ($\lambda > \lambda_o$) and the normal dispersion ($\lambda < \lambda_o$) regimes where the zero dispersion wavelength $\lambda_o$ can range from $1.3\mu\text{m}$ to $1.58\mu\text{m}$. The experiments presented here were performed in the normal dispersion regime. However, the integrability of the NLSE gives rise to interesting solutions in the form of solitons in both regimes. Soliton propagation occurs when the fiber nonlinearity balances the effect of dispersion and the pulse propagates without dispersive broadening. In the anomalous dispersion regime ($\beta^{(2)} < 0$) the fundamental soliton solution of the NLSE is in the form of hyperbolic secant pulses [1,12]. In the normal dispersion regime ($\beta^{(2)} > 0$) the fundamental soliton solution is in the form of a hyperbolic tangent, giving rise to dark solitons or dips in a continuous wave background [1,16]. In the context of the experiments presented, in the normal
dispersion regime with finite width pulses, a carrier pulse of finite width may support relatively stable propagation of dark pulses for short distances [16]. These are not 'proper' dark solitons however; the distance of stable propagation decreases with decreasing carrier width.

The split step Fourier method (SSFM), a pseudospectral technique, was used in this research [17]. Specifically, a symmetrized form of the SSFM was used [1], and the fast Fourier transform (FFT) routines were obtained from the IMSL mathematical libraries. An advantage of using the NLSE in the four wave mixing problem, is that integration is reduced to using the FFT. Modeling four wave mixing processes, for example, with a dual frequency input, the total complex field is represented by \( U \), the field envelope, and all frequency components are propagated using the single NLSE. However, care must be taken under conditions where many orders of sidebands are generated. As the number of sidebands increases, the size of the FFT must necessarily be increased.

For long pulses or continuous wave input, assuming monochromatic waves, the coupled amplitude equations for the pump waves and sidebands derived from the wave equation [13] are written below, normalizing all of the complex field amplitudes to the absolute value of the total amplitude of the pump with average frequency \( \omega_{ave} \) (which has total power \( P_0 \) at the input end of the fiber)

\[
\frac{dU_j}{dz} = i\gamma P_0 \left\{ U_j \left( 1 + \sum_{k \neq j} |U_k|^2 \right) U_j + \sum_{kmn} d_{kmn} U_k U_m U_n^* e^{i\varphi_{kmn}} \right\}
\]  

(3)

where \( j,k,m,n = 1,2,3,4... \) and \( k,m \neq n \). Here \( \Sigma_{kmn} \) denotes the permutations of the indices \( k, m \) and \( n \) such that \( \omega_k + \omega_n - \omega_m = \omega_j \), and the quantity \( \Delta \beta_{kmn} = \beta_k + \beta_m - \beta_n - \beta_j \) is the axial wavevector mismatch. The quantity \( d_{kmn} \) is a degeneracy factor that is unity when \( k = m \) and 2 when \( k \neq m \). The nonlinearity coefficient \( \gamma \) is given in equation (2). Comparing the coupled
amplitude equations (eqn. 3) with the NLSE (eqn. 1), the contributions to the evolution of the field $U_j$ are now separated into three sets of terms. On the right side of eqn. (3), from right to left, the contributions are due to self-phase modulation (SPM), cross-phase modulation (XPM) and four-wave-mixing (FWM).

The linear mismatches $\Delta \beta_{km}$ are simplified using the approximation that the material part of the index difference dominates the mismatch and the waveguiding contribution can be neglected. This approximation is justified for the frequency separations in these experiments, since the $v$-number characterizing the single transverse mode changes by less than 1 percent over the entire range of frequencies considered. By using the frequency relationships between the peaks and expanding the propagation constants $\beta_j$ in a Taylor series about average frequency, $\omega_{ave}$, all the mismatches are found to be integer multiples of the quantity $\Delta \kappa = \Omega^2 \beta^{(2)}$ where $\Omega$ is the frequency difference, or detuning, between the two pump waves and $\beta^{(2)}$ is the group velocity dispersion [18]. These amplitude equations can be solved numerically, and the power in each frequency component obtained as a function of distance along the fiber.

Choosing the scaled powers of the waves to be $\rho_m = i U_m^2$, then in reference [18] it was shown that the equations (3) display power conservation, as is expected. It was also shown that another conserved quantity

$$C(z) = (\rho_1(z) - \rho_2(z)) + (\rho_3(z) - \rho_4(z)) + (\rho_5(z) - \rho_6(z))$$

(4)

is obtained for the multiple four wave mixing processes that occur within the fiber. It was shown in [18] that the conservation of power and equation (4) are the only two conservation relations that involve linear combinations of the powers in the different frequency components. This
relation holds at any distance, $z$, of propagation in the nonlinear medium, and connects the asymmetries of the pump waves and sidebands. A more generalized form of equation (4) has been derived from the NLSE and is presented in reference [19]. Equation (4) has been verified experimentally for relatively short fiber lengths of less than 2 meters (see ref. [19]). The conservation of asymmetry (eqn. (4)) was used in all of the experimental measurements as a sensitive test for other competing processes not included in the models, for example, stimulated Raman scattering.

The initial parameter regime chosen for the experiments came from numerical simulations of the equations presented above. The dynamics of these equations for multiple waves copropagating in a fiber have been studied numerically for long fiber lengths [14,20]; however, experimental work has been limited to a few meters [13,15]. As the sidebands evolve along the fiber, there is exchange of power between the pumps and sidebands, the dynamics of which, are determined by the phase mismatch between the copropagating waves. The two key experimental parameters, for a given optical fiber and wavelength regime, that determine the dynamics of the power exchange are the pump detuning and the total input power. This can be seen in eqns. (3) where all terms on the right side of the equations are multiplied by $P_0$ and the FWM terms include oscillating terms with the argument proportional to $\Omega^2$. The FWM strength and dynamics are very sensitive to the pump detuning as well as $\gamma$ and $\beta^{(2)}$. The values used for $\beta^{(2)}$ and $\gamma$ are same throughout this research, $\beta^{(2)} = 55 \text{ ps}^2/\text{km}$ and $\gamma = 0.019 \text{ W}^{-1}\text{m}^{-1}$, and are consistent with the experimental regimes explored later in this paper.

To investigate the dependence of the evolution of the power in the first order sidebands ($\rho_3(z)$ and $\rho_4(z)$) on the pump power and detuning, the coupled amplitude equations are numerically solved using a fourth order Runge Kutta algorithm [21]. A comparison between the power in
the sidebands predicted by the NLSE and the coupled amplitude equations was made as a check on the numerical simulations. Simulations based on the NLSE with a continuous wave (CW) input and the coupled amplitude equations were performed and compared. The comparisons yielded the same predictions for the evolution of the power in the sidebands. However, comparison of the NLSE using a Gaussian pulse input with either the continuous wave input or the coupled amplitude equations showed a discrepancy between the models. The power generated in the first order sideband was found to be higher using the CW input than with the pulse input. It was found necessary to include a scale factor in the CW models, where \( \gamma P_0 \to \zeta \gamma P_0 \) with \( \zeta = 0.735 \). The value of \( \zeta \) was determined by comparing predictions from the NLSE for a Gaussian pulse input with a continuous wave input. Intuitively, as the pulse width approaches infinity, the CW and pulse inputs should agree. However, there is no analytic form for estimating this scale factor. Independent studies have also been performed comparing various pulse shapes input to the NLSE with the cw input, confirming the discrepancy between the two types of input.

The sensitivity of the sideband dynamics on the pump power is illustrated in Figure 1. The first order sideband evolution along 100 meters of fiber for a detuning of 300 GHz and different input pump powers is plotted in Figure 1, (a) 2 W, (b) 6 W and (c) 50 W. The coupled amplitude equations were truncated to six waves, including up to second order sidebands. In Figure 1 (a) the input power is low, generating relatively weak first order sidebands, and the pumps and sidebands exchange power periodically along the fiber. Using an undepleted pump approximation, eqns. (3) have an analytic solution which shows the power in the first order sidebands evolves as a sinusoid as a function of length [22]. In Figure 1 (b) and (c), as the pump input power is increased, higher order sidebands are generated and the power exchange between the pumps and sidebands becomes increasingly complex. In fact, the equations when truncated to include just a few orders of sidebands exhibit chaotic dynamics at high pump powers [13].
However, the NLSE is integrable, and does not exhibit chaos. In the case of the coupled amplitude equations the apparent chaos is induced by truncating the equations to include only a few frequency components [14].

Doubling the pump detuning to $\Omega = 600$ GHz, the phase mismatch is increased by a factor of four. As the phase mismatch increases the efficiency of power conversion from the pumps to the sidebands decreases. The evolution of the first order sidebands with length in the fiber for a detuning of 600 GHz is shown in Figure 2 with pump input power levels of (a) 2 W, (b) 6 W and (c) 50 W. Comparing the evolution with a pump input of 2 W, by doubling the detuning the maximum power in the first order sidebands is decreased by a factor of 10 (Figure 1 (a) and Figure 2 (a)). The period of the power exchange between the sidebands and pumps has also increased. In Figure 2(b) and (c), it takes much higher powers to generate higher order sidebands that impact the dynamics of the first order sidebands. Thus, increasing the detuning decreases the efficiency of the four wave mixing power conversion. Increasing the pump power increases the number of sidebands generated, and thus the dynamics becomes more complex.
III. Experimental Apparatus and Technique

The entire experimental set up used to study multiple four wave mixing processes along a length of optical fiber is shown in Figure 3. The laser system consists of two Littman type tunable dye lasers, pumped by the second harmonic of a Q-switched frequency doubled Nd:YAG laser. Pulses that are ~5 ns (FWHM) in length are generated. The outputs from the two dye lasers ($\lambda \sim 633$ nm) are amplified and then passed through the appropriate delays to ensure temporal overlap of the pulses at the input to the optical fiber. The telescope in the path of one laser controls the spot size and, thus, the coupling efficiency so that the relative power of the two lasers coupled into the fiber can be adjusted to the desired value. The two apertures ensure nearly colinear propagation of the two beams. The light is coupled into a single mode polarization maintaining optical fiber, after passage through a polarizer and half wave plate. The polarizer at the input to the fiber produces linearly polarized light while the half wave plate rotates the polarization of the light to coincide with a principal axis of the birefringent fiber. The fiber chosen for the experiments was developed by AT&T as an experimental fiber. The fiber is single mode at 633 nm and polarization maintaining. The AT&T fiber achieves high birefringence by deforming a circular fiber preform so that it is rectangular in shape, the cladding is elliptical, and the core is circular [23]. This fiber has a core diameter of 4 $\mu$m with a birefringence of $2.7 \times 10^{-4}$.

A beamsplitter cube, at the fiber output, is used to direct half of the light to an optical power meter to monitor the power in the pulses while the other half is input to a grating spectrometer. A computer controlled video camera is mounted on an output port of the spectrometer with a variable neutral density filter (VNDF) placed at the input port to regulate the amount of light incident on the camera. Spectra for individual pulses are digitized and stored in the memory of a microcomputer and a video monitor is used to display each spectrum.
For the experiments presented in this paper, images of the fiber output spectra were captured using a system based on advanced high speed, low noise, and high resolution charge coupled device (CCD) technology. The system uses a scientific CCD device developed by MIT Lincoln Laboratories [24]. The CCD device is backside illuminated with 420 x 420 pixels/frame. To increase the readout rate, there is a separate frame storage region which allows one image to be read as the next one is integrated. Each pixel has a dimension of 27 \( \mu \text{m} \times 27 \mu \text{m} \) with a full well depth or charge holding capacity of 100,000 electrons. Pixel nonuniformity has been measured to be 6% peak to peak for similar backside illuminated devices made by MIT Lincoln Laboratories [25]. The advantage of illuminating the CCD from the backside is that the quantum efficiency (QE) is high, for this device the peak QE is 90% near 600 nm. The CCD chip incorporates an on-chip readout amplifier which is the dominant source of noise in the device.

The camera system was built at Georgia Tech Research Institute (GTRI) for use in low light level astronomical imaging [26]. External to the CCD chip is a 14 bit A/D and controlling electronics for the CCD which run at a maximum rate of 1 Mpixel/sec. Using the full 420 x 420 array this translates to \(-5\) frames/second. The external electronics incorporate low noise design techniques such that the system noise is limited by the readout noise from the amplifier on the CCD chip. The CCD is liquid nitrogen cooled to -50°C, reducing the dark current to 0.04 electrons per pixel (at room temperature the dark current \(-700\) electrons per pixel). The minimum readout noise from the on-chip amplifier is 7.2 electrons per pixel rms at -50°C [26].

The camera system is controlled using a Macintosh computer running Labview control software. This software controls a Pulse Instruments PI5800A data generator. The PI5800A generates signals on 16 parallel programmable lines which control the camera. From the camera there is a fiber optic data link which transmits up to 8.3 MBytes per second. The data is then stored in a high speed 32 MByte ram buffer. From the buffer the data may be either stored on a high speed
video recorder which runs as fast as 4 MBytes per second, or, for small files it may be stored to a hard disk which is limited by the I/O of the computer system. Programs in Labview were developed with the capability to select a subarray at any location on the chip. For example, in the experiments presented here, a subarray of 10 x 256 pixels near the center of the chip was chosen. This decreases the size of required data storage and increases the maximum number of frames per second. In these experiments the frame rate is limited to the 10 Hz repetition rate of the Nd:YAG laser system. To achieve the slow rates the camera hardware is programmed to run at 10 Hz and a clock signal is generated which is used to control the laser system through the oscillator sync input. The resolution of the camera-spectrometer system is approximately 43 GHz and is limited by the resolution of the CCD.

The data acquisition method used in this research was to collect output spectra using the GTRI CCD. The pump lasers fluctuate from shot to shot. For statistical analysis, a total of many spectra for each data point are collected. Typically, 400 independent spectra are captured for each pump propagating alone in the fiber and the two pumps copropagating (FWM). The power in the individual sidebands is measured as a fraction of the total power, normalized to unity, in the fiber, and the total input power is determined based on measurements of each individual pump propagating alone in the fiber. Quantitative measurements are then made of the power in the pumps and sidebands, generated by FWM. Prior to data acquisition a set of "dark" frames (a set of frames with no light incident on the detector) is collected. An average "dark" frame is found and then subsequently used to remove the camera bias from the data frames using pixel by pixel subtraction. The power in each frequency component is distributed symmetrically about a central peak for that component. To calculate the power in the pumps and sidebands, we developed software to find the locations of the peaks in the spectrum and the power in each frequency component in two ways. The first is to take a linear cross-section along one row and integrate the power in each frequency component. The second is to integrate the power in the full
distribution for each component. The second method is insensitive to horizontal misalignment of the CCD detector with respect to the spectrometer. Both methods were employed in this research and agreed closely throughout. Quantitative values were obtained of the FWM pump and sideband power as well as the statistics. It will be seen later that the statistical information obtained played a crucial role in confirming the physical interpretation of the experiments.

The 14 bit dynamic range of the camera system allows for weak FWM signals to be detected. A typical linear cross-section of a FWM spectrum is shown in Figure 4 (a) linear scale and (b) logarithmic scale. The spectrum is plotted first on a linear scale which is comparable to the type of spectrum that would be obtained from a standard 8 bit video camera. The uniqueness of the GTRI CCD camera is shown in Figure 4 (b) where the spectrum is plotted on a log scale, the highest peaks in the spectrum are approximately four orders of magnitude above the noise. This spectrum shows many orders of sidebands, the highest orders just above the noise with a power less than 1% of the total pump power. The two central peaks are the pump waves at $\omega_1$, higher frequency (blue-shifted), and $\omega_2$ lower frequency (red-shifted). The first order sidebands are located on either side of the pumps at $\omega_3 = 2 \omega_1 - \omega_2$ and $\omega_4 = 2 \omega_2 - \omega_1$. The detection of weak FWM sidebands at the fiber output presented here would not have been possible without the exceptional performance of this CCD camera system.
IV. Experiments

The measurements of the dynamical evolution of four-wave-mixing processes along a length of single mode polarization maintaining optical fiber were performed using two different values of the pump power, 2.1 W and 5.5 W. The frequency separation between the pumps was held constant throughout the measurements at \(\Omega = 366\) GHz. The experiments began with 50.39 meters of AT&T birefringent optical fiber [23]. Starting at this initial length, measurements of the FWM spectrum at the output of the fiber were made using the GTRI CCD camera. From these measurements, conservation of total power and asymmetry (eqn. (4)) were tested for each data set. To check the conservation of these quantities, the total power and asymmetry in the single pumps propagating were calculated and compared with the power and asymmetry of the copropagating pumps. Data sets were accepted and kept if the conservation laws were preserved. In some cases, the presence of weak stimulated Raman scattering (SRS) was detected through the asymmetry relation. In the experiments tracing the evolution of the four wave mixing spectra along the fiber length no SRS was detected. After the initial measurements were made at the two input powers (2.1 W and 5.5 W), 1 to 1.5 meters of fiber was cut and cleaved. The fiber was cut and cleaved at the output side of the fiber, to maintain approximately constant pump coupling to the fiber throughout the experiments. This process was repeated until the four-wave-mixing spectrum had been traced along the full 50.39 meters of fiber for the two input power levels.

Figure 5 and Figure 6 show three dimensional plots of the average FWM output spectrum along the length of single mode birefringent optical fiber. The vertical axis represents the intensity, normalized to the peak power in one of the pumps, plotted on a logarithmic scale. The pump frequencies are centered on \(\pm\Omega/2\), and the fiber length is increasing into the page. In Figure 5 the input power to the fiber is 2.1 W and first order sidebands are clearly seen. Plotted on a log scale
the evolution of the power in the sidebands appears to evolve periodically with increasing fiber length. Figure 6 shows the evolution of the FWM spectrum for a pump input of 5.5 W. First order sidebands are generated as well as ‘weak’ second order sidebands. The first order sidebands appear to evolve periodically initially, and, with increasing fiber length, evolve to a constant value.

A clearer picture of the evolution of the first order sidebands is obtained by plotting the power in the sidebands as a function of length along the fiber. Figure 7 shows the evolution of the first order sidebands as a function of length. The two first order sidebands are plotted separately, where Figure 7 (a) shows the evolution of the blue sideband (blue-shifted from the pumps) and Figure 7 (b) shows the red sideband (red-shifted from the pumps). The solid line in the figure is generated by numerically solving the coupled amplitude equations truncated to six waves. The parameters $\beta^{(2)}$ and $\gamma$ were determined by finding the best fit of the numerical simulations to the experimental data. The values obtained were $\beta^{(2)} = 55 \text{ ps}^2/\text{km}$ and $\gamma = 0.019 \text{ m}^{-1}\text{W}^{-1}$, both well within the regime expected for a central wavelength of 633 nm [1]. The measured sideband power, normalized to the total power in the fiber, is periodic with length, but it appears to be damping to a constant value. Also, the first minimum of the blue sideband trajectory occurs at a shorter distance than the first minimum for the red sideband. This contradicts the predictions of the coupled amplitude equations (ODE) and NLSE. The models predict essentially the same evolution for each sideband. The other difference between the two sidebands is the magnitude of the first maximum. The blue sideband has a larger maximum than the red sideband.

The apparent damping of the periodic sideband trajectory is seen more dramatically in Figure 8 which shows the evolution of the first order sideband power along the fiber for an input power of 5.5 W. Again the two first order sidebands (blue and red) evolve with different trajectories. Furthermore, they also appear to damp to a constant value at a faster rate than for the case with a
pump input power of 2.1 W. Both sets of experiments are compared with the numerical simulations in Figure 7 and Figure 8. The standard theoretical models do not account for either the damping of the sideband power or the different trajectories of the blue and red sidebands.

The FWM spectrum in Figure 6 shows that first order sidebands as well as weak second order sidebands for a pump input power of 5.5 W. Figure 9 shows the evolution of the power in the second order sidebands with propagation length. The blue and red shifted sidebands are plotted separately and the power is normalized to the total input power. The measured sideband power has a maximum of 0.2% of the total input. The 14 bit A/D used in the camera system limits the resolution to $1/16384 = 0.07\%$. Figure 9 shows a complex evolution of the sidebands. The sidebands are weak and just above the limits of resolution imposed by the detection system. Comparison is made using simulations based on the nonlinear Schrodinger equation. The NLSE is used in these simulations because it was found necessary to include higher order sidebands (>second order) to model the dynamics.

A first set of experiments was performed using 20 meters of the AT&T birefringent fiber. In these earlier measurements, the evolution of the sidebands was traced along the fiber using an input power of 2.1 W and a pump detuning of 366 GHz. A direct comparison between the sideband power along the length of 20 meters of fiber with the sideband evolution along the 50.39 meters of fiber was made. The two sets of data were found to yield the same results. Thus, the observations of the damping of the sideband trajectory and the different evolutions of the individual sidebands are repeatable.

Another perspective on the evolution of the sidebands is gained through investigation of the sideband power dependence on the pump power [13]. Measurements were made of the sideband power as a function of pump power at a length of 50.39 meters for two different values of the
pump detuning. Figure 10 shows the power in the first order sidebands as a function of input power using a pump detuning \( \Omega = 366 \text{ GHz} \). The blue and red sidebands are plotted separately, Figure 10 (a) and (b) respectively. The input power was varied from approximately 2 W to 15 W, and the procedure outlined above was used for data collection. Pump depletion due to stimulated Raman scattering (SRS) was observed for pump powers greater than 10 W. For both the blue and red sideband, the measurement peaks around 12 W and then begins to decrease with increasing pump power. This decrease can be attributed to significant pump depletion associated with Raman scattering. The solid lines in Figure 10 were generated by numerically solving the coupled amplitude equations truncated to six waves using \( b^{(2)} = 55 \text{ ps}^2/\text{km} \) and \( \gamma = 0.019 \text{ m}^{-1} \text{W}^{-1} \). The numerical solutions yielded quite different dynamics than those observed experimentally.

The pump detuning was maintained at 366 GHz throughout the experiments probing the evolution along the fiber length. Prior to cutting the fiber, a series of measurements of the sideband power dependence on the pump power were performed with a detuning twice as large: \( \Omega = 722 \text{ GHz} \). Doubling the detuning resulted in a smaller conversion of power from the pumps to the sidebands. Figure 11 shows the results of these measurements. Only first order sidebands were detected for the range of pump powers explored. Consequently, essentially periodic dynamics were predicted by the theoretical models. As in the 366 GHz detuning case, the sideband power steadily increased with pump power until stimulated Raman scattering began to deplete the pumps. The numerical simulations again showed oscillations in the sideband power with increasing pump power, in marked contrast to the dynamics seen in the experiments.

To check some of the observed dynamics, a series of sideband power dependence measurements were performed at a length of 5.52 meters with a detuning of 366 GHz. Figure 12 shows the sideband power as a function of input power. Raman scattering was observed for pump powers
greater than 25 W. Comparison of the experimental measurements with numerical simulations shows very close agreement for pump powers less than 25 W. Thus, as the sidebands initially grow in the fiber, the numerical models can accurately predict the sideband dynamics. However, for longer fiber lengths the standard theory fails to predict the dynamical evolution of the pumps and sidebands as the pulses propagate through the fiber.

So far, only the dynamical evolution of the power in the sidebands has been discussed. It is also worth discussing the experimental FWM spectral envelope, which, resembles a hyperbolic secant shape at the output of 50.39 meters of fiber. The hyperbolic secant is an ubiquitous shape in nonlinear fiber optics and arises in the context of soliton propagation in fibers. Soliton propagation in the form of a hyperbolic secant pulse shape is found in the anomalous dispersion regime \( \beta^{(2)} < 0 \) \([12,27]\). However, the experiments in this research were performed in the normal dispersion regime. In the normal dispersion regime, dark-pulse solitons of the form of a hyperbolic tangent are predicted and have been observed \([16]\).

Figure 13 shows some of the experimental FWM output spectra at a fiber length of 50.39 meters, detuning \( \Omega = 366 \) GHz, with a range of input power levels (a) 2.1 W, (b) 5.5 W, (c) 8.3 W, and (d) 17.4 W. The solid line represents the experimental data and the dashed line is a curve fit to the spectral envelope. The curve is fit by \( y(\omega) = \text{Asech}(B\omega) \) where A and B are the fit parameters. The values used to generate the plots in Figure 13 are: (a) \( A=3.85, B=0.36 \), (b) \( A=2.26, B=0.27 \), (c) \( A=1.56, B=0.23 \), and (d) \( A=0.81, B=0.20 \). Figure 13 shows close agreement between the hyperbolic secant shape and the experimental spectral envelope. For the lower input powers, the peaks in the spectra are distinct. However, as the input power increases, the peaks broaden and the spectrum begins to fill in.
In Figure 13 (d) with an input power of 17.4 W, the pumps are depleted by Raman scattering. Furthermore, close examination of the spectrum shows an asymmetry even though the initial conditions on the pump waves were symmetric, i.e. $\rho_1(0) = \rho_2(0)$. As the waves copropagate in the fiber, photons from the band of frequencies generated through four wave mixing will be down-shifted by spontaneous and stimulated Raman scattering. The Raman gain spectrum ranges from 0 to tens of terahertz frequency shift from the pumps. For silica glass, maximum Raman gain occurs at a down-shifted frequency of 13.2 THz (several orders of magnitude larger than the pump detuning) [1]. However, the Raman gain is nonzero near zero frequency shift. Thus, in Figure 13 (d), the observed asymmetry in the spectrum arises from strong stimulated Raman scattering.

These experiments exposed several discrepancies in the comparison of experiment and theory and illustrated the inadequacy of the standard theoretical models to predict the observed dynamics over the full length of fiber investigated. The next section will present modifications to the theoretical models, to allow a quantitative comparison of experimental observations and numerical simulations. The key aspects of the experiments to be addressed are (1) the damping of the periodic sideband trajectories with length and (2) the difference between the red and blue sideband trajectories.
V. Theory vs. Experiment

This section develops a theoretical description which includes two effects which had not previously been considered. We consider the effect of a multimode pump laser at the fiber input and investigate the resulting dynamical evolution of the sidebands. By modeling one of the pump lasers as two closely spaced longitudinal modes, the subsequent dynamical evolution of the sidebands is altered dramatically. By introducing this asymmetry in the mode structure of the pump input, the resultant dynamics for the blue and red sidebands begin to approach the sideband dynamics observed in the experiments. However, the damping of the periodic trajectories seen in the experiments is still not explained with the simple multimode structure at the input. Building on the multimode analysis, one then introduces weak phase fluctuations to the pump waves propagating along the fiber. The combination of both the multimode pump input and weak phase perturbations along the fiber is found necessary to accurately predict the experimental observations. Excellent agreement is thus finally obtained on comparing predictions based on the stochastic multimode model with experiments.

V.I Multimode Pump Input

As mentioned previously, the dye laser systems used in the experiments were designed for narrowband operation. However, the resolution of our instrumentation limits the ability to measure the linewidth of the lasers and distinguish single versus multiple mode operation. Thus, either of the dye laser outputs may have consisted of several longitudinal modes. We examine the impact of multimode operation on the dynamical evolution of the sidebands by introducing a multimode pump input to the theoretical models. The sideband evolution predicted from numerical solutions of both the NLSE and the coupled amplitude equations with a multimode input is found to exhibit similar dynamics, when compared with the experimental observations.
To model wave propagation in the fiber using a multimode pump input, both the nonlinear Schrödinger equation and the coupled amplitude equations discussed earlier can be used. The NSLE requires only a modification of the input pulses. The input spectrum can be set for one, two, three, etc., modes in each pump laser. Thus, a variety of initial pump conditions can be investigated. Starting with the simplest case, Figure 14 shows an example (a) Gaussian input pulse and (b) corresponding spectrum, to the NLSE, with two modes in the blue shifted pump and one mode in the red shifted pump. In Figure 14 (b) the spectrum is plotted with the pump detuning normalized to unity and the pumps centered about zero frequency shift. The input is a Gaussian pulse modulated by the pump detuning and the longitudinal mode spacing. The longitudinal mode spacing ($\Delta v$) were chosen to be 0.5 GHz, consistent with the expected spacing from the experiments, and the pump detuning is 366 GHz. The initial conditions on the pumps were chosen so that the conservation relation for the asymmetry (eqn. 2.3) is zero, i.e. $\rho_1 = \rho_2$.

Figure 15 shows the FWM output spectrum generated from the multimode input. The sidebands and pumps now consist of many frequencies. To estimate the relative power in the pumps and sidebands, the power in the band of frequencies centered around the primary frequency is summed and then normalized to the total power in the spectrum, for example the blue pump power, $\rho_1$, is calculated from summing the power in the frequency components located between zero and one. For consistency the same notation used throughout this paper is retained to represent the power in the pumps and sidebands, e.g. $\rho_3$ represents the relative power in the blue first order sideband even though now it consists of multiple frequency components.

The split step Fourier method is used to propagate the pulses along the fiber [1]. Figure 15 shows a schematic representation of the FWM output spectrum after propagation through a length of fiber with the multimode input. The evolution of the four wave mixing processes is now more complex, not only is there mixing between the distantly spaced pump frequencies but
there is mixing between the closely spaced longitudinal modes. A difficulty of using the NLSE to model the multimode input is the size of the FFT which must be computed. Since the longitudinal mode spacing is several orders of magnitude smaller than the pump detuning the number of points necessary to represent the pulse spectrum is large $>=2^{15}$. The computation algorithm sets limits on the number of points used to represent the spectrum and the spectrum will necessarily be truncated. In general, the NLSE with single mode inputs yields the ability to work with a broad spectrum consisting of many orders of sidebands, a definite advantage over the coupled amplitude equations.

Extending the modeling of the multimode input to the coupled amplitude equations, the general form of the equations given in eqn. (3) is used to generate a new set of coupled amplitude equations. The frequencies in these equations now include the longitudinal mode spacing. Thus, the wavevector mismatch will now be proportional to $(\Omega \pm \Delta \nu/2)^2$, $(\Omega \pm 3\Delta \nu/2)^2$, etc, whereas in the single mode model, the mismatch was proportional to $\Omega^2$. The number of frequency components necessary to model the FWM dynamics, including up to second order sidebands, results in at least 100 terms in each equation. Using equation (3), a 'C' program was written to find the allowed combinations of $k$, $m$ and $n$. The multimode four wave mixing equations were then stored to a file in a subroutine format to be called from the integration programs.

For simplicity, the case of two longitudinal modes in the blue pump and one in the red pump is considered. The blue pump was initially chosen to be multimode because the fluctuations in experimental measurements of the linewidth were larger than those in the red pump. The single mode input standard model consists of 6 complex coupled field equations which includes the pumps, first order sidebands, and second order sidebands. Terms up to second order sidebands were included since they were observed in the experiments for a pump input of 5.5 W. With the
exhibited by the blue and red sidebands with this model are significantly different. Within the first 10 meters, the model follows the evolution quite well, and yet the damping observed in the experimental measurements with increasing length makes comparison difficult. Overall, the multimode model yields promising results for predicting the dynamical evolution of the sidebands. The modeling of the damping of the sideband trajectories will be discussed in the next section.

As another comparison of the multimode input model with experiment, Figure 18 shows the evolution of the second order sidebands with propagation distance. Terms including up to at least third order sidebands must be included in the model to properly predict the dynamics of the second order sidebands. The coupled amplitude equations including only up to second order were found inadequate. Thus, the nonlinear Schrodinger equation was used to easily include higher order sidebands. Comparison of the second order sideband and predictions based on the NLSE with a multimode input shows close agreement. The second order sidebands are weak and yet for fiber lengths less than 20 meters the simulations follow the experimental measurements closely. However, beyond 20 meters the blue second order sideband (Figure 18 (a)) appears to be damping to a constant value.

As mentioned earlier, the model can be extended to include various combinations of pump inputs, for example, three modes in one of the pumps and two or one in the other pump. No significant difference was found in the first order sideband evolution for the different combinations of asymmetric multimode input, for the parameter regimes investigated; $L_{\text{max}} = 50.4$ m, $P < 6W$, $\Omega = 366$ GHz, and $\Delta v = 0.5$ GHz. As the fiber length increases beyond 50.4 meters, differences in the trajectories arise between the various asymmetric combinations for the input. Referring to Figure 15, the spectrum broadens around the central frequency components due to FWM between the longitudinal modes. The longitudinal mode spacing used in these
simulations is small compared to the pump detuning. Furthermore, since the mode spacing is small, the FWM processes between adjacent modes will evolve with a period much longer than the fiber lengths considered in this research. Subtle differences in the sideband evolution will arise for different mode structures as the fiber length increases due to the different dynamics between adjacent modes. Only the simplest case of multimode input was considered for comparison with these experiments.

V.II Stochastic Phase Fluctuations

The previous theoretical analyses presented have been limited to deterministic models. We now turn to modeling of stochastic processes along the fiber length as well as including stochastic initial conditions on the pump inputs. The latter are included in the modeling to closely imitate the conditions present in the experiments. The former examines the impact on the dynamical evolution of the four wave mixing processes when weak fluctuations are added to the phase of each of the waves copropagating along the fiber. These phase fluctuations are found to damp the sideband periodic trajectories to a constant value. Comparison with the experimental observations is made and excellent agreement is found.

Consider a physical process which acts to perturb the phase of the waves propagating along the fiber. The physics of this phase noise could arise from a number of sources, such as; fiber medium inhomogeneities [29,30], Brillouin scattering, or Raman scattering [1]. In the experiments, there was no indication that these sources were present. However, the existence of these processes could have been lost in the background noise of the instrumentation.

Identification of the physical process generating the noise through both experiments and modeling is a promising area for future research. A strong candidate for the source of phase noise is stimulated Raman scattering that builds up from a spontaneous noise background.

Recalling the experiments probing the sideband power dependence on the pump input power, for
a 50 meter length of fiber significant Raman scattering was detected for pump inputs greater than
10 W. Thus, it is highly likely that very weak (< 4 orders of magnitude down from the pumps)
Raman scattering was present in measurements.

There are two theoretical models which may serve as the core set of equations to model the
nonlinear wave propagation along with stochastic processes in the fiber. The multimode coupled
amplitude equations, developed previously, were used for the stochastic modeling in this
research. A model incorporating the phase noise into the nonlinear Schrodinger equation is
desirable as well. However, algorithms to properly include the necessary stochastic terms in the
NLSE are not available at this time. Thus, the remainder of the research will use the coupled
amplitude equations. Integration of the amplitude equations proceeded as follows. After the
initial conditions on the input were set, the multiple waves were propagated in the fiber using a
fourth order Runge-Kutta integration [21] with a step size Δz (typically 10⁻³ meters). After each
integration step, the complex field amplitudes were converted to intensity and phase. The phase,
ϕj, of each wave at frequency, ωj, was modified according to:

\[ \phi_j(z + \Delta z) = \phi_j(z) + \delta \phi_j \]  

(5)

where the phase fluctuations are represented by δϕj. The intensity and phase were then
converted back to the complex field amplitudes. The field was then propagated another step Δz
and the process repeated for each integration step.

Since the exact source generating the noise is not known, the phase fluctuations are taken to be
delta correlated along the fiber and are considered to be independent sources for each wave. The
Box-Muller algorithm was used to generate Gaussian deviates from computer generated uniform
deviates [21,31]. The fluctuations are given by:

\[ \delta \phi_j = \sqrt{-2 \sigma_j \Delta z \ln(r_1)} \cos(2\pi r_2) \]  

(6)
and,
\[ \delta \phi_{\omega_1} = \sqrt{-2 \sigma_{\omega_1} \Delta z \ln(r_1) \sin(2\pi r_2)} \]  
(7)

where \( r_1 \) and \( r_2 \) are uniformly distributed random numbers on the interval \( (0,1) \) and \( \sigma_{\omega_1} \) is the standard deviation of the phase fluctuations for a given frequency component. For simplicity in the numerical computations, the phase fluctuations were added to only the frequency components associated with the two pump waves. However, computations were also performed adding phase noise to all components; there was no detectable difference in the resulting sideband dynamics for the parameters investigated. This is reasonable in the regime of primary interest, since for pump powers less than 6 W, the pump intensities are much larger than the sidebands and thus make the strongest contribution to the FWM dynamics. Typically, the noise strengths \( \sigma_{\omega_1} \) were chosen to be of the same order of magnitude for each pump.

V.III Stochastic Initial Conditions

Previous studies showed that fluctuations in the initial conditions of the pumps could have a significant impact on the dynamics of the FWM processes in the fiber [13,18]. To model the initial conditions of the experiments, measurements of pump fluctuations were included in the input to the integration of the equations. To measure the pump fluctuations, each pump was propagated alone in the fiber for each fiber length and pump power. The mean intensity, normalized to unity, and standard deviation were calculated from the output spectra. The intensity in the pumps was found to be Gaussian distributed. Figure 19 shows the measured standard deviation in the normalized pump power as a function of length along the fiber. The blue and red pump standard deviations are plotted separately. The blue laser has a higher mean intensity fluctuation than the red, this is probably associated with multiple longitudinal modes in the blue pump. The experimental measurements over the full length of optical fiber were
performed over a long period of time (approximately one year). As can be seen from Figure 19, the pump intensity standard deviation varied with time.

The numerical simulations were performed with fluctuations in the input pump intensity as well as fluctuations in the detuning. The measured frequency fluctuations had a magnitude of less than 1 GHz, several orders of magnitude smaller than the pump detuning of 366 GHz. Including the frequency fluctuations in the simulations was found to have no measurable impact on the resulting dynamical evolution of the sidebands. However, the pump power fluctuations were large (~10%) and could not be neglected. The pump input was of the form:

\[ U_j(0) = \sqrt{\rho_j(0)}e^{i2\pi r_j} \]  \hspace{1cm} (8)

where \( \rho_j \) is the intensity and \( r_j \) is a uniformly distributed random number in the interval (0,1), which selects a nonzero initial phase. For completeness \( r_j \) is included here. However, randomizing the initial phase had no measurable impact on the resulting evolution of the power in the waves. The pump intensity input to equation (8) for each component in the dual mode pump was set according to,

\[ \rho_{ik}(0) = \frac{1}{2} \rho_{ave} + \frac{1}{2} \delta \rho_1 \]  \hspace{1cm} (9)

where \( k \) represents each mode and for the single mode pump,

\[ \rho_k(0) = \rho_{ave} + \delta \rho_2 \]  \hspace{1cm} (10)

where \( \rho_{ave} \) is typically set to unity for both pumps and \( \delta \rho \) are the fluctuations in the pumps and are generated using the Box-Muller algorithm [21]. The fluctuations are generated using the measured values of the standard deviation in the pump intensities (see Figure 19) and, are given by, for the blue pump,

\[ \delta \rho_1 = \sqrt{-2\sigma^2_{\rho_1} \ln(r_1) \cos(2\pi r_2)} \]  \hspace{1cm} (11)

and for the red pump,
\[ \delta p_2 = \sqrt{-2\sigma_{p_2}^2 \ln(\tau_1) \sin(2\pi r_1)} \quad (12) \]

Computations with the multimode model (for the dual mode pump) used the same noise strength for each longitudinal mode. The next section will discuss the specific values used in the numerical simulations and compare with the experiments.

V.IV Numerical Simulations

The numerical simulations were performed using the multimode coupled amplitude equations along with the stochastic conditions discussed above. A complication arose when adding the phase fluctuations to the waves which resulted in a “noise-induced” drift [32]. This is a feature of multiplicative noise sources in which the noise added causes the sidebands to grow with propagation. Even though the noise is additive to the phase, the equations are cubic in the complex field and, thus, the phase noise is multiplicative when coupled back into the field equations. Including phase noise in the FWM calculations resulted in trajectories for the sideband power with length which were damped periodic trajectories with an increasing slope.

To remove this artifact of the computations, a linear loss term, \(-\alpha U_j\), was added to each of the complex field equations. The loss coefficient, \(\alpha\), was then set by finding the value which removed this increasing slope. In theory, the mathematical form of \(\alpha\) can be derived from the equations and is a function of the noise strength [32]. However, the size of the system of coupled propagation equations made the technique for estimation intractable, even for the simplest approximate form of the equations.

The strength of the phase noise used in all of the following simulations was determined by fitting the simulations to the experimental data. The values found to give the best fits were \(\sigma_{\phi_1} = 0.0067 \, \text{m}^{-1}\) and \(\sigma_{\phi_2} = 0.005 \, \text{m}^{-1}\) and \(\alpha = 0.0046 \, \text{m}^{-1}\). For comparison with the experiments
tracing the evolution of the sidebands along the fiber, the simulations were the result of two calculations, one from 0 to 20 meters and the other from 0 to 50.4 meters. The calculation from 0 to 20 meters replaces the first 20 meters of the 0 to 50.4 meter simulations. This was necessary since the initial conditions on the pump fluctuations were larger (due to the laser adjustments for the measurements, which took several months) for lengths less than 20 meters (see Figure 19).

The fluctuations in the pump intensities were set at $\sigma_{p_1} = 0.20$ (blue) and $\sigma_{p_2} = 0.11$ (red) to generate the curves from 0 to 20 meters and $\sigma_{p_1} = 0.12$ and $\sigma_{p_2} = 0.05$ to generate the curves from 20 to 50.4 meters. The other parameters were set at $\beta^{(2)} = 55 \text{ ps}^2/\text{km}$, $\gamma = 0.019 \text{ W}^{-1} \text{m}^{-1}$, $\Omega = 366 \text{ GHz}$, $\delta v = 0.5 \text{ GHz}$. The numerical simulations compute both an average and standard deviation from 50 trajectories. Simulations were done for 100 trajectories and it was determined that accurate statistics (the standard deviation was less than 5%) were obtained for as few as 50 trajectories. Thus, to reduce computation time the statistics are calculated from 50 runs.

Figure 20 (a) and Figure 21 (a) show the blue and red sideband trajectories, respectively, for an input power of 2.1 W. The experimental data are plotted with the numerical solution of the multimode coupled amplitude equations including both phase noise at each integration step and fluctuating the pump inputs. The multimode model with the inclusion of stochastic initial conditions and, most importantly, phase fluctuations along the fiber length, results in predictions which are very close to the experimental observations of the dynamical evolution of the sidebands. Figure 20 (b) and Figure 21 (b) show the measured standard deviation in the sideband power along the fiber length for the blue and red sidebands, respectively. The standard deviation was also calculated from the numerical simulations. Excellent agreement is found between the model and experimental measurements. Throughout the course of this research, many stochastic models have been investigated and this model is the only one found that reproduces the evolution of both the average power and fluctuations in the sidebands.
For an input power of 5.5 W, Figure 22 and Figure 23 show comparison between the numerical simulations and experimental data for the blue and red sidebands, respectively. The red sideband trajectories in Figure 23 show excellent agreement between the numerical model and experiment. The power in the red sideband from numerical solutions is periodic and appears to be damping at the appropriate rate. However, the blue sideband power trajectory shown in Figure 22 (a) does not reproduce the experimental measurements as closely. The numerical simulations at this pump power result in a blue sideband power evolution which does not damp as quickly as the experimental observations. However, the fluctuations measured in the experiments are fairly well predicted by the numerical simulations as shown in Figure 22 (b) and Figure 23 (b) for both the blue and red sidebands, respectively. The discrepancy in the damping seen between the experiment and model of the blue sideband power evolution could arise from several effects. The strength of the phase noise was the same for both the 2 W and 5.5 W calculations. Many simulations have been performed to optimize the values used for the phase noise strengths. The values used in these simulations were optimized in the sense they gave the best fit to the experimental data. A better approach would be, to identify the physical phenomena generating the phase fluctuations and with this knowledge, the magnitudes of $\sigma_{\varphi_1}$ and $\sigma_{\varphi_2}$ could be estimated from the physics. Another benefit of identifying the physics of the phase noise, the noise could be properly included in a model based on the nonlinear Schrodinger equation.

As a confirmation of the multimode model with phase noise, numerical simulations were performed examining the sideband power dependence on the input power at a length of 50.4 meters. Figure 24 and Figure 25 show the power in the sidebands as a function of input power for a pump detuning of 366 GHz and 722 GHz, respectively. The experimental measurements of the sideband powers are represented by closed circles in Figure 24 (a) and Figure 25 (a), blue sideband, and closed squares in Figure 24 (b) and Figure 25 (b) red sideband. The results of numerical simulations are represented by the open circles in Figure 24 (a) and Figure 25 (a) and
open squares in Figure 24 (b) and Figure 25 (b). The numerical simulations follow the general trend seen in the experiments. A large deviation occurs for pump powers >10 W where the pumps begin to be depleted by stimulated Raman scattering. Below 10 W, the experimental measurements of the red sidebands tend to be higher than the simulations, this increase arises from weak scattering of the blue photons to the red. With the smaller detuning (366 GHz), care must be taken to account for all orders of sidebands and for powers greater than 6 W probably third and fourth order sidebands are generated. However, with this model, including only up to second order sidebands, yields predictions in close agreement with experimental measurements especially when compared with the predictions based on the deterministic single mode input coupled amplitude equations.

We have presented a new approach to modeling the dynamical evolution of four wave mixing processes along an optical fiber. This modeling was motivated by the standard theoretical models inability to predict the results of experimental measurements presented in Section III. The two critical features of the model were a multimode pump input along with phase fluctuations added along the fiber length. The multimode pump input was found to alter the resulting sideband dynamics significantly. Due to an asymmetry introduced in the input, the blue and red sidebands evolved with different trajectories along the fiber. Furthermore, by adding weak phase fluctuations to the copropagating waves, the periodic sideband trajectories were found to damp out. Figure 20 through Figure 25 show comparisons between the experimental measurements of the sideband dynamics and the stochastic multimode model. The experimental observations brought to light several questions regarding the dynamics of the four wave mixing processes in the fiber.
VI. Conclusions

The dynamical evolution of four wave mixing (FWM) processes in an optical fiber has been investigated. This research consisted of experimental, theoretical, and numerical computations. The focus of this work was to experimentally trace the evolution of the sidebands, generated through FWM, along a length of optical fiber. Previous theoretical work suggested that, in certain parameter regimes, the sidebands exchange energy with the pumps periodically [13,14]. Specifically, in the undepleted pump regime [28], the sideband power evolves as a sinusoid with fiber length. Previous experiments had probed the dynamics for short fiber lengths (<2 m) [13], however, the periodic evolution had never been directly verified.

The FWM spectral evolution along 50 meters of fiber for two input pump power regimes was investigated. The experimental work consisted of measuring the FWM mixing spectrum output from an optical fiber at different lengths in the fiber. Specifically, a low noise, high resolution CCD camera made at Georgia Tech Research Institute, was used [26] to detect weak (<1% of the power in the pumps) sidebands. With this resolution, measurements of the power in the first order sidebands for input pump powers (2.1 W and 5.5 W) were made using a pump detuning of 366 GHz. In the case of a pump input of 2.1 W, the sideband power evolution is expected to follow a sinusoid along the length of the fiber. Experiments showed that the power in the sideband evolved periodically, but that the evolution followed a damped sinusoid. The experiments also found that the two first order sidebands (blue and red shifted from the two pumps) had different evolutions along the fiber. Neither the damping nor the different evolutions were predicted by theory. Using a pump input power of 5.5 W the evolution of both first and second order sidebands was also measured. For a pump input of 5.5 W the damping in the first order sidebands appeared to occur faster than in the 2.1 W case.
Experiments probing the dependence of the sideband power on the input power for two different values of the detuning (366 GHz and 722 GHz) were also performed at the output of 50 meters of fiber. With a detuning of 366 GHz, the sideband power for pump inputs ranging from 2 W to 17 W was measured. Comparison of theoretical predictions with the measurements showed a large discrepancy both quantitatively and qualitatively. The measurements of the sideband power as a function of pump input power with 722 GHz detuning showed the same discrepancies with the theoretical models as the 366 GHz detuning case. Another set of measurements were performed at a length of 5 meters with a pump detuning of 366 GHz. Comparisons of the measured sideband powers with theoretical predictions, for this case, showed excellent agreement up to a pump input power of 25 W. For higher powers, the deviation between experiment and theory was due to other competing processes (stimulated Raman scattering) not accounted for in the theoretical model. The results of the measurements show that the initial evolution of the FWM spectrum in the fiber is modeled well by the standard theory. However, beyond the initial growth of the spectrum the models do not predict, even qualitatively, the experimental observations.

Three dimensional plots of the evolution of four wave mixing spectrum in the fiber, indicate that the spectrum was evolving to a stable profile. Since, in the anomalous dispersion regime, soliton propagation in the form of a hyperbolic secant shape is known to be supported in an optical fiber [12], the envelope shape of the experimental FWM spectrum was investigated. It was found that at the output of 50 meters of fiber the spectral envelope could be fit by a hyperbolic secant shape. However, these experiments were performed in the normal dispersion regime, where the fundamental soliton shape is predicted to be a hyperbolic tangent. Furthermore, in the normal dispersion regime, true solitons are essentially dips in a continuous wave carrier. Theoretical and experimental research indicates that soliton-like pulses can be supported on carrier pulses where the length of stable propagation in the fiber is determined by the length of the carrier [16].

Further studies need to be done to determine if the FWM processes in the fiber evolved to a train
of stable soliton dips in the long pulse background for the fiber lengths investigated in the experiments [33].

The experimental results pointed to the need to modify the approach to the theoretical modeling of the four wave mixing processes. The experimental measurements tracing the sideband evolution along the fiber length, showed that the different first order sidebands evolved with different dynamics. This observation was not accounted for in the standard theoretical models. By imposing an asymmetry on the spectral structure of the pump inputs, the sidebands were found to follow different dynamical evolutions. Specifically, one of the pump inputs was modeled to consist of two closely spaced longitudinal modes. It is worth emphasizing that the inter-mode spacing is very small compared to the difference in wavelength of the two pump lasers, and is not resolvable with the spectrometer system used and had to be resolved with a higher resolution wavemeter. This multimode input was found to alter the sideband dynamics dramatically.

The experimental measurements of the sideband power with length along the fiber indicated that there was damping of the periodic evolution of sideband power with increasing fiber length. Again, this was not accounted for by the standard theory. One interpretation that gives insight to the damping of the sidebands is that the exchange of power between the pumps and sidebands copropagating in the fiber can be thought of as a coherent process. The experimental measurements showed the damping of sideband power, indicating that there was a mechanism along the fiber acting to remove the coherence of the power exchange between the pumps and sidebands. This mechanism was modeled by adding weak phase fluctuations to the waves as they propagated along the fiber, using the continuous wave model (coupled amplitude equations). These phase fluctuations were found to account for the damping of the sideband power evolution along the fiber. However, the physical source of these phase fluctuations has yet to be
determined and is an area for future research. Numerical simulations using the new approach, including a multimode input and phase fluctuations along the fiber length, were performed for the parameters of the experiments, and excellent quantitative and qualitative agreement was found.

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Figure Captions

Figure 1: **Dynamical evolution of first order sidebands as a function of fiber length; \( \Omega = 300 \) \( \text{GHz} \), and input pump powers of (a) 2 \( \text{W} \), (b) 6 \( \text{W} \), and (c) 50 \( \text{W} \). Note the different scales of each vertical axis.

Figure 2: **Dynamical evolution of first order sidebands as a function of fiber length; \( \Omega = 600 \) \( \text{GHz} \), and input pump powers of (a) 2 \( \text{W} \), (b) 6 \( \text{W} \), and (c) 50 \( \text{W} \). Note the different scales of each vertical axis.

Figure 3: **Experimental setup used to investigate four-wave-mixing in an optical fiber.

Figure 4: **Experimental FWM output spectrum (a) plotted on a linear scale and (b) plotted on a logarithmic scale.

Figure 5: **Evolution of the FWM spectrum along the fiber from experiments, \( P = 2.1 \text{ W} \), \( \Omega = 366 \) \( \text{GHz} \).

Figure 6: **Evolution of the FWM spectrum along the fiber from experiments, \( P = 5.5 \text{ W} \), \( \Omega = 366 \) \( \text{GHz} \).

Figure 7: **Comparison between the experimental measurements (symbols) and the standard theoretical models (solid line), of the sideband evolution as a function of fiber length; \( P = 2.1 \text{ W} \), \( \Omega = 366 \text{ GHz} \). **Dynamical evolution of the (a) blue shifted sideband and (b) red-shifted sideband.

Figure 8: **Comparison between the experimental measurements (symbols) and the standard theoretical models (solid line), of the sideband evolution as a function of fiber length; \( P = 5.5 \text{ W} \), \( \Omega = 366 \text{ GHz} \). **Dynamical evolution of the (a) blue shifted sideband and (b) red-shifted sideband.
Figure 9: Comparison between the experimental measurements (symbols) and the standard theoretical models (solid line), of the second order sideband evolution as a function of fiber length; $P=5.5$ W, $\Omega = 366$ GHz. Dynamical evolution of the (a) blue shifted sideband and (b) red-shifted sideband.

Figure 10: Comparison between the experimental measurements (symbols) and the standard theoretical models (solid line), of the sideband power versus pump input power; $L=50.39$ m, $\Omega = 366$ GHz. Power in the (a) blue shifted sideband and (b) red-shifted sideband.

Figure 11: Comparison between the experimental measurements (symbols) and the standard theoretical models (solid line), of the sideband power versus pump input power; $L=50.39$ m, $\Omega = 722$ GHz. Power in the (a) blue shifted sideband and (b) red-shifted sideband.

Figure 12: Comparison between the experimental measurements (symbols) and the standard theoretical models (solid line), of the sideband power with pump input; $L=5.52$ m, $\Omega = 366$ GHz. Power in the (a) blue shifted sideband and (b) red-shifted sideband.

Figure 13: Experimental FWM output spectrum (solid line) and hyperbolic secant envelope fit (dashed line) for pump input powers of: (a) $P=2.1$ W, (b) 5.5 W, (c) 8.3 W, and (d) 17.4 W. Fiber length $L=50.39$ m and detuning $\Omega = 366$ GHz.

Figure 14: Multimode pulse input to the NLSE, (a) input pulse in the time domain and (b) input spectrum.

Figure 15: Multimode output spectrum from the NLSE after propagation through several meters.

Figure 16: Comparison between the experimental measurements (symbols) and the multimode model (solid line), of the sideband evolution as a function of fiber length; $P=2.1$ W, $\Omega = 366$
GHz, $\Delta\nu=0.5$ GHz, $\gamma=0.019 W^{-1} m^{-1}$, and $\beta^{(2)}=55 ps^2/km$. Dynamical evolution of the (a) blue shifted sideband and (b) red-shifted sideband.

Figure 17: Comparison between the experimental measurements (symbols) and the multimode model (solid line), of the sideband evolution as a function of fiber length; $P=5.5$ W, $\Omega=366$ GHz, $\Delta\nu=0.5$ GHz, $\gamma=0.019 W^{-1} m^{-1}$, and $\beta^{(2)}=55 ps^2/km$. Dynamical evolution of the (a) blue shifted sideband and (b) red-shifted sideband.

Figure 18: Comparison between the experimental measurements (symbols) and the multimode model (solid line), of the second order sideband evolution as a function of fiber length; $P=5.5$ W, $\Omega=366$ GHz, $\Delta\nu=0.5$ GHz, $\gamma=0.019 W^{-1} m^{-1}$, and $\beta^{(2)}=55 ps^2/km$. Dynamical evolution of the (a) blue shifted sideband and (b) red-shifted sideband.

Figure 19: Measured input pump power standard deviation as a function of fiber length, (closed circles) blue shifted pump and (open squares) red shifted pump.

Figure 20: Comparison between the experimental measurements (symbols) and the stochastic multimode model (solid line), of the blue sideband evolution as a function of fiber length; $P=2.1$ W, $\Omega=366$ GHz, $\Delta\nu=0.5$ GHz, $\gamma=0.019 W^{-1} m^{-1}$, and $\beta^{(2)}=55 ps^2/km$. Dynamical evolution of (a) the power in the blue shifted sideband and (b) the measured fluctuations.

Figure 21: Comparison between the experimental measurements (symbols) and the stochastic multimode model (solid line), of the red sideband evolution as a function of fiber length; $P=2.1$ W, $\Omega=366$ GHz, $\Delta\nu=0.5$ GHz, $\gamma=0.019 W^{-1} m^{-1}$, and $\beta^{(2)}=55 ps^2/km$. Dynamical evolution of (a) the power in the red shifted sideband and (b) the measured fluctuations.

Figure 22: Comparison between the experimental measurements (symbols) and the stochastic multimode model (solid line), of the blue sideband evolution as a function of fiber length; $P=5.5$
W, $\Omega = 366$ GHz, $\Delta \nu = 0.5$ GHz, $\gamma = 0.019W^{-1}m^{-1}$, and $\beta^{(2)} = 55$ps$^2$/km. Dynamical evolution of
(a) the power in the blue shifted sideband and (b) the measured fluctuations.

Figure 23: Comparison between the experimental measurements (symbols) and the stochastic multimode model (solid line), of the red sideband evolution as a function of fiber length; $P = 5.5$ W, $\Omega = 366$ GHz, $\Delta \nu = 0.5$ GHz, $\gamma = 0.019W^{-1}m^{-1}$, and $\beta^{(2)} = 55$ps$^2$/km. Dynamical evolution of
(a) the power in the red shifted sideband and (b) the measured fluctuations.

Figure 24: Comparison between the experimental measurements (closed symbols) and the stochastic multimode model (open symbols), of the sideband power versus pump input power; $L = 50.39$ m, $\Omega = 366$ GHz. Power in the (a) blue shifted sideband and (b) red-shifted sideband.

Figure 25: Comparison between the experimental measurements (closed symbols) and the stochastic multimode model (open symbols), of the sideband power versus pump input power; $L = 50.39$ m, $\Omega = 722$ GHz. Power in the (a) blue shifted sideband and (b) red-shifted sideband.


Fig 1

(a) 

(b) 

(c) 

\( \rho_{34}(z) \) vs. Length (m)
Fig 8

(a)

$\rho_1(z)$

Length (m)

(b)

$\rho_4(z)$

Length (m)
Fig 10

Blue Sideband

\[ \rho_{\text{blue}}(z) \]

Red Sideband

\[ \rho_{\text{red}}(z) \]

Input Pump Power (W)
Fig 14
Fig 20

(a) 

(b)
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Summary

We have investigated the stability properties of two and three element laser arrays that are nearest neighbor coupled. A novel form of generalized synchronization has been discovered, where the outer elements of the three laser linear array are synchronized identically, but the middle one is not synchronized with the outer ones. Experiments on fiber ring lasers have lead to a model that employs delay equations coupled to a differential equation to describe the fast (nanosecond) dynamics of the polarized light output from these lasers. Four wave mixing of light beams at detuned frequencies has been studied both experimentally and theoretically and a unique set of measurements has been analyzed. Phase fluctuations of the light play an important role in the propagation of the sidebands through the fiber. The first experiments on optical communication with chaotic fiber lasers have been performed.
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NONLINEAR DYNAMICS OF COUPLED LASER SYSTEMS

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Final Technical Report

by

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This final technical report summarizes the results of research supported by the Office of Naval Research from October 1, 1995 to March 31, 1999, at the School of Physics, Georgia Institute of Technology.

The research has encompassed a broader range of topics than was originally in the original proposal. The topics investigated were as follows.

(a) **Dynamics of coupled laser systems and arrays.** The research focused on experiments and numerical modeling to explore the dynamics of synchronization of two and more coupled lasers. A new amplitude instability of coupled lasers was predicted and then experimentally verified (Phys. Rev. E 55, 3865 (1997)). This result is of significance to the design and fabrication of coupled laser arrays and describes the fundamental instabilities of amplitude and phase that may occur when lasers are coupled together through sharing of light between their cavities by evanescent leakage between waveguides. We showed that there can be phase coherence even in regimes of amplitude instability and chaos, a result that has been further investigated theoretically by Kurths and his group at Potsdam.

We also investigated the nature of synchronization and intermittency of the oscillations of coupled lasers in a collaboration with Peter Ashwin and his student John Terry, of the mathematics department of the University of Surrey (Phys. Rev. E 58, 7186 (1998)). This investigation of blowout bifurcations in the dynamics of coupled lasers was extended to studies of synchronization of a linear array of three coupled lasers (Phys. Rev. E 59, 4036 (1999)). This paper contained a first demonstration of generalized synchronization in this linear three laser system, where the two outer lasers were identically synchronized, but the middle laser was synchronized to the outer ones through a nonlinear functional relation. This paper also contains the first demonstration of subharmonic generalized
synchronization, where the oscillations of the central laser can occur at twice the average frequency of the outer ones.

(b) Wave propagation in nonlinear media and four wave mixing in optical fibers. A unique set of experiments was carried out to explore four wave mixing and its dynamical evolution in single mode optical fibers. These experiments revealed the importance and influence of spectral structure and phase fluctuations in the propagation of light at multiple frequencies through optical fiber. The results of these experiments were presented in a comprehensive paper (Phys. Rev. E57, 4757 (1998)). It was found that multiple wave interactions are significantly influenced by fine spectral structure of the lasers as well as by phase fluctuations of the light, and that these effects must be included in a model in order to make accurate predictions of the dynamical evolution of the waves along the length of fiber. The experiments consisted of hundreds of measurements at over fifty different lengths of fiber in order to obtain measurements of the spectral evolution of the waves. A specially built CCD camera developed for astronomical observations was used for these measurement, that provided a 14 bit dynamic range for detection of the sidebands generated by four wave mixing. The numerical investigations included integration of the nonlinear Schroedinger equation and of coupled mode equations with stochastic noise sources to simulate the phase fluctuations of the lasers.

(c) Transmission of polarized light through a single mode fiber with random fluctuations of birefringence. In the course of our work on propagation of light through fibers we discovered that it is possible to convey linearly polarized light through single mode optical fibers that possess random birefringence fluctuations along their length. The mathematical prediction of this possibility and its experimental demonstration were carried out (Applied Optics 38, 3888 (1999)), and are a very practical application of an earlier theory developed by us to describe the dynamics of a laser with nonlinear, birefringent
elements in its cavity. These results were quite surprising to many members of the nonlinear optics community.

(d) Nonlinear dynamics of erbium doped fiber lasers and application to synchronization and communication. A large part of our time was involved in the measurements of the fast dynamics of erbium doped fiber ring lasers (EDFRLs). These lasers had previously been examined by many different groups, but their focus had been on slow millisecond dynamics that originated in q-switching behavior due to the long decay time of the upper lasing level of erbium for the 1.55 micron transition. We showed that there was chaotic dynamics at the nanosecond and faster time scales that could be observed and that new models were required to describe these effects (Phys. Rev, A55, 2376 (1997). The inclusion of stochastic effects was an important part of our model, and the equations involved were delay-differential equations of a type developed earlier by Ikeda to describe optical bistability. These equations provide a very good qualitative and often quantitative description of the waveforms measured for different values of parameters of operation of the laser system. The development of the Ikeda model to include stochastic noise was carried out by us (Phys. Lett. A224, 51 (1996) and Phys. Lett. A229, 362 (1997)). The influence of noise as a precursor to bifurcations in laser systems described by these equations, predicted by us in these papers, has been observed Mozdy and Pollock at Cornell on sodium chloride lasers (Phys. Lett. A249, 218 (1998)).

The application of EDFRLs to chaotic communication has been developed in a sequence of experiments, including the first demonstration of optical chaotic communication (Science 279, 1198 (1998)). The development of more sophisticated schemes has been given in later papers (Phys. Rev. Lett. 81, 3547 (1998)) and in a paper accepted for publication (Int. J. Bif. and Chaos, 1999) that is attached with this report. These papers report optical communication with chaotic waveforms at hundreds of Mb/s, demonstrate the possibility of multiplexing of channels at different wavelengths, and show
that messages can be transmitted over tens of kilometers of fiber. The role of polarization variations in the light due to the fiber channel, and the design of a receiver to recover the information suitable for different configurations of the transmitter system are all examined in detail. Addition of a digital message to chaotic light, as well as direct modulation of the chaotic light, are shown to be effective in these communication schemes. The influence of parameter mismatches between transmitter and receiver is investigated, and it is shown that mismatches from a few percent to as much as fifty percent are possible for different parameters, in order to receive and decode the message carried by chaotic waveforms.

Our most recent studies have been concerned with the use of polarization dynamics of EDFRLs for communication of information. As a first step, we developed a technique for the measurement of fast polarization fluctuations in these lasers. Conventional polarization analyzers operate at millisecond time scales or slower. We developed a high speed fiber optic polarization analyzer that can measure polarization dynamics on nanosecond time scales (Opt. Comm. 164, 107 (1999)). The polarization dynamics of a chaotic erbium laser is displayed on a Poincare sphere. We discovered that there are several modes of polarization switching that occur in EDFRLs, between both orthogonal and non-orthogonal polarization states. It is now possible to measure the changes in degree of polarization of the light as well as the time variation of the Stokes parameters.

The final set of measurements that are now being written up for submission involve experiments that utilize the polarization of light for communication of information. Previous methods that use polarization for communication use designated polarization states to correspond to given symbols. We have used a dynamical encoding method in which it is the changes in polarization state that are relevant, and that carry the information. We have demonstrated such communication at a hundred Mbits.
The research described here has initiated work on similar themes by researchers in U.K., Spain, Germany, Australia and Japan. Different schemes and variations of our methods are being developed by these researchers, and preliminary experiments of Gbit/sec communication with chaotic waveforms have been submitted. We have many ideas that we hope to investigate in the near future that concern the transmission and storage of images as spatio-temporal chaotic waveforms, as well as the possibility of data compression through optical implementations of iterated function systems. We believe that nonlinear dynamics has much to contribute to practical applications in information processing and communications, and regard the results reported here a first stage in the development and demonstration of fundamental scientific concepts.
Synchronization of chaos in an array of three lasers

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Synchronization of the chaotic intensity fluctuations of three modulated Nd:YAG lasers oriented in a linear array with either a modulated pump or loss is investigated experimentally, numerically, and analytically. Experimentally, synchronization is only seen between the two outer lasers, with little synchrony between outer and inner lasers. Using a false nearest-neighbor method, we numerically estimate the experimental system dynamics to be five dimensional, which is in good agreement with analytical results. Numerically, synchronization is only seen between the two outer lasers, which matches the experimental data well. Lack of synchrony between outer and inner lasers, is explained analytically and then we numerically investigate loss of synchronization of the two outer lasers, observing the occurrence of a blowout bifurcation. Finally, the effects of noise and symmetry breaking are examined and discussed. [S1063-6511(99)03604-1]

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I. INTRODUCTION

Experimental and theoretical investigations of chaotic synchronization in coupled nonlinear systems have attracted much attention in recent years due to the possibility of practical applications of this fundamental phenomenon. Several papers have studied the synchronization of chaotic signals in the context of electronic circuits [1–3], secure communication [4–6], turbulence in fluids [7,8], chemical and biological systems [9], and laser dynamics [10–14]. Winful and Rahman have numerically investigated the possibility of synchronization of chaos in semiconductor laser arrays on a nanosecond time scale [10] and previously, we have also performed experimental measurements and demonstrated synchronization of two chaotic lasers [15]. To our knowledge, however, the experimental synchronization of chaos in laser arrays with more than two lasers has yet to be reported.

In this paper, the synchronization, both experimentally and numerically, of three coupled, chaotic, Nd:YAG (rivalent neodymium doped yttrium aluminum garnet) lasers in the separate cases of pump and loss modulation is reported. In a linear array of three lasers, a high degree of synchronization between the two outer lasers is seen, while little if any synchronization is observed between the outer and inner lasers. The experimental observations are in good agreement with analytical results, which clearly explain the lack of synchronization between outer and inner laser. Similar results were seen by Winful and Rahman [10] in a numerical model for three semiconductor lasers coupled in a linear array.

The numerical simulations show similar behavior in this coupled linear array of three lasers to that seen in a system of two coupled lasers [14] and we present numerical evidence to suggest that synchronization between the two outer lasers may be lost through a blowout bifurcation, where an attractor contained within the synchronized submanifold loses its transverse stability [16]. This indicates that as in the two laser case, forced symmetry breaking is not necessary for desynchronization of the two outer lasers to occur.

The rest of this paper is arranged as follows. In Sec. II we describe the experimental setup for a system of three Nd:YAG lasers coupled in a linear array and explain the techniques that we used in obtaining the experimental data. Section III describes the equations we used to model the laser system and investigates the occurrence of synchronization between the two outer lasers and also the lack of synchronization between the outer and inner laser. In Sec. IV, we describe how the numerical simulations were performed in the case of loss modulation and finally, in Sec. V, we discuss our findings and consider the implications for coupling large systems of lasers in a linear array.

II. EXPERIMENTAL SETUP

To study the dynamics of a pump or loss modulated three laser array we use the experimental system as shown in Fig. 1. This setup consists of three equal intensity, parallel and laterally separated beams created by pumping a Nd:YAG rod, 5 mm in both length and diameter in a plane parallel cavity. Three Ar+ pump beams (λ = 514.5 nm) are formed by passing a single beam through a fan-out grating designed to produce equal intensities for the zeroth- and first-order beams, and negligible intensities elsewhere. The separation and relative orientation of the three beams of interest are controlled using a simple telescope. The pump beams, in the end, are parallel and symmetric with respect to the axis of the YAG crystal. The optical cavity consists of one high reflection coated end face of the rod and of an external planar output coupler with 2% transmittance. The pump power for the pump modulation case is approximately 5.8 W, and 5.0 W for the loss modulation case. For these parameters, the relaxation oscillation frequency, νR, is of the order of 100 kHz. A thick etalon ensures single longitudinal mode operation. This etalon doubles as an intracavity acousto-optical modulator (AOM) for the loss modulation case. Pump modu-
FIG. 1. Experimental system for generating three laterally coupled lasers in a Nd:YAG crystal and observing the synchronization of chaotic laser intensities. A diffractive optic is used to split the argon laser into three beams with almost equal intensities. The three beams are made parallel by a telescope; changing the magnification of the telescope changes the separation \( d \) between each laser. An Acousto-Optic Modulator (AOM) is placed in position (a) in the case of loss modulation and in position (b), in the case of pump modulation. The Nd:YAG crystal is coated for high reflectivity (HR) on one side and antireflection coated (AR) on the other. The output coupler (OC) is 2% transmissive; both mirrors are flat. A charge-coupled device camera is used to measure the far-field intensity pattern of the array, while the three photodetectors PD1, PD2, and PD3 simultaneously measure each laser’s intensity dynamics, which are subsequently recorded on a digital sampling oscilloscope (DSO).

Synchronization is attained using an AOM positioned before the fanout grating.

Thermal lensing in the YAG rod, generated by Ar+ pump beams with waist radii \( \approx 20 \, \text{\mu m} \) allows the formation of three separate and stable cavities [11]. The \( \text{TEM}_{00} \) infrared laser beams generated in the YAG crystal have radii \( \approx 200 \, \text{\mu m} \). Radii are measured at \( 1/e^2 \) of the maximum intensity of the Gaussian profile. The coupling between the beams is determined by their nearest-neighbor separation, which can be shifted by adjusting the grating and the telescope lenses’ positions. The pump beam separations and profiles are measured directly using a rotating slit method. The minimum value for nearest-neighbor separation used was 0.64 mm, for which there is no appreciable overlap of the pump beams and coupling is entirely due to the spatial overlap of the infrared laser fields. The couplings and detunings were chosen such that, in the absence of modulation, the lasers exhibit an instability caused by the resonance of the phase dynamics with the relaxation oscillations as described, for example, in [13].

The three infrared beams produced by the Nd:YAG laser are separated using a sequence of non-polarizing cube beam splitters and prisms. The intensity dynamics of the individual lasers are recorded simultaneously using fast photodiodes and a four-channel digital oscilloscope. A scanning Fabry-Perot interferometer is utilized to ensure that the individual lasers have only a single longitudinal mode.

Experimental measurements for the pump modulated case are displayed in Fig. 2 for nearest-neighbor separations of approximately 0.975 mm. Chaotic synchronization between the two outer lasers is clearly seen, whereas there is no apparent synchronization between outer and inner lasers. In the case of loss modulation they are displayed in Fig. 3 for nearest-neighbor separations of approximately 0.64 mm. Despite additional noise present in the loss modulated experimental setup, chaotic synchronization between lasers 1 and 3 is readily apparent. Again, pairing intensities of lasers 1 and 3.
FIG. 4. Power spectrum of three linearly coupled lasers, in the case of loss modulation at a rate of 166 kHz. Here the nearest neighbor separations are again 0.64 mm. Notice the peak in the central beam close to 150 kHz, which is not present in the two outer beams. However, the side beams display a peak at approximately 80 kHz of a greater intensity than the corresponding peak in the central beam. The peak in all beams at 166 kHz corresponds to modulation at this rate.

2, as well as lasers 3 and 2, show little synchrony.

It is interesting to note the harmonic relationships between the side lasers, 1 and 3, and the center beam, laser 2. The intensity of laser 2 oscillates at a rate approaching twice the frequency of the side beam oscillations. Figure 4 compares the power spectrums of the individual beams. The dominant peak of the central beam approaches 150 kHz while the side beams display peaks at approximately 80 kHz. The sharp spike at 166 kHz is due to modulation at this frequency.

The intensity time series dynamics of all three lasers was numerically estimated to be five dimensional (Fig. 5), using a false nearest-neighbors method [17], with 25 000 time units considered. This result agrees with the dynamically invariant state labeled amplitude antisynchronized in Table I, corresponding to a system with amplitude synchronization and equal left and right detunings present.

II. EQUATIONS OF MOTION

The equations describing the time evolution of the slowly varying, complex electric field amplitude \( E_i \) and real gain \( G_i \), of laser \( i \) in an array of three spatially coupled, pump modulated single-mode Class B lasers are similar to those of the two-laser system [15] and are as follows:

\[
\frac{dE_1}{dT} = \tau_\epsilon^{-1}[(G_1 - \varepsilon_1(T))E_1 - \kappa E_2] + i \omega_1 E_1,
\]

\[
\frac{dG_1}{dT} = \tau_f^{-1}(p_1(T) - G_1 - G_1|E_1|^2),
\]

\[
\frac{dE_2}{dT} = \tau_\epsilon^{-1}[(G_2 - \varepsilon_2(T))E_2 - \kappa(E_1 + E_3)] + i \omega_2 E_2, \quad (1)
\]

\[
\frac{dG_2}{dT} = \tau_f^{-1}(p_2(T) - G_2 - G_2|E_2|^2),
\]

\[
\frac{dE_3}{dT} = \tau_\epsilon^{-1}[(G_3 - \varepsilon_3(T))E_3 - \kappa E_2] + i \omega_1 E_3,
\]

\[
\frac{dG_3}{dT} = \tau_f^{-1}(p_3(T) - G_3 - G_3|E_3|^2).
\]

In these equations, \( \tau_\epsilon \) is the cavity round-trip time, \( \tau_f \) is the fluorescence time of the upper lasing level of the Nd\(^{3+}\) ion, and \( p_i(T) = p_{i0} + p_1 \cos(\Omega T), \quad \varepsilon_i(T) = \varepsilon_{i0} + \varepsilon_i \cos(\Omega T), \) and \( \omega_1 \) are the modulated pump parameters, modulated losses, and detunings (from a common cavity mode), respectively.

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\[
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\]

\[
\frac{dG_2}{dT} = \tau_f^{-1}(p_2(T) - G_2 - G_2|E_2|^2),
\]

\[
\frac{dE_3}{dT} = \tau_\epsilon^{-1}[(G_3 - \varepsilon_3(T))E_3 - \kappa E_2] + i \omega_1 E_3,
\]

\[
\frac{dG_3}{dT} = \tau_f^{-1}(p_3(T) - G_3 - G_3|E_3|^2).
\]

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TABLE I. Dynamically invariant subspaces in Eqs. (3). A list of symmetry forced invariant subspaces of the equations for a system of three linearly coupled lasers. We have listed only those states that contained an attractor in the numerical simulations. Note that other states exist but are not seen as attracting for the system.

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>Representative point</th>
<th>Dim.</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z_2(\xi) \times Z_2(\mu) )</td>
<td>((X_+, F_+, X_2, F_2, 0, 0, 0, 0))</td>
<td>4</td>
<td>synchronized</td>
</tr>
<tr>
<td>( Z_2(\xi) \times Z_2(\mu)^{\sigma_\pi} )</td>
<td>((X_+, F_+, X_2, F_2, 0, 0, 0, \sigma_\pi, \pi))</td>
<td>4</td>
<td>antisynchronized</td>
</tr>
<tr>
<td>( Z_2(\mu) )</td>
<td>((X_+, F_+, X_2, F_2, 0, 0, 0, 0, \phi, - \phi))</td>
<td>5</td>
<td>amplitude antisynchronized</td>
</tr>
<tr>
<td>( Z_2(\mu)^{\sigma_\pi} )</td>
<td>((X_+, F_+, X_2, F_2, X_-, F_-, 0, 0, 0, \sigma_\pi, \pi))</td>
<td>6</td>
<td>phase synchronized</td>
</tr>
<tr>
<td>( Z_2(\mu)^{\sigma_\pi} )</td>
<td>((X_+, F_+, X_2, F_2, X_-, F_-, 0, 0, 0, \sigma_\pi, \pi))</td>
<td>6</td>
<td>phase antisynchronized</td>
</tr>
</tbody>
</table>
of laser $i$. It is assumed that not both the pump and the loss are modulated at the same time. In the Nd:YAG lasers considered in the experiments, the round-trip time of light in the cavity $\tau_c$ is $0.40-0.50$ ns, while the decay time of the upper lasing level $\tau_l$ is $\approx 240$ $\mu$s. $\Omega$ is the modulation frequency and is chosen to be near the relaxation frequency.

The lasers are coupled linearly to one another with strength $\kappa_{ij}$, assumed to be small. For laser beams of Gaussian intensity profile and $1/e^2$ beam radius $w_0$ the coupling strength, as determined from overlap integral of the two electric fields $i$ and $j$ is defined as

$$\kappa_{ij} = \exp\left(-\frac{(d_i-d_j)^2}{2w_0^2}\right).$$

The coupling strength is normalized such that $\kappa_{ij}=1$ if $d_i-d_j=0$. As the coupling between lasers 1 and 3 is assumed negligible, only nearest-neighbor coupling is considered in 1.

In the analysis that follows we only consider the case of loss modulation, i.e., $p_{11}=p_{22}=p_{13}=0$, but note that the analysis is equally valid in the case of pump modulation [18].

We first let $E_i = X_i e^{i\phi_i}$ where $X_i$ is the amplitude and $\phi_i$ the phase of laser $i$ and rescale time, expressed in units of the round-trip time of light around the cavity $\tau_c$. We subsequently introduce $\Phi_L = \phi_2 - \phi_1$ and $\Phi_R = \phi_3 - \phi_1$ (and similarly for $\Delta_L$ and $\Delta_R$), so that we may rewrite Eqs. (1) as the following system of ordinary differential equations defined on $\mathbb{R}^8$,

$$\frac{dX_1}{dt} = [F_1 - \epsilon_1(i)]X_1 - \kappa X_2 \cos(\Phi_L),$$

$$\frac{dF_1}{dt} = \gamma (A - F_1 - F_1 X_1^2),$$

$$\frac{dX_2}{dt} = [F_2 - \epsilon_2(i)]X_2 - \kappa (X_1 \cos(\Phi_L) + X_3 \cos(\Phi_R)), $$

$$\frac{dF_2}{dt} = \gamma (A - F_2 - F_2 X_2^2), $$

$$\frac{dX_3}{dt} = [F_3 - \epsilon_3(i)]X_3 - \kappa X_2 \cos(\Phi_R),$$

$$\frac{dF_3}{dt} = \gamma (A - F_3 - F_3 X_3^2), $$

$$\frac{d\Phi_L}{dt} = \Delta_L + \kappa \left(\frac{X_2}{X_1} + \frac{X_1}{X_2}\right) \sin(\Phi_L) + \frac{X_3}{X_2} \sin(\Phi_R),$$

$$\frac{d\Phi_R}{dt} = \Delta_R + \kappa \left(\frac{X_3}{X_2} + \frac{X_2}{X_3}\right) \sin(\Phi_R) + \frac{X_1}{X_2} \sin(\Phi_L).$$

The issue of synchronization between the two outer lasers may be addressed by introducing the sum and difference of these lasers and assuming that all three lasers are equally detuned, i.e., $\Delta_L = \Delta_R = 0$. Then, $X_{13} = \frac{1}{2}(X_1 + X_3), X_{13-} = \frac{1}{2}(X_1 - X_3), F_{13+} = \frac{1}{2}(F_1 + F_3), F_{13-} = \frac{1}{2}(F_1 - F_3)$, and synchronization between the two outer lasers occurs when $X_{13-} = F_{13-} = 0$. The transformed system is equivariant under the action of the following symmetries:

$$\xi(X_+, F_+, X_2, F_2, X_-, F_-, \Phi_L, \Phi_R)$$

$$=(X_+, F_+, X_2, F_2, -X_-, -F_-, \Phi_L, \Phi_R),$$

and

$$\mu(X_+, F_+, X_2, F_2, X_-, F_-, \Phi_L, \Phi_R)$$

$$=(X_+, F_+, X_2, F_2, X_-, F_-, -\Phi_L, -\Phi_R),$$

corresponding to interchanging the two outer lasers,

$$\nu(X_+, F_+, X_2, F_2, X_-, F_-, \Phi_L, \Phi_R)$$

$$=(X_+, F_+, X_2, F_2, X_-, -F_-, -\Phi_L, \Phi_R),$$

corresponding to conjugating the phases of the electric fields of all three lasers.

There is also a parameter symmetry involving the coupling parameter $\kappa$ that takes

$$(\kappa, \Phi_L, \Phi_R) \rightarrow (-\kappa, \Phi_L + \pi, \Phi_R + \pi),$$

which adds $\pi$ onto the phase of the middle laser while reversing the sign of $\kappa$. It is interesting to note that all three lasers are phase synchronized when $\kappa$ is negative, corresponding to $\Phi_L = \Phi_R = 0$. However, only the two outer lasers are phase synchronized when $\kappa$ is positive and this is the physically relevant situation since $\kappa$ is assumed positive in some sense.

Owing to these symmetries, the dynamically invariant subspaces illustrated in Table I exist. Notice, in particular, the five-dimensional subspace labeled amplitude antisynchronized, corresponding to the case where the $\mu$ symmetry has been broken, via equal detuning of the two outer beams from a common cavity mode. The dimensionality of the experimental system as calculated using the false nearest-neighbor method gives good agreement with this state and gives emphasis to our assumptions about the parameter regimes considered.

Note that although there are several invariant subspaces where the phases of all three lasers are locked, there are no invariant subspaces forced by symmetry such that all the amplitude and gains are equal, $X_+ = X_2$ and $F_+ = F_2$. We may examine this using two approaches; first by examining the set of such points in the phase space and showing that it is not invariant (cf. [19]) and second by reducing the system of three lasers to one of two lasers with unequal coupling.

To this end, we define the manifold

$$\mathcal{M}_{12} = \{(X_1, F_1, X_2, F_2, X_3, F_3, \Phi_L, \Phi_R) : X_1 = X_2, F_1 = F_2 \& \Phi_R = 0 \text{ or } \pi\},$$

corresponding to perfect (anti)synchronization between lasers 1 and 2 in terms of the original variables.

A. Noninvariance of $\mathcal{M}_{12}$

We demonstrate that if $\kappa \neq 0$, any nonzero trajectory can only be in $\mathcal{M}_{12}$ instantaneously, by assuming that $X_1$ and $X_2$ are nonzero and examining the evolution of the difference $x_+ = \frac{1}{2}(X_1 - X_2)$ and sum $x_+ = \frac{1}{2}(X_1 + X_2)$. Note that
\[
\frac{dx_-}{dt} = \frac{F_1 + F_2}{2} x_- + \frac{F_1 - F_2}{2} x_+ - \varepsilon(t) x_- + \kappa x_- \cos \Phi_L + \frac{1}{2} \kappa X_2 \cos \Phi_R.
\]

If the system state lies on \( \mathcal{M}_{12} \) this means that \( x_- = 0 \) and \( F_1 = F_2 \); so the trajectory at this point will have

\[
\frac{dx_-}{dt} = \frac{1}{2} \kappa X_3 \cos(\Phi_R).
\]

Thus the trajectory must leave \( \mathcal{M}_{12} \) unless \( \kappa = 0 \), \( X_3 = 0 \) and/or \( \Phi_R = (\pi/2) + k \pi, k \in \mathbb{Z} \). We eliminate the first possibility by assumption. If \( X_3 = 0 \) then we note that

\[
\frac{dX_3}{dt} = -\kappa X_2 \cos(\Phi_R)
\]

and so this will be nonzero as long as \( \Phi_R \neq (\pi/2) + k \pi \) for some \( k \in \mathbb{Z} \), but from our definition of \( \mathcal{M}_{12}, \Phi_R = 0 \) or \( \pi \), so any trajectory satisfying Eq. (4) will not be contained in \( \mathcal{M}_{12} \). For the same reason we rule out the case \( \Phi_R = (\pi/2) + k \pi \) and this implies that a trajectory can only be in \( \mathcal{M}_{12} \) for an instant in time. As a result, \( \mathcal{M}_{12} \) is only an invariant subspace for the ordinary differential equation if \( \kappa = 0 \) and the only trajectories that remain within \( \mathcal{M}_{12} \) for all time have \( X_1 = X_2 = X_3 = 0 \).

B. Reduction to a system of two lasers with unequal coupling

If we assume that we lie on one of the amplitude synchronized subspaces, where \( X_- = F_- = 0 \), i.e., \( X_1 = X_3 \) and \( F_1 = F_3 \), then the system (3) simplifies to a two laser system with unequal coupling between the two lasers.

\[
\frac{dX_1}{dt} = [F_1 - \varepsilon(t)] X_1 - \kappa X_2 \cos(\Phi),
\]

\[
\frac{dF_1}{dt} = \gamma (A - F_1 - F_1 X_1^2),
\]

\[
\frac{dX_2}{dt} = [F_2 - \varepsilon(t)] X_2 - 2 \kappa X_1 \cos(\Phi),
\]

\[
\frac{dF_2}{dt} = \gamma (A - F_2 - F_2 X_2^2),
\]

\[
\frac{d\Phi}{dt} = \kappa (X_2 X_1^{-1} + 2 X_1 X_2^{-1}) \sin(\Phi).
\]

Introducing sum and difference variables in this case gives us the transformed system,

\[
\frac{dX_+}{dt} = X_+ (F_+ - \varepsilon(t)) + F_- X_- - \kappa \cos(\Phi) (3 X_+ + X_-),
\]

\[
\frac{dF_+}{dt} = \gamma (A - F_+ (1 + X_+^2 + X_-^2) - 2 F_- X_+ X_-),
\]

\[
\frac{dX_-}{dt} = X_- (F_- - \varepsilon(t)) + F_+ X_+ + \kappa \cos(\Phi) (3 X_- + X_+),
\]

\[
\frac{dF_-}{dt} = -\gamma (F_- (1 + X_+^2 + X_-^2) + 2 F_+ X_- X_+),
\]

\[
\frac{d\Phi}{dt} = \kappa \left( \frac{3 (X_+^2 + \frac{2}{3} X_+ X_- + X_-^2)}{(X_+^2 - X_-^2)} \right) \sin(\Phi)
\]

If we assume that the two lasers \( X_1 \) and \( X_2 \) are synchronized then we find that

\[
\frac{dX_-}{dt} = \kappa \cos(\Phi) X_+,
\]

\[
\frac{dF_-}{dt} = 0,
\]

\[
\frac{d\Phi}{dt} = 3 \kappa \sin(\Phi),
\]

assuming that \( \kappa \neq 0, X_+ \neq 0 \), then we see that \( X_- = 0 \) for at most an instant in time. Since if \( \cos(\Phi) = 0 \) then \( \Phi = (\pi/2) + k \pi \) for some \( k \in \mathbb{Z} \) and so

\[
\frac{d\Phi}{dt} = 3 \kappa,
\]

which is nonzero and therefore \( \Phi \) moves away from \( (\pi/2) \) (mod \( \pi \)). Consequently \( dx_- / dt \) moves away from 0 and so \( X_- \) also moves from 0. Therefore synchronization is not achieved in the asymmetric two laser setup and thus not achieved in the original three laser system.

IV. NUMERICAL RESULTS

We carried out numerical simulations independently in both the loss modulation situation as well as modulation of the pump excitation. We concentrate on the loss modulated situation due to numerical considerations, but note that our results remain valid in the case of pump modulation [18].

A. Loss modulated case

For the loss modulated case, the simulations were performed using both Bulirsch-Stoer and Runge-Kutta integrators. Due to numerical considerations we were forced to consider more moderate values of the stiffness parameter \( \gamma \), which was of the order 0.01 and 0.001. The parameter regimes considered were also altered in order that the difference in \( \gamma \) was taken into account. In both the cases \( \gamma = 0.01 \) and \( \gamma = 0.001 \) we saw similar results, and although
FIG. 6. Lyapunov exponent diagram in the case of modulated loss. The parameter values for the lasers were assumed identical and were \( \sigma_0 = 0.9, \sigma_1 = 0.2, \rho_i = 1.2 \) (for \( i = 1,2,3 \)). We assumed the detunings of the lasers were such that \( \Delta_l = \Delta_r = 0 \). We have labeled the largest tangential Lyapunov exponent \( \lambda_1 \). Notice that this is positive for most values of the coupling strength \( \kappa \). The non-normality of \( \kappa \) is apparent through the windows of stability that arise when varying \( \kappa \). These correspond to the periods where \( \lambda_1 \) is negative. The blowout occurs when the normal Lyapunov exponent, \( \lambda_1 \) passes through 0. In this case this occurs for \( \kappa \approx 0.003 \) 125.

The experiments were carried out with \( \gamma = 10^{-6} \), the use of longer resonators would give a value of the stiffness parameter somewhat closer to that considered numerically. We carried out simulations for many values of the pump coefficient and various modulation strengths for the loss.

As in the model for a two laser system, in the case \( 0 < \gamma < 1 \), the system undergoes a period doubling cascade to chaos as the strength of loss modulation is increased. Typically we see that for small values of the coupling parameter \( \kappa \), there is no amplitude synchronization and the amplitude behavior of all three lasers appears to be independent, although with antiphase synchronization between adjacent lasers. As the coupling strength is increased, a period of on-off intermittent type behavior [20], is observed in the amplitude fluctuations of the two outer lasers. During this period there are times when the two outer lasers appear to be synchronized in both amplitude and phase, before being away from the amplitude synchronization, while remaining completely phase (anti)synchronized. Then as the coupling strength is increased still further, there is no more bursting away from synchrony and the two outer lasers remain amplitude synchronized for all time after an initial transient phase.

For the particular case where all losses are modulated equally at the rate, \( 0.9 + 0.2 \cos (0.045 t) \), the pump parameters were equal to 1.2 for each laser and \( \Delta_l = \Delta_r = 0 \), the behavior of a typical trajectory is as follows. Upon varying the strength of coupling \( \kappa \), we see that there exists a critical value \( \kappa_c \approx 0.003 \) 125 such that for values of \( \kappa < \kappa_c \), trajectories evolve on to the phase antisynergized state. For values of \( \kappa > \kappa_c \) trajectories evolve on to the amplitude antisynergized state. This transition at \( \kappa_c \) is strongly suggestive of a blowout bifurcation, as was the case in a system of two lasers [14].

A blowout bifurcation occurs when a normal Lyapunov exponent governing the exponential rate of change transverse to a submanifold of the total phase space passes through 0. In the case where there is more than one transverse Lyapunov exponent we need consider only the largest or normal Lyapunov exponent. If the normal exponent is negative, then on average nearby trajectories are attracted onto the submanifold and the attractor within the subspace is an attractor for the full system. If the exponent is positive then on average trajectories close to the submanifold are repelled away from it.

We have numerically computed the Lyapunov exponents of Eq. (3) by integrating the variational equations and examine the change that occurs in the exponents upon varying the coupling strength \( \kappa \). These are illustrated in the case of no detunings in Fig 6.

For this system, the blowout bifurcation does not occur at an isolated parameter value because the bifurcation parameter \( \kappa \) varies the dynamics tangentially within the antisynergized subspace as well as those in a transverse direction from it; it is not a normal parameter for the dynamics [21,22]. Because of this (and apparent fragility of the chaotic attractors) we do not expect the Lyapunov exponents to vary smoothly or even continuously with the parameter. Hence we observe a blurred blowout [22].

The tangential variation of the dynamics is clearly indicated in Figs. 6 and 7, where windows of stability arise as the coupling strength \( \kappa \) is increased. These windows of stability correspond to all Lyapunov exponents of system (3) being negative. In particular, there is a window of stability shortly after the bifurcation point.

In order to examine the branching behavior at blowout, we have simulated the behavior of typical trajectories that are not in any invariant subspace. Starting at \( \kappa_c \), there appears to exist a chaotic attractor \( A \) within the antisynergized subspace, since after an initial transient phase (which may be prolonged for some initial conditions), all trajectories eventually appear to converge to the antisynergized sub-
FIG. 8. Numerical simulated three laser model with pump modulation. The modulation rate was again chosen to be near the relaxation oscillation frequency of the lasers so as to induce chaotic fluctuations in the intensities.

space. Reducing $\kappa$ towards $\kappa_c$, we find regions of region of on-off intermittent type behavior, typical for a supercritical blowout.

After the blowout, we no longer observe any attractors in the antisynchronized subspace, but there is a new branch of attractors in the phase antisynchronized subspace are created at the bifurcation. Just after $\kappa_c$, these attractors are apparently on-off intermittent and close to the antisynchronized subspace. The average position of the trajectory moves away as $\kappa \rightarrow 0$. This is a strong indicator that the blowout is of supercritical, soft or nonhysteretic type [16].

We also performed simulations of three loss modulated lasers in situations where the detunings were equal, i.e., $\Delta_L = \Delta_R = \Delta$. We calculated the Lyapunov spectrum in this case and saw similar results to that of the purely symmetric case, with the main difference being a bifurcation from the amplitude antisynchronized subspace, rather than the antisynchronized subspace. Again the blowout appears to be soft with an extended period of on-off intermittent behavior.

For the particular case with parameters identical to those considered above and a value of the detuning, $\Delta = 0.001$, the Lyapunov spectrum upon varying $\kappa$ is illustrated in Fig. 7. Again a blurred blowout is evident, and the normal Lyapunov exponent passes through zero at $\kappa_c \approx 0.003175$.

B. Pump modulation

The numerical simulations in the case of modulation of the pump excitation were carried out using a Runge-Kutta integrator with a variable time step. Frequency of the depth of modulation was chosen so that the dynamics of the system was in a region of chaotic behavior and in this case was chosen to be 100.53 kHz (in the case of loss modulation it was 139.62 kHz). As in the case of loss modulation, excellent agreement between the experimental results and the numerical simulations are seen. A high degree of synchronization between the two outer lasers and no apparent synchronization between outer and inner laser. The transient behavior displayed similar characteristics when compared to the loss modulated simulations, such as bursts away from synchronization over short time scales, before settling onto the synchronized subspace after longer periods of time.

Some of the numerical simulations we performed are illustrated in Fig. 8. The bifurcation analysis is not performed here, since the simulations indicate similar bifurcation behavior to that of the loss modulated case, as would be expected [18].

V. DISCUSSION

Concluding this work, the synchronization of three class B Nd:YAG lasers, coupled in a straight line linear array, is investigated experimentally, analytically and numerically. We investigate the separate cases of pump modulation and loss modulation both experimentally and numerically. In the experiments, a high degree of synchronization is observed between the two outer lasers of the array, while no synchronization is observed between outer and inner lasers. This is in good agreement with the theory, which demonstrates this lack of synchronization between outer and inner lasers. In the case of loss modulation we see numerically how the loss of synchronization between the two outer lasers is lost in both the fully symmetric case and in the case with equal left and right detunings, via an apparent supercritical blowout bifurcation. This is achieved by varying the strength of coupling between the three lasers.

For the experimental system, noise and symmetry breaking are both inherent, but even with quite high levels of noise, we have demonstrated a good degree of synchronization particularly in the loss modulated case. In the numerical simulations, noise and symmetry breaking have similar effects; in the region of on-off intermittency, it is unlikely that there will be a noticeable change if the perturbations are
small. Low levels of noise and imperfections can result in bubbling type effects [23], which can resemble on-off intermittency in numerical simulations. Consequently, the effect of bubbling on systems such as ours is similar to the effects of on-off intermittency, namely bursts away from a synchronized state. Such bubbling persists up to a point known as a bubbling transition [24] (see also the related riddling bifurcation [25]). This situation arises when an orbit embedded in a symmetric chaotic attractor loses its transverse stability. A more detailed description of this situation may be found in [26].

It is interesting to see the harmonic relationships between the central and the outer beams. Particularly for the loss modulated case with small nearest-neighbor separations, the central beam appeared to be at a rate approaching twice that of the two outer beams. We conjecture that this surprising phenomenon may be caused by the central beam communicating a greater quantity of information than the two outer beams. One area of future research is to investigate these dynamics and examine the effect of parameter variation on the harmonic relationship.

Although we have shown that there will be no synchronization between the outer and inner lasers in a three laser array, the question of generalized synchronization [27] arises. As we have shown, assuming that the two outer lasers are synchronized allows us to simplify the model to a system of two lasers with unequal coupling between the two lasers. This does not immediately fall into the category of generalized synchronization, since there is feedback from the “response” system into the “driving” system. However, it may still be possible to make similar conclusions to those of generalized synchronization in the case where the feedback from the one system is small compared to the input from the other.

Numerical simulations of the model suggests that for small symmetry breaking perturbations of the amplitude synchronized state, an instability should arise in the phase locking of the three lasers as predicted analytically and numerically in a system of two lasers coupled in a linear straight line array [19]. Another interesting area of future experimental work would be to heterodyne the outer beams, examine the beat frequencies over time to investigate the phase-locking instability. Such an instability may have an important bearing on maximizing power output and coherence in larger arrays of coupled lasers.

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Transmission of linearly polarized light through a single-mode fiber with random fluctuations of birefringence

Gregory D. VanWiggeren and Rajarshi Roy

A simple theoretical formalism is developed to describe the effect of transmission on linearly polarized light through a fiber with random fluctuations of birefringence. We conclude that, for any optical fiber that does not experience polarization-dependent gain or loss, there exist two orientations for linearly polarized light input into the optical fiber that will also exit the fiber linearly polarized. We report experimental results that verify this prediction and also investigate its practical implications and limitations; in particular we investigate the stability of these linearly polarized output states in laboratory conditions. © 1999 Optical Society of America

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1. Introduction

When one works with fiber-optic systems in the laboratory, it is often desirable to couple linearly polarized light into a fiber and to obtain linearly polarized light at the output as well. One way to accomplish this is to use polarization-maintaining fibers. The high birefringence of these fibers allows for linearly polarized light launched along the proper axis to travel long distances without change of polarization state. In ordinary single-mode fiber, however, the polarization state evolves rapidly as the light propagates. The output polarization will appear uncorrelated to the input polarization after only a few meters of propagation. However, it can be shown that unperturbed single-mode fiber can perform the same function as polarization-maintaining fiber in certain situations. In this paper we predict theoretically and demonstrate experimentally that, for any ordinary single-mode optical fiber, two orientations of linearly polarized quasi-monochromatic input light will also exit the fiber linearly polarized. When the ordinary single-mode fiber is not perturbed by external stresses or temperature changes, these states are reasonably stable for hours—especially for shorter fiber lengths (100 m or less).

Earlier researchers have also observed linearly polarized input and output of light through a fiber. In Refs. 1 and 2 it is noted that linear light input into a fiber can be output linearly polarized as well, but no theoretical explanation is provided. In Refs. 3 and 4 the observations of linearly polarized input and output of light through a fiber were explained with a model that treats the birefringence of the optical fiber as constant in magnitude and orientation throughout the length of the fiber. This assumption is not, in general, correct. Much of the research into the propagation of polarized light in optical fiber has been in the context of polarization-mode dispersion, a subject that is not addressed in this paper.

Here the birefringence fluctuations along a fiber are treated as a concatenation of wave plates with each wave plate possessing an arbitrary birefringence. The simple Jones matrix formalism used to analyze such a concatenation provides a framework for understanding many polarization phenomena observed in optical fiber, including polarization-mode dispersion and four-wave mixing in single-mode fiber. The formalism can be used to show that the operation of randomly fluctuating birefringences in an optical fiber is the same as the operation of only one constant birefringence for the whole fiber, as was assumed in Refs. 3 and 4. The formalism provides a simple way of mathematically determining the particular orientation of linearly polarized input light that will also exit a fiber linearly polarized.
clearly indicates that only two such orientations can exist, that they are orthogonal, and that in general they are not eigenpolarizations. In fact, this formalism shows that the polarization state of eigenpolarized light can evolve as it propagates through the fiber. Finally, the framework offers a way of understanding the effects of polarization-dependent gain or loss in an optical fiber. Experiments verifying these predictions and demonstrating their usefulness in a laboratory setting are performed and the results given in Sections 3 and 4.

2. Theory

At any point along a single-mode fiber the local birefringence is typically of the order of $10^{-7} < (n_1 - n_2)/\sqrt{n_1 n_2} < 10^{-5}$. Even such a small birefringence can lead to large changes in the polarization state of light over 1 m. In real fibers, the magnitude of the birefringence is never constant throughout a length of fiber. Instead, it fluctuates according to whatever local stresses, internal or external, exist in the fiber. To complicate matters further, the orientation of the index ellipsoid rotates unpredictably from one point to the next in fiber and is sensitive to movement of the fiber or changes in temperature. The birefringence in an optical fiber is also a function of wavelength, but this wavelength dependence is not accounted for in the analysis given. Consequently, the results from the analysis presented in this paper are valid only in situations in which this wavelength dependence can be neglected. This condition is often well satisfied in laboratory settings where the bandwidths of the optical sources are typically 1 nm or less and the lengths of fiber are less than a few kilometers.

As mentioned above, a length of single-mode optical fiber is modeled here as a concatenation of differential elements, each element a wave plate possessing a birefringence of arbitrary orientation and magnitude. After passing through one element, the light passes into a second element, and so on, until it reaches the end of the fiber. In a Jones matrix representation, this process can be described by a series of phase-shift and rotation matrices.

A phase shift of $\delta$ is described by the matrix

$$C(\delta) = \begin{bmatrix} \exp(i\delta/2) & 0 \\ 0 & \exp(-i\delta/2) \end{bmatrix}. \quad (1)$$

In the model the phase shift is given by $\delta = 2\pi L(n_1 - n_2)/\lambda$, where $L$ is the length of the differential element and $n_1$ and $n_2$ are the indices of refraction along the fast and slow axes, respectively. A rotation of the index ellipsoid by an angle $\theta$ is represented by the matrix

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}. \quad (2)$$

The polarization properties of a length of fiber, then, can be represented as a product of many unknown phase-shift matrices and rotation matrices:

$$M = C(\delta)R(\theta)C(\epsilon)R(\phi). \ldots \quad (3)$$

The product of a phase-shift matrix and a rotation matrix can produce any arbitrary unitary matrix. Consequently, the form of $M$ is also the most general form of a unitary matrix:

$$M = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}, \quad (4)$$

where $|a|^2 + |b|^2 = 1$. The unitary nature of the matrix $M$ allows for it to be decomposed into only one appropriate phase-shift matrix $C$ multiplied by one rotation matrix $R$. Interestingly, this suggests that any fiber's net effect on the polarization state of light is identical to the effect of one particular wave plate with a constant phase shift and orientation of its axes.

A proof for the form of $M$ can be given quickly with the Pauli matrix

$$\Sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \quad (5)$$

Now $\Sigma$ has the following properties:

$$\Sigma^2 = -I,$$

where $I$ is the identity matrix,

$$\Sigma R(\theta) \Sigma = -R(\theta), \quad (6)$$

$$\Sigma C(\gamma) \Sigma = -C(\gamma),$$

where the elements of $C$ are the complex conjugates of the elements of $C$. If a general matrix form for $M$ is assumed, then

$$\Sigma M \Sigma = -\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}. \quad (6)$$

From Eq. (3) it is also true that

$$\Sigma M \Sigma = \Sigma C(\delta)R(\theta)C(\epsilon)R(\phi). \ldots C(\gamma)R(\psi) \Sigma$$

$$= -\Sigma C(\delta)\Sigma^2 R(\theta) \Sigma \ldots \Sigma^2 C(\gamma) \Sigma^2 R(\psi) \Sigma$$

$$= -C(\delta)R(\theta)C(\epsilon)R(\phi). \ldots C(\gamma)R(\psi)$$

$$= -C(\delta)R(\theta)C(\epsilon)R(\phi). \ldots C(\gamma)R(\psi)$$

$$= -M = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix}. \quad (7)$$

When we take the results from Eqs. (6) and (7), it is clear that $d = a^*$ and $c = -b^*$ and that the form of $M$ given in Eq. (4) has been proved.
Light in a fiber has electric-field components along two orthogonal transverse axes. These axes are chosen arbitrarily and denoted here as x and y. In the Jones matrix representation these complex components are represented as a vector $\mathbf{j}_{\text{in}} = (E_x, E_y)$. The output fields are represented by $\mathbf{j}_{\text{out}} = (E'_x, E'_y)$. The transformation of light as it propagates through the length of the fiber is thus given by

$$\mathbf{j}_{\text{out}} = \mathbf{M} \cdot \mathbf{j}_{\text{in}}. \quad (8)$$

To obtain linearly polarized output from the fiber, the ratio $E'_x/E'_y$ must be real, though both components are in general complex. In other words, the input electric-field components $E_x$ and $E_y$ must satisfy the condition

$$\text{Im} \left( \frac{E'_x}{E'_y} \right) = 0 = \frac{ae_y + be_x}{-b^*E_x + a^*E_y} - \text{c.c.}, \quad (9)$$

where c.c. is the complex conjugate.

The ratio of the input electric-field components $E_x/E_y$ is also real, because the input light is linearly polarized. Rewriting Eq. (9) in terms of the ratio $r = E_x/E_y$ and rationalizing gives

$$r^2 - \frac{\text{Im}(a^2 - b^2)}{\text{Im}(ab)} r - 1 = 0. \quad (10)$$

The two solutions for $r$ given by Eq. (10) correspond to orientations of linearly polarized input light that will also exit the fiber linearly polarized. By using these solutions for $r$ to construct the vectors $\mathbf{j}_{\text{in+}}$ and $\mathbf{j}_{\text{in-}}$ and then taking the dot product, we can show that the two solutions, and thus their corresponding input orientations, are orthogonal. Finally, solving Eq. (8) for $\mathbf{j}_{\text{out+}}$ and $\mathbf{j}_{\text{out-}}$, we can show that the output orientations are also orthogonal. It should be noted that the polarization states of these solutions are not maintained as they propagate; instead, they evolve continuously. The evolution of these particular orientations, however, is precisely such that the light exits the fiber linearly polarized. It is also important to note that neither $\mathbf{j}_{\text{in+}}$ nor $\mathbf{j}_{\text{in-}}$ is an eigenvalue. Although both the input and the output are linearly polarized, the angular orientation is not necessarily the same.

A less idealized model for a length of optical fiber would have to include effects such as gain or loss. For long fibers, loss may be significant. Gain and loss can be incorporated into the description given above by simple inclusion of an appropriate Jones matrix for a polarization-dependent gain or loss,

$$\mathbf{G}(c, d) = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}, \quad (11)$$

where $c$ and $d$ are real.

The gain or loss need not be oriented along the same direction as the birefringence, but an extra rotation matrix in each differential element would compensate for this. Thus

$$\mathbf{M} = \mathbf{C}(\delta)\mathbf{R}(\theta)\mathbf{G}(c, d)\mathbf{R}(\phi)\mathbf{C}(\epsilon)\mathbf{R}(\psi)\mathbf{G}(e, f)\mathbf{R}(\zeta). \ldots \quad (12)$$

By applying $\mathbf{G}$ to the framework developed above, we can show that if $c = d$ then the effect of such a loss or gain is simply to multiply $\mathbf{M}$ by a constant value. In this case the discussion above is unaffected. However, if the elements of $\mathbf{G}$ are not identical, $c \neq d$, in each differential element, the proof given above no longer holds. Stated another way, if a fiber has polarization-dependent gain or loss, $\mathbf{M}$ will not have the same form as in Eq. (4), and the treatment given above will no longer apply.

3. Experiment

Birefringences in a fiber, and thus the matrix $\mathbf{M}$ for that fiber, are highly sensitive to movement and to other environmental perturbations. It is important, therefore, to verify that the theory's predictions can be demonstrated experimentally and to investigate the conditions in which the theory applies.

Figure 1 shows the basic setup. A tunable diode laser is used to produce 4 mW of light with a wavelength of 1550 nm and a 150-kHz linewidth. The light propagates down a fiber and is coupled to free space with a graded-index (GRIN) lens where it passes through a sequence of three wave plates ($\lambda/2$, $\lambda/4$, $\lambda/2$). These three wave plates operate on the light from the tunable diode laser to ensure that the light is roughly circularly polarized when incident on the polarizer so that roughly equal power is transmitted through the polarizer as it is rotated. Another GRIN lens couples the light that passes through the polarizer into the test fiber.

In the first set of experiments the light transmitted by the first polarizer is launched into the test fiber.
and propagates until it reaches a second polarizer. The light that passes through the second polarizer is then input into an optical power meter. The two polarizers are adjusted to achieve maximum extinction as measured by the power meter. Experimentally, two orientations of the first polarizer were found, which allowed for maximum extinction. Thus the experiment demonstrates that two orientations of linearly polarized light gave linearly polarized light at the output of the test fiber. This was true for both the short, 8-m, and the long, 35-km, lengths of fiber that were tested. For both lengths it was also observed that the orientations of the two linearly polarized inputs were orthogonal to within a measurement error of ±0.5°. The linearly polarized output, to within the same error, was also orthogonal. All of this is as predicted by the proof developed above. The extinction ratio for the short test fiber was >45 dB, whereas for the 35-km sample it had fallen to 35 dB. In all cases the ratio is large enough to accurately determine the location of the extinction maxima. The decrease in the extinction ratio is most probably due to scattering phenomena that depolarize the propagating light.

As a measure of the wavelength dependence of this result, amplified spontaneous emission light from an erbium-doped fiber amplifier (EDFA) was used as the source. The amplified spontaneous emission light is very broadband, possessing a 3-dB bandwidth of approximately 5 nm centered at 1532 nm. Even with such broadband light, a >15-dB extinction ratio was obtained after propagation through an approximately 1.5-km optical fiber.

Another set of experiments was performed with a polarization analyzer. For the 8-m and the 35-km test fibers mentioned above, a series of measurements were made in which the input polarizer was rotated, and the resulting polarization states were tracked on the Poincaré sphere. As can be seen in Fig. 2, rotating the polarizer through 180° causes great-circle paths to be traced out on the Poincaré sphere. The paths intersect the equator twice, and on opposite sides of the sphere. Because the equator of a Poincaré sphere represents linear polarization ($s_3 = 0$), these paths indicate that two orientations of the input polarizer will result in linearly polarized light output from the fiber. The fact that the intersections with the equator occur on opposite sides of the sphere demonstrates that the two linearly polarized output states are orthogonal, as predicted by the theory. The different paths traced around the Poincaré sphere shown in Fig. 2 for the same fiber are the result of simple rearrangement of the way the fiber lay on the experimental table. This is simply a manifestation of the fact that the polarization properties of fiber are sensitive to changes in external stresses.

For single-mode fiber to be useful in transmitting linearly polarized light from one place to another, the polarization state of the output light should be relatively stable under laboratory conditions. The polarization state of light output from a fiber was tracked over time by a polarization analyzer. As seen in Fig. 2, the polarization state of the output light is represented as a point on the Poincaré sphere. Figure 3 shows the wandering of that point with time. In Fig. 3(a), the 8-m case, the fiber was coiled and lay on an optical table. The data shown were obtained over the course of 32 h. The relatively small excursions show that the polarization state was fairly sta-

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Fig. 2. Polarization state paths (thicker curves) traced at the output of the fiber as the polarizer at the input is rotated through 180°. (a) 8-m test fiber. (b) 35-km test fiber.

Fig. 3. Smear of points on the Poincaré sphere represent the evolution of the output polarization from an 8-m length of optical fiber during 32 h under typical laboratory conditions. (b) Same experiment showing the evolution of the output state of polarization of a 35-km length of fiber over only 4 h. Both experiments are intended to give some practical indication of the stability of the output polarization of a single-mode fiber when the input polarization is held constant.

Fig. 4. (a) Polarization path traced by output light after experience of a small polarization-dependent gain in an EDFA. (b) Path traced by output light experiencing polarization-dependent loss.

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ble over that entire time. Figure 3(b) was obtained over a period of 4 h with a 35-km length of fiber wrapped around a spool in the same laboratory setting. Not surprisingly, the length of the fiber seems to have an effect on the stability of the polarization state over time.

4. Effects of Loss and Gain

Two more experiments were performed to test the effect of polarization-dependent loss and gain on the polarization properties of a fiber. In the first experiment an EDFA was used as a source of polarization-dependent gain. The amplifier’s specifications suggest a polarization-dependent gain of <0.3 dB, which is consistent with the very small (difficult to measure) ~0.1 dB of polarization-dependent gain observed in the laboratory for a small signal gain of ~27 dB. Although the EDFA gives an almost polarization-independent amplification of signals, it seems plausible that the winding of the doped fiber and the polarization of the pump lasers would contribute to polarization-dependent gain within individual elements along the fiber. This would result in local differential element gain matrices, G(c, d), which have c ≠ d. Although the total output power might be almost polarization independent, the form of M would be different from Eq. (4). This polarization gain dependence results in the deviation from a great circle that is observed in Fig. 3(a).

Another experiment was conducted to investigate the effect of polarization-dependent loss. In the test fiber portion of the experiment the light was coupled to free space, again with a GRIN lens. From there it passed through a birefringent calcite crystal that acts as a polarizing beam splitter. When we controlled the angle of a glass plate behind the crystal, the amount of light from each polarization that was coupled through a GRIN lens and back into the test fiber could be controlled. Depending on the angle of the glass plate, one polarization or the other could be coupled preferentially back into the test fiber. This device acts as a polarization-dependent loss. When such a polarization-dependent loss is created, the path on the Poincaré sphere that results from rotation of the input polarizer is not a great circle, as is evident in Fig. 3(b).

5. Conclusion

Using the Jones matrix formalism, we have proved that two orthogonal orientations of linearly polarized light can be launched into any single-mode fiber such that linearly polarized light is output from the fiber. Experiments were performed that verified these predictions. They also revealed, in accordance with the model, that these same predictions cannot be extended to the case of fiber with polarization-dependent gain or loss.

Knowing that ordinary single-mode fiber can transmit linearly polarized light may have some practical applications in laboratory settings. As with polarization-maintaining fiber, the proper axes for input and output must be found before this property can be used. Unlike with polarization-maintaining fiber, when we obtain linearly polarized input and output light with ordinary single-mode fiber, precautions must be taken against perturbing the fiber once these axes have been found. Although the particular orientation of linearly polarized input and output light for an unperturbed length of fiber may drift on a time scale of minutes or hours, it seems plausible that a simple feedback algorithm could be developed for maintaining the proper axes indefinitely.

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References

Chaotic Communication using Time-Delayed Optical Systems

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Abstract

We discuss experimental demonstrations of chaotic communication in several optical systems. In each, an erbium-doped fiber ring laser (EDFRL) produces chaotic fluctuations of light intensity onto which is modulated a message consisting of a sequence of pseudorandom digital bits. This combination of chaos and message propagates at a wavelength of ~ 1.5 microns through standard single-mode optical fiber from the transmitter to a receiver, where the message is recovered from the chaos. We present evidence of the high-dimensional nature of the chaotic waveforms and demonstrate chaotic communications through 35 km of single-mode optical fiber at up to 250 Mbit/s, a rate that is, at present, limited only by the speed of our detector electronics.
I. Introduction

A chaotic waveform that serves as a carrier of information represents a generalization of the more traditional sinusoidal carrier and offers the potential for enhanced privacy in communications. In ordinary radio communication, a specific frequency sine-wave carrier is modulated with a message and transmitted. A radio receiver must be tuned to the particular frequency of the carrier sine wave in order to recover the message. In conceptually the same way, the experiments presented here demonstrate that information can be recovered from an optical chaotic carrier using a receiver that is synchronized or "tuned" to the chaotic dynamics of the transmitter.

The synchronization of chaotic systems plays an important role in chaotic communications. The application of chaotic synchronization to secret communication systems was suggested in early work by Pecora and Carroll [1990; 1991]. They discovered that a chaotic transmitter could consist of an electronic circuit that simulated the dynamics, for example, of the Lorenz model [Ditto & Pecora, 1993]. The message to be concealed, assumed small in magnitude, was added to the chaotic fluctuations, assumed to be much larger, of one of the variables (let us choose the z variable for this purpose) and transmitted to the receiver, while another chaotic variable (let us choose x) was separately transmitted. The receiver consisted of a subsystem of the circuits in the transmitter that generated the dynamics of the y and z variables, and was driven by the signal from the x variable of the transmitter. The receiver synchronized to the chaos of the transmitter if the conditional Lyapunov exponents for the systems were negative for the given operating parameters.
One could then recover the message from the chaos through a subtraction at the receiver.

Cuomo and Oppenheim [1993; see also Strogatz, 1994] introduced an elegant variation of the method above that did not require the separate transmission of a driving signal to the receiver. They showed that the receiver could actually synchronize to the chaotic dynamics of the transmitter even when a message was added to the chaotic driving signal from the transmitter. The synchronized output from the receiver was then used to subtract out the information from the transmitted signal. The synchronization was not perfect, and the message, treated as a perturbation of the chaotic signal, had to be small compared to the chaos [Cuomo et al., 1993]. Development of techniques in which the message actually drives the chaotic transmitter system, in addition to being transmitted, were made by Wu and Chua [1993], Volkovskii and Rulkov [1993], and by Parlitz et al. [1996]. The synchronization between receiver and transmitter can be exact, so message recovery can be very accurate in principle. The experiments reported in this paper are related in spirit to a method developed first in electronic systems by Volkovskii and Rulkov [1993], who suggested the use of an open-loop system in the receiver. A different, adaptive approach to synchronization and secure communications was introduced by Boccaletti, Farini and Arecchi [1997].

A proposal to use modulated unstable periodic orbits (UPOs) for secure communications and multiplexing was made by Abarbanel and Linsay [1993]. Multiplexing would be possible by using different UPOs to carry different messages. An alternate approach to chaotic communications with UPOs was developed by Hayes et al. [1993; 1994]. They symbolically encoded digital information into UPOs of a chaotic
system and used chaos control methods to switch between different orbits. This approach does not attempt to provide any privacy to the information being transmitted.

The issue of privacy, however, arises naturally in a discussion of chaotic communication and is an important motivation for chaotic communication research. In his pioneering paper, "Communication Theory of Secrecy Systems", Claude Shannon discussed three aspects of secret communications systems: concealment, privacy, and encryption [1949; see also Hellman, 1977; Welsh, 1988]. These aspects apply to systems that use chaotic waveforms for communication and can be interpreted in that context. Concealment of the information occurs because the chaotic carrier or masking waveform is irregular and aperiodic; it is not obvious to an eavesdropper that an encoded message is being transmitted at all. Privacy in chaotic communication systems results from the fact that an eavesdropper must have the proper hardware and parameter settings in order to decode and recover the message. In conventional encryption techniques, a key is used to alter the symbols used for conveying information. The transmitter and receiver share the key so that the information can be recovered. In a chaotic communication system, a transmitter that generates a time-evolving chaotic waveform acts as a "dynamical key" to transform the information symbols. The information can be recovered with a receiver possessing the same dynamical key, i.e., its configuration and parameter settings are matched to those of the transmitter. It is interesting to note that using a chaotic carrier to dynamically encode information does not preclude the use of more traditional digital encryption schemes as well. Dynamical encoding with a chaotic waveform can thus be considered an additional layer of encryption.
Two factors that are important to privacy considerations in chaotic communication systems are the dimensionality of the chaos and the effort required to obtain the necessary parameters for a matched receiver. Earlier work has shown that for certain chaotic communication techniques, particularly those involving additive masking of a message by a chaotic carrier, the message can be recovered from the transmitted signal by mathematically reconstructing the transmitter's chaotic attractor if the chaos is low dimensional [Short, 1994(a); Short, 1994(b); Perez & Cerdeira, 1995]. Higher dimensional signals, especially those involving hyperchaotic dynamics, are likely to provide improved security. The number of parameters that have to be matched for information recovery and the precision with which they must be matched are important aspects of receiver design. We will show how the configuration and operation of the receiver may be designed to make it more suitable for private communications. At this point we would like to emphasize that the security of communication techniques is a complex and involved issue. In the work reported here, we do not make any claims of secure communications. Indeed, we do not know of any systematic cryptographic approach that has been taken to examine the security of different chaotic communication systems. We regard this as a very important open problem for future analysis.

Most realizations of chaotic communications have occurred in electronic circuits that simulate the dynamics of simple model systems (Lorenz, Rössler, double scroll or Chua system, etc.), even for the case of hyperchaotic systems. Peng et al. [1996] theoretically examined the question of synchronization of hyperchaotic Rössler systems and showed that synchronization is successfully achieved over a wide parameter range by using a transmitted signal that is a linear combination of the original phase space
variables. Mensour and Longtin [1998] have studied the synchronization of hyperchaotic systems described by Mackey-Glass delay-differential equations, with their use for private communications.

Optical systems present a somewhat different situation; one often does not know a priori what the equations are that should be used to model the system. Rather, insight into the formulation of appropriate models must be gained through experimental observations of the system dynamics. We follow this approach throughout this paper. Our research into chaotic communications using optical systems began when we experimentally achieved chaotic synchronization of two mutually coupled Nd:YAG (neodymium doped yttrium aluminum garnet) lasers that were operated side by side in a single YAG crystal [Roy & Thornburg, 1994; Sugawara et al., 1994]. We then proposed a scheme for digital communication with a transmitter laser unidirectionally coupled to a distant laser used as a receiver [Colet & Roy, 1994]. The chaotic dynamics for these systems is not high-dimensional [Alsing et al., 1997], and we showed that the message could be recovered fairly easily by time-delay embedding of the signals, unless one introduced complex modulations to increase the dimensionality of the dynamics. The dynamics present in such lasers is also rather slow, on the microsecond time scale at best.

Our next step was to examine a laser system that would provide not only a high-dimensional dynamics, but also a much greater bandwidth and high speed for signal transmission and recovery. This led us to the study of erbium doped fiber ring lasers (EDFRL); we developed a simplified model for the operation of an EDFRL in [Williams et al., 1997]. These lasers emit at 1.53 - 1.55 microns, the wavelength regime of choice for optical communications in fibers, which have minimum loss in this range. The
emission is broadband (covering many nanometers), and we found the dynamics present in this system to be extremely fast. We estimate the bandwidth of the light intensity fluctuations to be in the range of many gigahertz, but the bandwidth of our electronic detection equipment prevents direct observation of the higher frequencies.

Ring cavity optical systems with nonlinear elements, as pointed out by Ikeda and coworkers many years ago [Ikeda et al., 1980; Ikeda & Matsumoto, 1987], can possess very high-dimensional chaos resulting from the operation of the intra-cavity nonlinear elements and time-delayed feedback. Depending on the setting of the operating parameters of the erbium doped fiber ring laser system, one can observe both low- and high-dimensional dynamics [Ikeda et al., 1980]. Simulations of Ikeda type ring systems by Abarbanel and Kennel reveal dimensions of order twenty five or higher [Abarbanel & Kennel, 1998]; they have also numerically demonstrated the synchronization of two ring cavities via unidirectional coupling. The dimensionality of the chaos in our experiments has been analyzed from the measured data using a false-nearest-neighbors (FNN) algorithm [Abarbanel, 1996] and found to be of order 10 or higher. Erbium doped fiber ring lasers therefore simultaneously offer the advantages of high-dimensional dynamics and high-speed communications.

Very recently, we reported the first experiments on all-optical chaotic communication of 10 MHz square waves using erbium doped fiber lasers and amplifiers [VanWiggeren & Roy, 1998]. A very interesting experiment on chaotic communications was reported simultaneously by Goedgebuer and colleagues [1998], who used a hybrid electro-optic system to encode a 2 kHz sine wave in the chaotic wavelength fluctuations of a tunable semiconductor laser and then decoded the information with an
open-loop receiver. They also used a time-delay system [Larger et al., 1998] to generate high-dimensional chaotic dynamics.

The experiments described in this paper more fully take advantage of the large bandwidth available in an EDFRL, demonstrating communication of return-to-zero (RZ) pseudo-random digital bits at rates of 125 Mb/s. Results from an experiment showing communication of a non-return-to-zero (NRZ) digital message at 250 Mbit/s are also discussed. Currently, the bit rate is limited by the bandwidth of our detection equipment.

Though the high-dimensional dynamics of an EDFRL system offer the potential for privacy, we note that many of the experiments and techniques discussed in this paper are unlikely to be useful for private communications; rather, descriptions of those experiments are included because they both established and furthered our understanding. The more recent experiments described in this paper, however, do seem to offer the possibility for enhanced privacy in communication. A discussion of the privacy aspects of each method is included.

All of the communication experiments described in this paper use an EDFRL to generate chaotic waveforms and employ unidirectional coupling of the transmitter to the receiver. Each technique described utilizes an open-ring receiver designed to mimic the dynamics of light propagating once through the transmitter ring. The message to be communicated modifies the chaotic dynamics of the transmitter, and by comparing the output dynamics from the transmitter to the open-ring receiver dynamics, the message can be recovered. The message itself drives the chaotic dynamics of the transmitter. Thus, the manner in which the message is dynamically encoded is at least partly dependent on earlier portions of the message.
We took two distinct approaches in developing chaotic communication in EDFRLs. The first approach involves coupling an optical message signal from an external laser directly into the EDFRL. In the second approach, an intensity modulator is inserted within the ring laser itself to encode the message directly onto the chaotic carrier. A development of this technique incorporates two time-delays into the dynamics of the transmitter, making the information transmission more private.

A feature of all of these communication systems is that there is no encoding of symbols into unstable periodic orbits of the chaotic system, as has been proposed and demonstrated recently [Hayes et al., 1993; Hayes et al. 1994]. Thus there is no loss of speed in communication that could occur through the use of higher order orbits for symbolic coding or transients to synchronization in closed-loop communication systems.

This paper is organized as follows. Section II contains a description of the experimental system used for message injection from an external tunable semiconductor laser system and dynamical encoding with a chaotic EDFRL. The technique used to recover the digital information is outlined. The results of experiments that use polarization or wavelength filters in the EDFRL are given in Section III.1(A) - III.1(C). These experiments enabled us to reduce the interference between message and chaotic lightwaves and revealed many features of the wavelength and polarization dynamics of the laser system. In section III.2(A) and III.2(B) the results of message injection experiments performed without any wavelength or polarization filters are given. The possibility of dynamical encoding and wavelength multiplexing are examined. Section IV contains a discussion of the privacy and consistency of information recovery of the message injection techniques.
A different approach for dynamical encoding with an intra-cavity modulator is described in Section V, and the results of these experiments are given in Section VI. No external laser for message injection is needed for this technique, and the encoded information is recovered by a division of signals in the receiver. A modified intra-cavity modulation technique that adds a second fiber loop (and hence a second time-delay) to significantly enhance the privacy of the information is presented in Section VII and the results are given in Section VIII. We show that information is dynamically encoded and consistently and clearly recovered at 125 Mbits/s (return-to-zero, or RZ format) or 250 Mbits/s (non-return-to-zero, or NRZ format) and examine the result of mismatch of receiver configuration or parameters. A discussion of the bit-error rate ($< 10^{-5}$) and results for communication through more than 35 km of fiber (including an eye-diagram) is given. Section IX contains a discussion of the modified intra-ring modulator technique including results of a false nearest neighbors estimate of the dimensionality of the chaotic dynamics for this technique. Section X concludes the paper with a brief recapitulation of goals and results. We finally mention some open questions that must be addressed before practical implementation of these ideas.

II. Experimental Setup and Operation Using Message Injection:

Several variations of the message injection approach are described first, but their general form is shown in Fig. 1. To create the optical message, light produced by an external-cavity tunable diode laser is amplitude modulated by a lithium niobate Mach-Zehnder interferometer to form a sequence of pseudo-random bits. After modulation, the lightwave is amplified in a controllable fashion by an erbium-doped fiber amplifier.
(EDFA) with a 13 dBm maximum output power. This step governs the amplitude of the message injected into the EDFRL. The amplified message then passes through a polarization controller consisting of a series of waveplates ($\lambda/4$, $\lambda/2$, $\lambda/4$) arranged to permit complete control over the polarization state of the message as it is coupled into the ring.

The message light is injected into the EDFRL through a 90/10 waveguide coupler, which allows 10 percent of the message light to be injected into the ring, while retaining 90 percent of the light already in the ring. The light propagates to a 50/50 output coupler that sends half the light in the ring to the receiver unit, while the other half passes through EDFA. This EDFA has 17 dBm maximum output power and 30 dB small signal gain. The active (doped) fiber in the EDFA is 17 m long and is pumped by diode lasers with a 980 nm wavelength. After passing through the EDFA, the light then travels through a filter consisting of a polarization controller, again a series of waveplates, and either a polarizer or bandpass filter depending on the variation of the method being used. The total length of the active and passive fiber in the ring totals approximately 40 m, which corresponds to a round trip time for light in the ring of about 200 ns. Isolators in the EDFA ensure that light propagates unidirectionally in the ring, as indicated by Fig. 1.

Unless otherwise specified in the descriptions of subsequent experiments, the transmitter EDFRL is operated at pump powers more than 10 times threshold. Typical optical powers in the ring are between 10-40 mW depending on the losses in the ring.

Light exiting the ring through the 50/50 output coupler propagates in an optical fiber to the receiver unit. Light entering the receiver unit is split at the 90/10 coupler. Ten percent passes through a variable attenuator that prevents photodiode A (125 MHz
bandwidth) from saturating. The other 90 percent of the light passes through EDFA 2 and another filter. EDFA 2 and the receiver filter are intended to be replicas of EDFA 1 and the filter in the transmitter so that the receiver can synchronize to the dynamics of the transmitter. After passing through EDFA 2 and the receiver filter, the light passes through an attenuator and is measured by photodiode B (125 MHz bandwidth). The signals from photodiodes A and B are recorded by a digital oscilloscope with a 1 GS/s sampling rate and 8-bit resolution.

A model for the EDFRL without message injection is given in [Williams et al., 1997]. It consists of two delay equations for the two polarizations of the electric field and one differential equation for the population inversion. The functional forms of these equations, including message injection, are

\[
E_T(t) = f[E_T(t - \tau_R), N(t), m(t - \tau_R)]
\]

and

\[
\dot{N}(t) = g[E_T(t), N(t)].
\]

\(f \) and \(g\) are nonlinear functions of the population inversion, \(N(t)\), and the complex slowly varying envelope of the chaotic electric field in the transmitter, \(E_T(t)\). The message is represented as \(m(t)\), and the time it takes light to make a round trip in the EDFRL is \(\tau_R\). Eq. 1 shows that the electric field \(E_T(t)\) at any point in the cavity is functionally dependent on the electric field and message, \(E_T(t - \tau_R)\) and \(m(t - \tau_R)\), at that point one round trip earlier.
Conceptually, the method we use is similar to one described by Volkovskii and Rulkov [1993], but it has been modified to incorporate time-delays, optical phase, and polarization effects. The light that is transmitted from the EDFRL to the receiver unit is $s(t) = E_T(t) + m(t)$. If the injected message power is not too large (typically on the order of a few milliwatts or less), the message is "masked" by the larger (10-40 mW unless otherwise specified) chaotic intensity fluctuations. For simplicity, imagine that photodiode A detects its fraction of $s(t)$ at the same moment that the remaining fraction of $s(t)$ is incident at EDFA 2. The amplifier and filter then operate on the remaining fraction (~90%) of $s(t)$ to produce the waveform $E_R(t + \tau_R)$, which arrives at photodiode B with a time-delay equal to one round trip time in the transmitter, $\tau_R$. At precisely this moment, photodiode A is detecting $s(t + \tau_R) = E_T(t + \tau_R) + m(t + \tau_R)$. Note that $E_T(t + \tau_R)$ is produced when EDFA 1 and the filter (in the EDFRL) operate on $s(t) = E_T(t) + m(t)$. Because EDFA 1 and the filter in the transmitter operate on light in the same way as EDFA 2 and the receiver filter, $E_R(t + \tau_R) = E_T(t + \tau_R)$.

Mathematically, this occurs because $f$ and $g$ are the same in both systems, the systems have negative conditional Lyapunov exponents, and they display synchronization in a global sense [Abarbanel & Kennel, 1998]. The two photodiodes, therefore, measure the intensities $|E_T(t + \tau_R) + m(t + \tau_R)|^2$ and $|E_R(t + \tau_R)|^2$ respectively. The difference of these two measurements at any time is $2 \text{Re}\{E_T^*(t + \tau_R) \cdot m(t + \tau_R)\} + |m(t + \tau_R)|^2$.

When no message is transmitted, it is clear from the preceding discussion that the transmitter and receiver should be synchronized. In other words, the two photodiodes should measure the same intensities, $|E_R(t + \tau_R)|^2 = |E_T(t + \tau_R)|^2$. The results of an
experiment in which no message was transmitted are shown in Fig. 2. Fig. 2A shows the signal from the transmitter with the laser operated far above threshold (greater than 10 times the pump power required for threshold). A time-delay embedding of the data in Fig. 2A is given in Fig. 2B; it shows that the chaos is not low-dimensional. Fig. 2C shows the signal recorded by photodiode B. The straight line shown in Fig. 2D demonstrates the synchronization of the light intensities as measured by photodiodes A and B. We used a numerical false-nearest-neighbors (FNN) algorithm [Abarbanel, 1996] in an attempt to estimate the dimensionality of the signal. The results are presented in Fig. 2E. The dimensionality of a time-series can be estimated by observing the dimension at which the percentage of false-nearest-neighbors goes to zero. Interestingly, we observe that for analysis of 250,000 points acquired at 1 GS/s, the FNN algorithm gives a dimension of 8. The limited bandwidth of our photodiodes does not allow measurement of the much higher frequencies that may be present in the real signal. Consequently, there may be additional attractor dimensions that do not manifest themselves in the data we obtain. Fig. 2E shows that the natural dynamics of the ring laser are not low-dimensional for the operating parameters used, even without the presence of an injected message to influence the dynamics of the ring laser.

III. Results for experiments using message injection

Two basic techniques using an injected optical message have been investigated. The first technique follows from the description above, but a generalized “filter” is used in both the transmitter and receiver units. Several variations of this technique are presented. The second technique is conceptually similar to the first, but no generalized filter is necessary.
III. 1(A)

In our first variant of this method, \( m(t) \) is injected into the EDFRL with a polarization orthogonal to \( E_{r}(t) \) but with the same wavelength. The polarization of the injected \( m(t) \) is determined using the polarization controller in the message modulation unit. The filter in the EDFRL transmitter is a polarizer. The polarization controller and polarizer in the EDFRL are aligned such that the \( m(t) \) in the EDFRL is “blocked” upon reaching the filter, while still allowing an orthogonally polarized \( E_{r}(t) \) to pass through relatively unchanged. In this manner, \( m(t) \) is prevented from circulating in the EDFRL and mixing with \( E_{r}(t) \) on more than one pass through the EDFA, though it is transmitted to the receiver unit. The filter in the receiver unit is also aligned to prevent light polarized in the same direction as \( m(t) \) from reaching photodiode B. Subtracting the signals at the photodiodes, as mentioned earlier, gives

\[
2 \Re\left\{ E_{r}^{*}(t) \cdot m(t) \right\} + |m(t)|^2.
\]

The cross term is eliminated in this technique because \( E_{r}(t) \) and \( m(t) \) are orthogonally polarized. This leaves just \( |m(t)|^2 \) after subtraction.

Fig. 3 shows the results of an experiment using this technique. Fig. 3A is a time-trace of the transmitted signal, \( s(t) \), as measured by photodiode A, and Fig. 3B is its power spectrum. No hint of the message can be discerned in the time-trace of the signal. The broadband nature of the transmitted signal is evident from the power spectrum. The spectrum’s gradual decline with increasing frequency matches the spectral response of our photodiodes (125 MHz bandwidth) and results from this limitation. Fig 3C is a time trace of the signal measured by photodiode B, and its power spectrum is shown in Fig 3D. The power spectrum of the received signal lacks the smaller, fine peaks of the
transmitted signal. As can be seen in our next figure, those fine peaks are just the power
spectrum of our repeating sequence of bits. Fig. 3E results from a subtraction of the
waveform data in Fig. 3C from the data in Fig. 3A. The "random" bits are clear and
match the 125 Mbit/s pattern used for this experiment:
110011011011010010010111101010. Fig. 3F is the optical spectrum of the
transmitted signal s(t) showing that both m(t) and E_r(t) have the same wavelength,
\sim 1.532 \mu m.

Fig. 4 is included as a measure of the accuracy of this technique. For comparison,
Fig. 4A shows a segment of the message sequence as detected by photodiode A in the
absence of chaos (EDFA 1 is turned off and E_r(t) = 0). Fig. 4B is simply the power
spectrum of the entire recorded message sequence. Fig. 4C shows a portion of the
message obtained through the chaotic subtraction. Clearly, the recovered message is
degraded somewhat, probably due to imprecision in matching the polarization-based
filters in both the transmitter and receiver. The power spectrum of the recovered
message, a segment of which is shown in Fig. 4C, is shown in Fig. 4D. The peak
corresponding to the 125 MHz bit-rate is evident in Fig. 4B.

III. 1(B)

In the preceding technique, the wavelengths of m(t) and E_r(t) were the same, but
their polarizations were orthogonal. In this method, however, the wavelengths are
different, but their polarizations are the same. The filter in the EDFRL again consists of
the polarization controller followed by a polarizer. The polarization controller in the
message modulation unit is adjusted so that the polarizations of m(t) and E_r(t) are
parallel as they are transmitted from the EDFRL to the receiver unit. Again, the filter's
purpose in the EDFRL is to "block" light associated with $m(t)$ from continuing to circulate in the ring. A moment of explanation will be useful here.

$m(t)$ and $E_r(t)$ have the same polarization at the coupler where the message is injected. As the light propagates around the ring, the difference in wavelength, typically ~20 nm, results in a different output polarization state for light at the two wavelengths. We observe that this phenomenon of polarization dispersion occurs primarily in the erbium-doped fiber in the EDFA. At the filter, $m(t)$ is almost completely orthogonal to $E_r(t)$. Thus, $m(t)$ can be "blocked" by the polarizer while $E_r(t)$ continues to circulate.

The method works as described above, but this time the cross-term averages to zero because $m(t)$ and $E_r(t)$ have different optical frequencies; once again, only $|m(t)|^2$ remains after subtraction of the two photodiode signals.

Fig 5A shows the signal transmitted from the EDFRL to the receiver unit as detected by photodiode A. Its power spectrum is shown in Fig 5B, and again, the spectrum is quite broadband. The signal recorded by photodiode B is shown in Fig 5C, and its corresponding power spectrum is given in Fig. 5D. Subtracting the two time-traces results in the trace displayed in Fig. 5E and shows excellent reproduction of the bits. Finally, Fig. 5F shows the optical spectrum of the transmitted signal, revealing a narrow peak corresponding to $m(t)$ at 1.532μm and a broader peak corresponding to $E_r(t)$ at 1.555 μm.

III. 1(C)

Another method investigated was the use of bandpass filters rather than polarizers as filters. Once again, the purpose of the filters was to prevent light associated with $m(t)$ from continuing to circulate around the ring. The wavelength of the message light could
be adjusted to -1 nm of the central wavelength of the filter and still be successfully “blocked”. The filter in the receiver was tuned to match the filter in the transmitter. The electric field polarization, because no polarizer was in the ring, fluctuates very rapidly and effectively has no polarization, and consequently, the polarization state of the message is not important. Because the wavelengths of the message, \( m(t) \), and the chaotic electric field, \( E_r(t) \), are still substantially different, the cross-term in the subtraction of the photodiode signals averages to zero. Again, only \( |m(t)|^2 \) remains.

Fig. 6A shows the transmitted signal, \( |s(t)|^2 \), recorded at photodiode A. Its power spectrum is shown in Fig. 6B. The signal detected at photodiode B is shown in Fig. 6C, and its power spectrum is shown in Fig. 6D. Because the chaotic output has become almost periodic in this example, a comparison of Fig. 6A and Fig. 6C clearly shows the effect of the bits on the transmitted signal. However, it is still not at all obvious from observing just the transmitted signal that a message is included. A subtraction of the two signals is shown in Fig. 6E. The optical spectrum of the transmitted signal is supplied in Fig. 6F. The wavelength separation in this case is much smaller than in the previous case, and is approximately 1 nm.

III. 2(A)

The techniques discussed in section III. 1 all required the use of a filter and required that a distinction of polarization or wavelength be made between the chaotic electric field, \( E_r(t) \), and the message signal, \( m(t) \). The techniques presented in this section will demonstrate message communication without the use of filters and with the same wavelength for the message and chaotic electric field.
Time-traces from such an experiment are shown in Fig. 7A. In this example, both EDFA 1 and EDFA 2 are pumped by their diode pump lasers at 10 mW, a level slightly less than twice the threshold pump power for the ring laser. This pumping results in $\sim -1$ dBm optical power circulating in the ring when no message is being injected. The injected message has a power of $\sim -4.5$ dBm and a 1553.01 nm wavelength. The first panel, Fig. 7A, shows data taken from photodiode A (the thin line) and from photodiode B (the thick line). As explained earlier, a subtraction of the two signals is equal to 

$$2 \text{Re}\left\{ E_r^*(t) \cdot m(t) \right\} + |m(t)|^2.$$ 

Fig. 7B shows that subtraction (thin line). For comparison, the thicker line in Fig. 7B is the message signal measured by photodiode A when the EDFRL is turned off to remove the chaotic masking. This is equivalent to measuring just $|m(t)|^2$. The greater amplitude of the “decoded” message is simply the result of the cross term $2 \text{Re}\left\{ E_r^*(t) \cdot m(t) \right\}$; in this case, the polarizations and phases of $E_r(t)$ and $m(t)$ (relevant in the cross term) have combined to improve the message reception. The fidelity is quite good. An optical spectrum of the transmitted light is shown in Fig. 7C. The message injection is strong enough that its optical frequency forces the EDFRL to have the same lasing frequency--1553.01 nm.

Fig. 8 provides power spectra for the transmitted signal (Fig. 8A) and for the signal at photodiode B (Fig. 8B); they have a very close resemblance. The narrowly spaced discrete spikes result from the repetitive 32-bit pattern used. The bit-rate was 125 Mbit/s; the corresponding spike is evident in both power spectra.

III. 2(B)

We also performed experiments with message injection at wavelengths which were not resonant with the lasing wavelength of the EDFRL. Fig 9A shows signals
measured by photodiodes A and B when the wavelength of the injected message is 1533.01 nm. In this case the EDFAs were pumped at about 85 mW, many times threshold. This resulted in an optical power in the ring of ~ 9.1 dBm without any message injection. The injected message power was ~ -3.1 dBm. The subtraction of the traces in Fig. 9A is seen in Fig. 9B. Once again, the same pattern of bits is obtained. The optical spectrum (Fig. 9C) shows two distinct peaks. The first of these peaks (1533 nm) corresponds to the message injection, whereas the second peak (1558 nm) corresponds to the natural lasing wavelength of the EDFRL. The very broad linewidth is characteristic of the EDFRL. The message light at 1533 nm stimulates the EDFRL to emit at the same wavelength and the fraction of the light that remains in the ring continues to circulate stimulating additional emission. Consequently, the light detected at 1533 nm consists of a combination of the message itself plus chaotic light produced by the EDFRL.

It was important to determine whether it would be possible simply to isolate the message wavelength (1533.01 nm) using a bandpass filter and observe the message directly. We performed that experiment and observed that the message was well obscured by the chaotic laser light. Fig. 10 shows one of these measurements. The sequence of bits is not visible even after isolating just the message wavelength. This experiment indicates that wavelength division multiplexing may be possible, while still using chaos to hide the information. In summary, the message wavelength can be varied around the natural lasing wavelength of the EDFRL and chaotic communication can still occur.
IV. Discussion of message injection approach

All of the variations demonstrated in section III. 1 were able to consistently recover a 125 Mbit/s digital message. Each variation uses a matched chaotic receiver to extract the message but the variations all have limitations in their ability to mask information. In section II. 1(A), the polarizations of $E_r(t)$ and $m(t)$ are orthogonal. At any point in the transmission channel they are both, in general, elliptically polarized. But by using a polarization controller, it is possible to change their polarization from elliptical to linear. Once that is accomplished, a properly oriented polarizer could remove the masking $E_r(t)$ leaving only $m(t)$. Varying the polarizations of $E_r(t)$ and $m(t)$ in time could make this unmasking more difficult.

In the experiment described in section III. 1(B), the wavelengths of the message and chaotic field are different, though they are polarized identically. Many optical devices exist to isolate particular wavelengths. Those devices could be used obtain the message without the masking. The technique described in III. 1(C) has the limitation that $m(t)$ and $E_r(t)$ are separated by ~1 nm in wavelength. The same wavelength dependent filters could possibly remove $E_r(t)$ without distorting $m(t)$ when III. 1(C) is used. This might be more difficult in III. 1(C) than in III. 1(B) as the wavelength separation is much smaller. To make isolating just the message wavelength more difficult, one could imagine a filter in the EDFRL itself which allows many wavelengths of $E_r(t)$ to be transmitted, thereby making it difficult to remove all of them without removing $m(t)$ as well. One could also imagine varying the wavelength of $m(t)$ or transmitting $m(t)$ as two or more wavelengths.
As stated earlier, chaotic communication does lend itself naturally to issues of encryption and privacy. From that perspective, the techniques of section III. 2 are probably to be preferred. But this method, too, has some limitations. Principally, we were unable to control the phase relationship of the electric field and the message field. Consequently, the cross-term, $2 \text{Re}\{E^*_r(t) \cdot m(t)\}$, had to be minimized for more consistent message recovery. Without a filter, this meant that $E_r(t)$ had to be kept as small as possible. The low pump powers used in section III. 2(A) were chosen for that reason. Even so, the message recovery was not consistent. This inconsistency became a greater problem as the pump power in EDFA 1 was increased or as the wavelength of the injected message light came closer to the resonant lasing wavelength of the EDFRL. In section III. 2(B), the message wavelength was separated from the EDFRL's natural lasing line. In this regime, larger pump powers in EDFA 1 could be used before the interference term mentioned earlier becomes too large for reasonably consistent message recovery.

V. Experimental Setup and Operation Using an Intra-ring Intensity Modulator

In the experiments described earlier, consistent message recovery is difficult to achieve because of interference effects between the chaotic light in the ring and the injected optical message. Without a filter, the interference term, $2 \text{Re}\{E^*_r(t) \cdot m(t)\}$, fluctuates due to changing relative phase and polarization between $E_r(t)$ and $m(t)$. A new approach is taken to overcome this difficulty. In this approach, the message is applied directly to the chaotic light in the EDFRL using an electro-optic modulator located within the EDFRL. The message, in this approach, is a modulation rather than an
injected lightwave; consequently, the message does not interfere with the chaotic electric field in the EDFRL.

Fig. 11 shows a transmitter consisting of an EDFRL and a LiNBO₃ intensity modulator. The erbium-doped fiber amplifier (EDFA) has a small signal gain of $\sim 30$ dB and a maximum output power of 17 dBm. Its erbium-doped fiber is the active medium for the ring laser. As before, the intensity modulator uses a waveguide Mach-Zehnder interferometer as the basis for its operation. The modulator also acts as a polarizer because its waveguides are polarizing. A 90/10 output coupler directs 10 percent of the light out of the ring and into the communication channel. The remaining 90 percent of the light continues to circulate around the ring.

In the receiver, the light is split again. The transmitted signal is directly measured at photodiode A. The remaining light passes through a variable time-delay device. A precision time-delay is achieved using a GRIN (graded index) lens to couple light from a fiber into free-space. The light traverses a distance, the source of the variable time-delay, before it is incident on another GRIN lens coupler which couples the light from free-space into fiber. By controlling the separation between the two GRIN lens couplers, the time-delay between photodiodes A and B can be precisely controlled. For this experiment, photodiode B measures the same signal as photodiode A, but with a time-delay matched to within 0.1 ns of the round trip time of the EDFRL, $\tau_R$.

The slowly varying envelope of the chaotic lightwave after the EDFA in the receiver can be represented as $E$. The lightwave propagates through the ring and is amplitude modulated as it passes through the modulator to create $m(t)E$, where $m(t)$ is the message signal. Note that in this case, the message, $m(t)$ is a scalar modulation rather
than a vector lightwave as in the previous experiments. Ninety percent of this light continues until it is amplified in the EDFA to create $E' \equiv m(t)E$, while the remaining fraction is output to the receiver. The lightwave exiting the amplifier has a significantly greater intensity, but its waveform is very similar to the input wave. For simplicity, we write for the lightwave after the amplifier $E' \equiv m(t)E$ because the relative amplitudes of $E'$ and $m(t)E$ are not as important to the operation as is the shape of the waveform. The shape of the waveform is only slightly distorted in passing through the amplifier due to noise and nonlinearities in the amplification. $E'$ also circulates through the ring and is modulated to produce $m(t + \tau_R)E'$, and a fraction is output to the receiver.

In the receiver, photodiode B is delayed relative to photodiode A by one round trip time, $\tau_R$, to within an accuracy of $\sim 0.1\text{ ns}$. Consequently, when photodiode A is measuring $m(t + \tau_R)E'$, photodiode B is measuring $m(t)E$. Since $E' \equiv m(t)E$, a division of the signal recorded at photodiode A by the signal recorded at photodiode B should give $m(t + \tau)$, thereby recovering the message from the chaotic carrier.

VI. Results of Intra-Ring approach

Fig. 12 gives results of such an experiment. A repeating digital message with a 32 bit length was applied to the modulator in the ring. The message had the pattern: 011111010110011011101101001. The recovered message shown in Fig. 12E replicates this pattern. Figs. 12A and 12C, however, show no sign of the message. The power-spectra of the two measured signals (Figs. 12B and 12D respectively) show many peaks; the peak at 125 MHz corresponds to the bit-rate. That peak is also visible in the power spectrum of the recovered message bits, Fig. 12F.
Only a small message modulation was used to obtain Fig. 12; the depth of message modulation that could be used is limited by the dynamics of the ring laser. If the message modulation is too small, bit recovery is impaired because the noise may be larger in amplitude than the communicated message. If the message modulation amplitude is too large, it drives the laser into an unstable spiking regime. In that spiking regime, the transmitted intensities are near zero much of the time. Because our message is recovered through a division process, any noise present in the signal detected by photodiode B when the signal is near zero has very detrimental effects. Longer sequences of pseudorandom bits tended to require even smaller modulation amplitudes to prevent the spiking behavior than employed in Fig. 12.

Once an appropriate adjustment has been made, the message recovery is very consistent because, unlike in the earlier experiments, no interference term exists to distort message recovery. The intra-ring modulator method also has additional privacy benefits when compared with the techniques described in section III 1. In this method, the message is incorporated as a part of the chaotic carrier lightwave, \( m(t)E \). Consequently, the message cannot be separated from the chaotic electric field using optical devices such as polarizers or bandpass filters.

As mentioned earlier, two additional factors that are important to privacy considerations are the dimensionality of the transmitted chaotic lightwave, and the effort required to obtain the necessary parameters for message recovery. The signal from the transmitter EDFRL has been analyzed using a false nearest neighbors algorithm [Abarbanel 1996]. Using 100,000 data points, the analysis indicates that the dimensionality is high, of order 10 or greater.
We have observed that only one parameter must be known in order to construct a receiver capable of recovering the message. Precise knowledge of the round-trip time, $\tau$, of the EDFRL is sufficient for message recovery. Though the nonlinear dynamics of the coupled light field and population inversion in the EDFA of the transmitter result in chaotic fluctuations of light intensity, the EDFA does not significantly alter a waveform’s shape after just one pass through the amplifier. Consequently, an EDFA in the receiver matched to the EDFA in the transmitter is not actually necessary to recover the message. Having only one parameter, $\tau$, to be matched limits the potential privacy of the communication method. The same reasoning can be applied to the methods described in section III. There, too, the matched EDFA is unnecessary, though it was included in the experiments.

VII. Modified Intra-ring Method

A new configuration that requires multiple matched parameters in the receiver and allows consistent and clear message communication is shown in Fig. 13. An additional outer-loop is added to the solitary EDFRL in the previous system. The outer-loop extracts a portion of the light in the inner-ring, delays it relative to the light in the inner ring, and reinjects it. With just a small amount of reinjected light, the spiking behavior observed with larger modulation amplitudes in the previous experiment is eliminated. The transmitter also becomes more stable. Unlike the consistently chaotic output of the previous experiment, the transmitter is now only intermittently chaotic without message modulation.
The outer loop in the transmitter consists of a variable time delay, another EDFA, and a polarization controller. It is approximately 34 m in length, while the inner-ring is approximately 42 m long. The variable time delay is created using a pair of Graded Index (GRIN) collimating lenses separated by an adjustable free-space distance. One GRIN lens couples light from the optical fiber into air, and the other GRIN lens couples the light back into the fiber. The EDFA in the outer loop is used to control the relative amplitude of the light that is sent back into the inner ring. The polarization controller is used to change the relative polarization and phase between light in the outer-loop and light in the inner-ring.

The outer loop in the receiver system has the same length as the outer loop in the transmitter system. Its EDFA and polarization controller have the same function as their counterparts in the transmitter. An additional variable time delay has been introduced into the receiver’s main-line to match the time delay between photodiodes A and B to the round trip time of the inner ring of the transmitter.

The slowly varying envelope of the chaotic lightwave in the transmitter, just after EDFA 1, can be represented as $E_T$. The lightwave is split upon reaching the 90/10 coupler with ten percent sent into the outer loop and ninety percent remaining in the inner ring. At the 50/50 coupler, the light in outer loop is added to the light in the inner ring giving $E_{T_{in}} + E_{T_{out}}$. The sum passes through the modulator and exits as $m(t)(E_{T_{in}} + E_{T_{out}})$. Ten percent of this signal is output to the communication channel. The remaining ninety percent continues to circulate in the transmitter. It passes through EDFA 1 and is amplified to produce $E_T' \equiv m(t)(E_{T_{in}} + E_{T_{out}})$. Again, we disregard the amplification for simplicity, and instead focus on the shape of the envelope. $E_T'$
propagates as above through the transmitter, ultimately sending $m(t + \tau)(E_{Tin}' + E_{Tout}')$ to the communication channel, where $\tau$ is the round trip time for light in the inner ring.

In the receiver, the transmitted signal is divided as before between photodiode A and photodiode B. Photodiode A measures the transmitted signal directly. Light sent toward photodiode B is split and recombined in such a way as to recreate the dynamics of the transmitter. The length of the outer loop in the receiver has been matched in length to the outer loop of the transmitter. The transmitted lightwave $E'_R = m(t)(E_{Tin} + E_{Tout})$ is split and recombined to become $E'_R_{in} + E'_R_{out}$. Experimentally, the length of the outer loop in the receiver is matched to an accuracy of ±3 cm. Thus, $|E_{Rin}'| \equiv |E_{Tin}|$ and $|E_{Rout}'| \equiv |E_{Tout}|$, but the relative optical phase between $E_{Rin}'$ and $E_{Rout}'$ is uncorrelated to the relative phase of $E_{Tin}$ and $E_{Tout}$. The time delay between photodiode A and photodiode B has also been matched to an accuracy of 0.1 ns.

Recovering the message is accomplished by dividing the signal measured at photodiode A by the signal measured at photodiode B. The time-delay between photodiode A and photodiode B has been adjusted to ensure that photodiode A measures the intensity

$$|m(t + \tau_R)(E_{Tin} + E_{Tout})|^2 =$$

$$|m(t + \tau_R)|^2 (|E_{Tin}|^2 + |E_{Tout}|^2 + 2|E_{Tin}||E_{Tout}| \cos \theta \cos \phi),$$

(3)

while at the same time, photodiode B measures
\[ |(E_{R_{in}} + E_{R_{out}})|^2 = (|E_{R_{in}}|^2 + |E_{R_{out}}|^2 + 2|E_{R_{in}}||E_{R_{out}}|\cos \theta_R \cos \phi_R). \] (4)

\( \theta_{R,T} \) and \( \phi_{R,T} \) are the relative polarization angle and relative phase angle between \( E_{R,T_{in}} \) and \( E_{R,T_{out}} \), respectively. If \( \cos \theta_T \cos \phi_T = \cos \theta_R \cos \phi_R \), a simple division obviously gives \( |m(t + \tau_R)|^2 \). However, we have already explained that the relative phase in the receiver, \( \phi_R \), is not correlated with the relative phase in the transmitter, \( \phi_T \). An inherent property of the transmitter allows this problem to be overcome.

The transmitter consists of two coupled erbium doped fiber ring lasers. One of the ring lasers is the inner ring. The second ring laser is formed by the perimeter of the transmitter including the outer loop. The two lasers have limited flexibility to adjust their lasing modes to maximize the intensity that passes through the modulator. The optimized laser modes possess a fixed value for \( \cos \theta_T \cos \phi_T \) when they are recombined. Although \( \cos \theta_T \cos \phi_T = 1 \) maximizes the intensity when the lightwaves from the inner ring and outer loop are recombined, that particular polarization and phase relationship may not be optimal for passing through the modulator, which also acts as a polarizer. The fixed value \( \cos \theta_T \cos \phi_T \) that results from optimization can be altered by adjustment of the polarization controller in the outer loop. Using the polarization controller in this way, initial observations suggest rough values for \( \cos \theta_T \cos \phi_T \) as low as \( \sim 0.3 \), or as high as 1.

Although the value of \( \cos \theta_T \cos \phi_T \) is relatively stable, the transmitter's lasing wavelength fluctuates on a time scale of \( \sim 5 \) ms. In the receiver unit, the wavelength fluctuation ensures that \( \cos \theta_R \cos \phi_R \) also fluctuates. The polarization controller in the outer loop can fix the value of \( \cos \theta_R \) to any value, but the relative phase angle, \( \phi_R \), is
free to fluctuate as the wavelength changes. Unlike the transmitter, the receiver has no feedback mechanism to fix $\phi_r$. Figure 14 shows two 100 ms time traces of signals detected by photodiode A and photodiode B. The transmitter was unmodulated during this experiment. The transmitted signal, because $\cos \beta \cos \phi_T$ is fairly constant, shows very little variation in amplitude. In contrast, the receiver signal makes significant jumps corresponding to fluctuations of $\phi_r$ due to changing wavelength in the transmitted signal. The relative phase angle between lightwaves in the main-line and outer loop of the receiver changes if the wavelength of those lightwaves changes. A close examination of the figure shows that the amplitude of the transmitted signal also changes slightly at each of these wavelength changes.

If $\cos \beta_r$ is set to 0 using the polarization controller, the fluctuations of $\phi_r$ do not affect the intensity. With this setting for $\cos \beta_r$, message recovery is consistent even when $\cos \beta_T \cos \phi_T \neq 0$. This somewhat surprising result can be understood by observing that $|E_{T_{in}}||E_{T_{out}}|$ as a function of time very closely resembles the function $|E_{T_{in}}|^2 + |E_{T_{out}}|^2$.

To observe this using experimental data, light from the main-line and outer loop in the receiver were directed into photodiodes rather than combined in a coupler. The relative lengths of the arms was the same as for the experiment (again $\pm$ 3 cm). One photodiode (main-line) measured $|E_{R_{in}}'|^2$ while the other photodiode (outer-loop) measured $|E_{R_{out}}'|^2$. This data allows us to calculate $|E_{R_{in}}|^2 + |E_{R_{out}}|^2 + 2|E_{R_{in}}||E_{R_{out}}|$ and present its time-series in Fig. 15A. Fig. 15B shows a time-series corresponding to $2(|E_{R_{in}}|^2 + |E_{R_{out}}|^2)$.

To obtain an estimate for the magnitude of the message recovery errors produced by having $\cos \beta_r = 0$, we divide the time-series $|E_{R_{in}}|^2 + |E_{R_{out}}|^2 + 2|E_{R_{in}}||E_{R_{out}}|$
(maximizing the cross-term in (3), setting \(m(t + \tau) = 1\), and using the approximation \(|E_{\text{in, out}}| \cong |E_{\text{in, out}}|\) by the time-series corresponding to \(2[|E_{\text{in}}|^2 + |E_{\text{out}}|^2]\) (proportional to (4) when \(\cos \delta_R = 0\)). The result is shown in Fig. 15C. Any deviation from a value of 1 is a message recovery error resulting from the zero cross-term in the denominator. A comparison of the magnitude of these errors to the amplitude of a typical digital message (as shown in Figs. 16-19) shows that they are small enough that message reception is not impaired even when the cross-term is maximized in (3), as in this case.

VIII. Results of the Modified Intra-ring Method

Results from this technique, using a communication channel \(~ 4\) m in length, are shown in Fig. 16. A message consisting of pseudorandom sequence of 100,000 bits at 125 Mbit/s was communicated in this experiment. Fig. 16A shows 400 ns of the signal transmitted through the communication to the receiver and detected at photodiode A. Fig. 16C shows the signal simultaneously detected by photodiode B. Fig. 16E shows the division of the signal shown in Fig. 16A by the signal in Fig. 16B. The recovered bits are clear in Fig.16E, but they are indiscernible in Figs. 16A and 16C. The power spectra of the signals are shown in the panels on the right. Fig. 16B shows the power spectra of the signal shown in Fig. 16A. The 125 MHz frequency peak, corresponding to the bit-rate, appears as just one of many small peaks in the spectrum. The same is true for the power spectra of the signal recorded by photodiode B, shown in Fig. 16D. Finally, Fig. 16F shows the power spectra of the recovered bits. Only one peak, at 125 MHz, is significant.

Initial tests show very consistent recovery of the data. Bit-error-rates of \(<10^{-5}\) are routinely achieved. The BER may be significantly lower, but memory limitations on
our digital oscilloscope limit the number of message bits that can be acquired for
analysis. Occasionally, however, the transmitter "bursts" into an unstable, high
frequency chaos. The amplitude and frequency of these bursts can be greatly diminished
by limiting back-reflection from the open leads of the waveguide couplers.
Environmental vibrations such as tapping on the table seem to induce bursting as well.
During these intermittent bursts, the BER can be much higher, though in most instances,
the error produced is small enough that message recovery is not impaired at all.

In order to recover the communicated message, the receiver must be "tuned" to
the dynamics of the transmitter. This requires matching certain transmitter parameters in
the receiver. In this system, the configuration, time-delays, and relative amplitudes for
the lightwaves in the main-line and outer loop must be properly matched in order to
recover the message. Fig. 17 shows the effect of various parameter mismatches.

Fig. 17A shows successful recovery of a repeating pseudorandom 40-bit message
sequence at 125 Mbit/s using the same experimental parameters as used to obtain the data
for Fig. 16. The proper sequence is clearly recovered. The other panels represent
attempted recovery of this message with just one parameter mismatched. Fig 17B shows
an attempted recovery of the same message, but without using the outer loop in the
receiver. The message is very distorted. Fig. 17C shows recovery without the main-line
part of the receiver. Again, the message is greatly distorted. Fig. 17D shows recovery of
the message with an outer loop that is one meter too long. When the time-delay between
photodiode A and photodiode B is incorrect by just 1 ns (8 inches of optical fiber) the
message is also severely distorted, as shown in Fig. 17E. Fig. 17F demonstrates that the
relative power levels of $E_{Rin}$ and $E_{Rout}$ must be properly set in order to accurately recover
the message. For Fig. 17F, the pump power on EDFA 3 was adjusted from its optimal setting of 21 mW (used to produce the data shown in Fig. 16) to 100 mW.

To determine this method's viability as a communication scheme, it is also important to investigate the effect of long communication channels on the technique. With this in mind, an experiment was performed using a communication channel consisting of another EDFA (to compensate for channel losses) followed by 35 km of ordinary single mode fiber. As in Fig. 16, a repeating message consisting of a sequence of 100,000 pseudorandom bits at 125 Mbit/s are communicated. Fig. 18A shows the transmitted signal as recorded at photodiode A after passage through the longer channel. Its power-spectrum is shown in Fig. 18B. Fig. 18C shows the signal measured by photodiode B. Its power-spectrum is shown in Fig. 18D. The power spectra of the signals measured by the photodiodes show peaks at 125 MHz corresponding to the 125 Mbit/s bit-rate, but these peaks are not obviously conspicuous. Fig. 18E shows the recovered message. The message does not reveal any significant distortion caused by the long communication channel. Fig. 18F gives the power-spectrum of the message.

The bit-rate in all of these experiments is limited by the bandwidth of the photodiodes (125 MHz 3-dB roll-off), but a doubled bit-rate can be obtained without introducing higher frequencies components by using a non-return-to-zero (NRZ) communication scheme rather than the return-to-zero (RZ) communication used thus far. Fig. 19 shows results for communication of an NRZ 100,000 bit pseudorandom sequence at a rate of 250 Mbit/s. The communication channel for this experiment consisted of an EDFA and 38 km of dispersion-shifted (dispersion zero at 1550 nm) fiber. Fig. 19A shows the transmitted signal measured by photodiode A. Fig. 19B shows its power-
spectrum. Fig. 19C shows the signal measured by photodiode B, and Fig. 19D shows its power-spectrum. Fig. 19E shows recovery of the 250 Mbit/s NRZ message. The message's power spectrum is shown in Fig. 19F.

An eye-diagram illustrating the quality of this communication is shown in Fig. 20. The eye is open indicating accurate recovery of the 250 Mbit/s signal. The slope of the lines around the eye result from the slew-rate of the photodiodes.

IX. Discussion of the Modified Intra-ring Method

We have demonstrated consistent and clear optical chaotic communication using a modulator in the transmitter to encode the message. Bit-rates of 125 Mbit/s and 250 Mbit/s were demonstrated. Taking full advantage of the large bandwidth available in the optical system would permit even faster rates, but these experiments were limited by the bandwidth of our photodiodes (125 MHz 3-dB roll-off) and oscilloscope (1 GS/s). The method works well even over long communication channels (~35 km), and in both ordinary and dispersion-shifted fibers.

It also offers enhanced privacy compared to the other methods discussed in this paper. As shown in Fig. 17, accurate recovery of the message requires multiple matched parameters in the receiver. The geometrical configuration of the receiver must be the same as in the transmitter. The lengths of the fiber in the outer loop and the time-delay between photodiode A and B must be matched fairly precisely. Finally, the relative power levels of $E_{R_{in}}$ and $E_{R_{out}}$ must be properly matched to the power levels in the transmitter. This method suggests that more complicated geometries and systems requiring additional parameters for message recovery may also be possible to construct using an EDFRL as a basic element.
Another privacy consideration is dimensionality. As mentioned earlier in this paper, it has proved possible to recover a message from certain lower-dimensional (3 dimensional) chaotic communication systems [Short, 1994(a); Short, 1994(b); Perez & Cerdeira, 1995]. A communication method utilizing higher dimensional chaos is likely to provide enhanced privacy. The transmitted signals produced in this experiment, as seen in Fig. 16, have also been analyzed using a false-nearest-neighbors algorithm [Abarbanel, 1996]. Using 100,000 points acquired at 1 GS/s, the algorithm indicated that the dimensionality of the attractor is of order 10 or higher, as seen in Fig. 21. This figure can be compared to the false nearest neighbors data (Fig. 2) for the transmitted light from an unmodulated EDFRL. The increase in the dimensionality can probably be attributed to the fact that the pseudorandom message modulation drives the chaotic dynamics of the transmitter in this experiment. The nonlinear operation of the transmitter transforms the digital message into a high-dimensional chaotic waveform. The shape of the chaotic carrier waveform at any time is a nonlinear time-delayed function of the previous digital message bits.

X. Conclusion

The experiments described in this paper trace the development of several techniques for dynamical coding and cryptography using time-delayed optical systems. Our goal has been to introduce the use of time-delayed systems for all-optical chaotic communication at high speeds through standard single-mode optical fiber, using erbium doped fiber ring lasers at 1.5 microns. Our demonstrations include communication at up to 250 Mbits/s through 35 km of single-mode fiber. These results are limited at present only by the speed of the detection electronics and not by the bandwidths or dynamics of
the optical systems. We hope to stimulate theoretical and experimental studies of dynamical encoding and decoding with time-delayed systems.

There remain many open questions to be answered regarding the information theoretic aspects of the dynamical encoding and cryptographic processes outlined here. Estimates of channel capacity and bit-error rates, robustness of the methods against noise and the development of error control and correction methods are all topics that need to be addressed before such systems find practical applications in the real world. Finally, as we have mentioned earlier, the question of security of chaotic communication techniques is an important one that needs to be addressed by the proper cryptographic methods of analysis. We hope our paper will stimulate research in these new and fertile areas.

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References


Figure Captions

Figure 1: Experimental system for optical chaotic communication consisting of three parts. In the message modulation unit, cw laser light is intensity modulated to produce a message signal consisting of a series of digital bits. The message signal is injected into the transmitter where it is mixed with the chaotic lightwaves produced by the erbium-doped fiber ring laser. The message, now masked by the chaotic light, propagates through the communication channel to the receiver where the message is recovered from the chaos.

Figure 2: Fig. 2A shows the transmitted signal measured by photodiode A when no message is injected into the transmitter. Fig. 2B shows a time-delay-embedding plot of the data in Fig. 2A; the lack of structure indicates that the data is not low-dimensional. Fig. 2C shows the signal simultaneously detected by photodiode B. The signals recorded by the photodiodes are clearly synchronized, as shown in Fig. 2D. Fig. 2E gives the results of a false-nearest-neighbors analysis. It indicates a dimension of ~8 for the dynamics of the transmitted signal.

Figure 3: Fig. 3A shows the transmitted signal measured by photodiode A when a message is injected into the transmitter. Its power-spectrum can be seen in Fig. 3B. Fig. 3C shows the detected signal at photodiode B, and its power spectrum is also given by Fig. 3D. Subtracting the signals shown in Fig. 3A and Fig. 3C results in the recovered message shown in Fig. 3E. An optical spectrum is shown in Fig. 3F revealing a single
peak. Fig. 3F demonstrates that both the message and chaotic light share the same wavelength.

Figure 4: A comparison showing the quality of the message recovery. Fig. 4A shows a portion of the message directly detected at photodiode A when the ring laser is turned off, i.e., $E_r(t) = 0$. No chaotic encryption is used. The power-spectrum of the directly detected message is shown in Fig. 4B. The same message recovered from the chaotic transmitted signal, i.e., $E_r(t) \neq 0$, is shown in Fig. 4C. Fig. 4D shows the power spectrum of the recovered signal. The recovered message is somewhat degraded but still decipherable.

Figure 5: Fig. 5A shows the transmitted signal measure by photodiode A, while Fig. 5C shows the signal measured simultaneously by photodiode B. Their power-spectra are given in Figs. 5B and 5D respectively. Again, subtraction of the two time-traces reveals the message, shown in Fig. 5E. An optical spectrum, Fig. 5F, shows that the message light and chaos have different wavelengths.

Figure 6: Figure 6A and 6C show simultaneous signals measured by photodiodes A and B respectively. Fig. 6A’s power-spectrum is shown in Fig. 6B, and Fig. 6C’s power-spectrum is given in Fig. 6D. Subtracting the signal shown in Fig. 6C from the signal shown in Fig. 6A gives the recovered message shown in Fig. 6E. In Fig. 6F, an optical spectrum shows that the wavelengths are not the same, but are much closer than they had been in the experiments performed for Fig. 5.
Figure 7: Fig. 7A shows both the transmitted signal measured by photodiode A (thin-line) and the signal measured by photodiode B (thick-line). The thick-line in Fig. 7B is the message signal directly detected by photodiode A when the transmitter ring-laser is turned off, and is included to verify that the message recovered from the chaos (thin-line) is indeed correct. An optical spectrum of the transmitted signal is shown in Fig. 7C to show that both the message and chaotic light have the same wavelength.

Figure 8: Power spectra corresponding to the signals measured in Fig. 7A. Fig. 8A is the power spectrum for the transmitted signal recorded at photodiode A. Fig. 8B shows the power spectrum for the signal measured by photodiode B.

Figure 9: Fig. 9A once again shows both the transmitted signal measured by photodiode A (thin-line) and the signal measured by photodiode B (thick-line). Fig. 9B shows the results of subtracting the thickline from the thin-line. Fig. 9C gives an optical spectrum showing the lasing wavelengths of the EDFRL. The message injection is occurring at a wavelength of $\sim 1.533 \mu m$.

Figure 10: The transmitted signal after passing through a 1 nm bandpass filter at 1.533 $\mu m$. The chaotic light at this wavelength still masks the message.

Figure 11: Experimental setup for the intra-ring modulator approach. An intensity modulator is used to encode a digital message onto chaotic lightwaves produced by the erbium-doped fiber ring laser. Chaotic light from the ring travels to a receiver where a
precise time-delay between the photodiodes allows for the message to be recovered from the chaos..

Figure 12: Fig. 12A shows the signal from the transmitter as recorded by photodiode A. Fig. 12C is the signal recorded by photodiode B in the receiver. A division of the two signals recovers the message as shown in Fig. 12E. Panels B, D, and F show the power spectra of the signals shown in A, C, and E respectively. Note that the 125 MHz peak in Figs. B and D is not the most prominent peak. In Fig. 12F, a peak at the bit-rate of 125 MHz is clearly visible.

Figure 13: Experimental setup of the improved intra-ring modulator approach. An erbium-doped fiber ring laser with an additional outer loop is used as a transmitter. Again, an intensity modulator is used to encode a digital message onto the chaotic optical carrier. The carrier and message propagate to a receiver constructed to reproduce the dynamics of the transmitter. Proper configuration, time-delays, and power levels in the receiver allow recovery of the message.

Figure 14: The more level signal is the transmitted signal detected by photodiode A when there is no intensity modulation in the transmitter. The fluctuating signal was detected by photodiode B. The fluctuations illustrate the very large effect that the fluctuating phase angle $\phi_R$ has on the signal measured at photodiode B. In the experiments, however, the polarization angle $\gamma_R$ was adjusted to eliminate the effect of the $\phi_R$ fluctuations.
Figure 15: The error resulting from setting $\theta_R = 0$ in (4) when the crossterm in (3) is not equal to zero is shown to be small enough that it does not interfere with message recovery. Fig. 15A shows the time-series corresponding to
\[ |E_{Rin}|^2 + |E_{Row}|^2 + 2|E_{Rin}|\|E_{Row}|. \] Fig. 15B shows time-series data corresponding to
\[ 2(|E_{Rin}|^2 + |E_{Row}|^2). \] Dividing the data in Fig. 15A by the data in Fig. 15B gives an estimate for the size of the error that results from eliminating the cross-term in (4) by setting $\theta_R = 0$. The result of that division is shown in Fig. 15C. Comparing Fig. 15C with Fig. 16E shows that such small errors will not hinder message recovery.

Figure 16: This figure shows successful recovery of a message signal from its chaotic carrier. The message signal consisted of a sequence of 100,000 pseudorandom bits transmitted at a rate of 125 Mbit/s. Fig. 16A shows the transmitted signal detected at photodiode A. Its power spectrum is shown in Fig. 16B. Note that a small peak is visible at 125 MHz corresponding to the bit-rate. The signal detected at photodiode B is shown in Fig. 16C. Its power spectrum is shown in Fig. 16D. Again, a small peak is visible at 125 MHz. The recovered message (formed by a division of the signals at photodiodes A and B) is shown in Fig. 16E. Fig. 16F shows the power spectrum of the recovered bit sequence, with a clear peak at 125 MHz. The message itself is not discernible in either of the signals shown in Figs. 16A and 16C, but the recovered message is very clear.
Figure 17: Recovery of the message requires that certain parameters in the transmitter be matched in the receiver. Fig. 17A shows recovery of the message with all appropriate parameters matched. The other panels show attempted recovery with just one mismatched parameter. Fig 17B shows attempted recovery with a geometrical configuration in the receiver that lacks the outer loop. In Fig. 17C, recovery is attempted without the main-line part of the receiver. An extra meter of optical fiber is used in the outer loop for the attempted recovery shown in Fig. 17D. Fig. 17E shows attempted recovery when the time-delay between photodiode A and B is mismatched by just 1 ns (±20 cm of optical fiber). Fig. 17F shows recovery when the amplitude of $E_{Row}$ is too large when it is recombined with $E_{Rin}$.

Figure 18: Successful recovery of a message after propagation through 35 km of ordinary single-mode optical fiber. Fig. 18A shows the transmitted signal detected at photodiode A. The signals power spectrum is shown in Fig. 18B. A small peak at 125 MHz corresponding to the 125 Mbit/s bit-rate is visible. Fig. 18C shows the signal measured by photodiode B, and its power spectrum is seen in Fig. 18D. Fig. 18E shows the clearly recovered message. Its power spectrum is shown in Fig 18F.

Figure 19: Successful recovery of a non-return-to-zero (NRZ) message at 250 Mbit/s after propagation through a 38 km dispersion-shifted (1550 nm dispersion zero) fiber. the transmitted signal detected by photodiode A is shown in Fig. 19A. Fig. 19B shows the power-spectrum of the transmitted signal. No peaks are visible corresponding to the bit-rate of the message. Fig. 19C shows the signal recorded by photodiode B after passing
through the receiver. Its power spectrum is shown in Fig. 19D. The recovered message bits are shown in Fig. 19E. Their power spectrum is shown in Fig. 19F.

Figure 20: An eye-diagram showing the quality of the recovery of NRZ digital bits after propagation through 38 km of dispersion-shifted fiber. The eye-diagram is open, indicating good message recovery. The slew-rate of the photodiodes is evident in the slope of the intersecting lines. Clearly, the bit-rate is limited by the detection equipment.

Figure 21: False-nearest neighbors data showing the transmitted signal to be high-dimensional, of order 10 or greater. The high-dimensional nature of the chaos can be attributed to the fact that the pseudorandom message modulation signal drives the chaotic dynamics of the transmitter.
Figure 1
Transmitted signal without message

Time-delay embedding of transmitted signal

Receiver output

Synchronization plot

False-nearest-neighbor analysis of attractor dimension

Fig. 2
Fig. 3
Fig. 4
Fig. 5
Transmitted and received signal

Photodiode voltage (V)

Time (ns)

Recovered message

Difference of photodiode signals (V)

Time (ns)

Optical spectrum

dBm

Wavelength (nm)

Fig. 7
Fig. 8
Fig. 9
Fig. 11
Figure 12
Intensity fluctuations due to changing $\phi_R$

Fig. 14
Intensity with maximized crossterm

Intensity with crossterm = zero

Small errors resulting from division of above signals

Fig. 15
Transmitted signal at photodiode A

Power spectrum of signal at photodiode A

Signal measured by photodiode B

Power spectrum of signal at photodiode B

Recovered message

Power spectrum of recovered message

Fig. 16
Recovery with matched receiver

Recovery with no outer loop in the receiver

Recovery with no main-line in the receiver

Recovery with 5 ns reinjection mismatch

Recovery with 1 ns receiver length mismatch

Recovery with reinjection power mismatch

Fig. 17
Fig 18
Fig. 19

Transmitted signal at photodiode A

Power spectrum of signal at photodiode A

Signal measured by photodiode B

Power spectrum of signal at photodiode B

Recovered message

Power spectrum of recovered message
False-nearest-neighbor analysis of attractor dimension

Percentage of FNNs

Dimension

Figure 21
Full length article

High-speed fiber-optic polarization analyzer: measurements of the polarization dynamics of an erbium-doped fiber ring laser

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Abstract

Accurate measurements of fluctuating states of polarization (SOP) require determinations of the Stokes parameters on shorter time-scales than those of the fluctuations. For light sources that generate very rapid polarization and intensity fluctuations, such as erbium-doped fiber ring lasers (EDFRLs), conventional polarization analyzers are not sufficient. We describe a technique for measuring rapidly fluctuating states of polarization using a fiber-optic polarization analyzer. SOP fluctuations at rates up to 125 MHz can be accurately measured, a rate limited by the detection equipment (photodiodes and oscilloscope). Using the polarization analyzer, experimental measurements are made which provide new insights into the rapid polarization dynamics of an EDFRL. © 1999 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

Devices to specify the state of polarization (SOP) of light have been commercially available for many years, but these devices are not designed to measure light with rapid fluctuations of its SOP. Indeed, these devices typically require stable SOPs for time-scales from milliseconds to seconds. Accurate measurements of some optical phenomena, however, require tracking a fluctuating SOP on much faster time-scales. The polarization analyzer described here is capable of measuring fluctuating SOPs with time-scales as short as several nanoseconds. Even faster measurements are conceptually possible; the bandwidth of the photodiodes (3-dB roll-off at 125 MHz) ultimately limits the speed of the SOP fluctuations that can be accurately measured.

In Section 2, we present an overview of the theory of optical polarization that will assist the reader in understanding the description of the apparatus and the interpretation of the experimental results. Section 3 briefly describes the experimental apparatus and technique while focusing on experimental results and measurements. Specifically, the polarization dynamics of chaotic and self-pulsing light from an erbium-doped fiber ring laser (EDFRL) are mea-

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2. Theoretical foundation

For a quasi-monochromatic lightwave, the electric field, $E$, can be described as the sum of two orthogonal components, $E(t) = E_x(t) + E_y(t)$, where

$$E_x(t) = iE_0 x(t) \cos[kz - \omega t + \varepsilon_x(t)]$$

and

$$E_y(t) = jE_0 y(t) \cos[kz - \omega t + \varepsilon_y(t)].$$

In these equations, $E_0$, $\varepsilon_x$, and $\varepsilon_y$ are real numbers. The phase of each component is given by $\varepsilon_x, \varepsilon_y$. 

Fig. 1. Diagram of experimental apparatus. During the calibration process, light from the tunable diode laser (TDL) is transmitted to the apparatus. Once calibrated, light from an EDFRL is sent to the apparatus for measurement of the Stokes parameters. The variable attenuators (VATs) placed before the photodiodes are intended to prevent saturation of the photodiodes. The polarization controllers (PCs) consist of a sequence of three waveplates and allow light from any input polarization to be adjusted to any output polarization. The polarizers (Pol.) ensure that the photodiodes measure only one component of the light. All of these free-space optical elements are placed between coupled graded-index (GRIN) lenses, which allow the light to be coupled out of and back into fiber. The digital sampling oscilloscope (DSO) records the intensities measured by the photodiodes.

Fig. 2. This figure is intended to illustrate the accuracy of our measurement of the SOP. The light analyzed is produced by the EDFRL and possesses rapid intensity and polarization fluctuations. A polarizer at 45° has been placed in the calibration area. Thus, the ideal measurement of the SOP should, in spite of the intensity fluctuations evident in (c), give $\tau = (1, 0.1, 0)$. The experimental measurement, as indicated by (d, f), is very close to this ideal. The DOP shown in (b) is close to 100%, as one would expect for light that passes through a polarizer.
A common way of specifying the SOP of a lightwave is to determine its Stokes parameters. As described in Refs. [1,2], the four Stokes parameters for such a lightwave can be defined as

\[ S_0 = \langle E^2_{0} \rangle + \langle E^2_{\theta} \rangle \]

\[ S_1 = \langle E^2_{0} \rangle - \langle E^2_{\theta} \rangle \]

\[ S_2 = \langle 2E_0E_{\theta}\cos(\varphi) \rangle \]

\[ S_3 = \langle 2E_0E_{\theta}\sin(\varphi) \rangle, \]

where \( \varphi \) is just the relative phase, \( \varphi = \varphi_0 - \varphi_\theta \), between the two electric field components. Clearly, \( S_0 \) represents just the total intensity of the light. \( S_1 \) reflects a tendency for the light to have its energy concentrated along either the \( x \) or \( y \) axes. \( S_2 \) and \( S_3 \) depend on the relative phase between these two components.

The \( \langle \rangle \) expectation-value brackets must be included in Eq. (3) because it is possible that the arguments within them can fluctuate on time scales that are shorter than the time during which the measurement is made. To fully measure the time-evolution of a lightwave's SOP, it is necessary to use a polarization analyzer that operates on time-scales as fast or faster than the SOP fluctuations. For example, sunlight is called unpolarized (\( S_1 = S_2 = S_3 = 0 \)) because any real polarization analyzer averages over sunlight's very rapid and statistically random polarization fluctuations. Yet, the electric field of a ray of sunlight at any point in space and time must possess an amplitude, orientation, and phase. While the time-scales for the SOP fluctuation of sunlight are much too fast to be measured with the technique presented here, some optical phenomena which were previously too fast to be fully measured using conventional polarimetry can now be observed. In some modes of operation, the polarization dynamics of an EDFRL occur on time scales of several nanoseconds. Such time scales are long enough to permit full measurement of the EDFRL polarization dynamics using the high-speed polarization analyzer presented here.

Light that is neither completely unpolarized nor completely polarized is called partially polarized. Partially polarized light can be thought of as the superposition of two components: a completely polarized lightwave and a completely unpolarized lightwave [3]. A convenient measure of the relative proportion of polarized to unpolarized light is the Degree of Polarization (DOP). The formula for the DOP is given by

\[ \text{DOP} = \frac{(S_2^2 + S_3^2)}{S_0}, \]

where \( 0 \leq \text{DOP} \leq 1 \). Because of the \( \langle \rangle \) brackets in Eq. (3), the calculated DOP (for a lightwave with fluctuating SOP) is affected by the speed with which the Stokes parameters can be measured.

For convenience, all of the Stokes parameters can be normalized by dividing them by \( S_0 \). For the remainder of the paper, these normalized Stokes parameters will be written in lower-case type. With this simplification, a normalized Stokes vector for unpolarized light is simply given by \( \vec{\mathbf{S}} = (1,0,0,0) \). Linearly polarized light oriented along the \( x \)-axis becomes \( \vec{\mathbf{S}} = (1,1,0,0) \), etc. With these normalized Stokes parameters, Eq. (4) can be rewritten as

\[ \text{DOP} = (s_2^2 + s_3^2)^{1/2}. \]

The Poincaré sphere [3], originally introduced by Poincaré in 1892, is a convenient way to visualize the SOP of a lightwave. The sphere is placed at the center of a three-dimensional cartesian coordinate system \( s_1, s_2, s_3 \), where the horizontal plane is defined by \( s_1 \) and \( s_2 \) axes, and the vertical direction is defined by the \( s_3 \)-axis. For a particular lightwave with \( \text{DOP} = 1 \), the values for the three Stokes parameters \( s_1, s_2, \) and \( s_3 \) can be plotted on the surface of the sphere with unit radius. Any point falling on the equator is linearly polarized (because \( s_3 = 0 \)), and any point falling on a pole is circularly polarized. Thus, the latitude of a particular point gives an indication of the relative ellipticity of the lightwave's.

Fig. 3. Light directly from the EDFRL is analyzed. Unlike the experiment described in Fig. 2, no polarizer was present in the calibration area. This means that all of the polarization and intensity fluctuations produced by the EDFRL are measured directly. (a) Clearly shows that the polarization of the light tends to fluctuate in a somewhat localized area on the Poincaré sphere. The relatively low DOP shown in (b) indicates that these fluctuations are actually faster than can be observed with even the technique used here.
Poincare sphere representation

I₀ -- the total intensity

Stokes parameter s₁

Stokes parameter s₂

Stokes parameter s₃
SOP, while the longitude provides information about the orientation of the ellipse.

In this three-dimensional space, the distance from the origin to a point \((s_1, s_2, s_3)\) is, from Eq. (5), equal to the DOP. Thus, the radius of a partially polarized lightwave will be \(< 1\). In the experimental results described in this paper, the DOP of the measured light may fluctuate in time from much less than one to very nearly one; even so, it is still convenient to use the Poincaré sphere representation to visualize the SOP. To do so, the Stokes parameters of only the completely polarized component of the partially polarized lightwave are plotted on the Poincaré sphere as before. Equivalently (as in Figs. 2–6), the normalized points \((s_1/DOP, s_2/DOP, s_3/DOP)\) are plotted instead of \((s_1, s_2, s_3)\) so that all points, regardless of DOP, will lie on the unit sphere.

3. Experiments

Light to be analyzed propagates from its source to the polarization analyzer in standard single-mode optical fiber. As shown in Fig. 1, a graded-index (GRIN) lens in the calibration area is used to couple the light from the fiber and into a collimated free-space beam. A polarizer or \(\lambda/4\) plate is used in this free-space area during calibration, but during SOP measurement, this area is empty. A second GRIN lens is used to couple the collimated beam back into optical filter. After passing through the calibration area, the light is split by optical couplers into four paths. Each of the four lightwaves passes through some free-space optical elements before they are measured by the photodiodes. Three of the paths include a polarization controller (PC), polarizer (Pol.), and variable attenuator (VAT) to prevent saturation of the photodiode, while the fourth includes just an attenuator. Through a calibration process, described in greater detail in Appendix A, the Stokes parameters of light in the calibration area can be determined in a unique way from the intensities measured by the photodiodes \((I_0, I_1, I_2\), and \(I_3\)).

The light source for the experiments that follow is an EDFRL, as shown in Fig. 1. This EDFRL consists of 17 m of erbium-doped fiber (the active medium) with its ends connected together by a length of ordinary single-mode fiber. A 90/10 coupler in the ring couples approximately 10% of the light out of the ring to be analyzed. Pump diodes with a 980 nm wavelength are used to create the population inversion. These pump diodes are operated at around 100 mW optical pump power, many times the pump power of the first lasing threshold. Two optical isolators, one on either side of the erbium-doped fiber, ensure unidirectional propagation of light in the ring. A mandrel-type fiber-optic polarization controller is used inside the ring to control the type of dynamics that are observed at the output. The total length of fiber in the ring is about 37 m.

Though all of the fiber in the EDFRL is single-mode fiber, two orthogonal polarizations can propagate simultaneously in an EDFRL. These polarizations can interact nonlinearly through third-order nonlinearities of the fiber medium, and both polarizations are nonlinearly coupled to the population inversion. EDFRLs are well-known to generate light having rapid (many GHz) and chaotic fluctuations in output intensity, and some investigators have reported observations of polarization dependent fluctuations on these fast time scales as well [4]. Additionally, a self-pulsing regime can be observed which is the result of at least partial mode-locking between the thousands of very closely spaced longitudinal modes present in the laser. Several laser instabilities which may contribute to the chaotic intensity dynamics as well as the self-pulsing behavior are described by van Tartwijk and Agrawal in Ref. [5]. Attempts to understand the origin of the self-pulsing behavior and its relation to the Risken–Nummedal–Graham–Haken instability are made in Refs. [6,7].
Poincare sphere representation

$P_{3/DOP}$

$I_0$ -- the total intensity

Stokes parameter $s_1$

Stokes parameter $s_2$

Stokes parameter $s_3$

Percentage polarized

DOP x 100

Time (ns)

Time (ns)

Time (ns)

Time (ns)
the chaotic intensity fluctuations and of the self-pulsing behavior are observed using the high-speed polarization analyzer.

Data illustrating the accuracy of the experimental method is shown in Fig. 2. To obtain this data, light from an EDFRL was passed through a polarizer oriented at 45° from the x-axis in the calibration area. The normalized Stokes parameters for such a situation are \( S = (1,0,1,0) \). Any deviation from these parameters in the experiment is a measure of experimental error, such as those which may occur during the calibration process. For example, the polarizer in the calibration area may not be aligned at precisely the correct orientation. Additional errors in this experiment may result from the fact that the photodiodes in this experiment are not identical. Their amplifiers may respond differently to the same measured signal. Finally, small changes in temperature or fiber stress can lead to changes in the fiber birefringences. These changes result in a calibration drift that reduces the measurement accuracy over time. From Fig. 2, it is clear that the measurement errors are reasonably small, even when measuring a signal whose intensity fluctuates rapidly and chaotically, as shown in Fig. 2(c). It should be noted that the fluctuations in Fig. 2(c) are much larger than the noise present in the detection equipment and should actually be interpreted as chaotic intensity fluctuations. The SOP measured is very close to the predicted values, \( s_1 = 1 \) while \( s_{1,2} = 0 \). The degree of polarization (DOP), given in Fig. 2(b) is close to its maximum value of 100%, as one would expect for light that has passed through a polarizer.

Fig. 3 shows results of a similar experiment. In this example, the polarizer in the calibration area has been removed. Thus, the actual SOP for the chaotic light from the EDFRL is being displayed in Fig. 3. Fig. 3(b) shows that the DOP is much lower than for the case where the light passed through the polarizer (Fig. 2). This suggests that the light from the EDFRL is making large polarization fluctuations at rates that are faster than the 125 MHz bandwidth of the photodiodes. Fig. 3(a) clearly shows that the polarization output from the EDFRL changes rapidly with time, but is localized around a certain area on the Poincaré sphere. Fig. 3(d–f) show the fluctuations in the normalized Stokes parameters.

For certain settings of the polarization controller in the ring, the EDFRL used in this experiment also possesses a mode of operation in which self-pulsing of the intensity is observed. The SOP measurement technique described in this paper is able to discern features of this self-pulsing behavior that could not otherwise be observed. Fig. 4 shows data from an experiment in which the EDFRL was self-pulsing. The total intensity time series is shown in Fig. 4(c) showing self-pulsing with a repetition rate equal to one round trip in the ring laser. As can be seen in Fig. 4(a), the measured SOP of the light alternates between two areas on the Poincaré sphere. The two areas are located on opposite sides of the Poincaré sphere, indicating that they represent roughly orthogonal SOPs. From Fig. 4(d–f), it is clear that one of these areas corresponds to the SOP of light during a pulse, while the other area corresponds to the SOP of light between the pulses. Fig. 4(b) shows that the light during the pulses has a higher degree of polarization than the light measured during the interval between pulses. Such information could not have been obtained by simply observing the intensity of two orthogonal polarizations of light, as is often done by using a polarizing beam-splitter. The information about the DOP could lead to a new insight about the dynamics that give rise to such self-pulsing in ring lasers.

Fig. 5 shows another measurement of light from the EDFRL. The total intensity of the light from the EDFRL, shown in Fig. 5(c), displays an irregular self-pulsing. In Fig. 4, the SOP of the light seemed to be localized around two areas. In Fig. 5(a), the SOP is also evolving between two localized areas, but unlike the data shown in Fig. 4, these areas are not orthogonal to one another. Consequently, the polarization dynamics shown in Fig. 5 could not

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**Fig. 5.** Light produced by an EDFRL is analyzed. (f) Shows the time-evolution of \( s_3 \) which is clearly different from the evolution of \( s_{1,2} \) shown in (d,e,g). Again, the DOP shown in (b) seems to increase during the pulses and be lower during the interval between them, though the difference is not as pronounced as in Fig. 4(b). The SOP shown in (a) seems to be evolving between two non-orthogonal states—such an observation could not be made by examining the orthogonal polarizations output from a simple polarizing beam-splitter.
Poincare sphere representation

$I_0$ -- the total intensity

Stokes parameter $s_1$

Stokes parameter $s_2$

Stokes parameter $s_3$
have been obtained using a polarizing beam-splitter to analyze two orthogonal polarization of the light from the EDFRL. Interestingly, each time series in Fig. 5(d–f) displays behavior that is different than from the others.

Fig. 6 shows a case in which the total intensity of the light, shown in Fig. 6(c), has relatively small intensity fluctuations, but the polarization state of the light is switching rapidly between approximately orthogonal polarization states. In such a case, a simple intensity measurement provides little insight into the laser dynamics; much more insight can be obtained by analyzing the polarization dynamics.

4. Conclusion

A high-speed polarization analyzer was developed to measure rapidly fluctuating polarization states of lightwaves. The high-speed of the measurements was required in order to more accurately measure the full polarization dynamics of lightwaves whose polarization states fluctuate on fast (several nanoseconds) time-scales. The technique was enabled by a four-channel digital sampling oscilloscope recording the intensity information from four photodiodes (125 MHz, 3-dB bandwidth) simultaneously at 500 MS/s for each channel. In some ways, the use of fiber optics simplified the usage of the apparatus. Beam alignment and beam shaping difficulties were eliminated. On the other hand, the birefringence of the optical fibers and its sensitivity to perturbations meant that the system must be calibrated (as described in Appendix A) before each set of experiments.

As indicated by the experiments performed using an EDFRL source, a measurement of the full polarization dynamics can often reveal important and otherwise hidden insight into an optical system. Fig. 3 showed that the polarization of the light from the EDFRL evolved rapidly in time. Fig. 4 showed that the DOP of the self-pulsing light was higher during the pulses than during the interval between the pulses, a phenomenon that had not been observed before.

The non-orthogonal switching of the SOP shown in Fig. 5 could not have been observed by measuring two orthogonal polarizations, as is often done with polarizing beam splitters. Fig. 6, with its relatively constant intensity, is an example of a situation in which high-speed measurements of the SOP can provide much more insight into the dynamics of the system than a simple intensity measurement.

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Appendix A. Calibration and measurement of the stokes parameters

As shown in Fig. 1, light that is to be analyzed propagates from its source to the polarization analyzer via single-mode optical fiber. Light is coupled out of the fiber by a GRIN lens and passes through the calibration area before being coupled back into the fiber by yet another GRIN lens. During the calibration procedure, a polarizer or λ/4 waveplate is used in the calibration area, but for an actual SOP measurement, this area is empty. After passing through the calibration area, the light is divided at the first coupler, and approximately 10% of the light is directed toward the photodiode that measures the intensity, I0. The remaining 90% of the light continues propagating to the 1 × 3 coupler where it is split into thirds. The light in all of these branches is again coupled out of the fiber to be operated on by the free-space optical elements shown in the figure—a
polarization controller, a polarizer, and a variable attenuator. As in the calibration area, a second GRIN lens couples the free-space light back into the fiber to guide it to a photodiode. An accurate measurement of the SOP of a lightwave requires that the lengths of fiber in each path be properly matched. Experimentally, the precision with which the lengths are matched is ±2 cm.

VATs are used in the experiment to prevent saturation of the photodiodes. The attenuation is accomplished by placing an anti-reflection coated glass plate between the two GRIN lenses. Refraction in the glass plate for orientations other than perpendicular to the beam results in a lateral translation of the beam. The lateral translation affects the coupling efficiency of the light back into the fiber (through the GRIN lens). Thus, the variable attenuation is controlled by adjusting the orientation of the glass plate.

In addition to this attenuation, the free-space beams propagating toward the photodiodes measuring $I_1$, $I_2$ and $I_3$ must also pass through a polarization controller and a polarizer. The polarization controller consists of three waveplates, $\lambda/4$, $\lambda/2$ and $\lambda/4$, respectively. With proper adjustment of their orientations, these three waveplates are able to transform light of any SOP to any other SOP that is desired. Immediately after the waveplates, the light passes through a polarizer. The photodiode that follows it is only able to measure the component of the light that passes through the polarizer.

Finally, the detection equipment consists of the four photodiodes and the DSO. The photodiodes used in this experiment have a 3-dB bandwidth of 125 MHz, but higher frequencies can be observed. These photodiodes are the components that limit the speed at which polarization fluctuations can be observed in this experiment. The intensity measured by a photodiode is recorded at a rate of 500 MS/s by one of the four channels of a digital sampling oscilloscope.

In order to find the SOP of a lightwave using this polarization analyzer, a relationship between the four intensities, $I_0$, $I_1$, $I_2$ and $I_3$, and the four Stokes parameters must be determined. An example of how such a relationship can be obtained is provided in Ref. [1] for a free-space, rather than fiber-optic, system. In the example given in Ref. [1], the following equation relates the intensities to Stokes parameters:

$$
\begin{bmatrix}
S_0 \\
S_1 \\
S_2 \\
S_3
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
-1 & 2 & 0 & 0 \\
-1 & 0 & 2 & 0 \\
-1 & 0 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
I_0 \\
I_1 \\
I_2 \\
I_3
\end{bmatrix}.
$$

Optical fiber, as used in the experiment presented here, possesses random birefringences that causes the polarization of a lightwave to evolve in an unpredictable manner as it propagates. Consequently, the transfer matrix that relates the Stokes parameters to the measured intensities cannot be known prior to calibration. Though the matrix is unknown before calibration, there must exist some transfer matrix for relating the measured values of $I_0$, $I_1$, $I_2$, and $I_3$ to the Stokes parameters.

The elements of the appropriate transfer matrix can be obtained through a calibration procedure. By placing appropriate optical elements in the calibration area, the Stokes parameters of the light as it is coupled back into the fiber can be completely defined. Measuring the values $I_0$, $I_1$, $I_2$, and $I_3$ when the SOP is defined enables a determination of the elements of an inverse transfer matrix. The new calibration equation takes the form (using the normalized Stokes parameters),

$$
\begin{bmatrix}
a_1 & b_1 & c_1 & d_1 \\
a_2 & b_2 & c_2 & d_2 \\
a_3 & b_3 & c_3 & d_3 \\
a_4 & b_4 & c_4 & d_4
\end{bmatrix}
\begin{bmatrix}
S_0 \\
S_1 \\
S_2 \\
S_3
\end{bmatrix} =
\begin{bmatrix}
I_0 \\
I_1 \\
I_2 \\
I_3
\end{bmatrix}.
$$

where the normalized Stokes vectors have been used.

Sending continuous-wave (CW) unpolarized light, $\mathbf{s} = (1,0,0,0)$, into the apparatus reduces Eq. (7) to the following equations, which allow the determination of the $a$ elements:

$$
\begin{align*}
a_1 &= I_0 \\
a_2 &= I_1 \\
a_3 &= I_2 \\
a_4 &= I_3.
\end{align*}
$$

However, no source of completely unpolarized light was conveniently available during the experiments described in this paper. An alternative method
was used to obtain the values of the \( a \) elements. Light from the tunable diode laser having the same wavelength as light from the EDFRL is passed through a polarizer in the calibration area and then coupled back into the fiber. The polarization controllers in the paths leading to the photodiodes that measure \( I_1, I_2, \) and \( I_3 \) are oriented to ensure maximum transmission through the polarizers. Values for \( I_1, I_2, \) and \( I_3 \) are measured, as is a value for \( I_0. \) If this light had been completely unpolarized, only half of the measured intensity would have been transmitted through the polarizers in front of \( I_1, I_2, \) and \( I_3, \) so the \( a' \)s are determined by

\[
\begin{align*}
a_1 &= I_0, \\
a_2 &= I_1/2, \\
a_3 &= I_2/2, \\
a_4 &= I_3/2. \\
\end{align*}
\tag{9}
\]

The next step in the calibration process is to create linearly polarized light along the \( x \)-axis in the calibration area and couple it back into the fiber. To do so, a polarizer with an extinction ratio > 45 dB is used. It is placed in an optical mount capable of 0.5° precision. The Stokes vector for such a light source linearly polarized along the \( x \)-axis is \( \vec{S} = (1,1,0,0). \) Multiplying this by the matrix above allows the determination of the \( b \) elements in the following way:

\[
\begin{align*}
b_1 &= I_0 - a_1, \\
b_2 &= I_1 - a_2, \\
b_3 &= I_2 - a_3, \\
b_4 &= I_3 - a_4. \\
\end{align*}
\tag{10}
\]

Since the \( a \) values have already been determined, the determination of the \( b \)'s is straightforward. Subsequently, linearly polarized light at 45°, \( \vec{S} = (1,0,1,0), \) is sent to the photodiodes to determine the values of the \( c \) elements. Determining the \( d \) values requires circularly polarized light, \( \vec{S} = (1,0,0,1) \)—a somewhat more difficult task to create.

The most direct method in this experiment to create circularly polarized light is to pass linearly polarized light at an orientation that is half-way between the fast and slow axes of a quarter-wave-plate. While the laboratory possessed a quarter-wave-plate, the orientation of its principal axes had to be empirically determined. First, the polarization controller immediately after the TDL and a polarizer in the calibration area are jointly aligned for maximum extinction of the light from the TDL-light from the TDL is completely polarized, i.e., DOP = 100%. Maximum extinction implies that the light coupled from the fiber and into free space in the calibration area is linearly polarized orthogonal to the polarizer. After maximum extinction is obtained, a quarter-wave plate is placed in front of the polarizer in the calibration area. With this configuration, total extinction will occur if one of the optic axes of the quarter-wave plate is aligned with the polarizer. Maximum transmission (in this configuration) occurs when the axes are oriented at 45° relative to the linearly polarized light incident on the quarter-wave-plate. With this orientation of the axes, circularly polarized light is output from the quarter-wave plate. After the polarizer is removed, circularly polarized light passes to the GRIN lens where it is coupled back into the fiber. Thus, light represented by the Stokes parameters \( \vec{S} = (1,0,0,1) \) is generated, thereby permitting the determination of the \( d \) elements of the matrix.

With the elements of the matrix now determined, it is possible to determine the Stokes parameters of the lightwave in the calibration area from any set of intensity values, \( I_0, I_1, I_2, \) and \( I_3. \) All that is required is to invert the matrix in Eq. (7) and multiply by the intensity vector. To minimize the effect of experimental errors, it is important that the matrix to be inverted not be ill-conditioned. The polarization controllers in front of the photodiodes measuring \( I_1, I_2, \) and \( I_3 \) were therefore adjusted prior to calibration so that the matrix elements obtained are roughly similar to those obtained by inverting the matrix in Eq. (6), but scaled by a factor of \( S_0 \) due to the use of the normalized Stokes parameters.

After these procedures have been followed, the optical elements are removed from calibration area. A light source to be analyzed is connected to the apparatus, and the oscilloscope records the values of \( I_0, I_1, I_2, \) and \( I_3, \) as they fluctuate in time. These vectors are then multiplied by the transfer matrix obtained with the procedure given above to find the Stokes parameters of the light source as a function of time.
References