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APPLICATION OF THE GALERKIN METHOD IN THE SOLUTION OF COMBUSTION-INSTABILITY PROBLEMS

by

B. T. ZINN and E. A. POWELL
Georgia Institute of Technology
Atlanta, Georgia, U.S.A.

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APPLICATION OF THE GALERKIN METHOD IN THE SOLUTION OF COMBUSTION-INSTABILITY PROBLEMS

B. T. Zinn* and E. A. Powell**
Georgia Institute of Technology
Atlanta, Georgia

Abstract

A new approach for the solution of combustion instability problems is presented. In this study the Galerkin method is used to analyze the behavior of the flow field in a rocket combustor with mass, momentum and energy sources present on its boundaries and in the flow field. The presence of sources on the boundaries results in mathematically complicated boundary conditions that must be satisfied by the solutions. The "classical" Galerkin method, which was previously used in cases where the solution of the problem had to satisfy relatively simple boundary conditions, is modified in order to satisfy the more complicated boundary conditions which are encountered in combustion instability problems. The applicability of the modified Galerkin method is demonstrated when it is used to solve a specific problem whose solution is available. It is shown that the results produced by the Galerkin method are in complete agreement with the available solution.

Introduction

The engineer who is trying to solve a combustion instability problem is usually faced with the difficult task of solving a system of nonlinear coupled partial differential equations whose solutions must satisfy a set of complicated boundary conditions. These boundary conditions may present a nonstationary combustion process at the combustor's boundaries, making it impossible to use solution methods which they may describe the nonsteadiness of the flow at the entrance to the nozzle. To overcome these mathematical difficulties this engineer is often forced to simplify the original problem to such an extent that it no longer resembles the physical situation that he had originally intended to analyze.

Over the past fifteen years numerous researchers have expended considerable effort toward overcoming these mathematical difficulties. As a result a number of mathematically sophisticated theories have been developed. These theories utilize complicated perturbation techniques which are sometimes coupled with eigenfunction expansions or numerical integration methods (e.g., refs. (1) through (4)); in some instances (5) the stability analysis is based on the use of a straight-forward numerical integration scheme. Although these theories are able to explain some of the experimentally-observed phenomena, their highly complicated nature makes their application by the development engineer highly improbable.

At present there is a definite need for a new theoretical approach that will enable the propulsion engineer to perform a stability analysis with relative ease. This technique should be applicable to both linear and nonlinear problems and should be capable of handling the complicated boundary conditions which so often arise in combustion-instability problems. The Galerkin method has the potential for satisfying these needs.

The Galerkin Method is a special application of the Method of Weighted Residuals (usually referred to as MWR). Since its development around the turn of the century, the Galerkin method has been extensively used in the solution of various stability and aeroelasticity problems (see ref. (6) for a review of this method and an extensive list of references). In these instances the Galerkin method proved itself as a useful tool for the solution of both linear and nonlinear problems. Although it is an approximate mathematical technique, it has nevertheless produced results which were in excellent agreement with available exact solutions. These approximate solutions are usually simpler in form than the exact solutions obtained by numerical integration, and their quantitative evaluation requires considerably less computation time.

To date the Galerkin method has been mainly used in situations were the solutions to the governing equations were required to satisfy a set of relatively simple boundary conditions (e.g., see presentation given in ref. (7)). In the following section the "classical" Galerkin approach is modified to accommodate mathematically complicated, although physically meaningful, boundary conditions (such are encountered in combustion instability problems). Then to demonstrate its use, the modified Galerkin method will be applied in the analysis of a linear combustion instability problem.

Development of the Modified Galerkin Method

In analyzing the unstable behavior of solid-propellant and some liquid-propellant rocket motors it is customary to assume that hot combustion products with time-dependent properties (i.e., pressure, temperature, and composition) are generated at the combustor's boundary. The derivation of appropriate expressions capable of describing the behavior of the nonsteady mass, momentum and energy sources at the combustor's boundaries requires a detailed analysis of the complex combustion zone dynamics (e.g., ref. (8)). By applying mass, momentum, and energy balances to an infinitesimally thin combustion zone (9), and assuming no accumulation within the zone, the following conservation laws for a concentrated combustion zone are derived:

\[ \rho_{\text{Mass}} \left( \frac{\partial \rho_{2}}{\partial t} \right) \rho_{2} + \left( \rho_{2} \right) \frac{\partial \rho_{1}}{\partial t} = 0 \] (1a)

\[ \rho_{\text{Mom}} \left[ \rho_{1} \left( \frac{\partial \rho_{2}}{\partial t} \right) \rho_{2} \right] + p_{2} \frac{\partial \rho_{2}}{\partial t} \]
\[ - \left[ \rho_{1} \left( \frac{\partial \rho_{1}}{\partial t} \right) \right] + p_{1} \rho_{1} = 0 \] (1b)

This work was partially supported by NASA Grants NGR-002-083 and NAG-657; * Associate Professor, School of Aerospace Engineering; ** NASA Trainee.
where \( \rho, p, q, \) and \( e \) respectively represent the density, pressure, velocity, and internal energy; \( \mathbf{n} \) is a unit vector in the direction of the outward normal to the combustor's boundary; and \( Q_{\text{Mass}}, Q_{\text{Mom}}, Q_{\text{E}} \) represent the mass, momentum, and energy sources at the boundary. For the problem at hand these quantities are assumed to be known. In Eqs. (1) the subscripts 1 and 2 respectively represent conditions on the "outside" and "inside" sides of the concentrated combustion zone (see Fig. (1)).

The Galerkin method will now be used in the analysis of the unsteady behavior of a rocket combustor with unsteady mass, momentum, and energy sources present over a portion of its boundaries and a nozzle present at one end of the chamber. For the sake of clarity the presence of viscosity, heat conduction, body forces, and heterogeneities will be neglected, but the possibility of sources of mass, momentum, and energy distributed throughout the volume of the combustor will be considered. Under these conditions the flow field of a calorically perfect gas can be described by the following set of conservation equations:

\[
E_1(\rho, q, \mathbf{v}) = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) - F_{\text{Mass}} = 0
\]

\[
E_2(\rho, q, \mathbf{v}) = \frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{v}) + \mathbf{F}_{\text{Mom}} = 0
\]

\[
E_3(\rho, q, \mathbf{v}) = \frac{\partial (\rho e)}{\partial t} + \nabla \cdot (\rho e \mathbf{v}) + \mathbf{F}_{\text{E}} = 0
\]

*As an alternative form the coefficients in the series could be considered functions of time with the function \( \mathbf{q} \) only spatially dependent.

Figure 1 - Schematic Diagram of Concentrated Combustion Zone

According to the Galerkin method the dependent variables are assumed to have the following series solutions:

\[
Q_1 = \sum_{n=0}^{N} A_n \mathbf{q}_n(x, t) ; \quad \mathbf{p} = \sum_{n=0}^{N} B_n \mathbf{q}_n(x, t) ;
\]

\[
\mathbf{p} = \sum_{n=0}^{N} C_n \mathbf{q}_n(x, t)
\]

where the \( \mathbf{q}_n(x, t) \) are known functions which form a complete set in the \((x, t)\) space, and \( A_n, B_n, \) and \( C_n \) are unknown constants that must be determined. In the classical approach the functions \( \mathbf{q}_n(x, t) \) are chosen in such a way as to satisfy the specified boundary conditions (i.e., Eqs. (1)). Fulfilling this requirement is a straightforward matter in some problems and impossible in others. Successful application of the Galerkin method depends on the proper choice of the functions \( \mathbf{q}_n(x, t) \), which requires some additional information about the problem at hand. Such information can be obtained from experimental data, from the solution of a linearized version of the same problem, or from the solutions of closely related problems.

For the moment, it will be assumed that the expansions of the dependent variables have been properly chosen and that the mass, momentum, and energy source boundary conditions described in Eqs. (1) are identically satisfied, that is:

\[
B_1(\rho \mathbf{q}, \mathbf{v}) = Q_{\text{Mass}}(\rho \mathbf{q}, \mathbf{v}) + \mathbf{F}_{\text{Mom}} - (p \mathbf{q}) = 0
\]

\[
B_2(\rho \mathbf{q}, \mathbf{v}) = Q_{\text{Mom}}(\rho \mathbf{q}, \mathbf{v}) + \mathbf{F}_{\text{Mom}} - \mathbf{F}_{\text{Mom}} = 0
\]

\[
B_3(\rho \mathbf{q}, \mathbf{v}) = Q_{\text{E}}(\rho \mathbf{q}, \mathbf{v}) + \mathbf{F}_{\text{E}} - \mathbf{F}_{\text{E}} = 0
\]

In the above equations, \( p_1, q_1, \) and \( p_2 \) are considered to be known. Next the expansions given in Eq. (5) are substituted into Eqs. (2) through (4) to yield the following expressions:

\[
E_1(\rho, q, \mathbf{v}) = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) - F_{\text{Mass}}(\rho \mathbf{q}, \mathbf{v}) = 0
\]

\[
E_2(\rho, q, \mathbf{v}) = \frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{v}) + \mathbf{F}_{\text{Mom}}(\rho \mathbf{q}, \mathbf{v}) = 0
\]

\[
E_3(\rho, q, \mathbf{v}) = \frac{\partial (\rho e)}{\partial t} + \nabla \cdot (\rho e \mathbf{v}) + \mathbf{F}_{\text{E}}(\rho \mathbf{q}, \mathbf{v}) = 0
\]
where the $E_i$ for $i = 1, 2, 3$ are referred to as residuals of the conservation equations.

To determine the unknown coefficients in the series expansions for the dependent variables, the residuals are required to be orthogonal to each of the functions $\varphi_n(x,t)$, that is,

$$
\int_0^{t^*} \int_V E_i(x,t) \varphi_n(x,t) dv dt = 0 \quad i = 1, 2, 3
$$

where $v$ represents the space under consideration and $t^*$ is a properly specified time. In problems involving periodic solutions $t^*$ is chosen to be equal to the period of oscillation. Performing the integrations indicated in Eqs. (12) yield $3(N+1)$ algebraic equations which are sufficient for determining the $3(N+1)$ unknown coefficients which appear in the expansions of the dependent variables.

The basic ideology behind the above procedure will now be described briefly. If the series given by Eqs. (5) represent exact solutions of the conservation equations, all the residuals vanish identically in the $(x,t)$ space, and the validity of Eqs. (12) is evident. If this is not the case then Eqs. (12) expresses the condition that the residuals $E_i$ are orthogonal to all the $\varphi_n$ for $n=0, ..., N$. In the limit as $N$ becomes infinite the $E_i$ are required to be orthogonal to a complete set of functions in $(x,t)$, hence by definition of completeness, the residuals are identically equal to zero. In this case the expansions of the independent variables become identical with the exact solution of the problem. The expressions given in Eq. (12) may be considered as means of minimizing the error that results from the use of the suggested series solutions.

The discussion will continue with the consideration of the continuity equation. Letting $i = 1$ in Eq. (12) yields

$$
\int_0^{t^*} \int_V \left( \frac{\partial \varphi_n}{\partial t} + \nabla \cdot (\rho \varphi_n - F_{\text{mass}} \varphi_n) \right) dv dt = 0
$$

Rewriting the divergence term and using the divergence theorem gives:

$$
\int_0^{t^*} \left( \int_V \left( \frac{\partial \varphi_n}{\partial t} \varphi_n - \rho \varphi_n \cdot \nabla \varphi_n - F_{\text{mass}} \varphi_n \right) dv + \int_{\partial V} \left[ \varphi_n (\rho \varphi_n \cdot \hat{n}) ds \right] dt = 0
$$

Substitution of the boundary condition given by Eq. (6) into the above equation yields

$$
\int_0^{t^*} \left( \int_V \left( \varphi_n + \rho \varphi_n \cdot \hat{n} - F_{\text{mass}} \varphi_n \right) dv \right) dt = 0
$$

Similar relations can be established for the momentum and energy equations.

Next the more realistic case in which the chosen series expansions do not satisfy the prescribed boundary conditions will be considered. The substitution of the expansions given by Eqs. (5) into Eq. (1a) then yields a nonzero boundary residual of the following form:

$$
E_1(x,t) = \varphi_{\text{mass}} + (\rho \varphi_n \cdot \hat{n}) - (\rho \varphi_n \cdot \hat{n})
$$

Similar residuals are obtained from the momentum and energy boundary conditions, Eqs. (1b) and (1c). In this case there is a need to minimize the residuals generated by both the differential equations and the boundary conditions; the latter yield the following relations:

$$
\int_0^{t^*} \int_S E_i(x,t) \varphi_n(x,t) dv dt = 0 \quad n = 0, 1, 2, ..., N
$$

A straightforward solution of Eqs. (12) and (15) would yield $6(N+1)$ equations for the determination of $3(N+1)$ unknowns; the analyst is then faced with the need to discard $3(N+1)$ redundant equations. A different and more complicated approach is suggested by Shuleshko who expands the dependent variables in terms of two sets of complete functions; each of which must satisfy a different set of conditions.

It will be shown here that the difficulties which appear when the assumed solutions do not satisfy the boundary conditions can be overcome by properly adding or subtracting the expressions which result from the orthogonalization of the various residuals. Similar procedures were advocated by Finlayson and Scriven who analyzed the heat conduction equation. Finlayson and Scriven justified their approach by showing the equivalence of their method to a variational principle. In the present study the proposed approach will be justified by showing that the resulting expressions are identical with those which are obtained in the case when the expansions of the dependent variables identically satisfy the specified boundary conditions (i.e., Eq. (13)).

To demonstrate the validity of the modified Galerkin method, Eq. (15) into which the residual of the mass source boundary condition has been substituted, is subtracted from Eq. (12) for the residual of the continuity equation. Performing this operation yields:

$$
\int_0^{t^*} \left( \int_V \left( \varphi_n + \rho \varphi_n \cdot \hat{n} - F_{\text{mass}} \varphi_n \right) dv \right) dt = 0
$$
which is identical to Eq. (13) which was obtained by assuming that the expansions of the dependent variables identically satisfy the mass source boundary condition. The motivation behind the above procedure becomes clear when \( \varphi_n \) is put equal to a constant. In this case the integration of the divergence term over the volume is equal to the integral of the mass flux term over the surface in question, and the latter can be replaced by an appropriate expression (i.e., the function \( Q_{\text{Mass}} \)). It would perhaps be more satisfying if the applicability of the expression given in Eq. (16) could be demonstrated by proving its equivalence with a properly derived variational principle; in the absence of such a proof the correctness of Eq. (16) is further demonstrated in the next section where its use yields proper solutions.

Applying the same procedure to the momentum and energy equations yields the following two expressions:

\[
\begin{align*}
\int_0^t & \left\{ \left[ \frac{\partial}{\partial t} \rho \varphi - \frac{\partial}{\partial x} \rho \varphi \right]_{\nu} - \rho \varphi - \rho \varphi - \frac{\partial}{\partial x} \varphi \frac{\partial}{\partial x} \varphi \right\} \varphi_n \, dv \\
& - \int_s \left[ \frac{\partial}{\partial t} \rho \varphi - \rho \varphi - \rho \varphi - \frac{\partial}{\partial x} \varphi \frac{\partial}{\partial x} \varphi \right] \varphi_n \, ds \\
& - \int_0^t \left[ \frac{\partial}{\partial t} \rho \varphi - \rho \varphi - \rho \varphi - \frac{\partial}{\partial x} \varphi \frac{\partial}{\partial x} \varphi \right] \varphi_n \, dv \\
& \quad \left[ Q_{\text{Mass}} - \rho_1 \varphi \right] \varphi_n \, ds \right\} \, dt = 0
\end{align*}
\]

or

\[
\int_0^t \left\{ \left[ \frac{\partial}{\partial t} \rho \varphi - \frac{\partial}{\partial x} \rho \varphi \right]_{\nu} - \rho \varphi - \rho \varphi - \frac{\partial}{\partial x} \varphi \frac{\partial}{\partial x} \varphi \right\} \varphi_n \, dv \\
- \int_s \left[ \frac{\partial}{\partial t} \rho \varphi - \rho \varphi - \rho \varphi - \frac{\partial}{\partial x} \varphi \frac{\partial}{\partial x} \varphi \right] \varphi_n \, ds \right\} \, dt = 0
\]

Application to a Linear Combustion Instability Problem

To demonstrate the applicability of the modified Galerkin method it will now be applied in the solution of a linear, axial high-frequency combustion instability problem whose exact solution is available (15). A liquid propellant rocket motor with combustion concentrated at the injector face and a short (quasi-steady) nozzle at the other end will be considered. It is assumed that the combustion process is described by Crocco’s time-lag hypothesis.

If the dependent variables are written as a sum of a mean flow property and a small amplitude perturbation, and are substituted into the one-dimensional forms of the continuity and Euler equations, and all products of perturbation quantities are neglected, the following linear non-dimensionalized equations are obtained:

\[
\begin{align*}
E_1 (\rho', \nu') &= \rho_t' + \rho \nu x' + u_0 x = 0 \quad \text{(20)} \\
E_2 (\rho', \nu') &= u_t' + \rho u_x' + \rho x' = 0 \quad \text{(21)}
\end{align*}
\]

where the isentropic relation \( \rho' = \rho_0 \) has been used to eliminate the pressure from Eq. (21). In the above equations \( \rho' \), \( \nu' \) and \( u_0' \) respectively represent the density, pressure, and axial velocity perturbations; \( \rho_0 \) is the steady flow velocity; and the subscripts \( x \) and \( t \), respectively, denote partial differentiation with respect to space and time.

The boundary conditions for this problem are given by a mass source at the injector face \( x=0 \) and an admittance condition at the nozzle entrance \( x=1 \). At \( x=0 \) the fractional increase of the difference in mass flow rates across the concentrated combustion zone equals the fractional increase in the burning rate at the concentrated combustion zone. With the aid of Crocco’s time-lag hypothesis the combustion zone boundary condition can be expressed in the following form:

\[
\frac{\rho' \nu' + \rho u_0'}{\rho_0} = \nu \frac{\rho'(0, t) - \rho'(0, t - \tau)}{\rho_0}
\]

where \( \nu \) is the steady state value of the sensitive time-lag \( \tau \) and \( n \) is the interaction index. Since the gas phase mass flux into the concentrated combustion zone from the injector face is assumed to be zero, we have \( \varphi_0 = u_0 = 0 \) so Eq. (1a) becomes (for \( \beta = 1 \)):

\[
\int_0^t \left\{ \left[ \frac{\partial}{\partial t} \rho \varphi - \frac{\partial}{\partial x} \rho \varphi \right]_{\nu} - \rho \varphi - \rho \varphi - \frac{\partial}{\partial x} \varphi \frac{\partial}{\partial x} \varphi \right\} \varphi_n \, dv \\
- \int_s \left[ \frac{\partial}{\partial t} \rho \varphi - \rho \varphi - \rho \varphi - \frac{\partial}{\partial x} \varphi \frac{\partial}{\partial x} \varphi \right] \varphi_n \, ds \right\} \, dt = 0
\]

\[
(16)
\]

For more details see Ref. (15), pp. 79-87.
(23)

where \( Q_{\text{mass}} = \gamma \mu \left[ \rho' \left(0, t \right) - \rho' \left(0, t - \tau \right) \right] \)

In the case of a short nozzle the Mach number at the nozzle entrance \((x = 0)\) may be assumed to be constant. This condition leads to the following relation between the density and velocity perturbations

\[
\frac{u^*}{u} = \frac{\gamma_{-1}}{2} \frac{\rho^*/p}{p^*}
\]

which can be rewritten as

\[
Q_{\text{mass}} + \left( \rho' \bar{u} + u' \right) \cdot \bar{n} = 0
\]

where \( Q_{\text{mass}} = - \frac{\gamma_{+1}}{2} \bar{u} \rho^*(1, t) \)

Applying the modified Galerkin method to the continuity equation (i.e., Eq.(20)) gives

\[
\int_0^1 \left\{ \frac{\partial \rho}{\partial t} + \bar{u} \cdot \frac{\partial \rho}{\partial x} \right\} \psi_n \, dx
- \int_0^{1} \left[ \rho^* \bar{u}^* \right]_{x=0}^{x=1} \psi_n \, dx
- \int_{s} \left[ \rho^* \bar{u}^* \right] \psi_n \, ds = 0
\]

where \( M \) represents a momentum source due to interactions between liquid drops and the gas in the boundary zone. The Galerkin condition for the momentum equation can be written (with the volume source terms equal to zero) as

\[
\int_0^1 \left\{ \int \left[ \frac{\partial \rho}{\partial t} + \bar{u} \cdot \frac{\partial \rho}{\partial x} \right] \psi_n \, dv
+ \int \left[ \frac{\partial \rho}{\partial t} + \bar{u} \cdot \frac{\partial \rho}{\partial x} \right] \psi_n \, dv
- \int_{s} \left[ M + \bar{P} \bar{n} - \bar{q}_l \bar{n} \right] \psi_n \, ds
\]

Using the expansions presented in Eq. (5) (with \( N = \infty \)) the product \( \psi \cdot \psi \) can be written in the following form:

\[
\psi_{n} \bar{q} = \psi_n \sum_{k=0}^{\infty} A_k \bar{q}_k(x, t) \xi_k
\]

where \( \xi_k \) is a unit vector in the \( k \)th direction. Since the functions \( \psi_n \) form a complete set, the product \( \psi \bar{q} \) can be expanded in terms of the functions \( \psi_j \):

\[
\psi_{n} \bar{q} \bar{k} = \sum_{j=0}^{\infty} a_{nj} \psi_j
\]

Substitution of Eq. (30) into Eq. (29) gives

\[
\psi_{n} \bar{q} \bar{k} = \psi_n \sum_{k=0}^{\infty} A_k \bar{q}_k \psi_j
\]

Applying the Galerkin method to the Euler equation Eq. (21) is more difficult, since the Euler equation is a combination of the momentum and continuity equations, and the boundary condition for the momentum equation does not hold for the Euler equation. Therefore the boundary residuals appropriate for the Euler equation will be derived. For this purpose we will write

\[
Q_{\text{Mom}} = Q_{\text{mass}} + M
\]

In order to specialize Eq. (33) to the linear problem being treated, two assumptions made by Crocco (13) must be used. First the effect of liquid drops in the combustion zone is neglected, so \( M = 0 \). Secondly the pressure is assumed continuous across the concentrated combustion zone, therefore \( P_1 = \bar{p} \). The same assumptions also apply at the nozzle entrance. Under these conditions the orthogonality condition for the one-dimensional momentum equation becomes:
Next, the above equation together with Eq. (26) will be used to determine the unknown coefficients in the expansions of \( \gamma \) and \( \tilde{v} \).

In order to apply the Galerkin method one must select a set of functions in which to expand the dependent variables \( \rho ' \) and \( u ' \), that is, the functions \( \phi_n \) which appear in the expansions presented in Eq. (3). Since only neutrally stable (periodic) oscillations are considered, it is required that the functions \( \phi_n(x,t) \) be periodic in time. Available experimental data indicates that in most cases of high-frequency instability the oscillations are similar to the acoustic modes that can be excited in the given combustor geometry. The acoustic modes are the solutions to Eqs. (20) and (21) with \( u_0 \); these solutions can be expressed in terms of the following complex exponential functions:

\[
\phi(x,t) = e^{i\omega(t-x)} ; \quad \tilde{\phi}(x,t) = e^{i\omega(t+x)}
\]  

The real parts of these functions respectively represent a wave moving to the right and a wave moving to the left. In the instability problem the presence of a mean flow, a nozzle and a combustion process is expected to change the frequency from the acoustic value \( kw \), where \( k \) is a positive integer, hence the frequency \( \omega \) is left unspecified and it will be determined by the solution. The foregoing suggests that \( u ' \) and \( \rho ' \) be represented by the following expansions

\[
\begin{align*}
\tilde{\rho}' &= \sum_{k=1}^{N} (a_k \phi_k + b_k \tilde{\phi}_k) ; \\
\tilde{u}' &= \sum_{k=1}^{N} (c_k \phi_k + d_k \tilde{\phi}_k)
\end{align*}
\]  

where

\[
\begin{align*}
\phi_k(x,t) &= e^{ikt} ; \\
\tilde{\phi}_k(x,t) &= e^{ikt}
\end{align*}
\]  

Keeping only the first term in the series (36) gives the first approximation:

\[
\begin{align*}
\tilde{\rho}' &= a_1 \phi_1 + b_1 \tilde{\phi}_1 ; \\
\tilde{u}' &= c_1 \phi_1 + d_1 \tilde{\phi}_1
\end{align*}
\]  

Equations relating the constants \( a_1, b_1, c_1, \) and \( d_1 \) are derived from the Galerkin orthogonality conditions (i.e., Eqs. (26) and (34) ) in which \( t^* \) is the period of oscillation \( 2\pi/\omega \). Since \( \tilde{\rho}' \) and \( \tilde{u}' \) are complex quantities, the complex conjugates \( \phi_1^* \) and \( \tilde{\phi}_1^* \) will be used as weighting functions in the orthogonality conditions. The modified Galerkin orthogonality conditions for this problem can now be written as:

\[
\begin{align*}
- \int_0^{2\pi/\omega} E_1(\tilde{\rho}',\tilde{\nu}') \phi_1(x,t) \text{d}x \text{d}t &= 0 \quad (39a) \\
- \int_0^{2\pi/\omega} E_1(\tilde{\rho}',\tilde{\nu}') \tilde{\phi}_1(x,t) \text{d}x \text{d}t &= 0 \\
- \int_0^{2\pi/\omega} E_0(\tilde{\rho}',\tilde{\nu}') \phi_1(0,t) \text{d}t &= 0 \quad (39b) \\
- \int_0^{2\pi/\omega} E_0(\tilde{\rho}',\tilde{\nu}') \tilde{\phi}_1(0,t) \text{d}t &= 0 \quad (39c)
\end{align*}
\]

Substituting the expansions (38) into Eqs. (39) and performing the indicated integrations yields a set of four homogeneous linear equations in the four unknown constants. After some algebraic manipulations these four equations are given by:

\[
\begin{align*}
K_1 a_1 + b_1 \Omega_1 + K_2 b_1 + (\Omega-2) c_1 &= 0 \\
K_1 c_1 + b_1 \Omega_1 - K_2 a_1 + \Omega c_1 &= 0 \\
B_0(t) c_1 &= \Omega_1 \\
B_1(t) &= \Omega_1 (1 - e^{-imT})
\end{align*}
\]  

where \( E_1, E_2, B_0, \) and \( B_1 \) are given by Eqs. (20) and (21) and \( \Omega_1 \) is given by

\[
\Omega_1 = \frac{1}{\omega} (1 - e^{-imT})
\]

In order for non-trivial solutions of Eqs. (41) to exist, the determinant of the coefficient matrix must vanish. Setting this determinant equal to zero gives a complex relation between \( \omega, n, \) and \( T \); separating the real and imaginary parts gives two real equations in the three variables \( \omega, n, \) and \( T \). Solving these for \( n(\omega) \) and \( T(\omega) \) (if solutions exist) gives the curves of neutral stability in the \( n-T \) plane, the only points where linearly neutral (periodic) oscillations exist.

Neglecting all terms of order \( \tilde{u}^2 \) or higher, the real and imaginary parts of the characteristic equation become:

\[
\begin{align*}
\gamma \omega - \frac{1}{2} \Omega_1 (1 - e^{-imT}) &= 0 \\
\gamma \omega + \frac{1}{2} \Omega_1 (1 - e^{-imT}) &= 0 \\
\gamma \omega (1 - e^{-imT}) &= 0
\end{align*}
\]  

In order for non-trivial solutions of Eqs. (41) to exist, the determinant of the coefficient matrix must vanish. Setting this determinant equal to zero gives a complex relation between \( \omega, n, \) and \( T \); separating the real and imaginary parts gives two real equations in the three variables \( \omega, n, \) and \( T \). Solving these for \( n(\omega) \) and \( T(\omega) \) (if solutions exist) gives the curves of neutral stability in the \( n-T \) plane, the only points where linearly neutral (periodic) oscillations exist.
Solving equations (43) and (44) for \(\psi\) and equating the results to eliminate \(n\) gives, after some trigonometric manipulations, an equation relating \(\psi\) and \(\omega\).

\[
\psi = (2m+1)\tan^{-1}\left(\frac{\tan\omega}{m+\frac{1}{2}}\right)
\]

\(m = 0, 1, 2, 3, \ldots \) (45)

Solving (43) for \(n\) and using some trigonometric identities gives

\[
n = \frac{\psi + 1}{2y} \left[ 1 - \cos\psi + \frac{\psi - 1}{2} \sin\psi \tan\psi \right]^{-1}
\]

(46)

This same problem is solved by exact mathematical techniques by Crocco (15) who obtains the following pair of equations (Eq. 3.02.01 in ref. (15)):

\[
y \cos \psi \omega = -(1-y) - \frac{\psi - 1}{2} \sin \psi \tan \psi
\]

\[
y \sin \psi \omega = \frac{\tan \psi}{\omega}
\]

(47)

The expressions presented in Eq. (47) can be manipulated to yield Eqs. (45) and (46). It has just been shown that the stability limit predicted by the modified Galerkin method, which is an approximate mathematical technique, is in complete agreement with the stability limit predicted, for the same problem, by the more rigorous analysis performed by Crocco and Cheng (15).

References


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APPLICATION OF THE GALERKIN METHOD IN
THE DESIGN OF STABLE LIQUID ROCKET MOTORS

SEMIANNUAL REPORT COVERING PERIOD
August 1, 1969 - January 31, 1970

Prepared By
Ben T. Zinn, Associate Professor
Warren C. Strahle, Associate Professor
Eugene A. Powell, Graduate Research Assistant

GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL OF AEROSPACE ENGINEERING
ATLANTA, GEORGIA

Project Monitor: Dr. Richard J. Priem
The work performed during the previous six months represents an extension and diversification of the research performed during the first year of research supported by this grant. During that period the Galerkin Method has been modified and successfully applied in the solution of a number of combustion instability problems. Special attention was given to the study of nonlinear effects. As part of this effort a nonlinear second-order theory was developed; the latter was used as a guide in the development of a computer program designed to predict the nonlinear behavior of unstable liquid propellant rocket motors.

During the first half of the second year effort the above-mentioned computer program was used to determine the dependence of the nonlinear stability limits upon the combustion process (i.e., upon the values of \( n \) (the interaction index) and \( \tau \) (the time lag)), upon the magnitude of the Mach number of the engine's mean flow and upon the engine's length to diameter ratio. The results of these calculations showed that the nature of the resulting instability is strongly dependent upon the parameters that describe the unsteady combustion process (i.e., \( n \) and \( \tau \)). It was found that the values of \( n \) and \( \tau \) have a strong effect upon the engine's triggering limits as well as the shape and the amplitude of the resulting nonlinear wave forms. In addition it was found that for a given engine and combustion process increasing the Mach number of the mean flow and/or decreasing the combustor's length to diameter ratio would cause an increase in the final amplitude attained by the engine's pressure oscillations.

In a separate study the effect of an initially spinning disturbance was investigated. It was found in this study that the resulting unstable engine behavior is qualitatively similar to the unstable behavior predicted in cases when the initial disturbance was a standing wave; the exception being that in nearly all cases
the final amplitude attained by the pressure oscillation was larger when the initial disturbance was a spinning wave.

The above studies concentrated on the investigation of the behavior of the first tangential mode. In a separate computer program the behavior of the first radial mode was investigated. This study was primarily devoted to the determination of nonlinear triggering limits. The calculated limits were found to be strongly dependent on the values of $n$ and $\tau$ and to be practically independent of the magnitude of the Mach number of the mean flow as well as the combustor's length to diameter ratio.

During the same period work has also been done on the development of a third order theory. This theory represents an attempt to relax some of the restrictions imposed on the second order theory. The latter included such restrictions as small Mach number mean flow, irrotationality of the flow and the presence of waves whose amplitudes are not too large. When some of these restrictions are removed the resulting conservation equations can no longer be reduced to a single equation governing the behavior of the velocity potential. Instead a system of partial differential equations must be solved. In the analysis performed to date no terms have been neglected in the conservation equations, the only approximations used being those related to the absence of droplet drag and constancy of droplet temperature; both of these assumptions were used in the second order theory.

Once the appropriate system of wave equations was derived the Galerkin Method was applied to derive a system of ordinary nonlinear differential equations that describe the behavior of the time dependent amplitudes that appear in the assumed series expansions for the various dependent variables. Because of the complexity of these equations, it was desired to investigate first the behavior of a single transverse mode by approximating each
dependent variable as a product of an amplitude function and the spatial dependence of that mode. To complete the theory higher order expressions for the burning rate term and nozzle admittance relation were needed. Unlike the expressions describing the gas dynamics of the problem these expressions contained terms of all orders and had to be truncated to include terms up to third order only. In deriving these expressions it was necessary to assume that the combustor's mean flow Mach number was small. These burning rate expressions were derived by using Crocco's time lag hypothesis.

As a check on the analysis, linear stability limits were computed using the linearized version of the system of equations derived for the third order theory. Except for small corrections of the order of Mach number squared these limits agreed with those computed from the linearized version of the second order theory.

Using the third order theory numerical solutions were obtained for the following two cases: (1) the approximate solutions consisting of the first tangential mode only, and (2) the approximate solutions consisting of the first radial mode only. The systems of differential equations governing these two cases differed in several respects. The equations governing the behavior of the first radial mode contained both quadratic and cubic nonlinearities while the equations governing the behavior of the first tangential mode contained only cubically nonlinear terms. The radial mode equations contained nonlinearities in the combustion mass source term whereas a nonlinear driving term was missing in the equations for the first tangential mode.

Contrary to the results obtained from the second order theory the third order theory predicted the possibility of triggering combustion oscillations in both cases. These predictions also show that the unstable behavior of the first radial and the first
tangential modes are qualitatively different. The main causes for these differences are believed to be related to the presence of nonlinear combustion terms in the differential equations that describe the behavior of the first radial mode. These terms have great influence upon the nature of the predicted stability limits. The computed results also suggest that combustion source nonlinearities may be more important than gas dynamical nonlinearities.

Some of the third order solutions predicted the anomalous result that under certain conditions the combustor's pressure may become negative. The occurrence of negative pressures in the approximate solutions is a result of the assumed spatial dependence of the series solutions. It is expected that this shortcoming will be remedied by the use of multi-mode series solutions at the cost of increased computation time.

All of the above studies were devoted to the examination of the nonlinear response of unstable liquid propellant engines. In all of these studies the Crocco time lag theory was used to describe the combustion driving force. Recognizing the limitations of the time lag approach separate studies aimed at deriving physically meaningful expressions for the combustion source were also conducted during the past several months. As part of these studies calculations were completed on a new model of vaporization response to acoustic waves; a paper on this subject is in preparation. The new feature of this work is the inclusion of a thermal wave in the liquid. The numerical results predict that a combustion response peak, in phase with the pressure, occurs in the proper frequency range. It has also been found that velocity effects are important in the description of this response peak. Since the usual theories of instability have specifically neglected velocity effects in the feedback function, a reexamination of linear stability theory was made including
these effects. Substantial alteration of stability criteria occurs. It is believed that these results have bearing on a reconciliation between linear analysis results and nonlinear computer results. A paper on velocity effects is in preparation.

Additional studies that are presently in progress include the development of a nonlinear third order theory in which the solutions for the dependent variables are described by multi-mode series expansions. This theory treats the case of transverse instability and it is expected to improve earlier results. In another study, the case of axial type instability, in a combustor with a distributed combustion process, is investigated. The axial type of instability is known to exhibit discontinuous wave behavior and its theoretical treatment may require modification of the previously-used Galerkin Method. The solutions of these problems and the development of more meaningful unsteady combustion models are the primary objectives of this program for the next six months.
Research Conducted Under
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Ben T. Zinn, Associate Professor
Eugene A. Powell, Instructor

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SCHOOL OF AEROSPACE ENGINEERING
ATLANTA, GEORGIA

Project Monitor: Dr. Richard J. Priem
Introduction

During the first two years of NASA support for this project, an approximate mathematical technique was developed and used to investigate the nonlinear behavior of unstable liquid propellant rocket motors.\textsuperscript{1,2} The work performed during the first six months of the third year of this investigation is essentially a continuation of work begun in the second year. During the period covered by this report the following investigations were conducted: (1) the study of moderate amplitude transverse instability based on the second order theory developed during the first year, (2) the development of a third order theory to study large amplitude transverse instabilities, and (3) the study of nonlinear axial mode instability.

A summary of results obtained with the aid of the second order theory was presented at the Seventh JANNAF Liquid Propellant Combustion Instability Meeting held at the Jet Propulsion Laboratory, Pasadena, California during October 27-29, 1970. Additional results appear in the following publications:


(2) Powell, E. A., and Zinn, B. T., "A Single Mode Approximation in the Solution of Nonlinear Combustion Instability Problems," AIAA paper No. 71-208 which was presented on January 27 in New York at the 9th AIAA Aerospace Sciences Meeting. This paper has been submitted for publication in Combustion Science and Technology.

Another paper "Stable Limit Cycles and Triggering Limits of the First Radial Mode in Unstable Liquid Rockets" has been accepted for presentation and publication in the proceedings of the Thirteenth Israel Annual Conference on Aviation and Astronautics.
A brief description of the results obtained in the above-mentioned investigations are provided in the following sections.

Second Order Investigations: Transverse Modes

Using the second order theory, the following tasks related to the study of transverse instabilities were performed: (1) the construction of nonlinear stability maps in the \((n,\tilde{\tau})\) plane, (2) the development of means for correlation of the nonlinear theory with available experimental data in order to determine the operating \((n,\tilde{\tau})\) values of an unstable engine, and (3) an investigation of the effect of the initial disturbance upon the behavior of the resulting instability. In addition considerable time has been spent in the preparation of a technical report that summarizes the results obtained in various second order studies.

Nonlinear Stability Maps

During this period work continued on the construction of nonlinear stability maps for the first tangential \((1T)\) mode. Such maps are needed for the purpose of correlating available experimental data as well as providing a better understanding of the nonlinear behavior of unstable liquid rockets. A typical stability map is shown in Fig. (1) where lines of constant pressure amplitude and lines of constant frequency are plotted on an \((n,\tilde{\tau})\) plane. Stability plots for the \(1T\) standing mode have been made for \(\bar{u}_e = 0.1, 0.2\) and \(0.3\) with \(z_e = 1.0\) where \(\bar{u}_e\) is the steady state flow Mach number at the nozzle entrance and \(z_e\) is a dimensionless chamber length \(\text{(length/radius)}\). The calculations involved in constructing these maps were made using a three-mode series containing the \(1T, 2T,\) and \(1R\) modes.

These stability maps reveal the following features of standing \(1T\) mode instability. If \(\delta\) represents the vertical distance measured above the neutral stability limit, then it is seen that the pressure
amplitude attained in a linearly unstable engine increases as $\delta$ increases. Also for a fixed displacement $\delta$ the amplitude increases as $\tau$ increases. However, for a fixed value of $n$ the resulting maximum amplitude decreases as one moves away from $\tau_{\text{min}}$. The frequency lines show that the frequency of oscillation is close to that of the acoustic $1T$ mode; there is a slight decrease in frequency with increasing $\tau$ or with increasing amplitude (or $n$). Comparing the charts for different values of $\bar{u}_e$ or $z_e$ (see Figs. (1) and (2)) show that increasing $\bar{u}_e$ or decreasing $z_e$ is generally destabilizing with respect to the $1T$ mode.

**Correlation of Theory with Experiment**

During the last six months much effort has been expended toward the development of a method of determining the operating $(n,T)$ values of an unstable engine by a comparison of theoretical predictions and test data. One possible method becomes apparent from an examination of the stability map shown in Fig. (1). Suppose a rocket engine is unstable with respect to the $1T$ standing mode and the maximum peak-to-peak chamber pressure amplitude and frequency are measured. Lines of constant amplitude and constant frequency corresponding to the measured values can be plotted on Fig. (1) by interpolation. The point of intersection of these lines determines theoretical $n$ and $\bar{\tau}$ values for the unstable engine. This method could also be used in the case of spinning instability. From Fig. (1), however, it is readily seen that a small percentage error in determining the engine's frequency would be reflected in a large error in the corresponding $(n,\bar{\tau})$ values. Thus it is doubtful that the frequency can be obtained experimentally to the accuracy required for determining the engine's $(n,\bar{\tau})$ values.

The possibility of using the nonlinear waveforms to determine the $(n,\bar{\tau})$ operating point of a tested engine was also investigated. Results obtained so far indicate that for standing waves two or
more pressure measurements at various locations in the chamber may be adequate for this purpose. This method will be illustrated by use of the following example. Suppose that pressure measurements at the chamber wall indicate that the standing 1T mode is present, and that the peak-to-peak amplitude at the acoustic anti-node \( A_1 = 0.40 \) and the amplitude at the node \( A_2 = 0.07 \). The second order theory predicts that the operating point must be on the curve of constant amplitude \( (A_1 = 0.4) \) shown in Fig. (3). For each point of this curve a node amplitude \( A_2 \) can be calculated; the variation of \( A_2 \) with \( \tau \) along this curve is shown in Fig. (4). Figures (3) and (4) together can be used to determine a theoretical value of \( n \) and \( \tau \) for the data given. For \( A_2 = 0.07 \) the value \( \tau = 1.65 \) is read from Fig. (4), while the corresponding value \( n = 0.577 \) is determined from Fig. (3) as shown. This method has two major disadvantages: (1) it is not applicable to spinning instability, and (2) a small error in measuring \( A_2 \) results in a relatively large error in the determination of \( n \) and \( \tau \). Alternative methods for correlating the second order theory are currently under investigation.

**Effect of Initial Disturbance**

During this period an investigation of the effect of initial conditions on the nature of the final solution was conducted. The initial disturbance studied was a combination of spinning and standing 1T modes. It was found that a standing oscillation could develop only from a pure standing initial disturbance. If any spinning component was originally present the final oscillation was always of the spinning form. This satisfactory result is in agreement with available experimental data that indicates that in the majority of known instances the instabilities are of the spinning type. This also appears to be a pure fluid mechanical effect since there is nothing in the combustion response function that will dictate such behavior.
Technical Report and Computer Programs

During the last three months work has begun on the preparation of a complete technical report describing the second order theory as well as second order results. This report will include a user's manual and program listings for the computer programs based on this theory.

Two programs were developed: Program NLCOEF which computes the coefficients of the nonlinear terms appearing in the governing differential equations and Program LIMCYC which integrates these equations to determine the nonlinear stability characteristics of the rocket combustor. Although these programs were developed during the initial phases of this project, extensive modifications were necessary to make these programs more useful to the practicing engineer. For example the original NLCOEF, which was developed in ALGOL, was translated into FORTRAN and checked out. A complete set of test cases has been run on the improved version of Program LIMCYC and to all indications the program is working properly.

Second Order Investigations: Axial Instability

Using the second order theory described in Ref. (2), investigations of axial type instability in a combustor with a distributed combustion process continued during the six months covered by this report. During this period work concentrated on solutions obtained when the velocity potential $\Phi$ is given by the following series expansion:

$$\Phi(z,t) = \sum_{n=1}^{N} B_n(t) \cos(n\pi z)$$

This series expansion satisfies the boundary conditions at the injector ($z = 0$) but not at the nozzle end (i.e., where $z = 1$).
To check out the computer program, the case of nonlinear acoustics (i.e., no mean flow or combustion) was investigated. The computed results showed the existence of sharply peaked pressure waveforms. Solutions obtained using ten and fifteen terms in Eq. (1) were compared. Although qualitatively similar waveforms were obtained, there existed significant quantitative differences in the magnitudes of the computed amplitudes. Thus the series appears to be slowly convergent.

The next stage of the investigation was the determination of the neutral stability limit in the \((n, \tilde{\tau})\) plane using the cosine expansion given by Eq. (1). For a one-term expansion it was possible to obtain a closed-form solution for the stability boundary. For multi-mode expansions linear coupling between modes made it necessary to determine the neutral stability limit numerically. Results are presented in Fig. (5) for a one-term expansion along with Mitchell's results. The discrepancies between the present results and Mitchell's values are probably due to the lack of mean flow effects in the series expansion and the use of acoustic eigenvalues \((n_\infty)\) in the assumed spatial dependence.

To further check the computer program, the behavior of the solutions at stable, neutrally stable, and unstable values of \(n\) and \(\tilde{\tau}\) was investigated. For \(\tilde{\tau} = 1.0\) a continuous wave form was found at the neutrally stable point (where \(n = 0.8\)) while the expected decay was observed for the stable point (\(n = 0.5\)). For the unstable point (\(n = 0.9\)) a discontinuous (sawtooth) wave form was found. These results were obtained using an eleven term series expansion.

Efforts are now being directed toward reducing the discrepancy between the approximate neutral stability limit and that of Mitchell as well as improving the convergence of the series expansion. At present it appears that the most promising approach is to replace the acoustic eigenvalues \(n_\infty\) in Eq. (1) with a factor
of the form \( n\pi + e_n(t) \). The parameter \( e_n(t) \) can be shown to be small. The short nozzle admittance condition was used to relate \( e_n(t) \) to the undetermined time dependent amplitudes \( B_n(t) \). In this manner the nozzle boundary condition is satisfied. Methods for including mean flow effects in the series expansion and other approaches to satisfying the boundary conditions are under investigation.

Third Order Investigations

During the second year of this project a third order theory was developed. This theory represents an attempt to relax some of the restrictions imposed on the second order theory; namely, small Mach number mean flow, irrotationality of the flow, and the presence of moderate amplitude waves. In this case the full system of conservation equations must be solved. Due to the complexity of these equations, each dependent variable was expressed in terms of a single transverse acoustic mode. Results obtained by this method are given in Ref. 2.

It soon became apparent, however, that the single-mode approximation was not adequate to describe nonlinear combustion instability. For instance some of the third order solutions predicted the anomalous result that under certain conditions the combustor's pressure may become negative. Furthermore results obtained with the second order theory indicate that the nonlinear coupling between different modes is also important. To overcome the above shortcomings work began on the development of a multi-mode third order theory.

In the multi-mode theory, the approximate solutions are expressed in terms of the three lowest frequency transverse modes (i.e., the 1T, 2T, and 1R modes). Again application of the Galerkin method yields a system of ordinary differential equations to be solved for the time-dependent amplitudes of these modes. These equations, however, are considerably more
complicated than those obtained with the single-mode expansions. For the single-mode case a system of four equations in four unknowns had to be solved, but in the three-mode case one must solve a 17 x 17 system. The multi-mode equations are further complicated by the fact that they contain both linear and nonlinear coupling terms involving the first derivatives (highest order derivative) of the unknown amplitudes. This system must be solved by a numerical procedure involving alternate matrix inversion and integration processes.

A computer program to solve the multi-mode third order equations has been developed, but many difficulties have been encountered. To date the program has been checked only for the special cases of when only one mode is present in the series expansion. Also the program requires excessive computer time. The problems associated with this program are presently under investigation, and it is expected that more comprehensive nonlinear solutions will be obtained in the near future.
References


Figure 1. Nonlinear Stability Map for $\bar{u}_e/z_e = 0.1$
Figure 2. Nonlinear Stability Map for $\bar{u}_e/z_e = 0.3$
Figure 3.

Interaction Index, $n$

Time-Lag, $\tau$

Figure 4.

Amplitude At Mall Node, $A_2$

Time-Lag, $\tau$
Figure 5. Axial Mode Stability Limits

\[ \gamma = 1.2 \]

\[ \bar{u}(z) = 0.2z \]
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Prepared By

Ben T. Zinn, Professor
Eugene A. Powell, Instructor
Manuel E. Lores, Graduate Research Assistant
S. Kalyanasundaram, Graduate Research Assistant

GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL OF AEROSPACE ENGINEERING
ATLANTA, GEORGIA

Project Monitor: Dr. Richard J. Priem
INTRODUCTION

During the period covered by this report work continued on investigations which were started in the second year.\(^1,2\) The study of moderate amplitude transverse instability based on the second order theory was concluded and a technical report describing the results of this investigation was published. Work continued on the development of a third order theory to study the behavior of large amplitude transverse instabilities and on the study of nonlinear axial mode instability. Also a study was initiated to determine the influence of the functional form of the unsteady combustion response function upon the stability characteristics of various combustor designs.

The following papers were published during this report period:

A brief description of the results obtained in the above-mentioned investigations is provided in the following sections.

SECOND ORDER INVESTIGATIONS: TRANSVERSE MODES

Using the second order theory, the following tasks related to the study of transverse instabilities were performed: (1) the construction of nonlinear stability maps in the \((n,\bar{r})\) plane, (2) the investigation of the convergence of the assumed series expansion, and (3) the preparation of a technical report summarizing the results obtained in the various second order studies. In addition work has begun on incorporating various combustion response functions into the second order theory.
**Studies Involving the \((n, \tau)\) Model**

**Nonlinear Stability Maps.** The second order theory was used to construct a nonlinear stability map for the spinning first tangential \((1T)\) mode. This map is shown in Fig. 1 where lines of constant limit-cycle pressure amplitude and constant frequency are plotted on an \((n, \tau)\) plane. This map can be compared with a similar map for the standing \(1T\) mode (see Fig. 2). These plots show that for most values of \(n\) and \(\tau\) the spinning wave has the larger limit-cycle amplitude. Along a line of constant \(\tau\) the frequency increases with increasing \(n\) for the spinning \(1T\) mode, while the opposite behavior is observed for the standing \(1T\) mode. Both of these plots were made using a series consisting of the first tangential \((1T)\), second tangential \((2T)\), and first radial \((1R)\) modes for a motor with a steady state Mach number \((\bar{u}_e)\) of 0.2 and a length-to-diameter ratio \((L/D)\) of 0.5.

**Convergence Study.** A convergence study was conducted to determine how many modes are necessary to obtain an accurate solution. In this study both standing and spinning modes were considered. The results of this investigation are presented in the table below for \(n = 0.60167\) and \(\tau = 1.70629\).

<table>
<thead>
<tr>
<th>Modes</th>
<th>Limit-cycle Amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Standing</td>
</tr>
<tr>
<td>1T, 2T, 1R</td>
<td>0.604</td>
</tr>
<tr>
<td>1T, 2T, 3T, 1R</td>
<td>0.608</td>
</tr>
<tr>
<td>1T, 2T, 3T, (4T), 1R</td>
<td>0.609</td>
</tr>
</tbody>
</table>

These results indicate that convergence is very rapid as additional tangential modes are added to the basic three mode (i.e., \(1T, 2T, 1R\))
series. Thus it appears that a series consisting of the 1T, 2T, and 1R modes is sufficient for all practical purposes.

Technical Report. A NASA Contractor Report (CR-72902) describing the second order theory was published under the title "The Prediction of the Nonlinear Behavior of Unstable Liquid Rockets". This report summarizes the principal results obtained during the first two years of this project. Also included are user's manuals for the computer programs developed during this investigation.

Combustion Response Functions

In all studies conducted to date under this program the combustion response was described with the aid of Crocco's pressure sensitive time lag hypothesis. During this report period a new study was initiated in which the time lag model will be replaced by the following two combustion response functions:

\[ W'_m = A p' + B \frac{dp'}{dt} \]

\[ W'_m = C p' + D (p')^2 \]

where A, B, C, and D are prescribed constants, t is the time and p' is the pressure perturbation. At present Eq. (1) has been incorporated into the second order theory, the appropriate differential equations have been derived, and a computer program is being developed to solve these equations. This program is essentially a modification of the program used in the second order studies using the (n,τ) model.
The longitudinal combustion instability investigation consisted of two programs. The first program, with which a majority of the effort was concerned, consisted of a second order analysis using the modified Galerkin method to obtain solutions of a nonlinear wave equation. The second program involved the development of a third order solution in which both the injector and quasi-steady short nozzle boundary conditions are satisfied by the expansions used to represent the flow variables.

**Second Order Analysis**

**Problem Formulation.** By considering moderate amplitude instabilities in liquid propellant rocket combustors having low Mach number mean flows and an unsteady combustion process that can be represented with the aid of Crocco's $\hat{h}-\tau$ theory, the system of conservation equations can be combined into the following single nonlinear wave equation. \(^3,^4\)

\[
\begin{align*}
\varphi_{zz} & - \varphi_{tt} - 2M(z)\varphi_{z t} - \gamma \frac{dM}{dz} \varphi_t - 2\varphi_z \varphi_{zt} - (\gamma-1)\varphi_{t} \varphi_{zz} \\
+ \gamma\hat{h} \frac{dM}{dz} & \left[ \varphi_t(z,t) - \varphi_t(z,t-\tau) \right] = 0
\end{align*}
\]

where $\varphi(z,t)$ is the velocity potential defined by $u'(z,t) = \varphi_z(z,t)$, $M(z)$ is the steady state Mach number, $\gamma$ is the ratio of specific heats, $\hat{h}$ is the interaction index, and $\tau$ is the sensitive time lag. In deriving Eq. (3) it has been assumed that the normalized amplitudes of the flow instabilities are of the order of magnitude of the mean flow Mach number, and that terms of higher than second order may be neglected.

The injector solid wall and the quasi-steady short nozzle boundary conditions are.\(^3,^4\)
The proper choice of trial functions is critical to the usefulness and accuracy of the Galerkin method. Since the behavior of the high frequency combustion instability oscillations is known to be similar to the behavior of acoustic waves in a closed-ended chamber, the velocity potential is expanded in terms of acoustic eigenfunctions and eigenvalues:

\[ \varphi(z,t) = \sum_{n=1}^{N} A_n(t) \cos(n\pi z) \]  

The summation index is varied from 1 to N, dropping the spatially independent ("chugging") mode. A similar expansion was used successfully by Temkin\(^5\) who studied the nonlinear behavior of piston-driven axial waves.

The choice of trial functions defined by Eq. (6) satisfies the solid wall boundary condition, Eq. (4), but not the quasi-steady short nozzle boundary condition, Eq. (5). Therefore, the modified version of the Galerkin method, developed by Zinn and Powell\(^6\), was used to obtain solutions of Eq. (3). Applying this technique to Eq. (3) with the nozzle boundary condition, Eq. (5), yields:

\[ \int_{0}^{1} \left( \varphi_{zz} - \varphi_{tt} - 2M(z) \varphi_{zt} - \gamma \frac{dM}{dz} \varphi_{t} - 2\varphi_{z} \varphi_{zt} - (\gamma-1) \varphi_{t} \varphi_{zz} \right) + \gamma n \frac{dM}{dz} \left[ \varphi_{t}(z,t) - \varphi_{t}(z,t-T) \right] \cos(n \pi z) dz \]

\[ - \left[ \frac{\gamma-1}{2} M_{e} \varphi_{t}(1,t) - \varphi_{z}(1,t) (\varphi_{t}(1,t) - 1) \right] \cos(n \pi) = 0 \]
where $\varphi(z,t)$ is given by Eq. (6). Substitution of Eq. (6) into Eq. (7) and performing the required space integrations yields a set of second order quasi-linear ordinary differential equations:

$$
A''_\ell = -(\lambda \pi)^2 A_\ell + 2 \sum_{n=1}^{N} \left[ - \gamma A'_n + \gamma n (A'_n - A'_n(t-\tau)) \right] I_1(n,t) \\
+ 2 \sum_{n=1}^{N} \left[ -(v-1) M_e (-1)^{n+\ell} A'_n \right] \\
+ \sum_{m=1}^{N} \left[ (\gamma-1)(m\pi)^2 A'_n A_m I_3(n,m,t) - 2(n\pi)(m\pi) A'_n A_m I_2(n,m,t) \right] \\
\ell = 1, \ldots, N
$$

where

$$
I_1(n,\ell) = \int_0^1 \frac{dM}{dz} \sin(n\pi z) \cos(\ell\pi z) \, dz
$$

$$
I_2(n,m,\ell) = \int_0^1 \sin(n\pi z) \sin(m\pi z) \cos(\ell\pi z) \, dz
$$

$$
I_3(n,m,\ell) = \int_0^1 \cos(n\pi z) \cos(m\pi z) \cos(\ell\pi z) \, dz
$$

$$
I_4(n,\ell) = \int_0^1 M(z) \sin(n\pi z) \cos(\ell\pi z) \, dz
$$

Linear stability limits have been established using the linear version of Eq. (8) by considering each mode separately. A combined 1L and 2L analysis has indicated that this approach can be taken with no loss in generality. The loci of points of linear neutral stability of the $\ell$th longitudinal mode are defined by:
Results for $M(z) = M_0 z$. In order to proceed with the analysis, the mean flow Mach number must be specified. Specifying the Mach number is equivalent to defining the steady state combustion distribution. A convenient and often used Mach number distribution is a linearly varying mean flow Mach number, that is

$$M(z) = M_0 z$$

A linear Mach number distribution as defined by Eq. (15) was used to obtain the results discussed in the following paragraphs.

Using Eqs. (13) and (14), linear stability limits in the $\hat{n}$-$\tau$ coordinate system have been established for a specific heat ratio of 1.2 and an exit Mach number of 0.2. These limits are presented in Fig. 3 for the first three longitudinal modes. As pointed out by Crocco\textsuperscript{7}, the stability limits of highest physical interest, that is those encountered in practice, are the limits corresponding to the smallest values of $\omega \tau$ which are solutions of Eq. (14). These limits, shown as solid lines in Fig. 3, will be referred to as primary zones of instability. The solution corresponding to the next higher value of $\omega \tau$ for the second longitudinal mode is shown in a broken line in Fig. 3. The linear stability map presented in Fig. 3 is in qualitative agreement with linear stability limits established by Sirignano\textsuperscript{8} and Mitchell\textsuperscript{9}.

The nonlinear combustion instability oscillations are found by performing a fourth order Runge-Kutta numerical integration
of Eq. (8). A ten term expansion of the velocity potential was used. Due to the presence of a retarded time variable, initial conditions must be specified for the period $t-\tau$ ahead of "zero" time. It has been assumed that the combustion is smooth until $t = 0$, at which time a pressure disturbance is impulsively introduced inside the combustor. The velocity perturbation is taken to be zero at time $t = 0$. Spatially continuous and discontinuous pressure waves have been used as initial conditions. Integration of Eq. (8) is continued until a periodic solution is found, if one exists. The limit cycles have been found to be independent of the form of the initial disturbance.

Typical limit-cycle pressure waveforms for values of $\tau$ representing conditions above, below, and at resonance for the first longitudinal mode (i.e., $\tau < 1$, $\tau > 1$, and $\tau = 1$, respectively) are presented in Fig. 4. These waveforms are qualitatively similar to those obtained by Chester for forced oscillations in closed tubes.

A limit-cycle amplitude map for first longitudinal instabilities is presented in Fig. 5. Here lines of constant pressure amplitude are plotted on an $(n, \tau)$ plane. This map shows that the second order theory does not predict triggering of the first longitudinal mode.

The limit cycle behavior obtained for given values of $n$ and $\tau$ is determined by the corresponding linear stability of the various modes that are present in the series solution. In general, if the first longitudinal mode is linearly unstable at a given point on the $(n, \tau)$ plane, then the resulting limit cycle will exhibit an oscillation in the $1L$ mode. On the other hand, if for given values of $n$ and $\tau$ the $2L$ mode is linearly unstable and the $1L$ mode is linearly stable, then the limit cycle will exhibit a $2L$ type oscillation. However, it has been found that if the $1L$ mode is "slightly" unstable, while the $2L$ mode is very unstable (for example, $\tau = .45$, $n = 2.12$ in Fig. 3), the limit cycle exhibits
2L characteristics. These conclusions are independent of the form or the amplitude of the initial disturbances.

**Third Order Analysis**

The trend in rocket design is towards smaller nozzle contraction ratios, resulting in larger combustor mean flow Mach numbers. It has also been shown experimentally that the amplitude of the combustion instability oscillations can be large. In order to relax the assumptions of low mean flow Mach number and moderate amplitude instabilities used in the second order investigation, a third order analysis will be conducted. Without these two assumptions, the conservation equations cannot be combined into a single wave equation and the whole system of conservation equations must be solved.

The Runge-Kutta method will be used to numerically integrate the ordinary differential equations that result from the application of the Galerkin technique. To facilitate the numerical integration, it is desirable that the ordinary differential equations be quasi-linear. This can be accomplished without restricting the amplitude of the instabilities by expressing the conservation equations in terms of specific volume, pressure, and velocity. Suitably non-dimensionalized, the perturbed conservation equations can be written as follows:

i) continuity

\[
\frac{\partial v'}{\partial t} = -\bar{u} \frac{\partial v'}{\partial z} - u \frac{\partial \bar{v}}{\partial z} - u' \frac{\partial v'}{\partial z} + \bar{v} \frac{\partial u'}{\partial z} + v' \frac{\partial u}{\partial z} + v' \frac{\partial u'}{\partial z} 
\]

\[
-\bar{v}^2 \frac{\partial v'}{\partial z} - \bar{v} \frac{\partial u'}{\partial z} + \bar{v} \frac{\partial w'}{\partial z} - v' \frac{\partial u'}{\partial z} + v' \frac{\partial u'}{\partial z} + v' \frac{\partial w'}{\partial z}
\]

\[
(16)
\]

ii) momentum

\[
\frac{\partial u'}{\partial t} = -\bar{u} \frac{\partial u'}{\partial z} - u \frac{\partial \bar{u}}{\partial z} - u' \frac{\partial u'}{\partial z} - \bar{v} \frac{\partial p}{\partial z} - v' \frac{\partial u}{\partial z} - v' \frac{\partial p'}{\partial z}
\]

\[
(17)
\]
iii) energy

\[
\frac{\partial \rho'}{\partial t} = -\bar{\rho} \frac{\partial \rho'}{\partial z} - \rho' \frac{\partial u}{\partial z} - \rho' \frac{\partial v}{\partial z} - \gamma' \frac{\partial u'}{\partial z} - \gamma' \frac{\partial \rho}{\partial z} - \gamma' \frac{\partial \rho'}{\partial z} \\
+ \gamma' \frac{\partial \rho'}{\partial z} - \frac{\gamma-1}{2} \gamma \left[ 2u' \rho' + u'^2 \right] \frac{\partial \rho}{\partial z} + \frac{\partial \rho'}{\partial z} \]

Equations (19) and (20) will be satisfied explicitly by the choice of expansions for \(u'\) and \(\rho'\). Application of the Galerkin method to Eqs. (16), (17) and (18) will result in the desired set of quasi-linear differential equations. In addition, significant simplification will result from the orthogonality of the trial functions used to represent the flow variables. The solution of these equations is in progress.

THIRD ORDER INVESTIGATIONS

During the period covered by this report studies involving both single-mode and multi-mode theories were conducted. Additional data was obtained for the first tangential (1T) and the first radial modes (1R) using the single-mode third order analysis developed during the second year of this project (see Ref. 1 for a description
of this theory). Work continued on the development of a multi-mode third order theory.

**Single-mode Theory**

The purpose of this study was to construct nonlinear stability maps for the 1T and 1R modes. Such a stability map has been completed for the 1T mode as shown in Fig. 6. Here lines of constant peak-to-peak wall pressure amplitude are plotted on an \((n, \bar{T})\) plane for \(1.1 \leq \bar{T} \leq 2.7\). These curves cross the neutral stability limit at approximately \(\bar{T} = 2.15\). For \(\bar{T} < 2.15\) these lines (solid) lie within the region of linear instability and hence represent limit-cycle amplitudes. For \(\bar{T} > 2.15\) the curves (dashed) lie in the linearly stable region and thus correspond to triggering limits. For amplitudes larger than 2.0 the theory predicts the anomalous result that "negative" absolute pressure occurs at some point in the chamber. It is expected that the use of a multi-mode series expansion in the third order theory will eliminate the negative pressures.

Work has also begun to obtain a complete stability map for the 1R mode for \(0.5 \leq \bar{T} \leq 1.2\). At present lines of constant triggering amplitude have been determined, and data is now being obtained to determine the lines of constant limit-cycle amplitude.

**Multi-mode Theory**

During the initial attempts to develop a multi-mode third order theory for the study of transverse instability the conservation equations were derived with density, pressure, velocity and specific enthalpy as the dependent variables. The following series expansion was used:

\[
x = \sum_{m=0}^{2} A_{mn}(t) \cos \beta J_{m} (s \cdot r)
\]
where \( x \) represents any of the dependent variables. Application of the Galerkin method reduced the original system of conservation equations to a system of coupled non-linear ordinary differential equations that described the behavior of the unknown time-dependent variables. Unfortunately the resulting system could only be solved by a combination of matrix inversion and a numerical integration scheme. As a result the numerical solution of the differential equations, using a 3-mode expansion, required excessive computer time; the primary cause being the need to invert a \( 17 \times 17 \) matrix at each step of the integration.

It was found that the troublesome matrix inversion could be eliminated by replacing the density by specific volume, as one of the dependent variables. In addition the energy and state equations were also combined, resulting in the following system of equations.

**Continuity Equation**

\[
- \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) - \nabla \cdot \mathbf{W}_m = 0
\]  
(21)

Boundary condition:

\[
\nabla^2 Q_{\text{mass}} + \mathbf{v} \cdot \mathbf{n} = 0
\]  
(22)

**Momentum Equation**

\[
\frac{\partial \mathbf{v}}{\partial t} + (\nabla \cdot \mathbf{v}) \mathbf{v} + \mathbf{v} \nabla \mathbf{p} = 0
\]  
(23)

**Energy-State Equation**

\[
\frac{\partial \mathbf{p}}{\partial t} + \nabla \cdot (\mathbf{p} \mathbf{v}) + (\gamma - 1) \rho \nabla \cdot \mathbf{v} - \mathbf{W}_m \mathbf{p} = 0
\]  
(24)

Boundary condition:

\[
\left[ \gamma Q_{\text{mass}} + \gamma (\rho / \nabla) \cdot \mathbf{n} \right] \mathbf{p} = 0
\]  
(25)
Work has now begun on the development of a generalized computer program that will allow the use of any number of terms in the series expansion previously mentioned, and also allow the possibility of spinning waves. The following series expansion was therefore used.

\[
x = \sum_{m=0}^{N} a_{mn}(t) \cos \beta \sum_{m=1}^{N} b_{mn}(t) \sin \beta
\]

Application of the Galerkin method to the perturbed form of the governing equations yielded a system of \(4(2N+1)\) nonlinear equations.

In the present computational scheme, the coefficients of the terms involving the dependent variables and their products are calculated by separate computer programs. These programs evaluate the trigonometric integrals arising from the integration of terms involving \(\cos \beta\) and \(\sin \beta\) terms as well as the Bessel function integrals arising from the integration of products of Bessel functions. The values obtained from these programs are used as input to the program which numerically integrates the system of differential equations.

The program is presently undergoing checkout procedures and will soon be used to obtain nonlinear stability data with the aid of 3-mode series expansions of the dependent variables.
REFERENCES


Figure 1. Limit-cycle amplitude map for a spinning mode.
Figure 2. Limit-cycle amplitude map for a standing mode.
Figure 3. Longitudinal Linear Stability Limits.
Above Resonant Conditions: $n = 1.30$, $\tau = 0.7$

Resonant Conditions: $n = 1.15$, $\tau = 1.0$

Below Resonant Conditions: $n = 1.33$, $\tau = 1.30$

Figure 4. Time Dependence of the Nonlinear Pressure Waveforms at the Injector.
Figure 5. Injector Peak-to-Peak Pressure Amplitudes
Figure 6. Nonlinear stability map for the 1T mode determined by the single mode third order analysis.
Research Conducted Under
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APPLICATION OF THE GALERKIN METHOD IN
THE DESIGN OF STABLE LIQUID ROCKET MOTORS

SEMIANNUAL REPORT COVERING PERIOD
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Prepared By
Ben T. Zinn, Professor
Eugene A. Powell, Instructor
Manuel E. Lores, Graduate Research Assistant
S. Kalyanasundaram, Graduate Research Assistant

GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL OF AEROSPACE ENGINEERING
ATLANTA, GEORGIA

Project Monitor: Dr. Richard J. Priem
INTRODUCTION

During the period covered by this report work continued in the following three areas: (1) the development of the third-order, multi-mode theory to study the behavior of large amplitude transverse instabilities, (2) the study of nonlinear axial mode instability, and (3) the study of the influence of the functional form of the unsteady combustion response function upon the stability characteristics of rocket motors. Also a study was begun to improve the second order potential theory in order to provide a better approximation to the nozzle boundary condition.

The following papers were presented during this report period:


A brief description of results obtained in the above-mentioned investigations is provided in the following sections.

SECOND ORDER INVESTIGATIONS

Three Dimensional Potential Theory

During the first phase of this research program a second order potential theory using the n-τ combustion model was developed. This theory was restricted to two-dimensional (pure transverse) modes of
instability and to rocket motors using a multi-orifice (quasi-steady) nozzle. The approximate solutions were expressed in terms of the acoustic modes for a cylindrical chamber with solid wall boundary conditions at both the injector and the nozzle ends. Consequently the approximation of the flow conditions at the nozzle entrance was relatively poor.

A study was begun to improve the second order potential theory by choosing a series expansion for the velocity potential that provides a better approximation to the flow at the nozzle entrance. This was done by expanding the velocity potential in terms of the acoustic eigenfunctions for a chamber with a solid wall boundary condition at the injector end and a nozzle admittance condition at the other end. This removes both the two-dimensionality and the quasi-steady nozzle restrictions of the previous theory.

**Acoustic Eigenfunctions and Eigenvalues.** The acoustic eigenfunctions and eigenvalues are determined by solving the acoustic wave equation:

\[ \ddot{\phi}_{rr} + \frac{1}{r} \ddot{\phi}_r + \frac{1}{r^2} \ddot{\phi}_{\theta \theta} + \ddot{\phi}_{zz} - \ddot{\phi}_{tt} = 0 \]  

subject to the boundary conditions:

\[ \begin{align*}
\ddot{\phi}_r &= 0 \quad \text{at} \quad r = 1 \\
\ddot{\phi}_z &= 0 \quad \text{at} \quad z = 0 \\
\ddot{\phi}_z &= -\gamma Y \dot{\phi}_t \quad \text{at} \quad z = z_e
\end{align*} \]  

In the above equations $\ddot{\phi}$ is the velocity potential; $r$, $\theta$, and $z$ are the dimensionless radial, angular, and axial coordinates respectively; $t$ is the dimensionless time; and $\gamma$ is the specific heat ratio. The nozzle admittance, $Y$, is complex and is defined by:
where $u'$ is the dimensionless axial velocity perturbation and $p'$ is the dimensionless pressure perturbation.

Using the method of separation of variables yields the following solution to Eqs. (1) and (2):

$$
\varphi_{l_{mn}} = \left\{ F(t) Z_1(z) + G(t) Z_2(z) \right\} \Theta(\theta) R(r)
$$

where

$$
Z_1(z) = \cos(\epsilon_{l_{mn}} z) \cosh(\eta_{l_{mn}} z),
$$

$$
Z_2(z) = \sin(\epsilon_{l_{mn}} z) \sinh(\eta_{l_{mn}} z),
$$

$$
\Theta(\theta) = \cos m\theta \text{ or } \sin m\theta
$$

$$
R(r) = J_m(S_{mn} r)
$$

$$
F(t) = e^{-A t} [C \cos \omega t - D \sin \omega t]
$$

$$
G(t) = e^{-A t} [D \cos \omega t + C \sin \omega t]
$$

In Eqs. (5) $\epsilon_{l_{mn}}$ and $\eta_{l_{mn}}$ are the real and imaginary parts of the complex eigenvalue $b_{l_{mn}}$, where $l$, $m$, and $n$ are the axial, tangential, and radial mode numbers, respectively. The functions $F(t)$ and $G(t)$ are seen to be damped sinusoids of equal amplitude and differing in phase by $90^\circ$. The frequency, $\omega$, and damping, $A$, are related to the eigenvalues $\epsilon_{l_{mn}}$, $\eta_{l_{mn}}$, and $S_{mn}$; the frequency is very close to the frequency of a closed-ended chamber and the damping is small.
The eigenvalues \( b_{\ell mn} = \epsilon_{\ell mn} + i\eta_{\ell mn} \) are the solutions of the following transcendental equation:

\[
b_{\ell mn}^2 \sin^2(b_{\ell mn} z_e) + \gamma^2 \left[ \left( s_{mn}^2 + b_{\ell mn}^2 \right) \cos^2(b_{\ell mn} z_e) \right] = 0 \tag{9}
\]

A computer program has been developed to solve Eq. (9), and it has been used to determine the eigenvalues for quasi-steady nozzles \( (y_r = \frac{1}{2\gamma} \bar{u}_e, y_i = 0) \). The real part, \( \epsilon_{\ell mn} \), is very close to \( \ln/z_e \), \( \ell = 0, 1, 2, \ldots \), while the imaginary part, \( \eta_{\ell mn} \), is a small positive number. Typical values are given in the following table for \( \gamma = 1.2, z_e = 1.0, \) and \( \bar{u}_e = 0.2 \) (\( y_r = 0.01667, y_i = 0.0 \)).

<table>
<thead>
<tr>
<th>( \ell )</th>
<th>( m )</th>
<th>( n )</th>
<th>Mode</th>
<th>( \epsilon_{\ell mn} )</th>
<th>( \eta_{\ell mn} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1T</td>
<td>0.1362</td>
<td>0.1352</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1T-1L</td>
<td>3.1416</td>
<td>0.0232</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1T-2L</td>
<td>6.2832</td>
<td>0.0209</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>2T</td>
<td>0.1763</td>
<td>0.1733</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2T-1L</td>
<td>3.1417</td>
<td>0.0279</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2T-2L</td>
<td>6.2832</td>
<td>0.0223</td>
</tr>
</tbody>
</table>

The axial acoustic eigenfunctions given by Eqs. (5) are plotted in Fig. 1 for the first tangential \( (\ell = 0, m = 1, n = 1) \) mode and the combined 1T-1L \( (\ell = 1, m = 1, n = 1) \) mode. Inspecting these curves and recalling that \( p' = -\gamma \Phi_t \) and \( u' = \phi_z \) reveals that \( Z_1 \) is the greatest contributor to the pressure perturbation while \( Z_2 \) has the greatest influence on the axial velocity perturbation. This observation implies that in order to properly describe both pressure and axial velocity perturbations, both eigenfunctions must be included in the series expansion of the velocity potential.
Series Expansion. Based on the above remarks, the following series expansion is used in the three-dimensional nonlinear theory:

$$
\tilde{\varphi} = \sum_{n} \sum_{m \in \mathbb{N}} \left[ A_{tnn}(t) \sin m\theta + B_{tnn}(t) \cos m\theta \right] \cos(e_{tnn}z) \cosh(\eta_{tnn}z) \\
+ \left[ C_{tnn}(t) \sin m\theta + D_{tnn}(t) \cos m\theta \right] \sin(e_{tnn}z) \sinh(\eta_{tnn}z) \right] J_m(s_{mn}r)
$$

where both $\sin m\theta$ and $\cos m\theta$ are included in order to describe both standing and spinning waves. In order to simplify the algebra involved in applying the Galerkin Method, the expansion of the velocity potential is written as a single summation as follows:

$$
\tilde{\varphi} = \sum_{p=1}^{N} A_p(t) \varphi_p(r, \theta, z)
$$

where the $A_p$'s are the unknown time-dependent coefficients and

$$
\varphi_p(r, \theta, z) = \Theta_p(\theta) Z_p(z) R_p(r)
$$

A correspondence must be established between the index, $p$, in Eq. (11) and the mode-numbers $t$, $m$, and $n$ in Eq. (10). In addition an integer $S$ is needed to relate Eqs. (10) and (11). This relationship is given in the following table:

<table>
<thead>
<tr>
<th>s(p)</th>
<th>$A_p(t)$</th>
<th>$\Theta_p(\theta)$</th>
<th>$Z_p(z)$</th>
<th>$R_p(r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A_{tnn}(t)$</td>
<td>$\sin m\theta$</td>
<td>$\cos(e_{tnn}z) \cosh(\eta_{tnn}z)$</td>
<td>$J_m(s_{mn}r)$</td>
</tr>
<tr>
<td>2</td>
<td>$B_{tnn}(t)$</td>
<td>$\cos m\theta$</td>
<td>$\cos(e_{tnn}z) \cosh(\eta_{tnn}z)$</td>
<td>$J_m(s_{mn}r)$</td>
</tr>
<tr>
<td>3</td>
<td>$C_{tnn}(t)$</td>
<td>$\sin m\theta$</td>
<td>$\sin(e_{tnn}z) \sinh(\eta_{tnn}z)$</td>
<td>$J_m(s_{mn}r)$</td>
</tr>
<tr>
<td>4</td>
<td>$D_{tnn}(t)$</td>
<td>$\cos m\theta$</td>
<td>$\sin(e_{tnn}z) \sinh(\eta_{tnn}z)$</td>
<td>$J_m(s_{mn}r)$</td>
</tr>
</tbody>
</table>
Application of the Galerkin Method. The Galerkin method is used to determine the set of ordinary differential equations which describe the unknown time-dependent functions which appear in Eq. (11). Substituting the approximate solution (i.e., Eq. (11)) into the nonlinear wave equation (see Ref. 1) and the nozzle boundary condition yields the equation residual \( \mathbf{E}(\vec{\psi}) \) and the boundary residual \( \mathbf{B}(\vec{\psi}) \). The nozzle boundary residual involves the nozzle admittance, \( Y \), which is a complex number, thus an equivalent formulation involving only real variables is needed. Thus \( \mathbf{B}(\vec{\psi}) \) is expressed in the following form:

\[
\mathbf{B}(\vec{\psi}) = \left[ \frac{\partial}{\partial z} + \gamma \frac{\partial}{\partial t} \right] \vec{\psi} + \gamma Y \sum_{p=1}^{N} \frac{d^2 A_p}{dt^2} \frac{\varphi_p(r, \theta, z)}{\omega_p} \Bigg|_{z=ze} \quad (13)
\]

The residuals \( \mathbf{E}(\vec{\psi}) \) and \( \mathbf{B}(\vec{\psi}) \) must satisfy the following orthogonality conditions:

\[
\int_{0}^{2\pi} \int_{0}^{2\pi} \mathbf{E}(\vec{\psi}) \varphi_j(r, \theta, z) r dr d\theta dz - \int_{0}^{2\pi} \int_{0}^{2\pi} \mathbf{B}(\vec{\psi}) \varphi_j(r, \theta, z_e) r dr d\theta = 0 \\
\quad j = 1, 2, 3, \ldots N \quad (14)
\]

Performing the spatial integrations indicated in Eqs. (14) yields the following system of differential equations to be solved for the \( A_p \)'s:

\[
\sum_{p=1}^{N} \left\{ C_1(j, p) \frac{d^2 A_p}{dt^2} + C_2(j, p) A_p(t) + \left[ C_3(j, p) - nC_4(j, p) \right] \frac{dA_p}{dt} \right\} + nC_4(j, p) \frac{d[A_p(t-\tau)]}{dt} = 0 \\
\quad j = 1, 2, 3, \ldots N \quad (15)
\]
where the coefficients \( C_1, C_2, C_3, C_4, \) and \( D \) are functions of the spatial integrals involving \( \Theta, Z, \) and \( R. \)

**Computer Programs.** A computer program, COEFFS, has been written to calculate the coefficients appearing in Eqs. (15). This program involves a number of subroutines whose functions are given below:

<table>
<thead>
<tr>
<th>Subroutine</th>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>UBAR</td>
<td>Calculates the steady state velocity distribution.</td>
</tr>
<tr>
<td>EIGVAL</td>
<td>Computes the axial acoustic eigenvalues ( \epsilon_{lmn} ) and ( \eta_{lmn}. )</td>
</tr>
<tr>
<td>FCNS</td>
<td>Evaluates certain functions needed by EIGVAL.</td>
</tr>
<tr>
<td>ZFUNCT</td>
<td>Calculates the axial eigenfunctions and their first and second derivatives (i.e., ( Z_p(z), Z_p'(z), ) and ( Z_p''(z). ))</td>
</tr>
<tr>
<td>AXIAL1</td>
<td>Evaluates integrals involving products of two axial eigenfunctions.</td>
</tr>
<tr>
<td>AXIAL2</td>
<td>Evaluates integrals involving products of three axial eigenfunctions.</td>
</tr>
<tr>
<td>AZIMTL</td>
<td>Evaluates integrals involving products of three tangential eigenfunctions.</td>
</tr>
<tr>
<td>RADIAL</td>
<td>Evaluates integrals involving products of three radial eigenfunctions.</td>
</tr>
<tr>
<td>JBES</td>
<td>Evaluates Bessel functions needed by RADIAL.</td>
</tr>
</tbody>
</table>

The individual subroutines as well as the complete program, COEFFS, have been checked out and have been found to be functioning properly.

In order to numerically integrate Eqs. (15) they must be transformed to a form in which only one second derivative term appears in each equation. This decoupling process is accomplished by program COEFFS by computing the coefficients of the transformed equations using standard matrix inversion techniques. The resulting
coefficients will be used as input to program LCYC3D, which numerically integrates the corresponding differential equations.

Program LCYC3D is currently under development. It is essentially a modified version of program LIMCYC of Ref. 1 and will be capable of calculating the transient behavior, limit-cycle amplitudes, and triggering amplitudes of three-dimensional instability in cylindrical rocket chambers with arbitrary nozzles.

Combustion Models

Work continued on the investigation of the influence of the functional form of the unsteady combustion response function upon the nonlinear stability characteristics of liquid-propellant rocket motors. This section describes the first phase of this study, in which the unsteady combustion mass source, $W_m'$, is given by:

$$W_m' = \frac{du}{dz} \left[ A p' + B \frac{dp'}{dt} \right]$$

where $A$ and $B$ are prescribed constants, $u$ is the steady state Mach number, and $p'$ is the dimensionless pressure perturbation. The response function given by Eq. (16) is then incorporated into the second order potential theory of Ref. 1.

Analysis. Relating the pressure perturbation to the velocity potential and using a cylindrical coordinate system, Eq. (16) yields:

$$W_m' = -\gamma \frac{du}{dz} \left[ A \left( \frac{\phi_t}{t} + \frac{1}{2} \left( \frac{\phi_r}{r} + \frac{1}{r^2} \frac{\phi_r^2}{\theta_r^2} - \frac{\phi_t^2}{t^2} \right) \right) \right]$$

$$+ B \left[ \frac{\phi_{tt}}{t^2} + \frac{\phi_t}{r^2} \phi_{rt} + \frac{1}{2} \frac{\phi_r^2}{\theta_r^2} - \phi_t \left( \frac{\phi_{rr}}{r^2} + \frac{1}{r} \frac{\phi_r}{r} + \frac{1}{2} \frac{\phi_{tt}}{t^2} \right) \right]$$

where only pure transverse (two-dimensional) solutions are considered. Introducing Eq. (17) into the nonlinear wave equation of Ref. 1 yields:
E(\$) = \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \theta}{\partial \theta^2} - (1 - \gamma B \frac{du}{dz}) \frac{\partial \theta}{\partial t} \\
- (2 - \gamma B \frac{du}{dz}) \left[ \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r^2} \frac{\partial \theta}{\partial \theta} \frac{\partial \theta}{\partial r} \right] \\
- \left[ \gamma (1 + B \frac{du}{dz}) - 1 \right] \frac{\partial \theta}{\partial t} \left( \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r^2} \frac{\partial \theta}{\partial \theta} \right) \\
- \gamma \frac{du}{dz} (1-A) \frac{\partial \theta}{\partial t} + \frac{\gamma A du}{2} \left( \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \theta}{\partial \theta^2} - \frac{\partial^2 \theta}{\partial t^2} \right) = 0 \quad (18)

which is restricted to combustors with small Mach number mean flow and moderate amplitude waves. For quasi-steady nozzles the nozzle boundary condition is given by:

B(\$) = \left[ \frac{\gamma - 1}{2} \frac{\partial u}{\partial z} + \frac{\partial \theta}{\partial z} \right]_{z=z_e} = 0 \quad (19)

Approximate solutions to Eqs. (18) and (19) are obtained by means of the Method of Weighted Residuals. Restricting attention to pure transverse modes the velocity potential is expanded in terms of the acoustic eigenfunctions as follows:

\tilde{\$} = \sum_{m=0}^{M} \sum_{n=1}^{N} \left[ A_{mn}(t) \sin m\theta + B_{mn}(t) \cos m\theta \right] J_m(S_{mn}) \quad (20)

where A_{mn}(t) and B_{mn}(t) are unknown functions of time. The assumed series expansion, Eq. (20), is introduced into Eqs. (18) and (19) to obtain the residuals E(\$) and B(\$), which are required to satisfy Galerkin orthogonality conditions similar to Eqs. (14). This process yields the following system of nonlinear, ordinary differential equations:
\[
\left(1 - \frac{\gamma \bar{u}}{z_e} B\right) \frac{d^2A_{jk}}{dt^2} + S^2_{jk}A_{jk} + K \frac{dA_{jk}}{dt} \\
+ \sum_{m,n} \sum_{\mu,\nu} \left\{ C_1(m,n;\mu,\nu;j,k) A_{mn} \frac{dB_{\mu\nu}}{dt} + C_2(m,n;\mu,\nu;j,k) B_{mn} \frac{dA_{\mu\nu}}{dt} \right. \\
+ D_1(m,n;\mu,\nu;j,k) A_{mn} B_{\mu\nu} + D_2(m,n;\mu,\nu;j,k) B_{mn} A_{\mu\nu} \\
+ E_1(m,n;\mu,\nu;j,k) \frac{dA_{mn}}{dt} \frac{dB_{\mu\nu}}{dt} + E_2(m,n;\mu,\nu;j,k) \frac{dB_{mn}}{dt} \frac{dA_{\mu\nu}}{dt} \right\} = 0 \quad (21)
\]

\[
\left(1 - \frac{\gamma \bar{u}}{z_e} B\right) \frac{d^2B_{jk}}{dt^2} + S^2_{jk}B_{jk} + K \frac{dB_{jk}}{dt} \\
+ \sum_{m,n} \sum_{\mu,\nu} \left\{ C_3(m,n;\mu,\nu;j,k) A_{mn} \frac{dA_{\mu\nu}}{dt} + C_4(m,n;\mu,\nu;j,k) B_{mn} \frac{dB_{\mu\nu}}{dt} \right. \\
+ D_3(m,n;\mu,\nu;j,k) A_{mn} A_{\mu\nu} + D_4(m,n;\mu,\nu;j,k) B_{mn} B_{\mu\nu} \\
+ E_3(m,n;\mu,\nu;j,k) \frac{dA_{mn}}{dt} \frac{dA_{\mu\nu}}{dt} + E_4(m,n;\mu,\nu;j,k) \frac{dB_{mn}}{dt} \frac{dB_{\mu\nu}}{dt} \right\} = 0 \quad (22)
\]

where

\[
K = \frac{\gamma \bar{u}}{z_e} \left(1 + \frac{\gamma - 1}{2\gamma} - A\right) \quad (23)
\]

The coefficients of the nonlinear terms (i.e., \(C_i\), \(D_i\), and \(E_i\) for \(i = 1, \ldots, 4\)) are determined by evaluating the various integrals of trigonometric and Bessel functions that arise from the spatial integrations required by the Galerkin method. These coefficients depend on the combustion parameters \(A\) and \(B\).
The unstable behavior of an engine is determined by specifying the form of the initial disturbance and then following the subsequent behavior of the individual modes (i.e., the functions $A_{jk}(t)$ and $B_{jk}(t)$) by numerically integrating Eqs. (21) and (22). Typical numerical solutions of these equations will be presented and discussed in the following paragraphs.

**Linear Solutions.** The linear stability limit is determined by substituting $B_{jk}(t) = be^{iwt}$ into Eq. (22) and neglecting the nonlinear terms. Separating real and imaginary parts in the resulting complex equation yields the following conditions for neutral stability:

$$A = 1 + \frac{\gamma - 1}{2\gamma}$$

$$w = \frac{S_{jk}}{\sqrt{1 - \frac{\gamma}{z_e} B}}$$

The neutral stability limit and the stable and unstable regions are shown in the upper plot of Fig. 2. It is easily shown that for small amplitude (i.e., sinusoidal) oscillations the combustion response function given by Eq. (16) is equivalent to

$$w' = \frac{d\bar{u}}{dz} \left[ Pe^{i\Psi} \right]^p$$

where $P$ is the amplitude factor and $\Psi$ is the phase shift. The lower plot of Fig. 2 shows how the stability map on the $(A,B)$ plane transforms to the $(P,\Psi)$ plane; the $P-\Psi$ plot resembles an $n-\bar{n}$ curve. In both plots the frequency increases as one moves to the right along the neutral stability curve.

The effects of chamber length, $z_e$, Mach number, $\bar{u}_e$, and the mode of oscillation on the linear stability limits are shown in
Figs. 3 and 4. These plots show that the neutral stability curves in both the (A,B) and (P,φ) planes are independent of \( \bar{u}_e \), \( z_e \), and \( S_{jk} \). However, the frequency corresponding to a given point on the (A,B) neutral stability curve does depend on these parameters; the frequency is proportional to \( S_{jk} \) and increases as the ratio \( \bar{u}_e / z_e \) increases. Because of this frequency effect, the mapping of points from the (A,B) plane to the (P,φ) plane depends upon \( \bar{u}_e / z_e \) and \( S_{jk} \) as shown in Figs. 3 and 4. These results indicate that the region of linear instability in the (A,B) plane is the same for all transverse modes.

The behavior of the linear solutions for points away from the (A,B) neutral stability limit is determined by assuming that

\[ B_{jk}(t) = b e^{(A+i\omega)t} \]

and proceeding as before. In this manner the following expression is obtained for the growth rate, \( \Lambda \):

\[
\Lambda = \frac{\gamma \bar{u}_e}{2z_e} \left[ A - (1 + \frac{\gamma-1}{2\gamma}) \right] \frac{1 - \frac{\gamma \bar{u}_e}{z_e} B}{1 - \frac{\gamma \bar{u}_e}{z_e} B} \tag{27}
\]

From Eq. (27) lines of constant growth rate are determined as shown in the upper plot of Fig. 5. This plot shows that the growth (or decay) rate increases with increasing displacement above (or below) the neutral stability limit and with increasing \( B \). A singular point occurs on the neutral stability limit at \( B = z_e / \gamma \bar{u}_e \) where the constant-\( \Lambda \) lines intersect. Further analysis shows that oscillatory growth or decay can only occur for values of \( A \) and \( B \) lying to the left of the parabola shown in the lower plot of Fig. 5. Solutions for points above the parabola or for \( B > z_e / \gamma \bar{u}_e \) exhibit non-oscillatory growth, while solutions below the parabola with \( B < z_e / \gamma \bar{u}_e \) exhibit non-oscillatory decay.

**Nonlinear Solutions.** Nonlinear solutions were obtained with a three-mode series expansion consisting of the following acoustic modes: the first tangential (1T), the second tangential
(2T), and the first radial (1R) modes. All results presented in this section were obtained from a pure standing 1T initial disturbance of the form:

\[ B_{11}(t) = b_{11} \cos(1.84118t) \quad t \leq 0 \]  
\[ B_{21}(t) = B_{01}(t) = 0 \]  

Numerical solutions were obtained for values of A and B corresponding to the points shown in Fig. 6. These points represent conditions of linearly stable, neutral, and unstable oscillations. In each case the motor parameters were: \( \gamma = 1.2, \bar{U_e} = 0.2, \) and \( \text{L/D} = 0.5 \) (length-to-diameter ratio, \( z_e/2 \)).

The first three cases (i.e., points a, b, and c of Fig. 6) were run to determine the transient behavior and the existence of limit-cycles for resonant combustion (i.e., \( B = 0 \) and \( \omega = S_{11} \)). The transient behavior is shown in Fig. 7 as plots of pressure perturbation, \( p' \), versus time. In each case the initial amplitude \( b_{11} \) was 0.2, and the calculations were terminated after 50 cycles. It is seen from Fig. 7 that the linearly stable case exhibits decay to zero amplitude, the neutrally stable case exhibits a slight growth, and the linearly unstable case exhibits slow growth initially followed by rapid growth after 40 cycles due to nonlinear coupling between unstable modes. The pressure peaks are seen to vary considerably in height, particularly for the linearly unstable case. It appears that if a limit-cycle is reached in the linearly unstable case, its amplitude will be too large to be described by the present second order theory.

The result described above is typical of results obtained with the \((n,T)\) model for regions of the \((n,T)\) plane for which all three modes in the series were linearly unstable. This result was expected since the region of linear instability in the \((A,B)\) plane is the same for all modes.
Points a, d, and e in Fig. 6 were considered in order to investigate the possibility of triggering the 1T mode for both resonant and off-resonant conditions. In each case calculations were made for initial amplitudes $b_{11}$ of 0.2, 0.5, 1.0, 2.0, and 5.0. Growth occurred only for amplitudes greater than about 2.0, thus triggering was only possible by the introduction of very large disturbance which violate the assumptions of the second order theory.

Two investigations concerning the (A,B) model are now in progress: (1) a comparison of the nonlinear growth or decay rates with those computed from linear theory (i.e., Eq. (27)) and (2) a study of the triggering characteristics of the 1R mode using a single mode series. Upon completion of these studies a complete evaluation of the (A,B) model will be made.

THIRD ORDER INVESTIGATIONS

**Single-Mode Theory**

The single-mode study was completed with the construction of a nonlinear stability map for the first radial mode. This map is shown in Fig. 8 as lines of constant triggering amplitude (broken lines) and lines of constant limit-cycle amplitude (solid lines) plotted on an $(n,\bar{T})$ plane. For each line the value of the peak-to-peak wall pressure amplitude is given. The envelope of all of the constant-amplitude lines is the nonlinear stability limit (not shown for clarity). Triggered instability is possible in the region between the linear and nonlinear stability limits, while the region below the nonlinear stability limit is dynamically stable. This figure shows that triggered 1R instability is more likely to occur for larger values of $\bar{T}$. For sufficiently large amplitudes the theory predicts "negative" absolute pressure, thus the theory cannot predict the limit-cycle amplitude of 1R instability for values of $n$ and $\bar{T}$ lying to the right of the curve labeled $p'_{\min} = -1.0$ in Fig. 8.
Multi-Mode Theory

During the previous report period, a new method of approach to the third order multi-mode theory was developed. Using specific volume instead of density, as one of the dependent variables, the governing system of equations (Equations 21-25 of Ref. 2) were derived. A computational scheme, similar to that described in Ref. 1, was developed. To check out this program, solutions were computed using a series consisting only of the 1-T mode. These solutions (i.e., limit-cycle amplitudes) were compared with previous solutions obtained with available single-mode third order theory obtained with density as one of the dependent variables. This comparison is presented in Fig. 9, for \( \bar{T} = 1.5 \) and \( \bar{T} = 1.70629 \), which shows good agreement between the two approaches. However, due to the approximate nature of the Galerkin method and the difference between the computational schemes for the two cases, exact agreement was not expected.

After this check-out, solutions were obtained with the third order multi-mode program using a three mode series consisting of the 1T, 2T and 1R modes. Limit-cycle amplitudes thus obtained are shown in Fig. 10 where they are compared with similar results obtained with the second order potential theory. These results show that a somewhat smaller limit-cycle amplitude is predicted when third order nonlinearities are included in the analysis.

Some limit-cycle pressure amplitudes for spinning modes were also obtained using a series consisting of the 1T, 2T and 1R modes. These results were compared with the previous results obtained for the standing mode oscillations. This comparison is shown in Fig. 11 for \( \bar{T} = 1.5 \) and \( \bar{T} = 1.70629 \). It is seen that for a given value of \( \bar{T} \) and a given value of \( \delta \) (or \( n \)), the limit cycle pressure amplitude is higher for a spinning mode than for the corresponding standing mode. This result is in agreement with results obtained with the second order potential theory.
Some parametric studies were conducted with the third order theory. First, the variation of the limit-cycle pressure amplitude with the Mach number at nozzle entrance, $\bar{u}_e$, for particular values of $\bar{f}$ and $\bar{\tau}$ was determined. The third order results for $n = 0.61817$; $\bar{\tau} = 1.5$ are compared to results obtained for the same value of $n$ and $\bar{\tau}$ using the second order theory (see Fig. 12). It is seen that, for a given value of $\bar{u}_e$, the smaller limit-cycle pressure amplitude is obtained with the third order theory. A similar study involving the chamber length-to-radius ratio, $z_e$, was conducted and the results are shown in Fig. 13 for three values of $n$ and $\bar{\tau}$. In all three cases, the limit-cycle pressure amplitude was seen to approach a constant value as $z_e$ became large. A similar result was obtained with the second order potential theory\(^1\).

A study of the triggering limits using the third order multimode theory is now in progress.

**NONLINEAR AXIAL INSTABILITY**

During this report period the longitudinal combustion instability investigation consisted of two programs. The first program was concerned with the development of engineering applications of the solutions generated in the second order analysis of moderate amplitude instabilities. The second, and more extensive, program involved the derivation of solution techniques capable of describing the behavior of large amplitude instabilities.

**Second Order Analysis: Engineering Applications**

The development of the second order (i.e., moderate amplitude) combustion instability solutions is discussed in Ref. 2. In this solution technique, the modified Galerkin method\(^3\) is used to find approximate solutions of a nonlinear wave equation that describes the behavior of moderate amplitude instabilities in rocket combustors having a low Mach number mean flow and a quasi-steady short nozzle.
Correlation with Experimental Data. The results of this analysis demonstrated that the pressure waveforms exhibit a characteristic dependence upon the engine operating conditions. Specifically, the pressure waveforms are dependent upon the proximity of the operating point to engine resonant conditions (i.e., to $n_{\text{min}}$ and $T_{\text{min}}$). The observed behavior of the stable limit cycle pressure oscillations can be used to correlate the analytical results with experimental data. To accomplish this task, two waveform parameters are defined in Fig. 14. In this figure, the solid line shows the numerically computed pressure waveform, and the broken line is the theoretical pressure waveform used to determine the correlation parameters $A_p'(z_r)$ and $t_o/T(z_r)$. The normalized axial location at which the experimental pressure data is available is $z = z_r$.

Once $z_r$ is specified, the analytical solution technique developed in Ref. 2 is used to generate both a limit-cycle amplitude map showing curves of constant peak-to-peak pressure, $A_{p_{\text{max}}}'$, and a plot of $t_o/T$ as a function of $T$. Typical graphs, for $z_r = 0.0$, are presented in Figs. 15 and 16. The value of $t_o/T$ can be determined from experimental data. The computed value may then be used in Fig. 16 to determine the value of $T$. The determined value of $T$ together with the experimentally determined value of $A_{p_{\text{max}}}'$ can then be used, in Fig. 15, to determine the value of the interaction index, $n$. This correlation procedure should enable rocket design engineers to determine the $n$ and $T$ values of various liquid propellant rocket motor designs.

Semi-Empirical Pressure Waveforms. At the suggestion of Dr. R. J. Priem, the feasibility of a semi-empirical method for predicting the pressure waveforms has been explored. The objective of the semi-empirical method is to provide design engineers with a straightforward technique, requiring relatively little computation time, for finding the pressure waveforms.
The semi-empirical correlation method is based on the observation\(^4\) that the velocity potential, \(\phi\), can be approximated, at least for resonant oscillations, by the following series expansion:

\[
\phi = A_1 \sum_{n=1}^{N} n^{-\alpha} \cos(nw_1 t) \cos(n\pi z) \tag{29}
\]

where \(A_1\), \(\alpha\), and \(w_1\) are found from computer-generated data. The nonlinear pressure waveform is found from the following nonlinear relationship:

\[
p' = \frac{Y}{2} \left[ \ddot{\phi}_t (\ddot{\phi}_t - 2) - \ddot{\phi}_z (\ddot{\phi}_z + 2\dot{\phi}_z) \right] \tag{30}
\]

The parameters \(A_1\), \(\alpha\), and \(w_1\) can be found from computer solutions generated using a five term series expansion for \(\phi\), whereas at least a ten term series is necessary to adequately describe discontinuous waveforms\(^2\). The computation time is approximately proportional to the square of the number of terms retained in the series expansion. Consequently, the computation time required for the semi-empirical method is much less than that required to solve directly for the pressure waveform using the series containing unknown time-dependent functions.

Semi-empirical pressure waveforms are compared with computer-generated solutions in Fig. 17. Ten terms were retained in Eq. (29) in the computation of the semi-empirical waveforms (i.e., \(N = 10\) in Eq. (29)). It is evident from the data shown in Fig. 17 that the semi-empirical method fails to reproduce the waveforms of off-resonant oscillations. The probable reasons for this failure are:

1. There is a slight phase shift between the various modes at off-resonant conditions.
2. For off-resonant oscillations, the higher harmonics are both frequency and amplitude modulated.

3. For off-resonant oscillations, the higher harmonics may not obey the amplitude power law found by considering the behavior of the first ten mode-amplitude functions.

Large Amplitude Instabilities

An understanding of the effect of large amplitude flow oscillations on the stability of rocket engines is necessary for the investigation of engine triggering. An analysis of large amplitude instabilities in low Mach number mean flows has been conducted\textsuperscript{5,6}. In the absence of a proven nonlinear unsteady combustion model, Crocco's linear time-lag theory is used to represent the unsteady combustion process. As a result only gasdynamic nonlinearities are taken into account. In the formulation of the problem it is assumed that terms involving the product of an $0(\bar{u}_e^2)$ quantity with a perturbed flow parameter are negligible. However, terms of the form $\bar{u}_e p' \bar{u}_e^2$ are retained in the governing equations.

Solution Technique

The equations describing the behavior of large amplitude oscillations in a combustor with low Mach number mean flow have been derived in Ref. 6. The undetermined function version of the Galerkin method is used to find approximate solutions of these equations\textsuperscript{5,6}. In this solution technique, both the injector solid wall boundary condition and the quasi-steady short nozzle boundary condition are identically satisfied by the series expansions used to represent the dependent variables.

The application of the Galerkin method reduces the original partial differential equations to a system of quasi-linear ordinary differential equations describing the behavior of the mode-amplitude functions. The resulting ordinary differential equations are numerically integrated using a fourth order Runge-Kutta algorithm that has been modified to account for the presence of a retarded
time variable. In order to find nonlinear solutions the engine operating conditions (i.e., $y$, $\bar{u}_e$, $\hat{n}$ and $\bar{T}$) and the initial conditions must be specified. The solutions have been found to be dependent upon the engine operating conditions, and independent of the initial conditions.

**Convergence Study.** In order to minimize the computational time required to find nonlinear solutions, it is desirable to retain as few terms as possible in the series expansions of the dependent variables. A convergence test, in which solutions were generated for five, seven, and ten term expansions, was conducted. The results of this study are summarized in Fig. 18. The convergence of the solutions with increasing number of terms is evident in this figure. The data shown in Fig. 18 also indicates that the behavior of the nonlinear oscillations can be investigated using five term series expansions. The use of five term expansions in lieu of ten term expansions results in approximately a five fold reduction in computation time. Consequently, five term series expansions were used to generate the data discussed in this report.

**Results.** A comparison of the results of the present investigation with second order solutions computed in Reference 2 is made in Fig. 19. These data show that, when the engine operating conditions are only moderately unstable, the limit-cycle amplitudes are smaller than those found in the second order analysis. It can be shown that these discrepancies are probably due to the different manner in which the short nozzle boundary condition is treated in the two analyses. From Fig. 19 it is also apparent that for more unstable engine operating conditions, the present analysis predicts peak-to-peak amplitudes that are larger than those resulting from the second order solutions. This result is probably due to the fact that the large amplitude oscillations invalidate the second order analysis.
The presence of large amplitude flow oscillations can broaden the region of unstable engine operation predicted by a linear analysis. This conclusion can be drawn from the data presented in Fig. 20. Here, the peak-to-peak amplitudes of the fundamental mode limit cycle oscillations at neutrally stable (in the linear sense) engine operating conditions are presented as a function of \( \bar{T} \). For \( \bar{T} > 2/3 \), the higher longitudinal modes are linearly stable along the 1L linear stability limit. Consequently, the finite amplitude oscillations along the 1L linearly stability limit are due to flow nonlinearities.

The behavior of the combustion instability oscillations in the linearly stable region is shown in Fig. 21. In this figure, the limit-cycle peak-to-peak amplitudes are shown as a solid line. The critical flow disturbance required for triggering unstable engine operation in a linearly stable region is shown as a broken line. The data presented in Fig. 21 indicate that the flow nonlinearities only slightly broaden the region of possible unstable engine operation.

For engine operating conditions at which the second axial mode is linearly unstable, the behavior of the combustion instability oscillations predicted by the present analysis is the same as that found in the second order wave equation solutions. That is, when both the 1L and 2L modes are linearly unstable, the instability will be in the 1L mode for engine operating conditions of primary interest (i.e., \( \hat{\nu} < 2.0 \)). On the other hand, when the 2L mode is linearly unstable while the 1L mode is linearly stable, the flow instability will exhibit 2-L characteristics. Typical flow oscillations are presented in Fig. 22.

Discussion and Conclusions

The data presented in the preceding sections, together with the results discussed in Reference 2, indicate that to second order accuracy, the regions of axially unstable engine operation on the \( \hat{\nu}-\bar{T} \) stability plane can be predicted by a linear analysis. The
presence of large amplitude flow oscillations slightly broadens the region of unstable engine operation. However, because of the extreme narrowness of the nonlinearly unstable regions, they can probably be neglected from a practical point of view.

The results of this investigation indicate that second order solutions adequately describe the behavior of axial mode instabilities over a broad range of engine operating conditions. The second order analysis is considerably simpler than the solution of the equations describing large amplitude oscillations, and the former approach requires significantly less computation time.

It is important to note that these conclusions are based on the use of a linear unsteady combustion model. Such a model, while rigorous to second order, does not include nonlinear unsteady combustion effects which may be important in the presence of large amplitude oscillations.

REFERENCES


4. Private communication from Dr. R. J. Priem.

Figure 1. Axial Acoustic Eigenfunctions

First Tangential Mode (1T)

\( \gamma = 1.2; \bar{u}_e = 0.2; z_e = 1.0 \)

Combined Mode (1T-1L)

\( \xi_{011} = 0.13617 \)
\( \eta_{011} = 0.13524 \)

\( \xi_{111} = 3.14164 \)
\( \eta_{111} = 0.02319 \)
Figure 2. Linear Stability Limits
\frac{\bar{W}_m'}{\bar{W}_m} = A \phi' + B \frac{d\phi'}{dt}

Figure 3. Effect of $\bar{u}_e$ and $z_e$ on Stability Limits
\[ \frac{W_m'}{W_m} = A_p + B \frac{dp'}{dt} \]

\[ \frac{W_m'}{W_m} = \text{Pe}^{i\phi} p' \]

1T Mode \( S_{11} = 1.84118 \)

2T Mode \( S_{21} = 3.05424 \)

Figure 4. Effect of Mode on Stability Limits
Figure 5. Linear Growth and Decay Characteristics of the 1T Mode.

\[ \gamma = 1.2 \]
\[ \bar{u}_e/z_e = 0.2 \]
\[ S_{11} = 1.84118 \]

\[ B_{11}(t) = b e^{(\lambda + i\omega)t} \]
Figure 6. Points Considered in the Nonlinear Study
Stable: $A = 1.0, B = 0$

Neutral: $A = 1.0833, B = 0$

Unstable: $A = 1.15, B = 0$

Figure 7. Typical Nonlinear Pressure-Time Histories
Figure 8. Nonlinear Stability Map for the First Radial Mode
Figure 9. Comparison of Single-Mode (1T) Solutions
Figure 10. Comparison of Second Order and Third Order Solutions
Figure 11. Comparison of Standing and Spinning Third Order Solutions
Figure 12. Effect of Chamber Mach Number

\[ \gamma = 1.2 \]
\[ z_e = 1.0 \]
\[ n = 0.61817 \]
\[ \tau = 1.5 \]
Figure 13. Effect of Chamber Length
Figure 14. Theoretical Pressure Waveform Used to Determine $t_0/T$
Figure 15. Injector Peak-to-Peak Pressure Amplitudes
Figure 16. Waveform Correlation Parameter $t_0/T$

- $\bar{u}_e = 0.2$
- $\gamma = 1.2$
- $z_r = 0.0$
Comparison of Numerical and Semi-Empirical Pressure Waveforms.

- Above Resonance: \( \hat{n} = 1.3, \tau = 0.7 \)
- Below Resonance: \( \hat{n} = 1.38, \tau = 1.3 \)

Figure 17. Comparison of Numerical and Semi-Empirical Pressure Waveforms.
Figure 18. The Effect of the Number of Terms in the Series on the Injector Face Pressure.

- 10 terms
- 7 terms
- 5 terms

\[ \hat{\eta} = 1.2 \quad \gamma = 1.2 \]
\[ \tau = 1.0 \quad \bar{u}_e = 0.2 \]
Figure 19. A Comparison of the Large Amplitude Analysis with Second Order Wave Equation Solutions

$$\delta \hat{n} = \hat{n} - \hat{n}_{LS}$$

- Second order wave equation potential analysis
- $\hat{n}_{LS} = 1.0$
- $\hat{r} = 1.0$
- $\gamma = 1.2$
- $\bar{u}_e = 0.2$

$\Delta_{p_{max}}$ is the maximum pressure change.
Figure 20. Peak-to-Peak Injector Face Pressure on the Linear Stability Limit

\[ z = 0.0 \]
\[ \gamma = 1.2 \]
\[ \bar{u}_e = 0.2 \]

Sensitive Time Lag, \( \tau \)
Vertical Displacement, $\delta \hat{h} = \hat{h} - \hat{h}_{LS}$

$z = 0.0 \quad \phi = 1.623$
$\gamma = 1.2 \quad \hat{h}_{LS} = 2.011$
$\bar{u}_e = 0.2$

Figure 21. The Effect of Large Amplitude Oscillations on Engine Stability
Figure 22. Dependence of Nonlinear Waveforms on $\hat{n}$ and $\tau$ (Large Amplitude Analysis)
APPLICATION OF THE GALERKIN METHOD IN
THE DESIGN OF STABLE LIQUID ROCKET MOTORS

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Prepared by
Ben T. Zinn, Professor
Eugene A. Powell, Instructor
S. Kalyanasundaram, Graduate Research Assistant

GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL OF AEROSPACE ENGINEERING
ATLANTA, GEORGIA

Project Monitor: Dr. Richard J. Priem
INTRODUCTION

During the period covered by this report work continued in the following four areas: (1) the development of a three-dimensional, second-order theory to provide a better approximation to the nozzle boundary condition, (2) the development of the third-order, multi-mode theory to study the behavior of large amplitude transverse instabilities, (3) the study of the influence of the functional form of the unsteady combustion response function upon the stability characteristics of rocket motors, and (4) the study of nonlinear axial mode instability. A brief description of the results obtained in the above-mentioned investigations is provided in the following sections.

THREE-DIMENSIONAL, SECOND-ORDER THEORY

During the first phase of this research program a second-order theory using the n - τ combustion model was developed. This theory can predict the nonlinear behavior of two-dimensional modes of instability in liquid propellant rocket motors with quasi-steady nozzles. This section is concerned with the modification of the available second-order theory and its associated computer programs to predict the behavior of three-dimensional nonlinear combustion instabilities in combustors with conventional nozzles.

During the previous report period the second-order theory was improved by choosing a series expansion for the velocity potential that provides a better approximation to the flow at the nozzle entrance. Based on this improved series expansion, Program LCYC3D was developed to calculate the transient and limit-cycle behavior of three-dimensional instabilities. During the period covered by this report, Program LCYC3D was further developed, checked out, and used to obtain data for first-tangential, first-longitudinal, and combined longitudinal-transverse modes of instability in rocket combustors with quasi-steady nozzles. Results obtained in this study are presented in the following paragraphs.

Linear Solutions

Two auxiliary programs, LINSOL and LSTB3D, were written to determine the linear stability characteristics of the three-dimensional solutions. These programs are based on an analytical solution of the linearized differential
equations. For given values of the interaction index, $n$, and the time-lag, $\tau$, program LINSOL computes the linear growth rate, $A$, and the frequency, $\omega$, of the solution. For a given value of $\tau$, program LSTB3D calculates the value of $n$ for neutral stability.

Using Program LSTB3D, a neutral stability limit was generated for the first tangential (1T) mode as shown in Fig. 1. Here the results of the present three-dimensional theory (solid curve) are compared with results obtained using the two-dimensional series expansion of the previous second-order theory (dashed curve). The two curves are in close agreement, which indicates that the presence of an axial velocity perturbation has only a small effect on the 1T stability limit, at least for quasi-steady nozzles.

As part of the check-out procedure for Program LCYC3D, linear solutions were generated by numerically integrating the linearized differential equations for $\tau = 1.7; n = 0.5, 0.55, \text{ and } 0.60$. In each case the linear growth rate, $A$, was determined from the numerical solutions and was compared with the value computed analytically using Program LINSOL. The values obtained by the two programs were in excellent agreement, thus confirming the accuracy of the numerical integration scheme.

To check the accuracy of the approximation of the nozzle boundary condition, Program LCYC3D was used to calculate linear wall pressure and axial velocity waveforms at the nozzle entrance. The error at the nozzle boundary ($z = z_e$) is shown for neutrally stable 1T mode oscillations ($n = 0.5405; \tau = 1.7$) in Fig. 2. Here the axial velocity perturbation, $u'$, and the product of the quasi-steady nozzle admittance and the pressure perturbation, $y_p'$, are plotted as a function of time. The latter quantity is the axial velocity perturbation that would be obtained at the nozzle entrance if the nozzle boundary condition were exactly satisfied (i.e., the nozzle admittance condition requires that $u' = y_p' p$ at $z = z_e$). The good agreement between these curves indicates that the approximation to the nozzle boundary condition for transverse instability is good. On the other hand, similar calculations for longitudinal or combined longitudinal-transverse modes yielded a very poor approximation to the nozzle boundary condition. This difficulty will be discussed in a later section.

Nonlinear Transverse Mode Solutions

The three-dimensional, second-order computer program has been used to calculate limit-cycle amplitudes and waveforms for both standing and
spinning first tangential instability. In this study a three-mode series expansion consisting of the first tangential, second tangential, and first radial modes was used. This required six series terms to describe standing instability and ten series terms to describe spinning instability. Typical computation times to reach a limit-cycle were one minute for a standing wave and two minutes for a spinning wave.

Wall pressure waveforms \((r = 1)\) were computed at the injector face \((z = 0)\) and at the nozzle entrance \((z = z_e)\) for three azimuthal locations, \(\alpha = 0^\circ\), \(\alpha = 45^\circ\), and \(\alpha = 90^\circ\). The initial conditions for standing waves were chosen such that a pressure anti-node occurred at \(\alpha = 0^\circ\). Injector pressure waveforms for both standing and spinning instability are shown in Fig. 3. These waveforms exhibit sharp peaks and shallow minima; they are nearly identical in shape to those calculated using the previous two-dimensional computer program. Injector and nozzle pressure waveforms \((\alpha = 0^\circ)\) are compared in Fig. 4, which shows that there is very little variation in pressure with axial position.

The error at the nozzle entrance is shown in Fig. 5 where \(u'\) and \(y_r p'\) at the nozzle entrance are plotted as a function of time. Most of the discrepancy between the two curves is due to a slight phase shift between pressure and velocity and the second harmonic distortion of the pressure waveform resulting from the nonlinearities of the system. The nozzle boundary condition is satisfied in an average sense, however, for the ratio of the velocity amplitude (peak-to-peak) to pressure amplitude (peak-to-peak) is very close to the required value, \(y_r\).

In another study, limit-cycle amplitudes were calculated as a function of \(n\) and \(\tau\) for standing LT mode instability. Values of \(n\) in the linearly unstable region were chosen for below resonant \((\tau = 1.9)\), resonant \((\tau = 1.706)\), and above resonant \((\tau = 1.5)\) conditions. The resulting amplitudes are compared with those obtained with the two-dimensional theory in Fig. 6. This figure shows that the three-dimensional theory predicts a slightly higher limit-cycle amplitude than the two-dimensional theory for chambers with quasi-steady nozzles. This study also shows that the present theory, like the previous one, cannot predict triggering of LT mode instability.

Longitudinal and Combined Longitudinal-Transverse Modes

As previously mentioned, a difficulty arises when the three-dimensional theory is used to describe the behavior of a longitudinal or combined
longitudinal-transverse mode. An extraneous solution appears which results in a very poor approximation of the nozzle boundary condition for these modes. The large error at the nozzle entrance is illustrated in Fig. 7 for the first longitudinal mode. A brief description of the problem is given in the following paragraphs.

Extraneous Solution

The problem is best illustrated by considering the linear behavior of a single mode, thus the velocity potential is given by:

$$\tilde{\phi}_{mn} = \left[ B_{mn}(t) \cos(\epsilon_{mn} z) \cosh(\eta_{mn} z) + D_{mn}(t) \sin(\epsilon_{mn} z) \sinh(\eta_{mn} z) \right]$$

$$\times \left[ \cos \alpha_m J_m(\epsilon_{mn} r) \right]$$

(1)

where $\epsilon_{mn}$ and $\eta_{mn}$ are the axial acoustic eigenvalues. Substituting Eq. 1 into the linearized wave equation and applying the Galerkin Method yields a coupled system of linear ordinary differential equations. The solutions were determined analytically to be of the following form:

$$B_{mn}(t) = e^{\frac{\Lambda_1}{2} t} \left[ b_1 \cos \omega_1 t + b_2 \sin \omega_1 t \right] + e^{\frac{\Lambda_2}{2} t} \left[ b_3 \cos \omega_2 t + b_4 \sin \omega_2 t \right]$$

(2)

$$D_{mn}(t) = e^{\frac{\Lambda_1}{2} t} \left[ d_1 \cos \omega_1 t + d_2 \sin \omega_1 t \right] + e^{\frac{\Lambda_2}{2} t} \left[ d_3 \cos \omega_2 t + d_4 \sin \omega_2 t \right]$$

(3)

where the $b$'s and $d$'s are constants. Both $B_{mn}$ and $D_{mn}$ are seen to consist of two parts. The first part has a frequency, $\omega_1$, which is close to the natural acoustic frequency of the mode under consideration, thus it will be referred to as the physical solution. The second part has a higher frequency, $\omega_2$, and can be associated with the error produced by the Galerkin Method; this solution will be called the extraneous solution. For transverse modes (i.e., $t = 0$) the extraneous solution was found to be heavily damped and caused no problem. For longitudinal and combined longitudinal-transverse modes (i.e., $t > 0$) the extraneous solution was unstable and caused rapid growth of the numerical solutions even for values of $n$ and $\overline{r}$ in the linearly stable region. This behavior is illustrated for the 1T-1L mode in Fig. 8.
where the peak values of the functions $B_{111}(t)$ and $D_{111}(t)$ are plotted versus time. Using initial conditions which satisfy the nozzle boundary condition (triangle symbols) the extraneous solution completely dominates the solution resulting in rapid growth.

**Suppression of Extraneous Solution**

It was found that by adjusting the initial conditions the amplitude of the extraneous solution could be forced to zero initially. Results obtained with this procedure are also shown in Fig. 8 (circle symbols). Using this procedure, only the physical solution is present initially and the linearly stable oscillations decay as expected. However, numerical errors result in the reappearance of the extraneous solution after approximately 15 cycles. In addition, the results obtained in this manner also provide a very poor approximation to the quasi-steady nozzle boundary condition. Thus, it is concluded that the series expansion given by Eq. (1) is not satisfactory for the study of longitudinal or longitudinal-transverse modes of instability.

**Other Series Expansions**

In an attempt to overcome the above difficulties, other choices for the axial eigenfunctions appearing in Eq. (1) were considered. The first choice, which will be called Series II, is based on an analogy with the solutions for axial waves in a chamber with constant steady state velocity. The velocity potential for a single mode ($\ell > 0$) is thus expressed as:

$$\tilde{\phi}_{II} = \left\{ \frac{1}{2} B_{mn}(t) \right\} \cos \epsilon_{mn} z \cosh \eta_{mn} z + \cos \epsilon_{mn} z \cosh \eta_{mn} z \right\}$$

$$x \left\{ \cos m \phi J_m (\zeta_{mn} r) \right\} \quad (4)$$

The second choice, Series III, is based on the product of a polynomial in $z$ and the acoustic eigenfunctions for a closed-end chamber. Thus, the velocity potential ($\ell > 0$) is given by:
\[ s_{III} = [B_{mn}(t) + z^2D_{mn}(t)] \cos \frac{\pi z}{z_e} \cos^3 J_m(s_{mn} \tau) \]  

(5)

In both cases Series I (i.e., Eq. (1)) is used for pure transverse modes \( \tau = 0 \).

Results obtained with these three series expansions are compared in Fig. 9 in which \( u' \) and \( y_r p' \) are plotted as a function of time. This figure shows that for each of the three series expansions the approximation of the nozzle boundary condition is very poor. The approximation is much better, however, for Series III. For Series III the phase lag between the velocity and pressure is small \( (\sigma = 4.5^\circ) \), but the ratio of velocity amplitude to pressure amplitude is about eight times too large \( (Au/Ap = 0.1354) \). In each case the initial conditions were chosen to eliminate the extraneous solutions.

An attempt to improve the results obtained with Series III (i.e., Eq. (5)) was made. Since the extraneous solution for this case was heavily damped \( (\Lambda_2 = -0.239) \), it was not necessary to eliminate it by adjusting the initial conditions. Therefore, the initial conditions were chosen to satisfy the nozzle boundary condition for \( \tau \leq t \leq 0 \). Unfortunately, the error at the nozzle entrance reappeared during the first cycle. After several cycles the error approached that given by the bottom curves of Fig. 9. Thus, the results obtained by using Series III cannot be improved by a proper choice of the initial conditions.

Concluding Remarks

Results obtained to date for combustors with quasi-steady nozzles indicate that the present three-dimensional computer program (LCYC3D) provides satisfactory results only for pure transverse modes of instability; difficulties arise when longitudinal or combined longitudinal-transverse modes are analyzed. More recent studies, however, indicate that these difficulties can be resolved by a modification of the present approach. In the modified procedure, the complex form of the axial acoustic eigenfunction is used in the series expansion and its complex conjugate is used as the weighting function when the Galerkin orthogonality conditions are applied. This modified approach is currently under investigation.

THIRD-ORDER, MULTI-MODE THEORY

The third-order, multi-mode investigation was completed during this report period. This work included a study of triggered IT mode instability and a convergence study.
Triggered Instability

A study of the triggering behavior of the first tangential mode was conducted using a series expansion consisting of the 1T, 2T, and 1R modes. Results of this study are shown in the limit-cycle amplitude map of Fig. 10 for standing 1T instability. Here lines of constant pressure amplitude (0.2, 0.4, and 0.6) are plotted on an (n, \( \tilde{\tau} \)) plane. These curves all intersect the neutral stability curve at approximately \( \tilde{\tau} = 2.13 \). For \( \tilde{\tau} < 2.13 \) these curves (solid) represent limit-cycle amplitudes, while for \( \tilde{\tau} > 2.13 \) they represent triggering amplitudes (dashed curves). Consider points A and B on the 0.2 amplitude line. For point A (\( \tilde{\tau} < 2.13 \)) all disturbances reach a limit-cycle amplitude of 0.2. For point B (\( \tilde{\tau} > 2.13 \)) an initial disturbance with amplitude less than 0.2 decays to zero, while a disturbance with amplitude greater than 0.2 grows to a higher limit-cycle amplitude. These results show that the third-order multi-mode theory can predict triggering of the 1T mode; however, the region of triggering is confined to \( \tilde{\tau} > 2.13 \) in a narrow strip immediately adjacent to the linearly unstable region.

The triggering behavior for \( \tilde{\tau} > 2.13 \) is further illustrated in Fig. 11, which shows pressure amplitude as a function of displacement, \( \delta \), below the neutral stability limit along lines of constant \( \tilde{\tau} \). These curves exhibit two branches; the lower branch represents the threshold amplitude necessary to trigger instability, and the upper branch represents the limit-cycle amplitude attained by the triggered instability. These curves also exhibit a minimum value of \( \delta \) (or \( n \)) for which all disturbances decay. The small values of \( \delta \) in this plot again emphasize the narrowness of the triggering region.

Convergence Study

A convergence study was conducted to determine how many series terms are necessary to describe 1T mode instability. The basic series consisted only of the 1T mode; additional modes were then added and the results compared with those obtained with the one-mode series. Three points in the (n - \( \tilde{\tau} \)) plane were chosen for this study: \( \tilde{\tau} = 1.5, 1.7, \) and 2.1 with \( n \) chosen such that the limit-cycle amplitude computed with the one-mode series was 0.2. The results are summarized in Table 1 on the following page.
TABLE 1. CONVERGENCE STUDY

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>Modes in Assumed Series</th>
<th>Limit-Cycle Amplitude of 1T Series Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>1T</td>
<td>0.200</td>
</tr>
<tr>
<td>1.5</td>
<td>1T, 2T</td>
<td>0.220</td>
</tr>
<tr>
<td>1.5</td>
<td>1T, 2T, 1R</td>
<td>0.192</td>
</tr>
<tr>
<td>1.5</td>
<td>1T, 2T, 3T</td>
<td>0.214</td>
</tr>
<tr>
<td>1.7</td>
<td>1T</td>
<td>0.200</td>
</tr>
<tr>
<td>1.7</td>
<td>1T, 2T</td>
<td>0.194</td>
</tr>
<tr>
<td>1.7</td>
<td>1T, 2T, 1R</td>
<td>0.192</td>
</tr>
<tr>
<td>1.7</td>
<td>1T, 2T, 3T</td>
<td>0.191</td>
</tr>
<tr>
<td>1.7</td>
<td>1T, 2T, 3T, 1R</td>
<td>0.193</td>
</tr>
<tr>
<td>2.1</td>
<td>1T</td>
<td>0.200</td>
</tr>
<tr>
<td>2.1</td>
<td>1T, 2T</td>
<td>0.192</td>
</tr>
<tr>
<td>2.1</td>
<td>1T, 2T, 1R</td>
<td>0.230</td>
</tr>
<tr>
<td>2.1</td>
<td>1T, 2T, 3T</td>
<td>0.187</td>
</tr>
<tr>
<td>2.1</td>
<td>1T, 2T, 3T, 1R</td>
<td>0.214</td>
</tr>
</tbody>
</table>

These limit-cycle amplitudes are in such close agreement that it appears that the three-mode series used in the previous studies is adequate.

Technical Report

Work has begun on the preparation of a technical report describing the third order theory. The computer program is being modified to make it useful to the engineer in the design of stable rocket engines. The report will include a listing and user's manual for the computer program.

COMBUSTION MODELS

Work continued on the investigation of the A-B combustion model, which was incorporated into the second order theory during the previous report period (see Ref. 2). In this model the unsteady combustion mass source, \( w_m' \), is given by:

\[
w_m' = \frac{d\bar{u}'}{dz} \left[ Ap' + B \frac{dp'}{dz} \right]
\]  

\[ (6) \]
where $A$ and $B$ are prescribed constants, $\bar{u}$ is the steady state Mach number, and $p'$ is the dimensionless pressure perturbation. During this period two studies were completed: (1) a comparison of the linear and nonlinear decay rates, and (2) a study of the triggering characteristics of the first radial ($1R$) mode.

**Linear and Nonlinear Decay Rates**

During the previous report period it was established that neither limit-cycles nor triggering limits could be obtained for the $1T$ mode using the A-B model. A three-mode series consisting of the $1T$, $2T$, and $1R$ modes was used in this study. To determine the effect of the nonlinearities a comparison of linear and nonlinear decay (or growth) rates was made. Results are shown in Figs. 12 and 13. The linearly stable point ($A = 0.6; B = 0.0$) is given in Fig. 12 which shows good agreement between the linear and nonlinear decay rates. Figure 13 gives the linearly unstable point ($A = 1.6; B = 0.0$) which shows that for large amplitudes the nonlinearities cause much more rapid growth than that predicted by linear theory. Data was also obtained in the region of non-oscillatory solutions (see Ref. 2). For the case of non-oscillatory growth ($A = 17.0; B = 0.0$) the nonlinear solutions grow much more rapidly than the linear solutions. For non-oscillatory decay ($A = -15.0; B = 0.0$) the linear and nonlinear solutions were in good agreement. With the A-B model, it appears that the chief effect of the nonlinearities is to increase the growth rate of the linearly unstable solutions.

**First Radial Mode Triggering**

Since the second order $n-7$ theory predicted triggering of the first radial mode, the A-B model was also used to investigate the triggering characteristics of the $1R$ mode. This was done using a series expansion consisting only of the $1R$ mode. The results of this study are shown in Fig. 14 where triggering amplitude is plotted as a function of $A$ for $B = 0.5$, 0.0, and -0.5. It is seen that the amplitude required to trigger $1R$ instability becomes very large only a short distance below the neutral stability limit ($A = 1.0833$). Thus, the region of nonlinear instability is very narrow and may be neglected for all practical purposes.
Report Preparation

The studies using the A-B model are complete and the results are currently being evaluated and compared with those obtained with the n-7 model. A technical report will then be published. This report will include a listing and user's manual for the second-order A-B computer program.

AXIAL INSTABILITY

During this report period the axial instability studies were completed and a contractor report was published. This report describes the analytical technique used to solve nonlinear longitudinal combustion instability problems associated with liquid propellant rocket motors. The analysis produces the transient and limit-cycle behavior of unstable motors and the threshold amplitude required to trigger a linearly stable motor into unstable operation. Limit-cycle waveforms with shock wave characteristics are predicted for most unstable engine operating conditions. This report gives a method of correlating the analytical solutions with experimental data. Program listings and a user's manual for the axial instability computer program are also included in this report.
REFERENCES


Figure 1. Comparison of Linear Stability Limits for 1T Mode.
Neutral Stability: $n = 0.5405$, $\tau = 1.7$, $\gamma = 1.2$, $\bar{u}_e = 0.2$, $z_e = 1.0$

Figure 2. Nozzle Boundary Condition for Linear IT Mode Solution.
Figure 3. Nonlinear Pressure Waveforms for the 1T Mode.

(a) Standing Wave: $n = 0.65$, $\bar{\tau} = 1.7$, $\gamma = 1.2$, $\bar{u}_e = 0.2$, $z_e = 1.0$

(b) Spinning Wave: $n = 0.58$, $\bar{\tau} = 1.7$, $\gamma = 1.2$, $\bar{u}_e = 0.2$, $z_e = 1.0$
Figure 4. Comparison of Nozzle and Injector Pressure Waveforms for the 1T Mode.
Figure 5. Nozzle Boundary Condition for Nonlinear 1T Mode Solutions.

(a) Standing Wave: \( n = 0.65, \bar{\tau} = 1.7, \gamma = 1.2, \bar{u}_e = 0.2, \bar{z}_e = 1.0 \)

(b) Spinning Wave: \( n = 0.58, \bar{\tau} = 1.7, \gamma = 1.2, \bar{u}_e = 0.2, \bar{z}_e = 1.0 \)
Figure 6. Limit-Cycle Amplitudes for the 1T Mode.
$n = 0.86008, \tau = 2.1, \gamma = 1.2, \bar{u}_e = 0.2, z_e = 2.0$

**Dimensionless Time, t**

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Figure 7. Nozzle Boundary Condition for the First Longitudinal Mode.
Initial conditions satisfy nozzle boundary conditions

Initial conditions eliminate extraneous solution

Figure 8. Numerical Solutions for the 1T-IL Mode.
Figure 9. LL Nozzle Boundary Conditions for Three Series Expansions.
Figure 10. Third-Order Limit-Cycle Amplitude Map for the 1T Mode.
Figure 11. Third-Order Triggering Limits for the 1T Mode.
Figure 12. Decay Rates for the 1T Mode Using the A-B Model.
Figure 13. Growth Rates for the 1T Mode Using the A-B Model.

$A = 1.6$, $B = 0.0$
$\gamma = 1.2$, $\bar{u}_e = 0.2$
$z_e = 1.0$
Figure 14. Triggering Limits for the 1R Mode Using the A-B Model.
Research Conducted Under

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APPLICATION OF THE GALERKIN METHOD IN
THE DESIGN OF STABLE LIQUID ROCKET MOTORS

SEMIANNUAL REPORT COVERING PERIOD
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Prepared By
Ben T. Zinn, Professor
Eugene A. Powell, Instructor

GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL OF AEROSPACE ENGINEERING
ATLANTA, GEORGIA

Project Monitor: Dr. Richard J. Priem
INTRODUCTION

During the period covered by this report work continued in the following two areas: (1) the development of a three-dimensional, second-order theory to provide a better approximation to the nozzle boundary condition and (2) the development of the third-order, multi-mode theory to study the behavior of large amplitude transverse instabilities.

The following papers were presented during this report period:


(2) "Nonlinear Longitudinal Combustion Instability in Rocket Motors," co-authored by M. E. Lores and B. T. Zinn and presented at the AIAA 11th Aerospace Sciences Meeting in Washington, D. C.

A brief description of the results obtained in the above-mentioned investigations is provided in the following sections.

THREE-DIMENSIONAL, SECOND-ORDER THEORY

Results obtained during the previous report period (see Ref. 1) for combustors with quasi-steady nozzles indicated that the three-dimensional computer program (LCYC3D) provided satisfactory results only for pure transverse modes of instability; difficulties arose when longitudinal or combined longitudinal-transverse modes were analyzed. This program was based on a series expansion using the real form of the axial acoustic eigenfunctions. In this section a modified procedure, in which the complex form of the axial acoustic eigenfunctions is used in the series expansion, is presented. This modified approach, which will be called the complex eigenfunction method (CEM), was found to eliminate the difficulties encountered with the previous approach or real eigenfunction method (REM). Furthermore, the CEM allows the analysis of combustors with complex nozzle admittances corresponding to conventional nozzle geometries.
Complex Eigenfunction Method

In the Complex Eigenfunction Method the velocity potential, \( \tilde{\Phi} \), is approximated by the following series expansion:

\[
\tilde{\Phi} = \sum_{p=1}^{N} A_p(t) Z_p(z) \Theta_p(\theta) R_p(r)
\]

where the \( A_p \)'s are unknown complex functions of time, the \( Z_p \)'s are the complex axial acoustic eigenfunctions, and \( \Theta_p(\theta) \) and \( R_p(r) \) are real functions. The complex form of the axial acoustic eigenfunctions is given by

\[
Z_p(z) = \cos \eta_p z \cosh \eta_p z - i \sin \eta_p z \sinh \eta_p z
\]

where \( \eta_p \) and \( \eta_p \) are the real and imaginary parts of the axial acoustic eigenvalues. To obtain solutions, the series expansion is substituted into the wave equation and nozzle boundary condition to form the equation residual, \( E(\tilde{\Phi}) \), and the boundary residual, \( B(\tilde{\Phi}) \). Applying the Galerkin orthogonality conditions gives:

\[
\int_{0}^{2\pi} \int_{0}^{1} \int_{0}^{L} E(\tilde{\Phi}) Z_j^*(z) \Theta_j(\theta) R_j(r) r dr d\theta dz = 0
\]

\[
\int_{0}^{2\pi} \int_{0}^{1} B(\tilde{\Phi}) Z_j^*(z_e) \Theta_j(\theta) R_j(r) r dr d\theta = 0
\]

where \( j = 1, 2, \ldots, N \)

The Complex Eigenfunction Method has been used to obtain linear solutions for various nozzle admittances. The linear analysis and the resulting solutions are discussed in the following paragraphs.
Linear Analysis

The linear solutions were obtained by using the CEM to solve the following linearized wave equation and nozzle boundary conditions:

\[ E(\xi) = \frac{1}{r^2} \frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r} \frac{\partial \xi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \xi}{\partial \theta^2} + \frac{1}{\rho_{zz}} \frac{\partial^2 \xi}{\partial t^2} - 2u^2 \frac{\partial \xi}{\partial t} - \gamma \frac{\partial^2 \xi}{\partial z^2} + \gamma_n \frac{\partial u}{\partial z} \left[ \frac{\partial \xi}{\partial t} - \frac{\partial \xi}{\partial (t - \tau)} \right] = 0 \]  

(4)

\[ B(\xi) = \left[ \frac{\partial \xi}{\partial z} + \gamma Y \frac{\partial \xi}{\partial t} \right]_{z = z_e} = 0 \]  

(5)

where \( Y = y_i + iy_1 \) is the complex nozzle admittance. Assuming a one-mode series expansion for \( \xi \), the Galerkin method yields the following complex differential equation:

\[ \frac{d^2 A}{dt^2} + C_1 A + C_2 \frac{dA}{dt} + C_3 \frac{dA}{dt} \left[ A(t - \tau) \right] = 0 \]  

(6)

where \( A(t) \) is the complex amplitude function and \( C_1, C_2, \) and \( C_3 \) are complex constants given by:

\[ C_1 = \frac{Z'(z_e)Z^*(z_e) - \int Z'Z^* \, dz}{\int ZZ^* \, dz} \quad \text{and} \quad \int_0^{z_e} \]  

(7a)

\[ C_2 = \frac{2 \int_0^{z_e} \tilde{u}(z)Z'Z^* \, dz + \gamma (1 - n) \int_0^{z_e} \frac{\partial \tilde{u}}{\partial z} ZZ^* \, dz + \gamma YZ(z_e)Z^*(z_e)}{\int_0^{z_e} ZZ^* \, dz} \]  

(7b)
In Eqs. (7) $S_{mn}$ is the transverse mode frequency and the primes indicate differentiation of the axial eigenfunctions with respect to $z$.

**Analytical Solutions.** Analytical solutions of Eq. (6) are readily obtained by assuming a solution of the form:

$$A(t) = s_0 e^{st}$$

where $a_0$ is a complex constant and $s = \Lambda + i\omega$ is the complex growth rate. Substituting Eq. (8) into Eq. (6) and separating real and imaginary parts yields the following two equations to be solved for the growth rate, $\Lambda$, and the frequency, $\omega$:

$$\omega^2 = C_{1r} + \Lambda^2 + C_{2r} \Lambda - C_{2i} \omega + C_3 e^{-\Lambda T} (\Lambda \cos \omega T + \omega \sin \omega T)$$

$$\Lambda = -\frac{C_{11} + C_{2r} \omega + C_3 e^{-\Lambda T} \omega \cos \omega T}{2\omega + C_{2i} - C_3 e^{-\Lambda T} \sin \omega T}$$

In Eqs. (9) $C_{1r}$, $C_{11}$, $C_{2r}$, and $C_{2i}$ are the real and imaginary parts of the coefficients given by Eqs. (7) and $C_3$ is always real. The linear stability limits are obtained by setting $\Lambda = 0$ in Eqs. (9). Introducing $C_{2r} = D - En$ and $C_3 = En$ into the resulting equations and rearranging yield the following conditions for neutral stability:

$$n = \frac{D + C_{11}/\omega}{E(1 - \cos \omega T)}$$

$$\omega^2 = C_{1r} + \omega(nE \sin \omega T - C_{2i})$$
Both Eqs. (9) and Eqs. (10) must be solved numerically by means of iterative techniques. Two computer programs have been developed to solve these equations; Program LINSOL computes $\Lambda$ and $\omega$ for given values of $n$ and $\tau$, while Program LSTB3D calculates the values of $n$ and $\tau$ along the neutral stability curve.

In order to check the validity of the approximate solutions generated by the CEM, the CEM was used to calculate acoustic solutions (i.e., no mean flow or combustion) and the results were compared with available exact acoustic solutions. Using $\bar{u} = 0$ in Eqs. (9) the CEM yields two solutions for $\Lambda$ and $\omega$; one of these has a positive frequency and the other has a negative frequency. The negative frequency solution cannot be eliminated on the basis of $\omega < 0$ since it is equivalent to a positive frequency solution with a phase shift. The three acoustic solutions are compared in Fig. 1 where $\Lambda$ and $\omega$ are plotted as a function of nozzle phase shift, $\theta$, (i.e., $Y = A \text{e}^{i\theta}$). The frequencies corresponding to the CEM positive frequency solution (CEM+) are in nearly exact agreement with those obtained by the exact solution, while the CEM negative frequency solutions (CEM-) are in complete disagreement. Thus the positive frequency solution appears to be the "proper" solution and the negative frequency solution should be disregarded. In the remainder of this report the positive frequency solution will be taken to be the "physical" solution, and the negative frequency solution will be considered as an "extraneous" solution which represents an error resulting from the use of the Galerkin method.

The results given in Fig. 1 also show that the acoustic growth rate and frequency are sinusoidal functions of $\theta$. For $0 \leq \theta \leq 90^\circ$ and $270^\circ < \theta < 360^\circ$ (i.e., $-90^\circ \leq \theta \leq 90^\circ$) $\Lambda$ is negative and the nozzle acts as a damping device. For $90^\circ \leq \theta \leq 270^\circ$ $\Lambda$ is positive and the nozzle is feeding energy into the system and is thus destabilizing. Although the results shown in Fig. 1 are for the LT mode, similar results were also obtained for the LL mode.

As a further check on the CEM linear analysis, neutral stability limits were calculated and compared with the results of the 2-D analysis (Ref. 2) and the previous 3-D analysis (REM). Since the previous analyses could not handle complex nozzle admittances,
these results are presented for a combustor with a quasi-steady nozzle \( Y = \frac{1}{\gamma - 1} \frac{1}{\gamma} \). Linear stability limits for the 1T mode are shown in Fig. 2 which shows excellent agreement between the results of the three analyses. Figure 3 gives the 1L mode stability limits for the same combustor. Here it is seen that the 2-D and CEM results are in very good agreement, while the REM predicts much more unstable behavior. The lower stability limit of the REM for the 1L mode is a consequence of the large extraneous solution which caused the difficulties indicated in Ref. 1. The above results show that for quasi-steady nozzles the 3-D theory using the Complex Eigenfunction Method predicts neutral stability limits for the 1T and 1L modes which are very close to the stability limits obtained by the 2-D theory. This result is a further indication of the validity of the present theory.

Numerical Solutions. Before proceeding with the nonlinear analysis, it was desired to obtain numerical solutions of the linear differential equation (i.e., Eq. (6)) for comparison with the results of the analytical treatment given above. This is done by assuming that \( A(t) = F(t) + iG(t) \) and separating Eq. (6) into its real and imaginary parts. This procedure yields the following coupled system of equations:

\[
\frac{d^2 F}{dt^2} + C_{1r} F - C_{1i} G + C_{2r} \frac{dF}{dt} - C_{2i} \frac{dG}{dt} + C_{3} \frac{dF}{dt} (t - \tau) = 0
\]

\[
\frac{d^2 G}{dt^2} + C_{1r} G - C_{1i} F + C_{2r} \frac{dG}{dt} + C_{2i} \frac{dF}{dt} + C_{3} \frac{dG}{dt} (t - \tau) = 0
\]

The above equations were integrated numerically by introducing the appropriate coefficients into the previously developed 3-D computer program (LCYC3D of Ref. 1). The integrations were started by assuming initial pressure and velocity waveforms such that the nozzle boundary condition is exactly satisfied for \(-\tau \leq t \leq 0\). Solutions were obtained for values of \( n \) and \( \tau \) on the neutral stability limit for various nozzle admittances. For most values of the nozzle phase shift, \( \theta \), the numerical solutions were in excellent agreement with the positive frequency solutions obtained by the analytical method.
However, for values of $\theta$ in the vicinity of 90° and 270° the influence of the negative frequency solution caused large variations in amplitude of the resulting pressure and velocity perturbations. Such amplitude variations or "beats" occur whenever two sinusoidal functions of comparable amplitude and slightly differing frequency are combined. Typical "beats" for the LT mode for $\theta = 90^\circ$ are shown in Fig. 4 where pressure amplitude is plotted versus time. These beats are undesirable since they are not observed in unstable rocket engines.

Program LCYC3D has been modified to eliminate the beats shown in Fig. 4. This was accomplished by choosing the initial conditions such that the amplitude of the negative frequency solution is forced to zero. A similar procedure was used with the REM of Ref. 1 and resulted in a very poor approximation to the nozzle boundary condition. However, with the CEM this procedure gives very good results as shown in Tables 1 and 2. These tables give numerical solutions, in which the negative frequency solution was eliminated, for values of $n$ and $\gamma$ on the neutral stability limit for several values of the nozzle phase shift, $\theta$, and a nozzle amplitude ratio of 0.02 ($\gamma = 0.02 e^{19}$). Values of the frequency ($\omega_n$), growth rate ($\Lambda_n$), amplitude ratio ($A_n^\prime$), and phase shift ($\theta_n^\prime$) were determined from the numerical solutions. These results show that the approximation to the nozzle boundary condition is very good; that is, the maximum error in amplitude is about 5% and the maximum error in phase is approximately 0.5 degree. The extremely small values of $\Lambda_n$ show that the solutions are indeed neutrally stable with no amplitude variations or beats.

Nonlinear Analysis

On the basis of the results of the linear analysis presented above, the Complex Eigenfunction Method will be used in the nonlinear analysis. Thus the series expansion given by Eq. (1) will be used to obtain approximate solutions of the nonlinear wave equation:

\[ v^2 \phi - \phi_{tt} = 2 \nabla \cdot \nabla \phi_t + \gamma (\nabla \cdot \nabla \phi^\prime_t + \phi^\prime) + \phi_{tt} = 2 \nabla \cdot \nabla \phi^\prime_t + \gamma (\nabla \cdot \nabla \phi^\prime_t + \phi^\prime) + \phi_{tt} = 2 \nabla \cdot \nabla \phi^\prime_t + (\gamma - 1) \phi_t v^2 \phi \]  

(12)
Table 1. 1T Mode Numerical Solutions

\( \gamma = 1.2, \tilde{u}_e = 0.2, \tilde{z}_e = 1.0, \theta = 0.02e^{i\theta} \)

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Table 2. IL Mode Numerical Solutions

\( \gamma = 1.2, \quad u_e = 0.2, \quad z_e = 1.0, \quad Y = 0.02 e^{i\theta} \)

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where \( \bar{v} \) is the steady state velocity and \( \dot{w}_m \) is the unsteady combustion mass source. Care must be exercised when assuming complex solutions of Eq. (12), since only the real part of the assumed solution is physically meaningful. It can easily be shown that if \( \ddot{\xi} = \varphi + i\psi \) is a solution to Eq. (12), the real part, \( \varphi \), is not a solution to Eq. (12). This failure of \( \varphi \) to satisfy Eq. (12) is a result of the nonlinear terms in this equation. It can also be shown, however, that a modified wave equation can be constructed for which the real part of its solution satisfies the original wave equation (i.e., Eq. (12)). This modified wave equation is given by:

\[
\nabla^2 \ddot{\xi} - \dddot{\xi} = 2\ddot{\xi} \cdot \nabla \dddot{\xi} + \gamma(\nabla \cdot \dddot{\xi}) + \dddot{\xi} + \gamma \left[ -\ddot{\xi} \cdot \nabla^2 \dddot{\xi} + \dddot{\xi} \cdot \nabla^2 \dddot{\xi} \right] + \frac{\gamma - 1}{2} \left[ \dddot{\xi} \cdot \nabla^2 \dddot{\xi} + \dddot{\xi} \cdot \nabla^2 \dddot{\xi} \right]
\]

(13)

where \( \dddot{\xi}^* \) is the complex conjugate of \( \dddot{\xi} \). Thus the CEM will be used to obtain approximate solutions to Eq. (13) (i.e., \( \dddot{\xi} = \varphi + i\psi \)) from which the real part, \( \varphi \), will be taken as the approximate solution of Eq. (12).

The nonlinear computer programs based on the above procedure are now under development. These programs will be capable of calculating the nonlinear behavior of three-dimensional instability in cylindrical rocket chambers with conventional nozzles.

THIRD-ORDER MULTI-MODE THEORY

During this report period much time was spent in the preparation of a contractor report describing the third-order theory. A first draft had been written and was being edited in December when serious errors were discovered in the analysis. As a result publication of the contractor report has been delayed.

Work is now in progress to correct the errors found in the third-order analysis. To date all of the basic conservation equations and boundary conditions have been corrected and carefully
checked. The Galerkin method has been applied to these conservation equations and the resulting set of ordinary differential equations has been derived. A computer program, COEFFS, is now being developed to calculate the various coefficients appearing in these equations. These coefficients will then be used as input to a modified version of an existing program (i.e., LCYC3D) which will numerically integrate these equations. As soon as valid data is obtained, the above-mentioned contractor report will be published.

References


Figure 1. Comparison of IT Acoustic Solutions.

- **Exact**
- Galerkin: Positive Frequency Solution
- Galerkin: Negative Frequency Solution
Figure 2. Comparison of 1T Mode Linear Stability Limits.
Figure 3. Comparison of 1L Mode Linear Stability Limits.
Figure 4. Beats Produced by Combination of Positive and Negative Frequency Solutions (1T Mode).
Research Conducted Under
NASA Grant No. NGR 11-002-083

APPLICATION OF THE GALERKIN METHOD IN
THE DESIGN OF STABLE LIQUID ROCKET MOTORS

ANNUAL REPORT COVERING PERIOD
August 1, 1968 - July 31, 1969

Prepared By
Ben T. Zinn, Associate Professor
Eugene A. Powell, NASA Fellow

GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL OF AEROSPACE ENGINEERING
ATLANTA, GEORGIA

Project Monitor: Dr. Richard J. Priem
This report contains a brief description of the work performed during the first year of financial support under NASA grant NGR 11-002-083. During the initial phase of the study, the classical Galerkin Method has been modified in order to be applicable in the solution of problems where the dependent variables were required to satisfy complex boundary conditions. The applicability and usefulness of the Modified Galerkin Method has been demonstrated by applying it in the analysis of a variety of linear combustion instability problems. In all these instances, the Modified Galerkin Method produced results that were in excellent agreement with available exact solutions. Upon completion of the linear studies, the Modified Galerkin Method has been used in the analysis of nonlinear transverse combustion instability in liquid propellant rocket motors. The results obtained in this study include limit cycles of linearly unstable disturbances; triggering limits for linearly stable regions as well as nonlinear pressure waveforms. Of special significance is the fact that the determination of these quantities required relatively little computer time. Results obtained during the first year of investigation demonstrated beyond doubt the ability of the Galerkin Method to analyze and solve complex combustion instability problems. Based on these results, a future course of investigation is proposed.
Introduction

The objective of this study was to theoretically investigate the nonlinear behavior of liquid-propellant rocket motors. More specifically it was the aim of this investigation to develop analytical tools that will enable engineers to a priori determine the expected nonlinear behavior of unstable rocket motors. Once developed these analytical tools were to be used in the quantitative determination of the limit cycles, triggering limits and nonlinear pressure waveforms of unstable liquid propellant rocket motors.

Studies conducted during a period prior to the initiation of the NASA support for this project indicated that an approximate mathematical technique commonly known as the Method of Weighted Residuals* (MWR) would be suitable for obtaining the desired information. Further studies revealed that applications of this method in the solution of combustion instability problems would require a modification of the existing mathematical technique. During the early part of the program the necessary modification had been accomplished and the modified technique was applied in the solution of a number of linear combustion instability problems, whose solutions had previously been obtained with the aid of other mathematical techniques. These studies produced results that were in excellent agreement with the available solutions. Various results obtained in these studies were presented in the following conferences:

(1) 19th Congress of the International Astronautical Federation, held in New York City during October 13-19, 1968.

(2) 5th ICRPG Combustion Conference held in Silver Spring, Maryland during October 1-3, 1968.

* The Galerkin Method is a special application of the Method of Weighted Residuals.
(3) 11th Israel Annual Conference on Aviation and Astronautics held in Haifa, Israel in March, 1969.

Results obtained in these early studies can be found in the following publications:


Upon successful completion of the linear studies the work progressed into the theoretical investigation of nonlinear combustion instability problems. Early nonlinear results were very encouraging and further work in this area is presently in progress.

Brief description of the mathematical modifications, the linear and nonlinear studies are provided in the sections that follow this introduction. These sections are followed by a brief description of proposed future research.

The Modified Galerkin Method

The Galerkin Method is a special application of the Method of Weighted Residuals which to date has been extensively and successfully used in the solution of various structural stability and aeroelasticity
problems. In these applications the solutions of the governing differential equations are required to satisfy a set of relatively simple boundary conditions. In order to apply the Galerkin Method in the solution of combustion instability problems the "classical" approach must be modified to accommodate the mathematically complicated boundary conditions which must be satisfied in the analysis of combustion instability problems.

According to the "classical" Galerkin Method the dependent variables are expanded in terms of a set of functions \( \hat{\phi}_n \) that identically satisfy the imposed boundary conditions. The proper choice of the functions \( \hat{\phi}_n \) is aided by information from such sources as experimental data, the solution of the linearized version of the same problem, or from solutions of closely related problems. Each of the \( \hat{\phi}_n \)'s is multiplied by an arbitrary constant yet to be determined. These expansions are then substituted into the governing differential equations to form residuals, and the arbitrary constants are determined by imposing the condition that the residuals of the differential equations be orthogonal to all the functions \( \hat{\phi}_n \).

For example consider the nonlinear differential equation

\[
N(u) = 0
\]  

(1)

where \( N(u) \) is a nonlinear differential operator and \( u \) is the dependent variable. An approximate solution of the following form is assumed

\[
\tilde{u}(x,t) = \sum_{n=1}^{N} k_n \hat{\phi}_n(x,t)
\]  

(2)

where \( \hat{\phi}_n(x,t) \) are properly chosen functions and \( k_n \) are the constants to be determined. Substituting this expansion into the governing equation gives the residual

\[
N(\tilde{u}) = R_N(x,t) \neq 0
\]  

(3)
This residual is then made orthogonal to each of the functions $\tilde{\psi}_n$, that is, the $k_n$'s are determined by the $N$ relations

$$\int \int_R(x,t) \tilde{\psi}_n(x,t) dx dt = 0$$

(4)

When the boundary conditions which the solutions must satisfy are complicated it is usually impossible to find a set of functions, $\tilde{\psi}_n$, that can identically satisfy these boundary conditions. When this occurs it is also necessary to apply the orthogonality conditions to the resulting boundary residuals. This operation results in the appearance of additional $N$ redundant equations. It has been shown in this investigation that by proper manipulation of the expressions that resulted from applying the orthogonalization conditions to the boundary and differential equation residuals the above-mentioned difficulty may be eliminated. In the present study such generalized orthogonality conditions have been derived for the conservation equations and the boundary conditions that describe the unsteady flow field inside a rocket's combustor. These generalized conditions are of the following form

$$\int_{t^*}^t \left\{ \int_V E_i(p, \vec{v}, \vec{p}) \tilde{\psi}_n dv - \int_S B_i(p, \vec{v}, \vec{p}) \tilde{\psi}_n ds \right\} dt = 0$$

(5)

$$i = 1, 2, 3$$

$$n = 0, 1, 2, \ldots N$$

where the $E_i$ are the residuals of the conservation equations and the $B_i$ are the residuals of the corresponding boundary conditions. The details of the analysis leading to derivation of Eq. (5) are available in Reference 1.

**Summary of Linear Studies**

Before tackling any nonlinear combustion instability problem,
several linear problems were solved using the Modified Galerkin Method, and the results were compared with available "exact" solutions. The cases considered were the low and high Mach number longitudinal instability, and the low Mach number transverse instability.

In the longitudinal study a combustion chamber with a concentrated combustion zone at the injector end and a short nozzle at the other end was considered. The combustion process was assumed to be described by Crocco's sensitive time lag hypothesis. In the low Mach number case the approximate solution was expressed as a combination of neutrally stable leftward and rightward-running waves that could be excited in a chamber with no steady state flow. In the high Mach number analysis the approximate solution was modified to include the effect of steady state velocity on leftward and rightward-running waves. The neutral stability limits obtained by applying the Modified Galerkin Method were found to closely agree with those predicted by available exact solutions of the governing differential equations. In fact exact agreement occurred in the low Mach number case. These results are presented in References 1 and 3.

In the analysis of transverse and three-dimensional instabilities the combustion process was assumed to be distributed throughout the rocket's combustion chamber. It has been shown in this study that under low Mach number mean flow conditions the unsteady flow in the combustor can be described by a single partial differential equation describing the behavior of the velocity potential. This equation was then used in the analysis of linear combustion instability in a cylindrical combustion chamber with uniform propellant injection at one end and a multiple orifice (quasi-steady) nozzle at the other end. As in the analysis of the axial instability, the combustion process was described by using Crocco's time-lag hypothesis.

The approximate solution for the velocity potential was expressed in the following form

\[ \tilde{\phi} = e^{i\omega t} y_{(m,n)}(r,\theta) \left[ a_0 + \sum_{k=2}^{N} a_k z^k \right] \]  

(6)
where, in the case of standing transverse mode oscillations,

\[ y_{(m,n)}(r,\theta) = \cos m\theta \left( S_{(m,n)}(r) \right) \tag{7} \]

and \( S_{(m,n)} \) is the \( n \)th nonzero root of the equation \( J'_m(x) = 0 \). The constants \( a_0, a_2, ... a_N \) are to be determined by the Galerkin Method.

Linear stability limits for the various transverse modes were then computed using one-term and two-term expansions. It was found that the two-term series predicted a slightly more unstable behavior. The effect of increasing the Mach number of the combustor's mean flow and decreasing the combustor's length to diameter ratio were both found to be destabilizing. These results were found to be in complete agreement with the results obtained by Reardon\(^{(4)}\) who treated an identical problem with the aid of more exact mathematical techniques. Further details about the method of analysis and the calculated results can be found in Ref. 3.

**Nonlinear Investigations**

The modified form of the Galerkin Method has been used to study the effect of nonlinearity on the stability of a liquid-propellant rocket engine in which the combustion process is described by Crocco's time lag hypothesis. Transverse instabilities in a cylindrical combustor with uniform propellant injection at one end and a quasi-steady nozzle at the other end have been investigated. It has been shown that under low Mach number mean flow conditions the waves in the chamber are irrotational to second order (in wave amplitude). Under these conditions it is possible to describe the wave motion in the chamber by a single partial differential equation that has the following form:

\[ \nabla^2 \hat{\phi} - \hat{\phi}_{tt} = 2\hat{\nabla} \cdot \hat{\nabla} \hat{\phi} + \gamma(\nabla \cdot \hat{\mathbf{u}}) \hat{\phi}_t + 2\hat{\nabla} \cdot \hat{\nabla} \hat{\phi}_t + (\gamma - 1) \hat{\phi}_t \nabla^2 \hat{\phi} + \nu_m \tag{8} \]
In the above equation, that is correct to second order in wave amplitude, \( \phi \) is a velocity potential defined by \( \nabla \phi = \nabla \phi, \) \( \bar{u} \) is the steady state velocity, and \( \omega_m \) is the mass source perturbation that results from the unsteady response of the burning process to the pressure oscillations. The above differential equation has the mathematical form of an inhomogeneous wave equation and it is similar to the one used by Maslen and Moore (5) in their nonlinear studies. The two linear terms involving \( \bar{u} \) describe, to second order, the effect of steady state flow on the wave motions, while the last term \( \omega_m \) describes the coupling between the gas oscillations and the burning process.

The remaining terms describe the second order nonlinearities of the flow, and arise in part from the convective terms in the momentum equation. The third order nonlinear terms which are used in Maslen and Moore's analysis were considered to be negligible in this analysis.

It has also been shown that the mass source perturbation \( \omega_m \) is described by the following expression:

\[
\omega_m = - \gamma n \frac{d\bar{u}}{dz} \left[ \bar{\phi}_t (r, \theta, z, t) - \bar{\phi}_t (r, \theta, z, t-\tau) \right]
\]

where \( \gamma \) is the pressure interaction index and \( \tau \) the steady state value of the time lag.

Solutions of Eq. (8) were used to investigate the nonlinear behavior of unstable liquid-propellant combustors. Based on available experimental evidence it has been assumed that the velocity potential can be expressed in the following form.

\[
\phi = \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0,1}^{m} \left[ A_{i mn} (t) \sin m\theta + B_{i mn} (t) \cos m\theta \right] \cos \frac{m \pi x}{L_x} J_m (S_{mn} r)
\]

In Eq. (10) \( \sin m\theta \) and \( \cos m\theta \) are the functions describing transverse acoustic oscillations, \( S_{mn} \) is the nth root of the equation \( J_m (x) = 0, \) and the n-summation begins with \( n = 0 \) only if \( m = 0. \) This expansion identically satisfies the boundary condition of zero normal
velocity at the injector face (i.e., z=0) and at the outer wall (i.e., r=1). Including in the expansion both \sin m\theta and \cos m\theta (with different coefficients) admits the possibility of either spinning wave or standing wave solutions or a combination of both. For example if either \(A_{lmn}(t)\) or \(B_{lmn}(t)\) is zero, the transverse wave is standing.

To obtain a solution of Eq. (8) with the aid of the Modified Galerkin Method the series solution presented in Eq. (10) was truncated and an approximate solution having the following form

\[
\tilde{\psi} = B_{001}(t)J_0(s_{01}r) + \left[A_{011}(t)\sin\theta + B_{011}(t)\cos\theta\right]J_1(s_{11}r) + \left[A_{021}(t)\sin2\theta + B_{021}(t)\cos2\theta\right]J_2(s_{21}r) \tag{11}
\]

was used in the analysis. This truncated series solution contains the most frequently observed unstable modes. Once \(\tilde{\psi}\) is known the pressure could be determined from the following integral of the momentum equation:

\[
p' = -\gamma\left[\frac{\partial \tilde{\psi}}{\partial t} + \tilde{\psi} \frac{\partial \mathbf{u}}{\partial z} + \frac{1}{2}\left(\frac{\partial^2 \tilde{\psi}}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{\psi}}{\partial r} + \frac{\partial^2 \tilde{\psi}}{\partial \theta^2} + \frac{\partial^2 \tilde{\psi}}{\partial z^2}\right) - \frac{1}{2} \frac{\partial^2 \tilde{\psi}}{\partial t^2}\right] \tag{12}
\]

Using the Modified Galerkin Method it has been shown that the behavior of the unknown time dependent coefficients is controlled by the following system of coupled nonlinear ordinary differential equations:

\[
\frac{d^2 B_{001}}{dt^2} + s_{01}^2 B_{001} + k_1 \frac{dB_{001}}{dt} + k_2 \frac{dB_{001}}{dt}(t-\tau) + c_1 B_{001} \frac{dB_{001}}{dt} + c_2 \left(A_{011} \frac{dB_{011}}{dt} + B_{011} \frac{dB_{011}}{dt}\right) + c_3 \left(A_{021} \frac{dB_{021}}{dt} + B_{021} \frac{dB_{021}}{dt}\right) = 0
\]
The equation governing each mode-amplitude consists of two parts; the linear part which describes the motion of a damped (or amplified) harmonic oscillator with undamped frequency $S_{m1}$ and a retarded damping term, and the nonlinear coupling terms. The coefficients of the damping terms are the same for all modes, and are given by

$$K = \frac{\gamma}{z_e} \left( 1 + \frac{\gamma - 1}{2\gamma} - n \right) \quad ; \quad K_\tau = \frac{\gamma}{z_e} n$$
The coefficients of the nonlinear terms are independent of \( n \) and \( \tau \) and are functions of various integrals of trigonometric and Bessel functions which result from applying the Galerkin Method.

**Summary of Results**

The nonlinear ordinary differential equations which resulted from the application of the Galerkin Method were integrated numerically by using a standard fourth order Runge-Kutta method. This method was easily adapted to handle the retarded variable in the equations.

In order to get the numerical solution started, an initial disturbance was assumed. The values and derivatives of all the time dependent mode-amplitude functions were specified at time \( t = 0 \), and because of the presence of a retarded variable the derivatives were specified as a function of time for \( -\tau < t < 0 \). It was assumed that a first tangential acoustic mode is initially present, and that at \( t = 0 \) the damping and nonlinearities are suddenly "turned on". The initial conditions for a standing and a spinning wave are as shown below:

**Standing:**

\[
A_{011}(t) = A_{011}(0) \cos S_{11} t \quad (-\tau < t < 0)
\]

\[
\frac{dA_{011}}{dt} = -S_{11} A_{011}(0) \sin S_{11} t
\]

**Spinning:**

\[
\ddot{x} = A \cos (\theta - S_{11} t)
\]

\[
A_{011}(t) = A \sin S_{11} t \quad B_{011}(t) = A \cos S_{11} t \quad (-\tau < t < 0)
\]

\[
\frac{dA_{011}}{dt} = AS_{11} \cos S_{11} t \quad \frac{dB_{011}}{dt} = -AS_{11} \sin S_{11} t
\]
Numerical solutions have been obtained for two cases; one in which the approximating series was specified by Eq. (11) and another in which the expansion contained the first radial mode only. In either case for given operating condition (i.e., given \( n \) and \( \theta \)) an initial disturbance was assumed and the computer was allowed to run until a periodic solution was reached (if one existed).

The behavior of the first tangential mode was investigated with the aid of the series given by Eq. (11). For a point in the \((n,\tau)\) plane within the linearly unstable region for the first tangential mode, but stable with respect to other modes that are present in the series, periodic finite amplitude solutions (stable limit cycles) have been found. The pressure waveforms for standing oscillations are continuous, have sharp peaks and flat minima, and they oscillate at a frequency close to that of the first tangential mode. As expected, the results obtained in these preliminary studies show that the final amplitudes attained by the various modes are independent of the magnitude of the initial disturbances. These results also show that under the conditions investigated the amplitude of the first tangential mode dominates the amplitudes of the other modes present in the expansion.

Using the three mode series the \(n-\tau\) plane was explored and curves of limiting pressure amplitude as a function of \( n \), along the lines of constant \( \tau \), were plotted. Typical results are shown in the sketch below.

![Sketch of pressure amplitude versus n and \( \tau \)](image-url)
where the arrows indicate expected behavior of initial disturbances. The lower branch of the curve represents the locus of stable limit cycles. The curve shows that the limiting final amplitude increases with increasing distance from the neutral curve, in nearly parabolic fashion. The upper branch represents unstable finite amplitude solutions and it occurs at values of the amplitude where the second order theory is no longer valid. Disturbances with amplitudes above this limit were found to grow rapidly and without limit.

This study also showed that the nonlinear behavior of the resulting oscillation was strongly dependent on the specified values of \( n \) and \( \tau \). For instance if the values of \( n \) and \( \tau \) were such that the first radial mode was linearly unstable while the remaining modes were linearly stable then for these values of \( n \) and \( \tau \) the resulting finite amplitude oscillation behaved like the first radial mode. It has also been found that the magnitude of the final amplitude attained by the disturbance was also dependent on the values of \( n \) and \( \tau \).

By assuming an initial condition corresponding to a first tangential spinning wave in the region where the 1-T mode is unstable, finite amplitude spinning wave solutions were obtained. In the limiting periodic solution the 1-R mode vanishes and the 1-T and 2-T modes remain. These spinning waves exhibit amplitudes which are higher than that of the corresponding standing mode solution. This result is in agreement with available experimental observations.

In a separate study the nonlinear behavior of the first radial mode was investigated. This was of interest because only the radial modes exhibit self-coupling in the differential equations which result from applying the Galerkin Method. Therefore nonlinear behavior may be obtained by using a one mode series expansion. The nonlinear behavior of the first radial mode was investigated for values of \( n \) and \( \tau \) in the vicinity of the linear stability limit of the 1-R mode, for four values of \( \tau \). Both stable limit cycles and triggering limits were found as shown in the sketches below.
\[ B_{001} \]

\[ \tau = 0.5418 \]

\[ B_{001} \]

\[ \tau = 0.7106 \]

\[ B_{001} \]

\[ \tau = 0.8199 \]

\[ B_{001} \]

\[ \tau = 0.9307 \]
In the above sketches the intersection of the curve with the abscissa represents the value of \( n \) at which the line \( \tau = \text{const.} \) intersects the neutral stability line. The above sketches clearly indicate the \((n, \tau)\) dependence of the nonlinear stability limits. These sketches also show that the possibility of triggering combustion oscillations increases as the values of \( \tau \) is increased.

**Future Investigations**

In addition to the work described in previous paragraphs a nonlinear second order analysis of axial-type instability has been completed. This analysis produced a system of coupled nonlinear ordinary differential equations that control the behavior of the amplitudes of the various axial modes. This system of differential equations will have to be solved numerically and an appropriate computer program has been developed for this purpose. Relevant numerical solutions are expected in the near future.

In addition to the investigation of axial instability the equations described in the previous sections will be used to determine the dependence of nonlinear transverse type combustion instability on various engine design parameters.

Additional future studies will include the development of a new nonlinear theory in which all nonlinear terms will be included and in which no restrictions upon the magnitude of the mean flow Mach number will be imposed. This theory will be used to investigate the behavior of axial, transverse and three-dimensional type instabilities.

While in studies conducted to date Crocco's time lag hypothesis was used to describe the combustion response, future studies will consider the possibility of using different functional forms to describe the unsteady energy and mass sources. The nonlinear behavior associated with the use of these various functional forms will be investigated. It is hoped that this investigation would yield much needed information about the correct functional form of the combustion response function. Once determined this functional form will be of great value in the a priori determination of nonlinear behavior of unstable liquid-propellant rocket motors.
References


REPORT 70-5

SUMMARY OF COMBUSTION INSTABILITY RESEARCH

Conducted During August 1, 1969 - July 31, 1970

At the Georgia Institute of Technology

B. T. Zinn, W. C. Strahle, E. A. Powell, M. E. Lores

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School of Aerospace Engineering
GEORGIA INSTITUTE OF TECHNOLOGY
Atlanta, Georgia 30332
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ABSTRACT

The status and summary of recent results obtained in various studies aimed at developing means for a priori predicting the nonlinear behavior of unstable liquid-propellant rocket motors are described. The studies under consideration include: (1) Determination of the nonlinear behavior of unstable rockets with the aid of a second order theory; (2) The development of a third order theory; (3) Investigation of nonlinear axial instability in liquid rockets; and (4) Investigation of the behavior of the unsteady combustion response function.
SUMMARY

Brief descriptions of combustion instability studies performed at Georgia Institute of Technology during the second year of financial support under NASA grant NGR 11-002-083 are provided.

This project is concerned with the application of the Galerkin method in the prediction of the nonlinear behavior of unstable liquid-propellant rocket motors. In a study of transverse instabilities, numerical results predicting stable limit cycles, triggering limits, and nonlinear pressure waveforms were obtained. The dependence of the engine's nonlinear stability characteristics upon various engine parameters was also studied. The importance of various types of nonlinearities present in the conservation equations was also investigated.

During this same period investigation of the nonlinear axial mode instability problem was continued. Three approximate methods of solution have been devised, and computer programs based on these methods are presently being developed.

A study of the behavior of the unsteady combustion response function was also initiated during this period.
INTRODUCTION

During the first year of NASA support for this project an approximate mathematical technique was successfully applied in the solution of a number of combustion instability problems. Special attention was given to the study of nonlinear effects. As part of this effort a nonlinear second-order theory, which formed the basis for a computer program designed to predict the nonlinear behavior of unstable liquid propellant rocket motors, was developed.

The work performed during the second year of this project represents an extension and diversification of the research performed during the first year. Several different aspects of nonlinear combustion instability were considered and the following investigations were conducted: (1) the study of moderate amplitude transverse instability based on the second order theory developed during the first year, (2) the development and application of a third order theory to study large amplitude transverse instabilities, (3) the study of nonlinear axial mode instability, and (4) the study of the behavior of unsteady combustion response functions.

Some of the results obtained with the aid of the second order theory were presented at the 6th ICRPG Combustion Conference held in Chicago during September 9-11, 1969. Additional results appear in the following publications:


(2) Powell, E. A., "Nonlinear Combustion Instability in Liquid Propellant Rocket Engines," Georgia Institute of Technology,


The following paper has been accepted for presentation (and for publication in the proceedings of the conference) at the Thirteenth International Symposium on Combustion, to be held at the University of Utah during August 23-29, 1970:


Brief description of the results obtained in the above-mentioned investigations are provided in the following sections. These sections are followed by a brief description of proposed future research.

**Results of the Second Order Transverse Instability Studies**

During the first year of this project it was shown that when the mean flow Mach number is small the wave motion inside the combustor of a liquid-propellant rocket engine can be described, to second order, by the following nonlinear partial differential equation:

\[ \nabla^2 \phi - \phi_{tt} = 2a_{m} \cdot \nabla \phi_t + \nu (\nabla \cdot \bar{u}) \phi_t \]

\[ + 2\nu \cdot \nabla \phi_t + (\gamma - 1) \rho \phi_t \nabla^2 \phi + W_m \]

(1)
In the above equation, $\Psi$ is the velocity potential defined by $V' = v\Psi$, $\bar{u}$ is the steady state velocity, and $W'_m$ is the mass source perturbation that results from the unsteady response of the burning process to the pressure oscillations. This section presents a summary of the results obtained when Eq. (1) was used to study the nonlinear stability characteristics of cylindrical combustion chambers in which the liquid propellants are injected uniformly across the injector face and the combustion process is distributed throughout the combustion chamber. Crocco's time-lag hypothesis was used to describe the unsteady combustion process; hence the unsteady mass source is given by:

$$W'_m = -\eta n \frac{du}{dz} \left[ \Psi_t(r, \theta, z, t) - \Phi_t(r, \theta, z, t - \tau) \right]$$  \hspace{1cm} (2)

where $n$ is the pressure interaction index and $\tau$ is the steady state value of the time-lag. It was also assumed that the hot combustion products leave the combustion chamber through a multi-orifice quasi-steady nozzle.

In order to study transverse combustion instability, an approximate solution for the velocity potential was constructed as a series expansion in the tangential acoustic modes, and each of these modes was multiplied by an undetermined time-dependent coefficient. Using this series expansion and following the mathematical procedure outlined in Ref. 2 the solution of the original partial differential equation was reduced to the solution of a system of coupled nonlinear ordinary differential equations that control the behavior of the unknown time-dependent amplitudes. These equations form the basis of a computer program for calculating the nonlinear transverse stability characteristics of liquid propellant rocket engines.

Extensive numerical computations using series solutions containing various combinations of the chamber's natural modes were performed. The calculated results indicated that a series expansion consisting of the first tangential (1T), second tangential (2T), and first radial (1R)
modes provides a good description of the engine stability characteristics. Thus the velocity potential was approximated by the following series expansion:

\[
\tilde{\phi} = B_{01}(t)J_0(S_{01}r) + [A_{11}(t)\sin\theta + B_{11}(t)\cos\theta]J_1(S_{11}r) + [A_{21}(t)\sin2\theta + B_{21}(t)\cos2\theta]J_2(S_{21}r)
\] (3)

Numerical calculations obtained using the above three-mode series are presented with the following objectives in mind: (1) the prediction of the final amplitude for transverse mode instability; (2) the determination of the waveform and frequency of the nonlinear oscillations; and (3) the determination of the dependence of the resulting oscillation upon (a) the initial disturbance, (b) the combustion parameters \( n \) and \( \tau \), (c) the magnitude of the steady state Mach number at the nozzle entrance, and (d) the combustor's length-to-diameter ratio.

For a given initial disturbance it was possible to follow the time evolution of the three modes included in the series expansion. The computations showed that in the region of the \((n, \tau)\) plane where the \(1T\) mode is the only linearly unstable mode in the series (i.e., Region I of Fig. 1) an initial disturbance develops into a finite-amplitude oscillation. The magnitude of the final amplitude depends on the nature of the initial disturbance (i.e., spinning or standing) but not on the magnitude of the initial disturbance. These studies showed that in this region of the \((n, \tau)\) plane, the magnitude of the final amplitude is limited by nonlinear coupling between modes, whereby energy is transferred from the linearly unstable \(1T\) mode to the linearly stable modes (\(2T\) and \(1R\)). More complex behavior was observed in regions of the \((n, \tau)\) plane where the \(1R\) mode is unstable and the \(1T\) mode is either stable or unstable. In such regions (i.e., Region III of Fig. 1) "back-and-forth" energy transfer between the modes produces a finite-amplitude, highly-modulated oscillation which exhibits some character-
istics of both modes.

As seen in Fig. 2, once a stable limit cycle is reached the time dependent coefficients (i.e., mode-amplitudes) appearing in Eq. (3) are nearly sinusoidal functions of time. In most of the cases computed the 1R and 2T modes had a much smaller amplitude than the 1T mode and were oscillating at twice the frequency of the 1T mode. Using the mode-amplitudes the nonlinear pressure waveforms at any location in the chamber were easily computed. Typical wall pressure waveforms for standing 1T oscillations are shown in Figs. (3) and (4). These waveforms exhibit a pronounced second harmonic distortion, resulting in sharp peaks and shallow minima. Also predicted is a small-amplitude, double-frequency oscillation at the locations for the (1T) pressure nodes (i.e., \( \theta = 0 \) and \( \theta = \pi \)).

The predicted dependence of the final amplitude upon the combustion parameters \( n \) and \( \tau \) is shown in Fig. (5) for a standing type instability. In this plot \( \delta \) is the displacement of the operating point \((n, \tau)\) from the neutral stability limit measured along a line of constant \( \tau \). Positive values of \( \delta \) indicate displacements into the linearly unstable region. From this figure it is seen that the limiting amplitude increases with both increasing vertical displacement (i.e., increasing the value of \( n \)) and with increasing time-lag. For the 1T spinning instability similar plots were obtained, and they are compared with those for standing oscillations in Fig. (6). In most of the cases considered an initially spinning wave disturbance resulted in instabilities with larger final amplitude oscillations. The frequency of oscillation was also found to depend on \( n \) and \( \tau \). As seen from Fig. (7) the frequency decreases with increasing amplitude (i.e., increasing \( n \)) and with increasing \( \tau \). The above results were used to construct nonlinear \((n, \tau)\) stability maps (see Fig. (8)) showing the dependence of the final amplitude upon the values of \( n \) and \( \tau \).

The second order theory was also used to determine the dependence of the final amplitude upon the engine's mean flow as well as the engine's
Figure 1. Regions of Interest in the Stability Plane.
Figure 2. Time Dependence of the Mode-Amplitude Functions at a Stable Limit Cycle.
Figure 3. Time Dependence of the Wall Pressure Waveforms for a Large Amplitude Standing IT Oscillation.
Figure 4. Angular Dependence of the Wall Pressure Waveforms for a Large Amplitude Standing 1T Oscillation.
Figure 5. Dependence of the Limiting Pressure Amplitude Upon Combustion Parameters.
Figure 6. Comparison of the Limiting Pressure Amplitude for Spinning and Standing Type Instabilities.
Figure 7. Dependence of the Frequency upon the Amplitude for Standing Type Instabilities.
Figure 8. Stability Map with Curves of Constant Pressure Amplitude.
Figure 9. Dependence of the Limiting Pressure Amplitude upon the Ratio $\bar{u}_e/z_e$. 

\[ \gamma = 1.2 \]
\[ \delta = 0.06 \]
Figure 10. Effect of Chamber Length upon the Limiting Pressure Amplitude.

\[ \gamma = 1.2 \]
\[ \bar{u}_e = 0.2 \]
\[ \delta = 0.06 \]
length. In these studies the values of n and τ were held fixed and the unstable behavior for various values of \( \bar{U}_e \) (i.e., Mach number at the nozzle entrance) and \( z_e \) (i.e., dimensionless length) was investigated. The results of this study are shown in Figs. (9) and (10). It is seen from Fig. (9) that an increase in \( \bar{U}_e \) usually resulted in an increase in the limiting pressure amplitude; the exception occurring at smaller values of \( \bar{U}_e \). Figure (10) shows that for fixed values of \( n, \tau, \) and \( \bar{U}_e \) increasing the length \( z_e \) resulted in a decrease in the limiting value of the pressure amplitude.

**Third Order Transverse Investigations**

During the second year of this project a third order theory was developed. This theory represents an attempt to relax some of the restrictions imposed on the second order theory. The latter included such restrictions as small Mach number mean flow, irrotationality of the flow, and the presence of moderate amplitude waves. In the third order analysis no terms were neglected in the conservation equations; the only approximations used being those related to the absence of droplet drag and constancy of droplet stagnation enthalpy, both of which were used in the second order theory.

Under the above assumptions the conservation equations can no longer be combined to obtain a single equation governing the behavior of the velocity potential. Instead a system of partial differential equations must be solved. The development of these equations will now be briefly described. Considering only transverse oscillations the velocity components can be defined as follows:

\[
v' = \frac{\partial \eta}{\partial r} \quad ; \quad w' = \frac{1}{r} \frac{\partial \zeta}{\partial \theta}
\]

where \( v' \) and \( w' \) respectively represent the radial and tangential velocity components and \( \eta \) and \( \zeta \) are quasi-potentials. Using Eqs. (4) the appropriate system of conservation equations becomes:
Continuity:

$$\frac{\partial p'}{\partial t} + \left[ \rho'(z) + \rho' \right] \left( \eta_{rr} + \frac{1}{r} \eta_r + \frac{1}{r^2} \zeta_{\theta \theta} \right)$$

$$+ \eta_r p' + \frac{1}{r^2} \zeta_{\theta \theta} p' + u \frac{d \delta}{dz} - \omega'_m = 0$$

(5)

Radial momentum:

$$(\delta + \rho') \left( \eta_{rt} + \eta_r \eta_{rr} + \frac{1}{r^2} \zeta_{\theta \theta} \eta_r - \frac{1}{r^3} \zeta_{\theta \theta}^2 \right) + \frac{1}{r} \frac{d}{dr} \frac{r p'}{r} = 0$$

(6)

Tangential momentum:

$$(\delta + \rho') \left[ \Theta_{\theta t} + \eta_r \zeta_{\theta r} + \frac{1}{r^2} \zeta_{\theta \theta} \zeta_{\theta \theta} \right] + \frac{1}{r} \frac{d}{dr} (r p') = 0$$

(7)

Energy:

$$(\delta + \rho') \left[ \frac{\partial \delta}{\partial t} + \eta_r \frac{\partial \delta}{\partial r} + \frac{1}{r^2} \zeta_{\theta \theta} \frac{\partial \delta_{\theta \theta}}{\partial r} \right] - \frac{v-1}{v} \frac{d}{dr} (r p') \frac{(r \delta)}{r} + \frac{d}{dz} \left( \delta u \right) + \omega' \delta = 0$$

(8)

Equation of State:

$$p' = \delta h'_s + \rho' h'_s + \rho' h'_s - \frac{v-1}{2} \left( \delta + \rho' \right) \left( \frac{\eta_r}{r} + \frac{1}{r^2} \zeta_{\theta \theta}^2 \right) + \omega' u$$

(9)

In the above equations $\rho$ is the density, $p$ is the pressure, and $h_s$ is the stagnation enthalpy.

To complete the theory higher order expressions for the burning rate term and nozzle admittance relation are needed. Unlike the expressions describing the gas dynamics of the problem these expressions
contain terms of all orders and they need to be truncated to include terms up to third order only. In deriving these expressions it was assumed that the combustor’s mean flow Mach number was small. Using Crocco’s time-lag hypothesis the third order burning rate expression is given by the following expression:

\[ \frac{\dot{w}}{n} = n \frac{d}{dz} \left\{ (p' - p'_{\tau}) + \frac{n-1}{2} (p')^2 - n p' p'_{\tau} \right\} 
+ \frac{n+1}{2} (p')^2 - n \frac{\partial p'_{\tau}}{\partial t} \int_{t-\tau}^{t} p'(t') dt' \]  

(10)

where \( p' = p'(r, \theta, t-\tau) \).

Due to the complexity of Eqs. (5) through (9), the nonlinear behavior of a single transverse mode was investigated by approximating each dependent variable as the product of an amplitude function and the spatial dependence of that mode. The approximate solutions are expressed in the following form:

\[ \bar{\xi}' = A_{\bar{\xi}}(t) \psi_{mn}(r, \theta) \]

\[ \bar{\eta}' = A_{\bar{\eta}}(t) \psi_{mn}(r, \theta) \]

\[ \bar{\zeta}' = A_{\bar{\zeta}}(t) \psi_{mn}(r, \theta) \]

\[ p' = A_p(t) \psi_{mn}(r, \theta) \]

\[ h_s' = A_h(t) \psi_{mn}(r, \theta) \]  

(11)
where $\Psi_{mn}(r,\theta) = \cos m \theta J_n(s_{mn} r)$. Introducing the approximate solutions into Eqs. (5) through (9) and applying the mathematical technique described in Ref. 2 yields a system of nonlinear ordinary differential equations to be solved for the unknown $A_i$. A computer program was developed to determine these amplitude functions numerically.

As a check on the analysis, linear stability limits were computed using the linearized version of the system of equations derived for the third order theory. Except for small corrections of the order of $u_e^2$, these limits agreed with those computed from the linearized version of the second order theory.

Using the third order theory numerical solutions were obtained for the following two cases: (1) the approximate solutions consisted of the first tangential mode only, and (2) the approximate solutions consisted of the first radial mode only. The systems of differential equations governing these two cases differed in several respects. The equations governing the behavior of the LR mode contained both quadratic and cubic nonlinearities while those describing the behavior of the LT mode contained only cubically nonlinear terms. The radial mode equations also contained nonlinearities in the combustion mass source terms whereas a nonlinear driving term did not appear in the equations for the LT mode.

The numerical results show that the above-mentioned differences are important. The important characteristics of the LT mode solutions will now be summarized. As seen from Fig. (11) the pressure and velocity waveforms are nearly sinusoidal in shape, a result in contrast to the results of the second order theory. The effect of the combustion parameters $n$ and $\tau$ upon the final amplitude of the pressure oscillation is shown in Figs. (12) and (13). For values of $\tau \leq 2.1$ in the linearly unstable region, stable limit cycles were found as shown in Fig. (12). In each case the final amplitude attained for given values of $n$ and $\tau$ was considerably larger than the amplitude predicted by the second order theory. The possibility of "triggering" combustion oscillations
was found to exist for $\xi > 2.1$. Figure (13) shows the dependence of the triggering amplitude with $n$ and $T$ for standing $1T$ initial disturbances. For given values of $n$ and $T$ an initial disturbance with amplitude slightly less than the threshold value, given by the curve, will decay to zero amplitude. No stable limit cycles could be found in this region. A disturbance with amplitude slightly above the critical value was found to grow without limit. The differences between these third order results and those obtained from the second order theory are attributed to the lack of coupling between modes which results from using a one-mode expansion.

The results obtained using an expansion consisting only of the $1R$ mode will now be summarized. As seen from Fig. (14) the pressure waveforms resemble sinusoids which are shifted up (for radial positions near the axis, $r = 0$) or down (for stations near the wall, $r = 1$). Figure (15) shows the dependence of the limit cycles upon $n$ and $T$. Unlike the $1T$ mode, stable limit cycles for $1R$ instability were found in the vicinity of the neutral stability limit for both linearly stable and linearly unstable values of $n$ and $T$. For an engine operating in the linearly stable region an unstable limit cycle (i.e., triggering limit) was found with an amplitude below that of the stable limit cycle. Also predicted is the minimum value of $n$ (for a given $T$) below which it is impossible to trigger combustion oscillations. The curves shown in Fig. (15) show that as $T$ is increased, combustion oscillations are more easily triggered and the amplitude of the resulting oscillations is higher. It is also seen that the unstable range in $n$, for which triggering of combustion instability is possible, becomes larger with increasing values of the time-lag. These solutions are qualitatively similar to those obtained for the $1R$ mode using the second order theory and a one-mode expansion (see Ref. 1).

In the study of the $1R$ mode, the effect of various nonlinear terms appearing in the governing equations was evaluated. From the results of this study it was concluded that the nonlinearities in the combustion mass source are very important in determining the limiting
Figure 11. Third Order Wall Pressure and Velocity Waveforms for the 1T Mode at an Unstable Limit Cycle.

- Wall Pressure Perturbation, \( p'(1, 0, t) \)
- Wall Velocity Perturbation, \( w'(1, \pi/2, t) \)

\[ n = 0.81050 \]
\[ \tau = 2.5 \]
\[ \nu = 1.2; \ \theta_e = 0.2; \ z_e = 1.0 \]
Figure 12. Third Order Stable Limit Cycles for the First Tangential Mode.
Figure 13. Third Order Triggering Limits for the First Tangential Mode.
Figure 14. Third Order Pressure and Velocity Waveforms for the 1R Mode at a Stable Limit Cycle.
Figure 15. Third Order Stable and Unstable Limit Cycles for the First Radial Mode.
amplitude of triggered LR mode instability. On the other hand the
cubically nonlinear terms originating from the gasdynamics of the
problem have only a minor effect.

Some of the third order solutions predicted the anomalous result
that under certain conditions the combustor's pressure may become nega-
tive. The occurrence of negative pressures in the approximate solutions
is a result of the assumed spatial dependence of the series solutions.
To overcome this shortcoming work is presently in progress to develop
a multi-mode third order theory.

**Nonlinear Axial Mode Instability**

In a separate study, the case of axial type instability in a
combustor with a distributed combustion process is currently being
investigated. Difficulties were encountered in the early stages of
the study, but it now appears that they have been resolved. The work
done to date will now be briefly summarized.

The nonlinear partial differential equation governing longitudi-
nal combustion oscillations, correct to second order, is given by:

\[
\frac{\partial^2 \phi}{\partial z^2} - \frac{\partial^2 \phi}{\partial t^2} - 2 \frac{\partial \phi}{\partial z \cdot \frac{\partial \phi}{\partial t} - (\gamma - 1) \frac{\partial \phi}{\partial z} \frac{\partial \phi}{\partial z} - 2 \frac{\partial \phi}{\partial z} \frac{\partial \phi}{\partial t} - \gamma \frac{\partial U}{\partial z} \frac{\partial \phi}{\partial t} - \bar{W} = 0
\]  (12)

In Eq. (12) the mass source term is again described by Crocco's time-
lag hypothesis to give:

\[
\bar{W} = - \gamma n \frac{dU}{dz} \left[ \frac{\partial \phi}{\partial t}(z,t) - \frac{\partial \phi}{\partial t}(z,t-T) \right]
\]  (13)

The appropriate boundary conditions arise from the presence of a solid
wall boundary condition at the injector face \(z = 0\) and a quasi-steady
nozzle at \(z = 1\). The boundary conditions are given by:
In order to obtain an approximate solution of Eq. (12) a series expansion must be specified for \( \phi(z,t) \); that is, 
\[
\phi(t) = \sum_{k=0}^{N} B_k(t)\varphi_k(z).
\]
The initial phases of this investigation have been primarily concerned with the proper selection of the approximating functions \( \varphi_k(z) \).

If the trial functions \( \varphi_k \) do not satisfy the boundary conditions, then the resulting error at the boundary condition must be minimized, in some sense, in combination with the error arising from the fact that the trial function does not, in general, satisfy the differential equation. This is accomplished by imposing the following restriction (see Ref. 2):

\[
\int_V R_E \varphi_dV - \int_S R_B \varphi_dS = 0 \quad (16)
\]

where the residuals \( R_E \) and \( R_B \) are the errors incurred by substituting the approximate solutions into the differential equations and boundary conditions respectively. If the trial functions satisfy the boundary conditions then the residual \( R_B \) vanishes. Both approaches are currently under study, although most of the work done to date has been with trial solutions that do not satisfy the boundary conditions.

Obviously, a proper choice of the trial functions is a prerequisite for obtaining valid results. The first trial series for \( \phi(z,t) \) was taken to be composed of the following acoustic modes:

\[
\phi(z,t) = \sum_{k=0}^{N} \left[ A_k(t)\sin(k\pi z) + B_k(t)\cos(k\pi z) \right] \quad (17)
\]
This expansion does not satisfy either the injector or the nozzle boundary conditions. Using Eq. (16) yields a set of coupled nonlinear ordinary differential equations to be solved for the $A_k's$ and $B_k's$. Numerical calculations proved this trial series to be divergent for all the investigated cases (i.e., all investigated values of $n$, $\pi$).

Considerations of the expected physical behavior of the resulting pressure oscillations suggested the omission of the sine terms from the expansion given by Eq. (17), hence:

$$\Psi(z,t) = \sum_{k=0}^{N} B_k(t) \cos(k\pi z)$$

Preliminary results using this cosine series are favorable. The time dependent coefficient, $B_0(t)$ corresponding to the spatially uniform term in the series (i.e., $\varphi_0 = 1$) was found to oscillate with a lower frequency than the remaining terms, but with a larger amplitude. This term corresponds to chugging-type instability. Results with and without this term present in the series expansion will be obtained; a comparison of these results with available experimental data will determine whether $B_0(t)$ should be included in the approximate series solution.

The second approach taken in selecting an approximate solution was to construct expressions that satisfy both the boundary condition at the injector face (i.e., $z = 0$) and that at the nozzle entrance (i.e., $z = 1$). Two methods are presently under consideration. One such method is to select an approximate solution of the form:

$$\Psi(z,t) = A(t)F(z) + G(z) \sum_{k=0}^{N} B_k(t) \cos(k\pi z)$$

where the summation term satisfies homogeneous boundary conditions and the remaining term $A(t)F(z)$ satisfies both boundary conditions. For example, one such solution was found to be:
\[ \psi(z, t) = z^2 \exp\left[-\frac{2t}{\beta}\right] + z^2(z - 1) \sum_{k=0}^{N} B_k(t) \cos(k\pi z) \quad (20) \]

where \[ \beta = \frac{\sqrt{2}}{2} \bar{u}_e. \] It should be noted that the first term is necessary to satisfy the nozzle admittance condition, but its effect on \( \psi(z, t) \) decreases with time.

The second method of satisfying the boundary conditions is to use a cosine series with time dependent "eigenvalues" as follows:

\[ \psi(z, t) = \sum_{k=0}^{N} B_k(t) \cos\left(\frac{k\pi z + \epsilon_k(t)}{z}\right) \quad (21) \]

The above expression satisfies the boundary condition at \( z = 0 \). Applying the boundary condition at \( z = 1 \) (i.e., Eq. (15)) yields the following relation between \( B_k(t) \) and \( \epsilon_k(t) \):

\[ \left[ k\pi + \epsilon_k(t) + \beta \frac{d\epsilon_k(t)}{dt}\right] B_k(t) \tan\epsilon_k(t) - \beta \frac{dB_k(t)}{dt} = 0 \quad (22) \]

Combining Eqs. (22) with the equations resulting from applying Eq. (16) results in a system of \( 2(N+1) \) nonlinear ordinary differential equations for the \( 2(N+1) \) unknowns, \( \epsilon_k(t) \) and \( B_k(t) \).

At present this investigation is at a preliminary stage, hence no numerical results are available. Computer programs are being developed based upon the three approaches discussed above. These will be used to determine both the linear and nonlinear stability limits in the \( (n, \tau) \) plane and to study the possibility of triggering axial combustion oscillations.

**Combustion Response Studies**

With a view in mind to determine a more realistic combustion
response function for incorporation into the Galerkin method two
detailed studies were carried out concerning combustion processes.
The first study concerned the linear acoustic response of the vaporiza-
tion process in the vicinity of the stagnation point; the major
difference from previous treatments was the assumption of no internal
liquid circulation and the consequent existence of a thermal wave in
the liquid in the steady state. The reasons for doing this study were
that the thermal wave assumption had never been used before and it is
precisely the thermal wave which is responsible for combustion response
peaks in the appropriate frequency range in solid propellant response
theory. A sample frequency response plot for two fuels is shown in
Fig. (16). To draw this plot three items concerning the acoustics must
be specified - the mode type, the position of the droplet in the chamber
and the relative velocity direction between the chamber gases and the
droplet. While Fig. (16) is a representative average for a longitudinal
mode, significant differences can appear for transverse modes. The
major item, however, is that the frequency response is fairly flat even
though there is mild variation in the frequency range of interest to
instability. This relatively flat behavior is in accord with treatments
of the vaporization process using assumptions different than those used
in the present study.

A second study concerned the effects of combustion process
velocity sensitivity on stability. There has been a general lack of
recognition of the velocity effect's importance and it is believed
that a reconciliation between two presently accepted but different
instability theories can be affected by more detailed understanding of
the velocity effect. Figure (17) demonstrates that a combustion rate
instantaneously proportional to velocity raised to some power \( n \) can
cause longitudinal instability; \( \Psi \) is the location of a concentrated
combustion front expressed as a fraction of the chamber length. While
the required \( n \) value is high (> 1) the demonstration is that velocity
sensitivity is at least an important contribution to stability criteria.

The outcome of the above studies has been the recommendation of
Figure 16. Frequency Response of Vaporization Process for Two Fuels.
Figure 17. Stability Boundaries and Acoustic Frequencies for Velocity Effect Instability.
a new response function for incorporation into the Galerkin method. This function consists of the sum of two terms - one describing the variable vaporization rate, which can include velocity effects, and one involving the variable combustion rate. It is shown in Ref. 3 that this formulation removes a rather critical assumption in the Crocco time lag theory.

**Future Investigations**

Future research efforts will be devoted to the following investigations: (1) determination of nonlinear stability limits using the second order theory, (2) determination of the characteristics of large amplitude transverse wave instability, (3) determination of the characteristics of nonlinear axial instability, and (4) investigation of the unsteady combustion response functions.

The second order computer programs that were developed to date are now being used to determine stability maps (similar to the one presented in Fig. 8) for rocket combustors characterized by various values of the Mach number and different length-to-diameter ratios. The possibility of using such maps together with experimental data, to determine the operating point (i.e., values of $n$ and $\tau$) of actual rocket motors will be explored. Additional investigations concerning the effect of initial conditions and the convergence of the assumed series expansions will be conducted. A report of the second order results, which includes a fully documented computer program, will be prepared in the near future.

The behavior of large amplitude transverse instability will be studied using a third order multi-mode theory which is currently under development. Results similar to those obtained with the aid of the second order theory will be obtained, and a comparison will be made with the second order results to determine the range of applicability of the second order theory.
Present efforts to determine the best analytical approach for the solution of the axial instability problem are expected to be completed shortly. Upon completion of this investigation the chosen method will be used to determine the nonlinear stability characteristics of various rocket combustors.

A nonlinear response function based on instantaneous high Reynolds number vaporization response will be incorporated into the second order theory. One of the objectives of this study is to compare triggering limits obtained by using the Galerkin method with the results obtained in Ref. (5). These results will also be compared with the previous second order results in order to determine differences in mode-excitation, magnitude of triggering amplitude, and limit cycle amplitude.
REFERENCES


THE PREDICTION OF NONLINEAR THREE
DIMENSIONAL COMBUSTION INSTABILITY IN LIQUID
ROCKETS WITH CONVENTIONAL NOZZLES

by

E. A. Powell and B. E. Zinn

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ABSTRACT

An analytical technique is developed to solve nonlinear three-dimensional, transverse and axial combustion instability problems associated with liquid-propellant rocket motors. The Method of Weighted Residuals is used to determine the nonlinear stability characteristics of a cylindrical combustor with uniform injection of propellants at one end and a conventional DeLaval nozzle at the other end. Crocco's pressure sensitive time-lag model is used to describe the unsteady combustion process. The developed model predicts the transient behavior and nonlinear wave shapes as well as limit-cycle amplitudes and frequencies typical of unstable motor operation. The limit-cycle amplitude increases with increasing sensitivity of the combustion process to pressure oscillations. For transverse instabilities, calculated pressure waveforms exhibit sharp peaks and shallow minima, and the frequency of oscillation is within a few percent of the pure acoustic mode frequency. For axial instabilities, the theory predicts a steep-fronted wave moving back and forth along the combustor.
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SUMMARY

An approximate analytical technique has been developed for the solution of nonlinear three-dimensional, transverse and axial combustion instability problems that are frequently observed in liquid-propellant rocket motors. This theory is an extension and generalization of previous analyses, which could analyze either transverse or axial instabilities in liquid combustors with quasi-steady nozzles, to the practical situations of three-dimensional instabilities in combustors with conventional DeLaval nozzles. Unlike the quasi-steady nozzle, the presence of a conventional nozzle imposes restrictions upon the behavior of both the amplitudes and phases of the oscillations at the nozzle entrance plane. The Method of Weighted Residuals is used to determine the nonlinear stability characteristics of a cylindrical combustor with uniform injection of propellants at one end and a conventional nozzle at the other end. Crocco's pressure sensitive time-lag model is used to describe the unsteady combustion process. The developed model can predict the transient behavior and nonlinear wave shapes as well as limit-cycle amplitudes and frequencies typical of unstable motor operation. These results establish the relationship that exists between the resulting instability (i.e., waveform, final amplitude and final frequency), the combustion parameters (i.e., interaction index, n, and time-lag, \( \tau \)), and the chamber Mach number and length-to-diameter ratio. Results indicate that the limit-cycle amplitude increases with increasing sensitivity of the combustion process to pressure oscillations. For transverse instabilities, calculated pressure waveforms exhibit sharp peaks and shallow minima, and the frequency of oscillation is always within a few percent of the frequency of one of the chamber's acoustic modes. For axial instabilities, the theory predicts the presence of a steep-fronted wave moving back and forth along the combustor. In both cases calculations of pressure and velocity perturbations at the nozzle entrance plane show that the approximation to the nozzle boundary condition is very good. The theory described in this report represents the final stage in the development of a unified nonlinear theory for the solution of general three-dimensional, transverse and axial combustion instability problems.
INTRODUCTION

Observation of the behavior of unstable rocket motors indicates that combustion instability can be divided into two categories; that is, linear and nonlinear instabilities. Linear instabilities are spontaneous in nature, and they are usually an outgrowth of the random combustion and flow fluctuations present in the system. On the other hand, nonlinearly unstable motors require the introduction of a finite amplitude disturbance to produce (or trigger) combustion instability. In either case the instability, after a transient period, reaches a limiting maximum amplitude (i.e., limit-cycle amplitude) at which it oscillates with a given frequency that is usually close to the frequency of one of the chamber's acoustic modes. Pressure measurements taken during test firings of unstable motors indicate that the limit-cycle waveforms of transverse instabilities are non-sinusoidal; that is, they exhibit sharp peaks and flattened minima. On the other hand, experimental observations of axial instabilities indicate the presence of shock-like steep-fronted waves in the chamber. These results indicate that nonlinearities need to be considered in the theoretical treatment of combustion instability.

Any analytical treatment of combustion instability should be capable of solving nonlinear multi-dimensional combustion instability problems without exceeding memory core limitations of current computers and without requiring excessive computation time. To be of practical use, such a solution technique should be conceptually simple and easily adaptable for use by industry. This report describes the development and use of such a numerical solution technique.

Work on this problem has been in progress during the past several years and due to its complexity, the problem had to be tackled in stages. In earlier investigations by these authors theories describing the nonlinear behavior of longitudinal and transverse instabilities in liquid combustors with quasi-steady nozzles were developed. These theories, which were based upon the application of the Method of Weighted Residuals (MWR), successfully
predicted the transient behavior, nonlinear waveforms, and limit-cycle amplitudes of longitudinal and transverse instabilities in unstable liquid rockets. This report is concerned with the development of a generalized nonlinear theory that will be capable of analyzing three-dimensional, transverse and axial instabilities in the more practical situations where the combustors are attached to conventional nozzles. Obviously, this generalized theory will encompass the above-mentioned investigations as special cases. Contrary to the quasi-steady nozzle case, the presence of a conventional nozzle imposes both amplitude and phase boundary conditions that must be satisfied by the solutions of the problem at the nozzle entrance plane. The generalized theory presented herein also provides a better description of the unsteady flow field in the vicinity of the nozzle entrance plane.

The application of the theory presented herein will be demonstrated by considering the nonlinear stability of a liquid-propellant rocket combustor with uniform injection of propellants at one end and a conventional nozzle at the other end. Crocco's pressure sensitive time lag model$^7$ is used to describe the unsteady combustion process. In the sections to follow, the development of the wave equation for the analysis of nonlinear combustion instability in liquid rockets will be briefly described, the solution of this nonlinear wave equation will be outlined, and typical results will be presented and discussed. User's Manuals and program listings for the computer programs used to solve these problems are included as appendices to this report.

**SYMBOLS**

\[ A_{lmn}(t), B_{lmn}(t) \] time-dependent amplitudes in series given by Eq. (6)

\[ A_p(t) \] time-dependent amplitudes in series given by Eq. (9)

\[ B(\xi) \] boundary residual

\[ b_{lmn} \] complex axial acoustic eigenvalue

\[ c^* \] velocity of sound, ft/sec
<table>
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<td>$C_0, C_1, C_2, C_3$</td>
<td>coefficients of linear terms in Eqs. (12)</td>
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<tr>
<td>$D_1, D_2, D_3, D_4$</td>
<td>coefficients of nonlinear terms in Eqs. (12)</td>
</tr>
<tr>
<td>$E(Y)$</td>
<td>residual of Eq. (10)</td>
</tr>
<tr>
<td>$i$</td>
<td>imaginary unit, $\sqrt{-1}$</td>
</tr>
<tr>
<td>$J_m$</td>
<td>Bessel function of the first kind, order $m$</td>
</tr>
<tr>
<td>$\ell, m$</td>
<td>axial and tangential mode numbers, respectively</td>
</tr>
<tr>
<td>$n$</td>
<td>pressure interaction index</td>
</tr>
<tr>
<td>$p$</td>
<td>dimensionless pressure, $\gamma_P/\rho_0/c_0^2$</td>
</tr>
<tr>
<td>$r$</td>
<td>dimensionless radial coordinate, $r^<em>/R_c^</em>$</td>
</tr>
<tr>
<td>$R_c^*$</td>
<td>chamber radius, ft</td>
</tr>
<tr>
<td>$R_p(r)$</td>
<td>radial acoustic eigenfunction in Eq. (9)</td>
</tr>
<tr>
<td>$S_{mn}$</td>
<td>dimensionless transverse mode frequency</td>
</tr>
<tr>
<td>$t^*$</td>
<td>dimensionless time, $t^<em>/(R_c^</em>/c_0^*)$</td>
</tr>
<tr>
<td>$u^*$</td>
<td>dimensionless axial velocity, $u^<em>/c_0^</em>$</td>
</tr>
<tr>
<td>$Y^*$</td>
<td>dimensionless velocity vector, $Y^<em>/c_0^</em>$</td>
</tr>
<tr>
<td>$W'_m$</td>
<td>unsteady combustion mass source</td>
</tr>
<tr>
<td>$Y$</td>
<td>complex nozzle admittance</td>
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z  
- dimensionless axial coordinate, $z^*/R_c$

$Z_{lmn}(z), Z_p(z)$  
- axial acoustic eigenfunctions

γ  
- ratio of specific heats

ε  
- ordering parameter

θ  
- azimuthal coordinate

$\Theta_p(\theta)$  
- tangential acoustic eigenfunction in Eq. (9)

ρ  
- dimensionless density, $\rho^*/\rho_o$

τ  
- dimensionless pressure sensitive time lag, $\tau^* = \frac{T^*}{(R^*/c_s^*)}$

φ  
- velocity potential

Subscripts:

e  
- evaluated at the nozzle entrance

n  
- radial mode number

r, t, z, θ  
- partial differentiation with respect to r, t, z, or θ respectively

r, i  
- real and imaginary parts of a complex quantity, respectively

o  
- stagnation quantity

Superscripts:

′  
- perturbation quantity, differentiation with respect to argument
ANALYSIS

Development of the Wave Equation

To keep the problem as simple as possible, yet still physically meaningful, the following assumptions are made. The gas phase in the combustor is assumed to consist of a single constituent which is thermally and calorically perfect. Transport phenomena, such as diffusion, viscosity, and heat conduction are neglected. The momentum interchange between the liquid and gas phases is neglected (see Appendix A for a discussion of this assumption), and the specific stagnation enthalpy of the unburned propellant is assumed constant throughout the chamber. The presence of burning propellant drops is represented by a distribution of unsteady mass sources and it is also assumed that the Mach number of the combustor's mean flow is small and that the waves have moderate amplitudes.

As a result of the last two assumptions, the governing conservation equations may be combined and the unsteady flow in the combustor can be described by a single nonlinear wave equation. The derivation of this equation appears in Refs. 8 and 9, where it was assumed that each perturbation quantity and the mean flow Mach number were of $O(\varepsilon)$, where $\varepsilon$ is an ordering parameter that is a measure of the wave amplitude. After neglecting all terms of $O(\varepsilon^3)$ or higher and combining equations, one obtains the following nonlinear partial differential equation that describes the behavior of the velocity potential, $\tilde{v}$, of the combustor disturbance:

$$\nabla^2 \tilde{v} - \tilde{v}_{tt} = 2V \cdot \nabla \tilde{v} + \gamma (\nabla \cdot \nabla \tilde{v}) \tilde{v}_t + 2 \nabla \cdot \nabla \tilde{v}_t + (\gamma - 1) \tilde{v}_t \nabla^2 \tilde{v} + W_m' \quad (1)$$

Equation (1) is the desired wave equation, and it is similar to the inhomogeneous wave equation solved by Maslen and Moore in a related study on nonlinear acoustics. This equation accounts for the following effects: (1) th
effect of a steady state flow on the wave motion (viz., the first two terms on the right-hand side), (2) the coupling between the gas dynamical oscillations and the unsteady combustion process (viz., the last term on the right-hand side), and (3) the second order nonlinearities of the gas dynamical processes (viz., the third and fourth terms on the right-hand side).

In addition to satisfying Eq. (1), the desired solutions must satisfy rigid wall boundary conditions at the injector end of the chamber and at the chamber walls, while a nozzle admittance condition must be satisfied at the nozzle entrance. These boundary conditions are given (in a cylindrical coordinate system) by:

\[
\begin{align*}
\phi_r &= 0 \text{ at } r = 1 \\
\phi_z &= 0 \text{ at } z = 0 \\
B(\phi) &= \phi_z + \gamma Y_t = 0 \text{ at } z = z_e
\end{align*}
\]  

(2)

The nozzle admittance, \( Y \), is a complex number defined by

\[
Y = Y_r + iY_i = (u'/p')_{z = z_e}
\]  

(3)

where \( u' \) is the dimensionless axial velocity perturbation and \( p' \) is the dimensionless pressure perturbation.

It should be pointed out that due to the absence of an appropriate nonlinear nozzle admittance boundary condition, the solutions of the problem are required to satisfy a linear nozzle admittance. Although inconsistent with the nonlinear wave equation, the linear nozzle admittance condition is used herein with the hope that the solution techniques developed herein will also be applicable when nonlinear nozzle admittance conditions become available. Also, the relative importance of nozzle nonlinearities is not known at the moment and it is quite possible that the linear nozzle boundary condition used herein adequately describes the flow conditions at the nozzle entrance.

The unsteady combustion process is represented by mass sources distributed throughout the volume of the chamber, and the response of the mass sources to pressure oscillations is assumed to be described by Crocco's pressure sensitive time-lag hypothesis. 7 The mass source perturbation, \( W'_m \), is then given by: 5,8
where n is the pressure "interaction index" that describes the sensitivity of the combustion process to pressure oscillations, and \( \tau \), commonly referred to as the sensitive time-lag, is the part of the total combustion time-lag during which the combustion process is sensitive to pressure oscillations. The unsteady combustion response described by Eq. (4) is linear and the comments made above regarding the use of a linear nozzle admittance boundary condition are also applicable to this case.

Substituting Eq. (4) into Eq. (1) and expressing the resulting equation in a cylindrical coordinate system yields the following wave equation:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial z^2} - \frac{\partial^2 \Phi}{\partial t^2} = -\frac{n}{m} \frac{\partial u}{\partial z} \left( \Phi_t (r, \theta, z, t) - \Phi_t (r, \theta, z, t - \tau) \right)
\]

(5)

The combustor and nozzle geometries considered in this study, as well as the cylindrical coordinate system used in writing Eq. (5), are shown in Fig. 1.

**Method of Solution**

Since Eq. (5) has no known closed-form mathematical solution, it is necessary to resort to the use of either exact numerical solution techniques or approximate analytical techniques. For multi-dimensional problems, the exact numerical solution techniques generally exceed the computer storage capacities, therefore an approximate solution technique is used herein.

The experience of previous investigators in the fields of structural stability and aeroelasticity indicates that an approximate solution technique known as the Method of Weighted Residuals\(^{(11, 12)}\) may be effective in the solution of this nonlinear wave equation.

In order to employ the Method of Weighted Residuals in the solution on Eq. (5), it is first necessary to express the velocity potential, \( \Phi \), as an
Figure 1. Combustor Configuration and Coordinate System.
approximating series expansion, $\psi$. The question naturally arises as to what form of series expansion should be used. Inasmuch as the experimentally observed pressure oscillations during combustion instability usually resemble the natural acoustic modes of the chamber, the velocity potential, $\psi$, is expanded in terms of the natural acoustic modes of the chamber with unknown time-dependent amplitudes.

In previous analyses of related problems the approximate solutions were expressed in terms of the acoustic modes for a cylindrical chamber with solid wall boundary conditions at both the injector and the nozzle ends. Consequently, the approximation of the flow conditions at the nozzle entrance was poor. In the present analysis a better approximation to the flow at the nozzle entrance is obtained by expanding the velocity potential in terms of the acoustic eigenfunctions for a chamber with a solid wall boundary condition at the injector end and a nozzle admittance condition at the other end. This removes both the two-dimensionality and the quasi-steady nozzle restrictions imposed upon the previous investigations.

The velocity potential, $\psi$, is therefore approximated by the following series expansion:

$$\psi = \sum_{l} \sum_{m} \sum_{n} \left\{ A_{lmn}(t) \sin m\theta + B_{lmn}(t) \cos m\theta \right\} Z_{lmn}(z) J_m(S_{mn} r)$$

(6)

where the $A$'s and $B$'s are unknown complex functions of time, and the $Z$'s are the complex axial acoustic eigenfunctions. The complex form of the axial acoustic eigenfunctions is given by

$$Z_{lmn}(z) = \cosh(ib_{lmn} z)$$

(7)

where the $b_{lmn}$ are the axial acoustic eigenvalues which must satisfy the following transcendental equation:

$$b_{lmn}^2 \sin^2(b_{lmn} z_e) + \gamma^2 Y^2(S_{mn}^2 + b_{lmn}^2) \cos^2(b_{lmn} z_e) = 0$$

(8)

Equations (7) and (8) are obtained by linearizing Eq. (5) and solving the resulting equation for the case of no mean flow or combustion (i.e., the acoustic case) subject to the boundary conditions specified in Eq. (2). Each term in the above expansion exactly satisfies the solid wall boundary conditions at the injector end (i.e., at $z = 0$) and at the chamber wall (i.e., at
However, due to the unknown time dependence of Eq. (6) the nozzle admittance condition imposed at $z = z_e$ is not exactly satisfied by the individual terms. Including both the $\sin m\theta$ and $\cos m\theta$ terms in the expansion for $\Psi$ allows for the possibility of either spinning or standing wave solutions.

In order to simplify the algebra involved in the application of the Method of Weighted Residuals, the development of the associated computer program, and the presentation of the results; the expansion of the velocity potential is written as a single summation as follows:

$$\tilde{\Psi} = \sum_{p=1}^{N} A_p(t)Z_p(z)\Theta_p(\theta)R_p(r)$$  

(9)

where the $A_p$'s are the unknown time-dependent amplitudes. In order to use Eq. (9) a correspondence must be established between the index, $p$, in Eq. (9) and the mode-numbers $l$, $m$, and $n$ in Eq. (6). Such a correspondence is given in Table 1 for a three mode series consisting of the spinning first tangential ($1T$) mode ($l = 0$, $m = 1$, $n = 1$), the spinning second tangential ($2T$) mode ($l = 0$, $m = 2$, $n = 1$), and the first radial ($1R$) mode ($l = 0$, $m = 0$, $n = 1$).

<table>
<thead>
<tr>
<th>$p$</th>
<th>Mode</th>
<th>$l(p)$</th>
<th>$m(p)$</th>
<th>$n(p)$</th>
<th>$A_p$</th>
<th>$\Theta_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1T$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$A_{011}(t)$</td>
<td>$\sin \theta$</td>
</tr>
<tr>
<td>2</td>
<td>$1T$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$B_{011}(t)$</td>
<td>$\cos \theta$</td>
</tr>
<tr>
<td>3</td>
<td>$2T$</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>$A_{021}(t)$</td>
<td>$\sin 2\theta$</td>
</tr>
<tr>
<td>4</td>
<td>$2T$</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>$B_{021}(t)$</td>
<td>$\cos 2\theta$</td>
</tr>
<tr>
<td>5</td>
<td>$1R$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$B_{001}(t)$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1
Correspondence Between Eq. (6) and (9) for a Three-Mode Series

Before proceeding with the analysis, the wave equation (i.e., Eq. (1)) must be modified for use with the assumed complex solution given by Equation (9). This modification is necessary because only the real part of the assumed solution is physically meaningful. It can easily be shown that if $\tilde{\Psi} = \varphi + i\Psi$ is a solution to Eq. (1), the real part, $\varphi$, is not a solution to Eq. (1).
This failure of $\varphi$ to satisfy Eq. (1) is due to the presence of the nonlinear terms in this equation. It can also be shown, however, that a modified wave equation can be constructed for which the real part of its solution satisfies the original wave equation (i.e., Eq. (1)). This modified wave equation is given by:

$$
\begin{align*}
E(\varphi) &= \nabla^2 \varphi - \varphi_{tt} - 2V \cdot \nabla \varphi_t - \gamma(\nabla \cdot \nabla) \varphi_t - W_m \\
&\quad - \frac{1 - i}{2} \left[ \nabla \varphi \cdot \nabla \varphi_t + \nabla \varphi^* \nabla \varphi^*_t \right] - \frac{1 + i}{2} \left[ \nabla \varphi^* \cdot \nabla \varphi^*_t + \nabla \varphi^* \cdot \nabla \varphi^*_t \right] \\
&\quad - \frac{\gamma - 1}{4} \left\{ (1 - i) \left[ \varphi_t \nabla^2 \varphi + \varphi^*_t \nabla^2 \varphi^* \right] \
&\quad + (1 + i) \left[ \nabla \varphi^* \nabla \varphi^*_t + \nabla \varphi^* \nabla \varphi^*_t \right] \right\} = 0
\end{align*}
$$

(10)

where $\varphi^*$ is the complex conjugate of $\varphi$. The derivation of this equation is discussed in Appendix B. Thus, the Method of Weighted Residuals will be used to obtain approximate solutions to Eq. (10) (i.e., $\tilde{\varphi} = \varphi + i\psi$) from which the real part, $\tilde{\varphi}$, will be taken as the approximate solution of Eq. (1).

In order to obtain a solution, the unknown time-dependent mode-amplitude (i.e., $A_j(t)$) are determined by the following mathematical procedure. The assumed series expansion, $\varphi$, (i.e., Eq. (9)) is substituted into the wave equation (i.e., Eq. (10)) to form the equation residual, $E(\varphi)$. Similarly, substituting the series expansion into the nozzle boundary condition (i.e., the last of Eq. (2)) yields the boundary residual, $B(\varphi)$. In the event that these residuals are both identically zero, the solution is an exact solution. The residuals $E(\tilde{\varphi})$ and $B(\tilde{\varphi})$ represent the errors incurred by using the approximate solution, $\tilde{\varphi}$.

According to the modified version of the Method of Weighted Residuals, developed by the authors in Refs. 5 and 8, the residuals $E(\tilde{\varphi})$ and $B(\tilde{\varphi})$ must satisfy the following orthogonality conditions:

$$
\begin{align*}
\int_0^{2\pi} \int_0^1 \int_0^{2\pi} E(\varphi)Z_j^*(z)\Theta_j(\theta)R_j(r)rdrd\theta dz &= 0 \\
\int_0^{2\pi} \int_0^1 B(\varphi)Z_j^*(z_e)\Theta_j(\theta)R_j(r)rdrd\theta &= 0
\end{align*}
$$

(11)

$j = 1, 2, \ldots N$
where in the present study the complex conjugate of the axial eigenfunction, \( Z_j^* \), is used in the weighting functions. The chosen weighting functions must correspond to the terms that appear in the assumed series solution; that is, Eq. (9).

Evaluating the spatial integrals in Eq. (11) yields the following system of \( N \) complex ordinary differential equations to be solved for the unknown complex amplitude functions, \( A_p(t) \):

\[
\sum_{p=1}^{N} \left\{ C_0(j,p) \frac{d^2 A_p}{dt^2} + C_1(j,p) A_p(t) + \left[ C_2(j,p) - nC_3(j,p) \right] \frac{dA_p}{dt} \right\} \\
+ nC_3(j,p) \frac{d[A_p(t - \bar{T})]}{dt} \right\} + \sum_{p=1}^{N} \sum_{q=1}^{N} \left\{ D_1(j,p,q) A_p \frac{dA_q}{dt} + D_2(j,p,q) A_p \frac{dA_q^*}{dt} \right\} = 0 \]

\( j = 1, 2, \ldots, N \) (12)

The coefficients appearing in the above equations are determined by evaluating the various integrals of hyperbolic, trigonometric, and Bessel functions that arise from the spatial integrations indicated in Eq. (11). A user's manual for the computer program COEFFS3D used to calculate these coefficients is given in Appendix C.

The time-dependent behavior of an engine following the introduction of a disturbance is determined by specifying the form of the initial disturbance and then following the subsequent behavior of the individual modes by numerically integrating Eqs. (12). Once the time-dependence of the individual modes is known, the velocity potential, \( \tilde{\psi} \), is calculated from Eq. (9). The pressure perturbation at any location within the chamber is related to the real part of \( \tilde{\psi} \) (i.e., \( \tilde{\varphi} \)) by the following second-order momentum equation (see Refs. 5 and 8):

\[
\tilde{p}' = -\gamma \left[ \tilde{\varphi}_t + \tilde{u}(z) \tilde{\varphi}_z + \frac{1}{r^2} \left( \tilde{\varphi}_r^2 + \frac{1}{r^2} \tilde{\varphi}_\theta^2 + \tilde{\varphi}_z^2 \right) - \frac{1}{r^2} \tilde{\varphi}_t^2 \right] \]

(13)
A user's manual for the computer program, LCYC3D, which obtains numerical solutions of Eqs. (12) and (13) is given in Appendix D.

In summary, the theory presented in this section represents a two-stag simplification of the original problem. In the first stage the problem has been reduced to the solution of a single nonlinear, partial differential equation (i.e., Eq. (1)). In the second stage the solution was expanded in a series of acoustic modes with time-dependent coefficients and the Method of Weighted Residuals was used to replace the solution of the nonlinear partial differential equation with the solution of a system of nonlinear, ordinary differential equations (i.e., Eq. (12)). Typical numerical solutions of these equations will be presented and discussed in the following section.

RESULTS AND DISCUSSION

The generalized three-dimensional theory introduced in the previous section has been used to obtain both linear and nonlinear data for pure transverse modes and pure longitudinal modes for rocket motors with conventional nozzles. Nonlinear data for the first tangential (1T) mode and the first longitudinal (1L) mode has also been obtained for combustors with quasi-steady nozzles for comparison with the results of the previous two-dimensional theory.

Linear Solutions

Before proceeding with the nonlinear analysis, it was desired to obtain numerical solutions of the linearized equations (i.e., Eqs. (12) with \( D_1 = D_2 = D_3 = D_4 = 0 \)) in order to determine how closely the approximate solutions satisfied the nozzle boundary condition. The linear solution is also needed for comparison with the corresponding nonlinear results. The linear solutions were obtained by assuming a one-mode series expansion consisting only of the mode under consideration. Due to the presence of the retarded variables (i.e., \( d[A_p(t - \tilde{T})]/dt \) in Eqs. (12)), it is necessary to specify the initial amplitudes over the interval \(-\tilde{T} \leq t \leq 0\). In this study the initial values were chosen such that the nozzle boundary condition was exactly satisfied during this initial time period. Solutions were obtained for values of \( n \) and \( \tilde{T} \) on the neutral stability limit (see Appendix E for the determination of neutral stability limits) for various conventional nozzle configurations. The nozzle admittance was expressed in the form, \( Y = Ae^{i\phi} \), where \( A \) is the amplitude fac
tor and $\phi$ is the phase shift. The pressure perturbation, $p'$, and the axial velocity perturbation, $u'$, at the nozzle entrance were calculated numerically for several values of the nozzle phase shift, $\phi$. These calculated values were then used to compute the ratios $(u'\mid p')_{z=e}$, which were then compared with the specified nozzle admittance values. These results are shown in Tables (2) and (3) where $A_n$ and $\phi_n$ are the computed values of the amplitude factor and phase shift, respectively. These results show that the approximation to the nozzle boundary condition is very good for both the $\text{IT}$ and $\text{IL}$ modes; that is, the maximum error in the amplitude ratio is about 5% and the maximum error in phase is approximately 0.5 degree. These results are in contrast with previous theoretical investigations where the representation of the unsteady flow conditions in the vicinity of the nozzle entrance was very poor.

Table 2. IT Mode Linear Solutions (Numerical).

<table>
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<th>$\phi$ (Degrees)</th>
<th>$\tau$</th>
<th>$n$</th>
<th>$A_n - A$</th>
<th>$\phi_n - \phi$ (Degrees)</th>
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<td>0.004</td>
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Table 3. LL Mode Linear Solutions (Numerical).

<table>
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<th>( \phi ) (Degrees)</th>
<th>( \tau )</th>
<th>( n )</th>
<th>( A_n - A )</th>
<th>( \phi_n - \phi ) (Degrees)</th>
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</table>
Nonlinear Solutions

Nonlinear solutions have been computed for both the LT mode and the LL mode. For the LT mode calculations a three mode series expansion consisting of the LT, 2T (second tangential), and LR (first radial) modes was used. These are the same modes that were included in the series expansion used in the previous two-dimensional transverse instability studies. In these studies it was shown that convergence was obtained with this three mode series; that is, the addition of higher transverse modes (i.e., 3T, 4T, etc.) to the basic series had little effect on the solution. The LL mode computations were made using a series consisting of the first five longitudinal modes (i.e., 1L, 2L, 3L, 4L, and 5L). It has been shown by Lores and Zinn that convergence is obtained with this five-mode series.

Transverse Mode Solutions. Nonlinear solutions have been computed for rocket combustors with quasi-steady nozzles (i.e., real admittances) and also for nozzles with complex admittances. The quasi-steady nozzle solutions were generated for comparison with the results of the previous two-dimensional theory. For this case the nozzle admittance is given by:

$$ Y = \frac{1}{2y} v_e $$

For nozzles with complex admittances the admittance was expressed in the form,

$$ Y = \alpha e^{i\theta} $$

For both cases limit-cycle amplitudes and waveforms have been computed for both standing and spinning first tangential instability. This required three series terms to describe standing instability and five series terms to describe spinning instability. Typical computation times on a Univac 1108 computer to reach a limit-cycle were one minute for a standing wave and two minutes for a spinning wave.

Wall pressure waveforms (r = 1) were computed at the injector face (z = 0) and at the nozzle entrance (z = z_e) for three azimuthal locations, θ = 0°, θ = 45°, and θ = 90°. The initial conditions for standing waves were chosen such that a pressure anti-node occurred at θ = 0°. Injector pressure waveforms for both standing and spinning instability are shown in Fig. 2 for combustors with quasi-steady nozzles. These waveforms exhibit sharp peaks.
Figure 2. Nonlinear Pressure Waveforms for the 1T Mode.
and shallow minima; they are nearly identical in shape to those calculated using the previous two-dimensional theory.\textsuperscript{5,6} Comparison of injector and nozzle pressure waveforms (\( \theta = \theta^0 \)) shows that there is very little variation in pressure with axial position. These waveforms are in qualitative agreement with the results of pressure measurements taken during test firings of unstable rocket motors.\textsuperscript{1}

To check the accuracy of the approximation of the nozzle boundary condition, wall pressure and axial velocity waveforms were calculated at the nozzle entrance. The error at the nozzle boundary \((z = z_e)\) is shown for nonlinear standing and spinning LT mode instabilities in Fig. 6. Here the axial velocity perturbation, \( u' \), and the product of the quasi-steady nozzle admittance and the pressure perturbation, \( Y_p' \), are plotted as a function of time. The latter quantity is the axial velocity perturbation that would be obtained at the nozzle entrance if the nozzle-boundary condition were exactly satisfied (i.e., the nozzle admittance condition requires that \( u' = Y_p' \) at \( z = z_e \)). Most of the discrepancy between the two curves is due to a slight phase shift between pressure and velocity and the second harmonic distortion of the pressure waveform resulting from the nonlinearities of the system. The nozzle boundary condition is satisfied in an average sense; however, for the ratio of the velocity amplitude (peak-to-peak) to pressure amplitude (peak-to-peak) is very close to the required value, \( Y_r \).

In another study, limit-cycle amplitudes were calculated as a function of \( n \) and \( \tau \) for standing LT mode instability. Values of \( n \) in the linearly unstable region were chosen for below resonant (\( \tau = 1.9 \)), resonant (\( \tau = 1.706 \)), and above-resonant (\( \tau = 1.6 \)) conditions. The resulting amplitudes are compared with those obtained with the two-dimensional theory in Fig. 4. This figure shows that the three-dimensional theory predicts a slightly higher limit-cycle amplitude than the two-dimensional theory for chambers with quasi-steady nozzles.

Figure 4 also shows that the three-dimensional theory, like the previous two-dimensional one, cannot predict triggering of LT mode instability by the introduction of finite amplitude disturbances. This result was expected since it was shown in Refs. 6 and 8 that the second order (i.e., \( O(e^2) \)) theory can predict triggering only for pure radial modes \((m = 0, n = 1, 2, \ldots)\). Such triggering limits for the LR mode are discussed in Ref. 9. It has also been
Figure 3. Nozzle Boundary Condition for Nonlinear 1T Mode Solutions for Quasi-Steady Nozzles.
Figure 4. Limit-Cycle Amplitudes for the 1T Mode.
shown, however, that triggering of 1T mode instability can be described when the $O(e^3)$ terms are retained in the analysis.\textsuperscript{8,14} The third order theory given in Refs. 8 and 14 is limited to a single mode in the approximating series expansions. A more general multi-mode third-order theory is now under development and the results will be presented in a future publication. It is also suspected that nonlinear unsteady combustion effects (not included in the present analysis) may play an important role in the triggering phenomenon.

For nozzles with complex admittances a study was conducted to determine the effect of the nozzle phase shift, $\varphi$, upon the limit-cycle amplitudes and waveforms for both standing and spinning 1T mode instability. The effect of nozzle phase shift on the nonlinear pressure and velocity waveforms at the nozzle entrance plane is shown in Fig. 5 for spinning waves. This figure shows that, while $\varphi$ has little or no effect on the pressure waveforms, the phase and shape of the velocity waveforms is strongly dependent on $\varphi$. The effect of $\varphi$ on the limit-cycle amplitude for standing 1T mode instability is shown in Fig. 6. For a given value of $n$ and $\tau$ (in the linearly unstable region for the 1T mode), Fig 6 shows a sinusoidal variation of limit-cycle amplitude with $\varphi$ having a maximum amplitude at about $\varphi = 200^\circ$ and a minimum amplitude at about $\varphi = 20^\circ$. In this connection, it should be pointed out that according to linear results nozzle damping is a maximum at $\varphi = 0^\circ$ and a minimum at $\varphi = 180^\circ$; thus the observed shifts must be due to nonlinearities.

In order to determine how well the solutions approximate the nozzle boundary condition, the amplitude ratio and phase shift between pressure and velocity at the nozzle entrance have been calculated from the nonlinear solutions and have been compared with the specified nozzle admittance condition. Since the waveforms are non-sinusoidal, an approximate amplitude ratio, $A_c$, was calculated by taking the ratio of peak-to-peak velocity amplitude to peak-to-peak pressure amplitude. The approximate phase shift, $\varphi_c$, was calculated from the following formula:

$$\varphi_c = \left[ \frac{t_p - t_u}{t} \right] \times 360 \quad (15)$$

where $t_p$ is the average of an ascending zero-crossing and the following descending zero-crossing for the pressure perturbation, $t_u$ is a similar average
Figure 5. Effect of Nozzle Phase Shift, $\phi$, on Nozzle Waveforms for Spinning LT Modes.
Figure 6. Effect of Nozzle Phase Shift on Limit-Cycle Amplitudes for the Standing 1T Mode.
for the velocity perturbation, and $T$ is the period of oscillation. The results of this study are shown in Fig. 7 for both standing and spinning waves. For standing waves the calculated amplitude ratios are seen to be consistently higher than required by the nozzle admittance condition (dashed line), while for spinning waves the calculated amplitude ratios are lower than required. For both standing and spinning waves the calculated phase shifts are in excellent agreement with the imposed phase shifts. This study shows that the three-dimensional theory provides a good approximation to the nozzle boundary condition for the IT mode, considering that the nonlinear solutions are being forced to satisfy a linear boundary condition.

**Longitudinal Mode Solutions.** Letting $m$ and $n$ equal zero in Eq. (6) and using a series consisting of the first five longitudinal modes (i.e., $l = 1, 2, \ldots, 5$), limit-cycle solutions were calculated for quasi-steady nozzles as well as for nozzles with complex admittances. The longitudinal mode solutions required somewhat longer computation times than the transverse mode solutions; the time required to reach a limit cycle was from three to four minutes on the Univac 1108 computer.

Longitudinal mode solutions for chambers with quasi-steady nozzles were compared with the solutions previously obtained by Lores and Zinn using a one-dimensional theory. Pressure waveforms at the injector face are compared for both resonant and off-resonant conditions in Fig. 8 which shows excellent agreement between the two theories. Pressure and velocity waveforms at the nozzle entrance as well as injector face pressure waveforms are shown in Fig. 9 for quasi-steady nozzles, while Fig. 10 shows waveforms at the nozzle entrance for nozzles with complex admittance ($\phi = 45^\circ$ and $\phi = 90^\circ$). In each case the results indicate the presence of a steep-fronted pressure wave moving back and forth in the chamber. This behavior is in agreement with experimental observations of axial instabilities. The relation between pressure and velocity waveforms at the nozzle entrance is a fairly good approximation to the nozzle admittance condition (see Figs. 9 and 10) in spite of the highly nonlinear waveforms. The results of this investigation indicate that the three-dimensional nonlinear theory is applicable to longitudinal instabilities as well as transverse instabilities. The theory can also be used to investigate the nonlinear behavior of combined
Figure 7. Nozzle Boundary Condition for Nonlinear 1T Mode Solutions.
Figure 8. Comparison of Nonlinear IL Mode Solutions for Quasi-Steady Nozzles.
Figure 9. Longitudinal Mode Waveforms for Quasi-Steady Nozzles.
Figure 10. Longitudinal Mode Waveforms for Nozzles with Complex Admittances.
longitudinal-transverse instabilities, although no results for instabilities of this type are presented in this report.

**CONCLUDING REMARKS**

A general three-dimensional second-order nonlinear theory has been developed for predicting the linear and nonlinear behavior of combustion instability in liquid-propellant rocket combustors. This theory contains previous analyses of transverse and longitudinal instabilities as special cases. Furthermore, it extends the previous analyses which were applicable only to combustors with quasi-steady nozzles, to the more practical cases of combustors with conventional DeLaval nozzles. The present theory can be used to predict the stability characteristics of longitudinal, transverse and combined longitudinal-transverse modes for various liquid-propellant rocket motor designs.

Results obtained for combustors with quasi-steady nozzles are in excellent agreement with the predictions of previous theories for both transverse and longitudinal instabilities. For combustors with conventional nozzles, the limit-cycle amplitude varies sinusoidally with nozzle phase shift, $\phi$, having a maximum value at $\phi = 200^\circ$ and a minimum value at $\phi = 20^\circ$. The nozzle phase shift has a strong effect on the axial velocity waveforms at the nozzle entrance while having only a minor influence on the nonlinear pressure waveforms. In both cases, the nonlinear theory developed in this paper provides a good approximation to the unsteady flow conditions at the nozzle entrance plane. This is in contrast to the previous theories which provided a relatively poor approximation to the nozzle boundary condition.

The results presented in this report establish the relationship that exists between the resulting instability (i.e., waveform, final amplitude, and final frequency), the combustion parameters (i.e., interaction index, $n$, and time-lag $\tau$), and the chamber Mach number and length-to-diameter ratio. These results indicate that the limit-cycle amplitude increases with increasing sensitivity of the combustion process to pressure oscillations. For transverse instabilities, calculated pressure waveforms exhibit sharp peaks and shallow minima, and the frequency of oscillation is always within a few percent of the frequency of one of the chamber's acoustic modes. For axial instabilities, the theory predicts the presence of a steep-fronted wave moving back and forth along the combustor. In both cases the calculated pressure waveforms are in
good qualitative agreement with available experimental data.

MOMENTUM INTERCHANGING BETWEEN LIQUID AND GAS PHASES

The research presented in this report were obtained under the assumption that the momentum interchange between the liquid and gas phases is negligible. This assumption will not be relaxed for this special case of the present study. The assumption will be shown that this momentum interchange is an important qualitative effect.

The momentum equation for two phases flow can be written as (1) and is

\[ (\mathbf{u} + \mathbf{c})(\mathbf{v} - \mathbf{c}) = \rho \frac{d\mathbf{v}}{dt} + \left[ \nabla \cdot \mathbf{v} \mathbf{v} + \frac{\nabla \mathbf{v}}{\rho} \right] \]

where \( \mathbf{u} \) and \( \mathbf{c} \) are the gas and liquid velocities, respectively. The term on the right-hand side of the equation represents the momentum transfer due to the interaction between the two phases (1) and is the sum of the momentum terms of the pressure gradient (S) and (V - V) \( \nabla \mathbf{v} \mathbf{v} \) due to the gas velocity gradient (i.e., the rate of change of the gas velocity with respect to space).

In order to derive a formula for the velocity potential \( \phi \), it is necessary to note the following equations:

(1) \[ \mathbf{v} = \nabla \phi \]

(2) \[ \frac{d}{dt} \left( \frac{1}{2} \mathbf{v} \mathbf{v} \right) = \nabla \cdot (\phi \mathbf{v}) \]

(3) \[ \nabla \cdot (\mathbf{c} \mathbf{v}) = 0 \]

(4) \[ \frac{d}{dt} \left( \frac{1}{2} \mathbf{c} \mathbf{c} \right) = \nabla \cdot (\phi \mathbf{c}) \]

With these, the following expression for the momentum source characteristics gives

\[ \frac{d}{dt} \left( \frac{1}{2} \mathbf{c} \mathbf{c} \right) = \nabla \cdot (\phi \mathbf{c}) \]
APPENDIX A

MOMENTUM INTERCHANGE BETWEEN LIQUID AND GAS PHASES

The results presented in this report were obtained under the assumption that the momentum interchange between the liquid droplets and the burned gases is negligible. This assumption will now be relaxed for the special case of uniformly distributed combustion, and it will be shown that this momentum interchange is an important stabilizing effect.

Analysis

The momentum equation for two-phase flow was derived in Ref. 8 and is given by:

\[ \rho \left[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right] + \frac{1}{\rho} \nabla p = -(\mathbf{v} - \mathbf{v}_L)(C + \mathbf{w}_m) \quad (A-1) \]

where \( \mathbf{v} \) and \( \mathbf{v}_L \) are the gas and liquid velocity, respectively. The term on the right-hand-side of Eq. (A-1) represents a momentum source to the gas produced by the burning liquid drops. This momentum source consists of two parts: (1) the force necessary to accelerate the evolved gases from the droplet velocity to the gas velocity (i.e., the term \(-\mathbf{w}_m(\mathbf{v} - \mathbf{v}_L)\)) and (2) the aerodynamic drag of the droplets (i.e., the term \(-C(\mathbf{v} - \mathbf{v}_L)\)).

In order to derive a wave equation for the velocity potential \( \Phi \) it is necessary to make the following assumptions: (1) the drag term is negligible compared with the acceleration term, (2) liquid velocity fluctuations are negligible, and (3) the combustion is uniformly distributed throughout the chamber. Neglecting the drag term, perturbing, and neglecting third order quantities gives the following expression for the momentum source perturbation, \( \mathbf{M}^p \):

\[ \mathbf{M}^p = -(\mathbf{v}' - \mathbf{v}_L')\mathbf{w}_m \quad (A-2) \]

This is simplified further by neglecting the liquid velocity perturbation.
introducing the velocity potential, and using the steady-state relation, \( \dot{\theta}_m = \frac{d\theta}{dz} \), to obtain:

\[
\dot{M}' = - \frac{d\theta}{dz} \nabla \phi
\]  

(A-3)

Finally, the assumption of uniformly distributed combustion gives \( \frac{d\theta}{dz} \) constant which yields:

\[
\dot{M}' = - \nabla \left[ \frac{d\theta}{dz} \phi \right]
\]  

(A-4)

Perturbing the left-hand-side of Eq. (A-1), introducing the velocity potential, and combining with Eq. (A-4) gives:

\[
\nabla \left[ \frac{\partial \phi}{\partial t} + \frac{1}{r} \nabla' + \bar{u}\nabla_z + \frac{d\theta}{dz} \phi + \frac{1}{2} \nabla \phi \cdot \nabla \phi - \frac{1}{2} \dot{\phi}^2 \right] = 0
\]  

(A-5)

which can be integrated to obtain:

\[
\dot{p}' = -\nabla \left[ \bar{u}\nabla_z + \frac{d\theta}{dz} \phi + \frac{1}{2} \nabla \phi \cdot \nabla \phi - \frac{1}{2} \dot{\phi}^2 \right]
\]  

(A-6)

Equation (A-6) is similar to Eq. (13), where the additional term \( \left( \frac{d\theta}{dz} \right) \phi \) arises from the droplet momentum source. Following the procedure outlined in Ref. 8, the momentum equation given by Eq. (A-6) is combined with the continuity and energy equations to obtain the desired wave equation:

\[
\nabla^2 \phi - \dot{\phi}_{tt} = 2\bar{u}\dot{\phi} z_t + (r + 1) \frac{d\theta}{dz} \phi_t + 2\nabla \phi \cdot \nabla \phi_t + (r - 1) \dot{\phi}_t \nabla^2 \phi + \dot{\theta}_m
\]  

(A-7)

Comparing Eq. (A-7) with Eq. (1) shows that the droplet momentum source
appears only in the second term on the right-hand-side of this equation, where the factor $\gamma$ in Eq. (1) becomes $(\gamma + 1)$ in Eq. (A-7).

Applying the Method of Weighted Residuals to obtain approximate solutions to Eq. (A-7) yields a set of ordinary differential equations identical to Eq. (12) where the coefficient $C_2(j,p)$ is now given by:

$$C_2(j,p) = \left\{ \int_0^Z \bar{u}(z)Z_j^* dz + (\gamma + 1)\int_0^Z \frac{\partial \bar{u}}{\partial z} Z_j^* dz + \gamma \bar{z}(z) Z_j^*(z) \right\} X$$

Equation (A-8) is readily obtained from Eq. (C-3) by replacing $\gamma$ in the second term by $\gamma + 1$.

Linear Stability Limits

Linear stability limits for the IL mode were calculated by the method described in Appendix E for the following two cases: (1) the droplet momentum source was included in the analysis and (2) the droplet momentum source was neglected. The results were compared with the linear stability limit calculated by Mitchell\textsuperscript{15} on a plot of interaction index, $n$, versus stretched time-lag, $\mu$, where $\mu = \omega \tau / \pi$ (see Fig. A-1). This figure shows excellent agreement between the results of Mitchell (solid curve) and the present theory (circle symbols) when the droplet momentum source is included. Neglecting the droplet momentum source shifts the stability curve to much lower values of $n$ (dashed curve), which indicates that the droplet momentum source is an important stabilizing effect.

Nonlinear Solutions

In the second-order analysis presented in this report, the droplet momentum source affects the nonlinear solutions primarily by increasing the linear stability of the system. This is readily shown in Fig. (A-2) where the limit cycle amplitude is plotted as a function of the displacement, $\delta n$, above the neutral stability limit. By plotting the limit-cycle amplitudes in this manner...
Figure A-1. Effect of Droplet Momentum Source on Linear Stability Limits for the 1L Mode.
Figure A-2: Effect of Droplet Momentum Source on Limit-Cycle Amplitude

- Without Droplet Momentum Source
- With Droplet Momentum Source

\[ \tau = 1.7, \gamma = 1.2, \bar{u}_e = 0.2, L/D = 0.5 \]

A = 0.02, \( \phi = 0^\circ \)

Standing lT Mode

Displacement Above Neutral Stability Limit, 6n
the effect of the shift in the neutral stability curves is removed so that only the nonlinear effect of the momentum source is seen. Figure A-2 shows that, for equal displacements above the neutral stability limits, including the droplet momentum source results in a slightly smaller limit-cycle amplitude. This difference in limit-cycle amplitude is negligible for most practical purposes.

For combustors with uniformly distributed combustion it has been shown that the droplet momentum source is an important effect which is easily incorporated into the present analysis. Consequently the computer programs based on this theory include the droplet momentum source as an optional feature (see Appendices C, D, and E).

For chambers with non-uniform combustion distributions, Eqs. (A-6) and (A-7) are no longer applicable; however, the droplet momentum source can be taken into account in the following manner. Using the present theory with the droplet momentum source omitted, the neutral stability limit, \( n_1(T) \), is calculated and the limit-cycle amplitudes are determined as a function of \( \delta n \) as in Fig. A-2. In addition, the linear stability limit, \( n_2(T) \), is calculated using a linear theory which includes the droplet momentum source and is not restricted to uniformly distributed combustion (such as in Ref. (15)). Assuming that the nonlinear effect of the droplet momentum source is also small for non-uniformly distributed combustion and using the values of \( \delta n \) and \( n_2(T) \) calculated above, the desired plot of limit-cycle amplitude as a function of \( n \) is readily obtained.
USE OF COMPLEX VARIABLES IN THE SOLUTION OF NONLINEAR DIFFERENTIAL EQUATIONS

It is often convenient to use complex variables in the solution of the linear equations which arise in acoustics, combustion instability and related fields. In this case the solution is expressed in complex form, and the real part represents the physically meaningful solution. However, care must be used when applying this technique in the solution of nonlinear equations. The difficulties that are encountered in applying the complex variable technique to nonlinear problems will be illustrated by analyzing the following simplified example. Consider the nonlinear wave equation given by:

\[ \nabla^2 \psi - \psi_{tt} = \psi \psi_t \]  \hspace{1cm} (B-1)

A complex solution of Eq. (B-1) of the form \( \psi = \varphi + i\gamma \) would be useful only if its real part, \( \varphi \), satisfies Eq. (B-1), which would be the case if the equation were linear. However, straightforward substitution of \( \psi = \varphi + i\gamma \) into Eq. (B-1) and separating its real and imaginary parts yields the following equation for \( \varphi \):

\[ \nabla^2 \varphi - \varphi_{tt} = \varphi \varphi_t - \varphi \gamma_t \] \hspace{1cm} (B-1)

indicating that the real part, \( \varphi \), does not satisfy Eq. (B-1) because of the extra term, \(-\varphi \gamma_t\), appearing on the right hand side. In order to eliminate this extra term, the form of the original differential equation (i.e., Eq. (B-1)) must be modified.

Since Eq. (B-1) supposedly describes some physical phenomenon, and since only the real part of the complex solution is physically meaningful, then the nonlinear term \( \varphi \varphi_t \) should really be expressed as the product \( \text{Re}(\bar{\varphi}) \cdot \text{Re}(\bar{\varphi}) \) which is equivalent to \( (\varphi \varphi_t + \varphi \varphi_t^* + \bar{\varphi} \bar{\varphi}_t + \bar{\varphi} \bar{\varphi}_t^*)/4 \). Substituting this expression into Eq. (B-1) yields:
Substituting $\tilde{\phi} = \phi + i\psi$ into Eq. (B-3) and separating its real and imaginary parts yield:

\[\nabla^2 \tilde{\phi} - \tilde{\phi}_{tt} = \frac{1}{u}[\tilde{\psi}\tilde{\phi}_t + \tilde{\phi}\tilde{\phi}_t^{*} + \tilde{\phi}^{*}\tilde{\phi}_t + \tilde{\phi}_t^{*}\tilde{\phi}_t^{*}] \tag{B-3}\]

Substituting $\tilde{\phi} = \phi + i\psi$ into Eq. (B-3) and separating its real and imaginary parts yield:

\[\nabla^2 \phi - \phi_{tt} = \phi_{tt} \tag{B-4}\]
\[\nabla^2 \psi - \psi_{tt} = 0 \tag{B-5}\]

which shows that the real part of the solution of Eq. (B-3) satisfies the desired equation (i.e., Eq. (B-1)) and the imaginary part satisfies a homogeneous linear wave equation. This technique was applied to the solution of nonlinear combustion instability problems (i.e., to Eq. (1)), and the resulting modified wave equation was solved using the Method of Weighted Residuals. Due to the approximate nature of the Method of Weighted Residuals, however, the resulting solution contained an error term which grew without limit. Consequently, the above procedure had to be modified in order to obtain satisfactory solutions of Eq. (1) using the Method of Weighted Residuals.

An alternate technique is to modify Eq. (B-1) such that both the real and imaginary parts satisfy the original equation. This can be done by replacing terms of the form $\tilde{\phi}\tilde{\phi}_t$ with $\text{Re}(\tilde{\phi})\text{Re}(\tilde{\phi}_t) + i\text{Im}(\tilde{\phi})\text{Im}(\tilde{\phi}_t)$; using the relations:

\[\text{Re}(\tilde{\phi})\text{Re}(\tilde{\phi}_t) = \frac{(\tilde{\phi} + \tilde{\phi}^{*})}{2}\left(\frac{\tilde{\phi}_t + \tilde{\phi}_t^{*}}{2}\right) = \frac{1}{u}[\tilde{\phi}\tilde{\phi}_t + \tilde{\phi}^{*}\tilde{\phi}_t + \tilde{\phi}\tilde{\phi}_t^{*} + \tilde{\phi}_t^{*}\tilde{\phi}_t^{*}] \tag{B-5}\]
\[i\text{Im}(\tilde{\phi})\text{Im}(\tilde{\phi}_t) = -i\left(\frac{\tilde{\phi} - \tilde{\phi}^{*}}{2}\right)\left(\frac{\tilde{\phi}_t - \tilde{\phi}_t^{*}}{2}\right) = -\frac{1}{u}[\tilde{\phi}\tilde{\phi}_t - \tilde{\phi}^{*}\tilde{\phi}_t - \tilde{\phi}\tilde{\phi}_t^{*} + \tilde{\phi}_t^{*}\tilde{\phi}_t^{*}] \tag{B-5}\]

in Eq. (B-1) gives:

\[\nabla^2 \tilde{\phi} - \tilde{\phi}_{tt} = \frac{1}{u}[(1 - i)(\tilde{\phi}\tilde{\phi}_t + \tilde{\phi}^{*}\tilde{\phi}_t^{*}) + (1 + i)(\tilde{\phi}\tilde{\phi}_t^{*} + \tilde{\phi}_t^{*}\tilde{\phi}_t^{*})] \tag{B-6}\]

Substituting $\tilde{\phi} = \phi + i\psi$ into Eq. (B-6) and separating into its real and imaginary parts yields:
nary parts gives:

\[ \nabla^2 \varphi - \varphi_{tt} = \varphi \varphi_t \]

\[ \nabla^2 \psi - \psi_{tt} = \psi \psi_t \]

which shows that both \( \varphi \) and \( \psi \) satisfy Eq. (B-1). Applying this method to the solution of Eq. (1) yields the modified wave equation (i.e., Eq. (10)) used in the present investigation.
Statement of the Problem

Program COEFFS3D calculates the coefficients of both the linear and non-linear terms which appear in Eqs. (12). These coefficients are required as input for Program LCYC3D (see Appendix D) which numerically integrates this system of equations. The coefficients that are required depend on the choice of terms to be included in the series solution for \( \tilde{\varphi} \) (see Eq. (9)), therefore this information must be provided as input to Program COEFFS3D. The output of Program COEFFS3D is either punched onto cards or stored on drum (FASTRAND) for input to Program LCYC3D.

The coefficients to be calculated are functions of various integrals of hyperbolic, trigonometric, and Bessel functions and are given by the following expressions:

\[
C_0(j,p) = \int_0^{z_e} Z_j z^* \int_0^{2\pi} \int_0^1 R R j r d r d \theta d \phi
\]  \hspace{1cm} (C-1)

\[
C_1(j,p) = \left\{ S_{m,n}^2 (p) \int_0^{z_e} Z_j z^* \int_0^{2\pi} \int_0^1 R R j r d r d \theta d \phi \right\}
\]  \hspace{1cm} (C-2)

\[
C_2(j,p) = \left\{ 2 \int_0^{z_e} \tilde{u}(z) Z_j z^* \int_0^{2\pi} \int_0^1 R R j r d r d \theta d \phi \right\}
\]  \hspace{1cm} (C-3)

\[
C_3(j,p) = \left\{ \int_0^{z_e} \frac{d\tilde{v}}{dz} Z_j z^* \int_0^{2\pi} \int_0^1 R R j r d r d \theta d \phi \right\}
\]  \hspace{1cm} (C-4)
\[ D_1(j,p,q) = \frac{1}{2} (1 - i) \left\{ T_1 \int_{p}^{q} Z^* Z^* dz + T_2 \left[ \int_{p}^{q} Z Z^* dz + \frac{\gamma - 1}{2} \int_{p}^{q} \frac{\partial^2 \phi}{\partial z^2} dz \right] \right\} \]

\[ D_2(j,p,q) = \frac{1}{2} (1 + i) \left\{ T_1 \int_{p}^{q} Z Z^* dz + T_2 \left[ \int_{p}^{q} \frac{\partial \phi}{\partial z} dz + \frac{\gamma - 1}{2} \int_{p}^{q} \frac{\partial^2 \phi}{\partial z^2} dz - \frac{1}{2} \gamma \int_{p}^{q} (z^*)^2 dz \right] \right\} \]

\[ D_3(j,p,q) = \frac{1}{2} (1 + i) \left\{ T_1 \int_{p}^{q} Z^* Z^* dz + T_2 \left[ \int_{p}^{q} (Z^*)^2 dz + \frac{\gamma - 1}{2} \int_{p}^{q} \frac{\partial^2 \phi}{\partial z^2} dz \right] \right\} \]

\[ D_4(j,p,q) = \frac{1}{2} (1 - i) \left\{ T_1 \int_{p}^{q} Z Z^* dz + T_2 \left[ \int_{p}^{q} \frac{\partial \phi}{\partial z} dz + \frac{\gamma - 1}{2} \int_{p}^{q} \frac{\partial^2 \phi}{\partial z^2} dz + \frac{1}{2} \gamma \int_{p}^{q} (z^*)^2 dz \right] \right\} \]

where

\[ T_1 = \int_{p}^{q} \chi_{\Theta} \Theta_0 \Theta_0 \, dz + \int_{p}^{q} \Theta_0 \chi_{\Theta} \Theta_0 \, dz + \int_{p}^{q} \Theta_0 \Theta_0 \chi_{\Theta} \, dz + \int_{p}^{q} \Theta_0 \Theta_0 \chi_{\Theta} \, dz \]

\[ T_2 = \int_{p}^{q} \Theta_0 \Theta_0 \Theta_0 \, dz + \int_{p}^{q} \Theta_0 \Theta_0 \Theta_0 \, dz + \int_{p}^{q} \Theta_0 \Theta_0 \Theta_0 \, dz + \int_{p}^{q} \Theta_0 \Theta_0 \Theta_0 \, dz \]
In the equations on the prior page the notation of Eq. (9) is used; that is, a single index (i.e., j, p, or q) is used to identify a particular series term rather than the mode numbers used in Eq. (6). The index j identifies the equations in which a given coefficient appears which corresponds to the weighting function used in deriving that equation. For the coefficients of the linear terms (i.e., the C's) the index p identifies the amplitude function which the coefficient multiplies. For coefficients of the nonlinear terms, (i.e., the D's) p identifies the factor which is not differentiated with respect to time, (i.e., \( A_p \) or \( A_p^* \)), while q identifies the differentiated factor (i.e. \( dA_p/dt \) or \( dA_p^*/dt \). Due to the complex nature of the axial eigenfunctions, the above coefficients are complex numbers.

Structure of the Numerical Calculations

A flow chart for Program COEFFS3D is shown in Figure (C-1). The program can be divided into five major sections: (1) input, (2) calculation of the complex linear coefficients, (3) calculation of the complex nonlinear coefficients, (4) obtaining coefficients of the equivalent uncoupled real system, and (5) output.

The inputs to the program include the various parameters describing the chamber geometry, the nozzle boundary condition, the modes included in the approximating series expansion, and various control numbers, as well as the roots of the Bessel functions.

In the second section the axial acoustic eigenvalues are calculated by means of Subroutines EIGVAL and FCNS, and the integrals of the products of two axial eigenfunctions are computed by means of Subroutines AXIAL1 and UBAR. The integrals involving radial and tangential eigenfunctions are evaluated by using the orthogonality properties of these functions. The complex linear coefficients are then calculated according to Eqs. (C-1) through (C-4) and are normalized by dividing by \( C_0(j, j) \).

In the third section the integrals of products of three Bessel functions are calculated using Subroutines RADIAL and JBES, while similar integrals involving azimuthal eigenfunctions and axial eigenfunctions are computed using Subroutines AZIMTL and AXIAL2 respectively. The normalized complex nonlinear coefficients are obtained from Eqs. (C-5) through (C-8) by dividing by \( C_0(j, j) \).

In the fourth section the normalized complex coefficients are used to
Figure C-1. Flow Chart for Program COEFFS3D
obtain the coefficients for the equivalent system of real differential equations obtained by separating the real and imaginary parts of the complex equations. Since the axial eigenfunctions are non-orthogonal, the resulting system of equations may be coupled in the second derivative terms. Therefore, a matrix inversion procedure is used to obtain the coefficients of an equivalent system which is not coupled in the second derivatives.

In the last section the computed values of the coefficients are either printed out, punched onto cards, or stored on drum (FASTRAND file) as desired.

**Input Data**

The input data consists of the chamber parameters (i.e., ratio of specific heats, steady state Mach number, and length-to-diameter ratio), the nozzle admittance ratio, various control numbers, and information indicating which modes are included in the approximate series expansion. Regarding the latter information, each term in the series is identified by the integer variable J. The nature of each term is specified by the four integers L(J), M(J), N(J), and NS(J), and each term is given a four character name NAME(J). In this manner the coefficients are identified by the integers J associated with the modes involved rather than the corresponding axial, azimuthal, and radial mode numbers.

The following comments pertain to the detailed description of the input. The location number refers to columns of the card. Three formats are used for input: "A" indicates alphanumeric characters, "I" indicates integers, and "F" indicates real numbers with a decimal point. For the "I" and "F" formats the values are placed in fields of five and ten locations, respectively, and the numbers must be placed in the rightmost locations of the allocated field.

<table>
<thead>
<tr>
<th>No. of Cards</th>
<th>Location</th>
<th>Type</th>
<th>Input Item</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1-72</td>
<td>A</td>
<td>TITLE</td>
<td>Title of Case</td>
</tr>
<tr>
<td>1</td>
<td>1-10</td>
<td>F</td>
<td>GAMMA</td>
<td>Ratio of specific heats, γ.</td>
</tr>
<tr>
<td>11-20</td>
<td>F</td>
<td>UE</td>
<td></td>
<td>Steady state Mach number at nozzle entrance, u_e.</td>
</tr>
<tr>
<td>21-30</td>
<td>F</td>
<td>RLD</td>
<td></td>
<td>Length-to-diameter ratio, L/D = z_c/2</td>
</tr>
<tr>
<td>31-40</td>
<td>F</td>
<td>ZCOMB</td>
<td></td>
<td>Length of combustion zone, z_c/z_e.</td>
</tr>
<tr>
<td>Cards</td>
<td>Location</td>
<td>Type</td>
<td>Input Item</td>
<td>Comments</td>
</tr>
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<td>----------</td>
<td>------</td>
<td>------------</td>
<td>----------</td>
</tr>
<tr>
<td>11-45</td>
<td>I</td>
<td>NDROPS</td>
<td></td>
<td>If 0: droplet momentum source neglected. If 1: droplet momentum source included.</td>
</tr>
<tr>
<td>16-50</td>
<td>I</td>
<td>NOZZLE</td>
<td></td>
<td>If 0: quasi-steady nozz. If 1: conventional nozz.</td>
</tr>
<tr>
<td>1-5</td>
<td>I</td>
<td>NJMAX</td>
<td></td>
<td>Number of series terms (complex). (NJMAX ≤ 10)</td>
</tr>
<tr>
<td>6-10</td>
<td>I</td>
<td>NONLIN</td>
<td></td>
<td>If 0: linear terms only. If 1: linear and nonlinear terms.</td>
</tr>
<tr>
<td>11-15</td>
<td>I</td>
<td>NEGL</td>
<td></td>
<td>If 0: No nonzero coefficients calculated. If 1: small coefficients neglected.</td>
</tr>
<tr>
<td>16-20</td>
<td>I</td>
<td>NOUT</td>
<td></td>
<td>If 0: printed output on paper. If 1: printed and written into FASTRAND file. If 2: FASTRAND only. If 3: card output only.</td>
</tr>
<tr>
<td>6-15</td>
<td>F</td>
<td>AMPL(J)</td>
<td></td>
<td>Linear coefficients with absolute value less than SM1 neglected.</td>
</tr>
<tr>
<td>16-25</td>
<td>F</td>
<td>PHASE(J)</td>
<td></td>
<td>Nonlinear coefficients with absolute value less than SM2 neglected.</td>
</tr>
<tr>
<td>6-10</td>
<td>I</td>
<td>L(J)</td>
<td></td>
<td>Integer which identifies series term.</td>
</tr>
<tr>
<td>46</td>
<td></td>
<td></td>
<td></td>
<td>Amplitude factor of nozzle admittance, ( A ).</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Phase of nozzle admittance, ( \Phi ).</td>
</tr>
<tr>
<td>1-5</td>
<td>I</td>
<td>JM</td>
<td></td>
<td>Integer which identifies series term.</td>
</tr>
<tr>
<td>6-10</td>
<td>I</td>
<td>L(J)</td>
<td></td>
<td>Axial mode number, ( l ). (0 ≤ l(J) ≤ 10)</td>
</tr>
<tr>
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<td>Location</td>
<td>Type</td>
<td>Input Item</td>
<td>Comments</td>
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<td>------</td>
<td>------------</td>
<td>----------</td>
</tr>
<tr>
<td>11-15</td>
<td>I</td>
<td>M(J)</td>
<td>Tangential mode number, m. (0≤L(J)≤8)</td>
<td></td>
</tr>
<tr>
<td>16-20</td>
<td>I</td>
<td>N(J)</td>
<td>Radial mode number, n. (0≤N(J)≤5)</td>
<td></td>
</tr>
</tbody>
</table>
| 21-25       | I        | NS(J) | NS(J) = 1: θ = sin(mθ)  
|             |          |      | NS(J) = 2: θ = cos(mθ) |
| 26-30       | A        | NAME(J) | Four character name. |

The first card gives a title (maximum 72 characters) used to identify the run. The second card gives the chamber parameters (i.e., γ, u_e, L/D, z_c), determines whether the droplet momentum source is included in the analysis (see Appendix A), and specifies the type of nozzle (quasi-steady or conventional). If a quasi-steady nozzle is specified the nozzle admittance is calculated using Eqs. (14), and no further information concerning the nozzle is required. The control numbers are given on the third card. Due to computer storage limitations the series expansion is limited to ten terms, thus NJMAX ≤ 10. The control number NEGL gives the option to neglect all coefficients with absolute value smaller than a given number, thus allowing a considerable saving in computation time when the equations are numerically integrated by Program LCYC3D. It has been found that neglecting coefficients with absolute value smaller than 0.1 (i.e., SM1 = SM2 = 0.1) reduces the computation time by half and has a negligible effect on the resulting solutions. For conventional nozzles a series of NJMAX cards is read which gives the nozzle admittance (amplitude and phase) for each term in the series. This is followed by another series of NJMAX cards giving the mode numbers for each series term.

The proper input for program COEFFS3D will be illustrated with the following example. Suppose the velocity potential Φ is expressed in terms of the first tangential (1T), the second tangential (2T), and the first radial (1R) modes. It is also desired to investigate instability of the spinning type, therefore both sin(mθ) and cos(mθ) terms are included in the series. However, for the 1R mode (m=0) there is no corresponding sin(mθ) term, therefore the resulting series will contain five terms. A nozzle admittance of A = 0.02 and φ = 45° will be assumed for each term in the series, and coefficients smaller than 0.1 as well as the droplet momentum source will be neglected.
The output data will be punched on cards. A sample input for this case is given in Table (C-1) below.

Table C-1. Sample Input

<table>
<thead>
<tr>
<th>Sample Input</th>
<th>Sample Input</th>
<th>Sample Input</th>
<th>Sample Input</th>
<th>Sample Input</th>
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<tbody>
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<td>1, 2, 4, 8</td>
<td>1, 2, 4, 8</td>
<td>1, 2, 4, 8</td>
<td>1, 2, 4, 8</td>
</tr>
</tbody>
</table>

After the last card in the sequence described above is read, the program is executed and control returns to the input section. Thus, several cases may be executed on the same run. If no further cards are given, the run is terminated.

In addition to the above card input, roots of the Bessel functions \( J_m(r) \) which give zero slope at \( r = 1 \) and the associated values \( J_m(r^n) \) are needed for these calculations. These values were taken from Ref. (16) for \( m = 0, 1, \ldots, 8 \) and \( n = 1, 2, \ldots, 7 \); they are automatically put into the program by means of a preprogrammed statement, which is an integral part of the program.
Complex Linear Coefficients

For NDROPS = 0 the complex linear coefficients are computed from Eqs. (C-1) through (C-4) and are stored in the complex array CC(KC,NJ,NP). For NDROPS = 1 the coefficients $C_2(j,p)$ are computed from Eq. (A-8).

In order to calculate these coefficients the following information is needed: (1) the axial acoustic eigenvalues, $b_{\ell mn}$, (2) the steady state Mach number distribution, $\tilde{u}(z)$, (3) the orthogonality properties of the transverse eigenfunctions, and (4) the integrals of products of two axial eigenfunctions. The calculation of these quantities is described below.

Axial Acoustic Eigenvalues. The axial acoustic eigenvalues are determined by numerically solving the transcendental equation given by Eq. (8). This is done by first substituting $b_{\ell mn} = \epsilon_{\ell mn} + i\eta_{\ell mn}$ and $Y = Y_r + iY_i$ into Eq. (8) and separating real and imaginary parts. This yields a pair of simultaneous equations of the form:

$$f(\epsilon, \eta) = 0$$
$$g(\epsilon, \eta) = 0$$

where

$$f(\epsilon, \eta) = (\epsilon^2 - \eta^2) F(\epsilon, \eta) - 4 \epsilon \eta h(\epsilon, \eta)$$

$$+ \gamma^2 \left\{ \left( Y_r^2 - Y_i^2 \right) (S_{mn}^2 + \epsilon^2 - \eta^2) - 4 Y_r Y_i \epsilon \eta \right\} G(\epsilon, \eta)$$

$$+ 4 \left\{ Y_r Y_i \left( S_{mn}^2 + \epsilon^2 - \eta^2 \right) + (Y_r^2 - Y_i^2) \epsilon \eta \right\} h(\epsilon, \eta)$$

(C-10)
\[ g(\varepsilon, \eta) = (\varepsilon^2 - \eta^2)H(\varepsilon, \eta) + \varepsilon \eta F(\varepsilon, \eta) \]

In the above equations the subscripts on \( \varepsilon \) and \( \eta \) have been omitted.

Equations (C-9) are solved by Subroutine EIGVAL using Newton's Method for two unknowns. In this method successive approximations to the roots are generated by the recursion formulas:

\[ \varepsilon_{i+1} = \varepsilon_i - \left[ \frac{g_f - f_g}{J(f,g)} \right] \]

\[ \eta_{i+1} = \eta_i - \left[ \frac{g_f - f_g}{J(f,g)} \right] \]

where the Jacobian \( J(f,g) \) is given by:

\[ J(f,g) = f \varepsilon \eta - g \eta \varepsilon \]
and the subscripts indicate partial differentiation with respect to \( e \) and \( \eta \). The quantities \( f, g, f_e, f_\eta, g_e, g_\eta \) are calculated by the Subroutine FCNS. The iteration is started by assuming the following values for \( e \) and \( \eta \):

\[
\begin{align*}
\varepsilon_0 &= \varepsilon_m + a \cos(\beta) \\
\eta_0 &= a \sin(\beta)
\end{align*}
\]

where for \( \ell = 0 \):

\[
\begin{align*}
\varepsilon_m &= 0 \\
a &= 10A/z_e \\
\beta &= \Phi/2 + 45 \text{ (degrees)}
\end{align*}
\]

and for \( \ell \neq 0 \):

\[
\begin{align*}
\varepsilon_m &= \ell \pi/z_e \\
a &= A/z_e \\
\beta &= \Phi + 90 \text{ (degrees)}
\end{align*}
\]

The iteration is terminated when the errors \( \Delta \varepsilon \) and \( \Delta \eta \) are smaller than \( 10^{-7} \). If the iteration fails to converge after 40 iterations or the Jacobian vanishes a warning message is printed. FORTRAN listings of Subroutines EIGVAL and FCNS are given at the end of this appendix.

**Steady State Mach Number Distribution.** The steady state Mach number distribution is calculated by means of Subroutine UBAR which must be supplied by the user. This distribution must be of the form shown in Fig. (C-2) where the Mach number varies from zero at the injector face \( (z = 0) \) to its maximum value at the end of the combustion zone \( (z = z_c) \) and remains constant until the nozzle entrance \( (z = z_e) \) is reached. Thus the Mach number is given by

\[
\overline{U}(z) = \overline{U}(z)\overline{u}_e \quad (0 \leq z \leq z_c)
\]

(C-17)
where \( U(0) = 0 \) and \( U(z_c) = 1 \). Although the function \( U(z) \) may be arbitrary, the results presented in this report were obtained using a linear Mach number distribution in the combustion zone (i.e., uniformly distributed combustion). Thus the function \( U(z) \) in the listing of \( \bar{U} \) provided herein is given by:

\[
U(z) = \frac{z}{z_c} \quad \text{(C-1)}
\]

In addition to the Mach number distribution (\( NOPT = 1 \)), the first (\( NOPT = 2 \)) and second (\( NOPT = 3 \)) derivatives are also calculated.

Figure C-2. Steady-State Mach Number Distribution.
Orthogonality of Transverse Eigenfunctions. The tangential eigenfunctions have the following orthogonality properties:

\[
\int_{0}^{2\pi} \sin(m_\theta) \sin(m_\theta') d\theta = \int_{0}^{2\pi} \cos(m_\theta) \cos(m_\theta') d\theta = 0 \quad m_p \neq m_j
\]

\[
= \pi \quad m_p = m_j \neq 0
\]

\[
\int_{0}^{2\pi} \cos(m_\theta) \cos(m_\theta') d\theta = 2\pi \quad m_p = m_j = 0
\]

\[
\int_{0}^{2\pi} \sin(m_\theta) \cos(m_\theta') d\theta = 0 \quad \text{for all } m_p \text{ and } m_j
\]

For the special case of \( m_p = m_j = 0 \) the integral involving sines vanishes.

The orthogonality property of the radial eigenfunctions is given by:

\[
\int_{0}^{1} R_p R_j r dr = 0 \quad n_p \neq n_j \quad (m_p = m_j)
\]

\[
\int_{0}^{1} R_p R_j r dr = \frac{3^{2m} - m^2}{28^{2m}} \left[ J_m(S_{mn}) \right]^2 \quad n_p = n_j \quad (m_p = m_j)
\]

Since the tangential integrals vanish when \( m_p \neq m_j \) it is not necessary to calculate the radial integrals for \( m_p \neq m_j \). These orthogonality properties are used to calculate the integrals, \( \int_{0}^{2\pi} \phi_p \phi_j d\theta \) and \( \int_{0}^{1} R_p R_j r dr \), which appear in Eqs. (C-1) through (C-4). For a series containing pure transverse modes only \((\ell = 0)\), it is easily seen that all of the linear coefficients vanish except those corresponding to \( p = j \), yielding a system of equations which are not coupled in the linear terms.

Axial Integrals. The integrals of products of two axial eigenfunctions are calculated by Subroutine AXIAL1. According to the value of the input parameter NOPT these integrals are calculated as follows:
The last two integrals, which involve the mean flow Mach number, are evaluated by means of Simpson's Rule. A FORTRAN listing of AXIAL1 is provided at the end of this appendix.

Complex Nonlinear Coefficients.

The complex nonlinear coefficients are calculated from Eqs. (C-5) through (C-8) and are stored in the complex arrays, CD1(NJ,NP,NQ), CD2(NJ,NP,NQ), CD3(NJ,NP,NQ), and CD4(NJ,NP,NQ).

In order to calculate these coefficients, the various integrals of axial, azimuthal, and radial eigenfunctions must be evaluated. Since many of the azimuthal integrals are zero they are evaluated first, and the remaining integrals are computed only if the corresponding azimuthal integral is nonzero. The subroutines used to calculate these integrals are described in the following paragraphs.

Azimuthal Integrals. The azimuthal integrals are calculated by Subroutine AZIMTL according to the value of NOPT as follows:

\[
\text{NOPT = 1: } \int_0^Z e^{i(b_p + b_j^*)z} dz = \frac{1}{2} \left\{ \frac{\sinh(i(b_p + b_j^*)Z_e)}{i(b_p + b_j^*)} \right\} + \frac{\sinh(i(b_p - b_j^*)Z_e)}{i(b_p - b_j^*)}
\]

\[
\text{NOPT = 2: } \int_0^Z e^{i(b_p - b_j^*)z} dz = -\frac{2}{b_p} \int_0^Z e^{i(b_p - b_j^*)z} dz
\]

\[
\text{NOPT = 3: } \int_0^Z u \left( \frac{du}{dz} \right) e^{i(b_p - b_j^*)z} dz \quad \text{(evaluated numerically)}
\]

\[
\text{NOPT = 4: } \int_0^Z u(z) e^{i(b_p - b_j^*)z} dz \quad \text{(evaluated numerically)}
\]
These integrals are easily evaluated analytically; for most values of \( p, q, \) and \( j \) they are zero. The nonzero integrals are readily expressed in terms of the following integrals:

\[
\int_0^{2\pi} \cos(m_\theta) \cos(m_\theta) \cos(m_\theta) d\theta = \pi/2 \quad \text{for} \quad m_j = m_p + m_q, \\
\quad m_p = m_j + m_q, \quad \text{or} \\
\quad m_q = m_j + m_p \quad \text{(C-23)}
\]

\[
\int_0^{2\pi} \cos(m_\theta) \sin(m_\theta) \sin(m_\theta) d\theta = \pi/2 \quad \text{for} \quad m_q = m_p + m_j, \quad \text{or} \\
\quad m_j = m_p + m_q \quad \text{(C-24)}
\]

\[
\int_0^{2\pi} \cos(m_\theta) \sin(m_\theta) \sin(m_\theta) d\theta = -\pi/2 \quad \text{for} \quad m_p = m_q + m_j \quad \text{(C-25)}
\]

where \( m_p, m_q, \) and \( m_j \) are nonzero. If any one of the tangential mode numbers is zero (corresponding to a radial mode) the following values are obtained:

\[
\int_0^{2\pi} \cos(m_\theta) \cos(m_\theta) \cos(m_\theta) d\theta = 2\pi \quad m_p = m_q = m_j = 0 \\
= \pi \quad m_p = 0, \quad m_q = m_j; \\
= \pi \quad m_q = 0, \quad m_p = m_j; \\
= \pi \quad m_j = 0, \quad m_p = m_q \quad \text{(C-26)}
\]
Subroutine AZIMTL consists of two sections. In the first section the azimuthal integral is expressed as the product of a constant factor and one of the basic forms given in Eqs. (C-23) and (C-24). The second section is essentially a series of logical tests to determine if the mode numbers, \( m_p \), \( m_q \), and \( m_j \) satisfy any of the conditions for Eqs. (C-23) through (C-27). If any of these conditions is satisfied the appropriate value is multiplied by the corresponding factor determined in the first section and the product is assigned to the output variable (i.e., RESULT), otherwise the value zero is assigned.

Radial Integrals. Subroutine RADIAL calculates the radial integrals which appear in Eqs. (C-5) through (C-8) according to NOPT as follows:

\[
\begin{align*}
\text{NOPT} &= 1 : \int_0^1 R R R_j r dr \\
\text{NOPT} &= 2 : \int_0^1 R R R_j^2 r dr \\
\text{NOPT} &= 3 : \int_0^1 R R R_j^3 r dr
\end{align*}
\]

where the \( R \)'s are the Bessel functions, \( J_m(\hat{S}_{mn} r) \). These integrals are computed numerically using Simpson's Rule with 100 subdivisions. In calculating the integrands the derivatives of the Bessel functions are given by:

\[
\begin{align*}
J_0'(S_{mn} r) &= -J_1(S_{mn} r) \\
J_m'(S_{mn} r) &= \frac{1}{2} \left[ J_{m-1}(S_{mn} r) - J_{m+1}(S_{mn} r) \right] \quad \text{for } m = 1, 2, 3, \ldots
\end{align*}
\]
The integrand of the second integral \((\text{NOPT} = 2)\) is indeterminate at the lower limit of integration. However a limit exists, denoted by \(L\), which vanishes with the following exceptions:

\[
L = S_{mn}^{(p)/2} \quad \text{for} \quad m_p = 1, \ m_q = m_j = 0
\]

\[
L = S_{mn}^{(q)/2} \quad \text{for} \quad m_q = 1, \ m_p = m_j = 0 \quad (C-29)
\]

\[
L = S_{mn}^{(j)/2} \quad \text{for} \quad m_j = 1, \ m_p = m_q = 0
\]

All of the calculations in Subroutine RADIAL are carried out in double precision arithmetic. The results are given as a single precision number.

Subroutine JBES computes the double precision Bessel functions which are needed for the above calculations. A description of this subroutine and a program listing are given in Chapter 23 of Ref. (18).

**Axial Integrals.** The integrals of the products of three axial eigenfunctions (see Eqs. (C-5) through (C-8)) are computed by Subroutine AXIAL2 according to the input parameters NOPT and NCONJ. The three basic forms are specified by NOPT as follows:

\[
\text{NOPT} = 1 : \quad \int_0^{z_e} Z_p Z_q Z_j^* dz
\]

\[
\text{NOPT} = 2 : \quad \int_0^{z_e} Z'_p Z'_q Z'_j^* dz
\]

\[
\text{NOPT} = 3 : \quad \int_0^{z_e} Z'_p Z'_q Z'_j dz
\]

When NCONJ = 1 these basic forms are calculated; these are the forms appearing in the expression for \(D_1(j, p, q)\) (see Eq. (C-5)). For NCONJ = 2 the second function in the integrand is replaced by its complex conjugate to obtain the
integrals appearing in the expression for $D_3(j, p, q)$. The integrals appearing in the expressions for $D_3(j, p, q)$ and $D_4(j, p, q)$ are obtained by setting $NCONJ = 3$ and $NCONJ = 4$ respectively.

The basic forms are calculated from the following analytical formulas:

\[
\int_0^\infty Z^e Z_3^* Z_4^* dz = \frac{1}{4} \left\{ \frac{\sinh[i(b_p + b_q + b_j)z]}{i(b_p + b_q + b_j)} \right. \\
+ \frac{\sinh[i(b_p + b_q - b_j)z]}{i(b_p + b_q - b_j)} \\
+ \frac{\sinh[i(b_p - b_q + b_j)z]}{i(b_p - b_q + b_j)} \\
+ \left. \frac{\sinh[i(b_p - b_q - b_j)z]}{i(b_p - b_q - b_j)} \right\} 
\]  

\[
\int_0^\infty Z^e Z_3^* Z_4^* dz = -\frac{1}{4} b_p b_q \left\{ \frac{\sinh[i(b_p + b_q + b_j)z]}{i(b_p + b_q + b_j)} \right. \\
+ \frac{\sinh[i(b_p + b_q - b_j)z]}{i(b_p + b_q - b_j)} \\
+ \frac{\sinh[i(b_p - b_q + b_j)z]}{i(b_p - b_q + b_j)} \\
- \left. \frac{\sinh[i(b_p - b_q - b_j)z]}{i(b_p - b_q - b_j)} \right\} 
\]  

\[
\int_0^\infty Z^e Z_3^* Z_4^* dz = -\frac{1}{2} \int_0^\infty Z^e Z_3^* Z_4^* dz 
\]
The remaining forms are obtained from Eqs. (C-30) through (C-32) by replacing the appropriate eigenvalues with their complex conjugates; thus, for $\text{NOPT} = 2$ $b_q$ is replaced by $b_q^*$, for $\text{NOPT} = 3$ $b_p$ is replaced with $b_p^*$, and both $b_p$ and $b_q$ are replaced by their conjugates for $\text{NOPT} = 4$.

FORTRAN listings for Subroutines AZIMTL, RADIAL, and AXIAL2 are given at the end of this appendix.

**Coefficients for Equivalent Real System.**

Equations (12) are a system of complex differential equations to be solved for the unknown complex amplitude functions, $A_p(t)$. In order to solve these equations numerically they must first be separated into their real and imaginary parts. This is done by assuming that $A_p(t) = F_p(t) + iG_p(t)$, substituting into Eqs. (12), and separating real and imaginary parts to obtain an equivalent system of real differential equations that describe the behavior of the $F_p$'s and $G_p$'s. Since these equations contain twice as many unknown functions (i.e., $F_p(t)$ and $G_p(t)$) as Eqs. (12), it is convenient to re-index the unknown functions and their coefficients as follows:

\[
\begin{align*}
F_p(t) &= B_{2p-1}(t) \\
G_p(t) &= B_{2p}(t)
\end{align*}
\]  

(C-33)

Thus the $B$'s with odd indices correspond to the real parts, $F_p(t)$, and the $B$'s with even indices correspond to the imaginary parts, $G_p(t)$. The corresponding set of differential equations is given by:

\[
\sum_{p=1}^{2N} \left\{ C_{0j}^p(j,p) \frac{d^2B_p}{dt^2} + C_{1j}^p(j,p)B_p(t) + \left[ C_{2j}^p(j,p) - nC_{3j}^p(j,p) \right] \frac{dB_p}{dt} + ... \right\}
\]
The real coefficients in Eqs. (C-34) (i.e., $C_0, C_1, C_3, C_4,$ and $D$) are relate to the complex coefficients in Eqs. (12) (i.e., $C_0, C_1, C_3, C_4, D_1, \ldots D_4$) as follows:

$$C_k(2j-1, 2p-1) = \text{Re} \left[ C_k(j, p) \right]$$

for $k = 0, 1, 2, 3, 4$ and $j = 1, 2, \ldots N$, $p = 1, 2, \ldots N$.

$$C_k'(2j-1, 2p-1) = \text{Im} \left[ C_k(j, p) \right]$$

$$C_k(2j, 2p) = \text{Re} \left[ C_k(j, p) \right]$$

$$C_k'(2j, 2p) = \text{Im} \left[ C_k(j, p) \right]$$

$$D'(2j-1, 2p-1, 2q-1) = \text{Re} \left[ D_1(j, p, q) + D_2(j, p, q) + D_3(j, p, q) + D_4(j, p, q) \right]$$

$$D'(2j-1, 2p-1, 2q) = \text{Im} \left[ -D_1(j, p, q) + D_2(j, p, q) - D_3(j, p, q) + D_4(j, p, q) \right]$$

$$D'(2j-1, 2p, 2q-1) = \text{Im} \left[ -D_1(j, p, q) - D_2(j, p, q) + D_3(j, p, q) + D_4(j, p, q) \right]$$

$$D'(2j-1, 2p, 2q) = \text{Re} \left[ -D_1(j, p, q) + D_2(j, p, q) + D_3(j, p, q) - D_4(j, p, q) \right]$$

(C-3)
\[ D'(2j,2p-1,2q-1) = \text{Im} \left[ D_1(j,p,q) + D_2(j,p,q) + D_3(j,p,q) + D_4(j,p,q) \right] \]
\[ D'(2j,2p-1,2q) = \text{Re} \left[ D_1(j,p,q) - D_2(j,p,q) + D_3(j,p,q) - D_4(j,p,q) \right] \]
\[ D'(2j,2p,2q-1) = \text{Re} \left[ D_1(j,p,q) + D_2(j,p,q) - D_3(j,p,q) - D_4(j,p,q) \right] \]
\[ D'(2j,2p,2q) = \text{Im} \left[ -D_1(j,p,q) + D_2(j,p,q) + D_3(j,p,q) - D_4(j,p,q) \right] \]

for \( j = 1, 2, \ldots, N, \quad p = 1, 2, \ldots, N, \quad q = 1, 2, \ldots, N \). The linear coefficients are stored in the arrays \( C_1(NJ,NP) \) for \( k = 0 \) and \( C(KC,NJ,NP) \) for \( k = 1, 2, 3 \). The nonlinear coefficients are stored in the array \( D(NJ,NP,NQ) \).

In general Eqs. (C-34) are coupled in the second derivatives; that is, they are of the form:

\[
\sum_{p=1}^{2N} \left\{ C'_0(j,p) \frac{d^2 B^p}{dt^2} \right\} = g_j(3_1, B_2, \ldots, B_{2N})
\]  

(C-37)

where there are two or more \( C'_0 \) terms in each equation. This coupling results from the non-orthogonality of the axial eigenfunctions. In order to numerically integrate Eqs. (C-34), they must be decoupled by transforming to the form:

\[
\frac{d^2 B^1}{dt^2} = f^1_j(B_1, B_2, \ldots, B_{2N})
\]  

(C-38)

in which only one second derivative appears in each equation. Using Eq. (C-38), it is seen that Eq. (C-37) can be expressed as

\[
C'_0 f = g.
\]  

(C-39)

where \( C'_0 \) is the \( 2N \times 2N \) matrix of coefficients of the coupled system, \( f \) is
the column matrix corresponding to the right-hand-side of the decoupled system, and \( g \) is the column matrix corresponding to the right-hand-side of the coupled system. To decouple Eqs. (C-37), therefore, Eq. (C-39) is solved for \( f \), thus:

\[
f = C_0^{-1} g
\]  

where \( C_0^{-1} \) is the inverse of the matrix \( C_0' \). Performing these operations and equating the coefficients of like terms in \( f \) and \( C_0^{-1} g \) gives the following relations:

\[
\begin{align*}
\tilde{C}_i(j,p) & = \sum_{k=1}^{2N} C_0^{-1}(j,k)C'_i(k,p) & i = 1, 2, 3 \\
\tilde{D}(j,p,q) & = \sum_{k=1}^{2N} C_0^{-1}(j,k)D'(k,p,q)
\end{align*}
\]

where \( \tilde{C}_i \) and \( \tilde{D} \) are the corresponding coefficients of the decoupled system. The matrix inverse, \( C_0^{-1} \), is computed by the subroutine GJR, which is a standard Univac 1108 library program, and is stored in the array \( C(NJ,NP) \). A listing of GJR and instructions for its use are given in Ref. (19).

The calculation of \( \tilde{C}_i(j,p) \) and \( \tilde{D}(j,p,q) \), which are the coefficients for the equivalent set of real, decoupled equations, is the final step in the computations performed by COEFFS3D. The coefficients are stored in the arrays \( C(KC,NJ,NP) \) and \( D(NJ,NP,NQ) \), replacing those computed from Eqs. (C-35) and (C-36). The output of these coefficients is described below.

Output

According to the value of the control number NOUT, the coefficients calculated by Program COEFFS3D are printed, punched onto cards, or stored on drum (FASTRAND). These three output modes will now be discussed individually.

Printed Output. Since the printed output cannot be used as input to
Program LCYC3D, the option "printed output only" (NOUT = 0) is only used for checkout purposes. Printed output can also be obtained in conjunction with the drum storage mode (NOUT = 1). Since the printed output format can only accommodate five series terms (complex), it should only be used for NJMAX ≤ 5.

The first page of printed output gives a restatement of the input parameters. This page is headed by the title of the case (TITLE) which is followed by the ratio of specific heats (GAMMA), the steady-state Mach number at the nozzle entrance (UE), the length-to-diameter ratio (L/D), and the length of the combustion zone as a fraction of the chamber length (ZCOMB). After statements concerning the presence or absence of the liquid droplet momentum source and the type of nozzle considered, a restatement of the input parameters J, L(J), M(J), N(J), NS(J), and NAME(J) which describe the terms in the series expansion of \( \varphi \) is given. This tabulation also includes additional parameters needed by Program LCYC3D: \( S_{mn} \), the dimensionless frequency of the mode (SMN); \( J_m(S_{mn}) \), the associated value of the Bessel function (JM(SMN)); the real part (EPS) and the imaginary part (ETA) of the axial acoustic eigenvalue; and the real part (YR) and imaginary part (YI) of the nozzle admittance.

The next three pages give the decoupled linear coefficients, \( \tilde{C}_1(j,p) \), \( \tilde{C}_2(j,p) \), and \( \tilde{C}_3(j,p) \). These coefficients are presented in the matrix format with the rows corresponding to the index j and the columns corresponding to the index p. The remaining pages give the decoupled nonlinear coefficients \( \tilde{D}(j,p,q) \) for each value of j. Here the rows correspond to the index p and the columns correspond to the index q.

A sample printed output for the five term series used in the sample input is given in Tables (C-2) through (C-4).

Drum Storage. When available drum storage, such as the FASTRAND system used with the Univac 1108, is the most convenient means of storing the output of Program COEFFS3D. In the absence of such a system, the program can be easily modified to store the coefficients on magnetic tape. In either case magnetic tape can be used as a back-up file or for permanent storage of the data. The control statements needed to execute these procedures depend upon the computer facilities being used and cannot be described in
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<td>.10617</td>
<td>.25115</td>
<td>.01414</td>
<td>.01414</td>
</tr>
<tr>
<td>B021</td>
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<td>0</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3.05424</td>
<td>.48650</td>
<td>.10617</td>
<td>.25115</td>
<td>.01414</td>
<td>.01414</td>
</tr>
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<td>B001</td>
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<td>.11993</td>
<td>.28170</td>
<td>.01414</td>
<td>.01414</td>
</tr>
</tbody>
</table>
Table C-3. Sample Printed Output, Page 2.

<table>
<thead>
<tr>
<th>J</th>
<th>1</th>
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<th>3</th>
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<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
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<td>1</td>
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<td>.000000</td>
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<td>.000000</td>
</tr>
<tr>
<td>2</td>
<td>-.000000</td>
<td>3.390599</td>
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</tr>
<tr>
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<td>.000000</td>
<td>.000000</td>
</tr>
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<td>.000000</td>
<td>.000000</td>
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<td>.000000</td>
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<td>.000000</td>
<td>.000000</td>
<td>9.330212</td>
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</tr>
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<td>.000000</td>
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<td>.000000</td>
<td>.000000</td>
<td>14.684905</td>
</tr>
</tbody>
</table>
Table C-4. Sample Printed Output, Page 5.

DECOUPLED COEFFICIENT OF B(P) = DB(Q)/DT IN EQUATION FOR B(1):

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>-1.73538</td>
<td>-0.15812</td>
<td>-2.33857</td>
<td>-0.026796</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>-0.010850</td>
<td>-0.00834</td>
<td>-0.013979</td>
<td>0.011946</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>1.73538</td>
<td>0.015812</td>
<td>-0.00834</td>
<td>-0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.010850</td>
<td>-0.00834</td>
<td>-0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>0.00000</td>
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<td>0.00000</td>
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<td>0.00000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.015813</td>
<td>0.000000</td>
<td>0.00000</td>
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<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>-1.497831</td>
<td>-0.00834</td>
<td>0.000000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>0.015813</td>
<td>0.007594</td>
<td>0.000000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>-1.962611</td>
<td>-0.010850</td>
<td>0.000000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>-0.024780</td>
<td>0.09986</td>
<td>0.000000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>
Card Output. When a drum or magnetic tape storage is not available, punched card output can be used (NOUT = 3). This method becomes unwieldy, however, when a large number of coefficients is involved since only one coefficient can be punched on a card. The format for both drum and card output is the same and is given below:

<table>
<thead>
<tr>
<th>Number of Cards</th>
<th>Location</th>
<th>Type</th>
<th>Output Item</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1-10</td>
<td>F</td>
<td>GAMMA</td>
<td>Same as for input.</td>
</tr>
<tr>
<td></td>
<td>11-20</td>
<td>F</td>
<td>UE</td>
<td>Same as for input.</td>
</tr>
<tr>
<td></td>
<td>21-30</td>
<td>F</td>
<td>ZE</td>
<td>Dimensionless chamber length, ((2L/D)).</td>
</tr>
<tr>
<td></td>
<td>31-40</td>
<td>F</td>
<td>ZCOMB</td>
<td>Same as for input.</td>
</tr>
<tr>
<td></td>
<td>41-45</td>
<td>I</td>
<td>NDROPS</td>
<td>Same as for input.</td>
</tr>
<tr>
<td></td>
<td>46-50</td>
<td>I</td>
<td>NJMAX</td>
<td>Number of unknown functions, (B_p(t)) (see Eq. (C-34)).</td>
</tr>
<tr>
<td>(NJMAX/2)</td>
<td>1-5</td>
<td>I</td>
<td>J</td>
<td>Same as input.</td>
</tr>
<tr>
<td></td>
<td>6-10</td>
<td>I</td>
<td>L(J)</td>
<td>&quot;</td>
</tr>
<tr>
<td></td>
<td>11-15</td>
<td>I</td>
<td>M(J)</td>
<td>&quot;</td>
</tr>
<tr>
<td></td>
<td>16-20</td>
<td>I</td>
<td>N(J)</td>
<td>&quot;</td>
</tr>
<tr>
<td></td>
<td>21-25</td>
<td>I</td>
<td>NS(J)</td>
<td>&quot;</td>
</tr>
<tr>
<td></td>
<td>26-35</td>
<td>F</td>
<td>S(J)</td>
<td>Root of Bessel function, (S_{mn}).</td>
</tr>
<tr>
<td></td>
<td>36-45</td>
<td>F</td>
<td>SJ(J)</td>
<td>Associated value of Bessel function, (J_m(S_{mn})).</td>
</tr>
<tr>
<td></td>
<td>46-50</td>
<td>A</td>
<td>NAME(J)</td>
<td>Same as input.</td>
</tr>
<tr>
<td>(NJMAX/2)</td>
<td>1-5</td>
<td>I</td>
<td>J</td>
<td>Same as input.</td>
</tr>
<tr>
<td></td>
<td>6-15</td>
<td>F</td>
<td>YR</td>
<td>Real part of nozzle admittance, (Y_r).</td>
</tr>
<tr>
<td></td>
<td>16-25</td>
<td>F</td>
<td>YI</td>
<td>Imaginary part of nozzle admittance, (Y_i).</td>
</tr>
<tr>
<td></td>
<td>26-35</td>
<td>F</td>
<td>EPS</td>
<td>Real part of axial eigenvalue, (e).</td>
</tr>
<tr>
<td></td>
<td>36-45</td>
<td>F</td>
<td>ETA</td>
<td>Imaginary part of axial eigenvalue, (\eta).</td>
</tr>
<tr>
<td>Number of Cards</td>
<td>Location</td>
<td>Type</td>
<td>Output Item</td>
<td>Comments</td>
</tr>
<tr>
<td>-----------------</td>
<td>----------</td>
<td>------</td>
<td>-------------</td>
<td>----------</td>
</tr>
<tr>
<td>1</td>
<td>1-5</td>
<td>I</td>
<td>KMAX(1)</td>
<td>Number of nonzero linear coefficients of type $\tilde{C}_1(j,p)$</td>
</tr>
<tr>
<td>KMAX(1)</td>
<td>1-5</td>
<td>I</td>
<td>NJ</td>
<td>Index, j</td>
</tr>
<tr>
<td></td>
<td>6-10</td>
<td>I</td>
<td>NP</td>
<td>Index, p</td>
</tr>
<tr>
<td></td>
<td>11-25</td>
<td>F</td>
<td>C(1,NJ,NP)</td>
<td>Linear coefficient, $\tilde{C}_1(j,p)$</td>
</tr>
<tr>
<td>1</td>
<td>1-5</td>
<td>I</td>
<td>KMAX(2)</td>
<td>Number of nonzero linear coefficients of type $\tilde{C}_2(j,p)$</td>
</tr>
<tr>
<td>KMAX(2)</td>
<td>1-5</td>
<td>I</td>
<td>NJ</td>
<td>Index, j</td>
</tr>
<tr>
<td></td>
<td>6-10</td>
<td>I</td>
<td>NP</td>
<td>Index, p</td>
</tr>
<tr>
<td></td>
<td>11-25</td>
<td>F</td>
<td>C(2,NJ,NP)</td>
<td>Linear coefficient, $\tilde{C}_2(j,p)$</td>
</tr>
<tr>
<td>1</td>
<td>1-5</td>
<td>I</td>
<td>KMAX(3)</td>
<td>Number of nonzero linear coefficients of type $\tilde{C}_3(j,p)$</td>
</tr>
<tr>
<td>KMAX(3)</td>
<td>1-5</td>
<td>I</td>
<td>NJ</td>
<td>Index, j</td>
</tr>
<tr>
<td></td>
<td>6-10</td>
<td>I</td>
<td>NP</td>
<td>Index, p</td>
</tr>
<tr>
<td></td>
<td>11-25</td>
<td>F</td>
<td>C(3,NJ,NP)</td>
<td>Linear coefficient, $\tilde{C}_3(j,p)$</td>
</tr>
<tr>
<td>1</td>
<td>1-5</td>
<td>I</td>
<td>KMAX(4)</td>
<td>Number of nonzero nonlinear coefficients</td>
</tr>
<tr>
<td>KMAX(4)</td>
<td>1-5</td>
<td>I</td>
<td>NJ</td>
<td>Index, j</td>
</tr>
<tr>
<td></td>
<td>6-10</td>
<td>I</td>
<td>NP</td>
<td>Index, p</td>
</tr>
<tr>
<td></td>
<td>11-15</td>
<td>I</td>
<td>NQ</td>
<td>Index, q</td>
</tr>
<tr>
<td></td>
<td>16-30</td>
<td>F</td>
<td>D(NJ,NP,NQ)</td>
<td>Nonlinear coefficient, $D(j,p,q)$</td>
</tr>
</tbody>
</table>

The first card of output gives the chamber parameters $\gamma$, $\bar{u}_e$, $L/D$, and $z_c/z_e$; the droplet momentum source control number, NDROPS; and the number of unknown real functions (i.e., $B_p(t)$), NJMAX. This is followed by $NJMAX/2$ cards (the number of unknown complex functions, $A_p(t)$) describing the terms included in the series expansion of $\tilde{\phi}$. The next $NJMAX/2$ cards gives the complex nozzle admittance ($Y_r$ and $Y_i$) and the corresponding complex axial eigenvalue ($\epsilon$ and $\eta$) for each complex series term. The linear coefficients are given in three sets of cards. The first card in the set gives the number of coefficients of the given type, while the remaining
cards give the indices \( j \) and \( p \) and the coefficient \( \tilde{c}_1(j,p) \). The next card gives the number of nonlinear coefficients and is followed by cards giving the indices \( j, p, q \) and the corresponding coefficient \( D(j,p,q) \). Both linear and nonlinear coefficients are given in a field of 15 spaces with six decimal places. For \( \text{NEGL} = 0 \) only the nonzero coefficients (absolute value greater than \( 10^{-5} \)) are given, while for \( \text{NEGL} = 1 \) only linear coefficients with absolute value greater than \( \text{SM1} \) and nonlinear coefficients with absolute value greater than \( \text{SM2} \) are given.

A sample card output produced by the sample input of Table (C-1) is given in Table (C-5) below.

**Table C-5. Sample Card Output.**

<table>
<thead>
<tr>
<th>Card</th>
<th>( j )</th>
<th>( p )</th>
<th>( q )</th>
<th>( \tilde{c}_1 )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1.84118</td>
<td>.58187</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1.84118</td>
<td>.58187</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>3.05424</td>
<td>.48650</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>3.05424</td>
<td>.48650</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3.83171</td>
<td>-.40276</td>
</tr>
<tr>
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<td>0.01414</td>
<td>.01414</td>
<td>.08422</td>
<td>.19451</td>
<td></td>
</tr>
<tr>
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<td>0.01414</td>
<td>.01414</td>
<td>.08422</td>
<td>.19451</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.01414</td>
<td>.01414</td>
<td>.10617</td>
<td>.25115</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.01414</td>
<td>.01414</td>
<td>.10617</td>
<td>.25115</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.01414</td>
<td>.01414</td>
<td>.11993</td>
<td>.28170</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3.59059</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>14.684905</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.61527</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>2.66538</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
C
C ****************************PROGRAM COEFFS3D**************************
C
C THIS PROGRAM COMPUTES THE COEFFICIENTS WHICH APPEAR
C IN THE DIFFERENTIAL EQUATIONS WHICH GOVERN THE MODE-AMPLITUDE
C FUNCTIONS. THESE COEFFICIENTS ARE STORED ON DRUM OR
C PUNCHED ONTO CARDS FOR INPUT INTO PROGRAM LCYC3D.*
C
C THE FOLLOWING INPUTS ARE REQUIRED:
C THE TITLE OF THE CASE*
C GAMMA IS THE SPECIFIC HEAT RATIO.
C UE IS THE STEADY STATE MACH NUMBER AT THE NOZZLE ENTRANCE.
C RLD IS THE LENGTH-TO-DIAMETER RATIO.
C ZCOMB IS THE LENGTH OF THE REGION OF UNIFORMLY DISTRIBUTED
C COMBUSTION, EXPRESSED AS A FRACTION OF THE CHAMBER LENGTH.
C NDROPS DETERMINES THE PRESENCE OF DROPLET MOMENTUM SOURCES:
C NDROPS = 0 DROPLET MOMENTUM SOURCE NEGLECTED.
C NDROPS = 1 DROPLET MOMENTUM SOURCE INCLUDED.
C NOZZLE SPECIFIES THE TYPE OF NOZZLE USED:
C NOZZLE = 0 QUASI-STEADY.
C NOZZLE = 1 CONVENTIONAL NOZZLE.
C FOR CONVENTIONAL NOZZLE:
C AMPL IS THE NOZZLE AMPLITUDE RATIO.
C PHASE IS THE NOZZLE PHASE SHIFT.
C NJMAX IS THE NUMBER OF MODE-AMPLITUDE FUNCTIONS IN THE ASSUMED
C SERIES SOLUTION. NJMAX MUST NOT EXCEED 10.
C THE COEFFICIENTS COMPUTED ARE DETERMINED BY NONLIN AS FOLLOWS:
C NONLIN = 0 LINEAR COEFFICIENTS ONLY.
C NONLIN = 1 BOTH LINEAR AND NONLINEAR COEFFICIENTS.
C COEFFICIENTS TO BE NEGLECTED ARE DETERMINED BY NEGL
C AS FOLLOWS:
C NEGL = 0 TERMS SMALLER THAN 0.00001 ARE NEGLECTED.
C NEGL = 1 LINEAR TERMS SMALLER THAN SM1 AND NONLINEAR
C TERMS SMALLER THAN SM2 ARE NEGLECTED.
C THE OUTPUT IS DETERMINED BY NOUT AS FOLLOWS:
C NOUT = 0 PRINTED OUTPUT ONLY.
C NOUT = 1 PRINTED AND STORED ON DRUM (FASTRAND FILE).
C NOUT = 2 FASTRAND FILE ONLY.
C NOUT = 3 CARD OUTPUT ONLY.
C EACH MODE-AMPLITUDE IS ASSIGNED AN INTEGER J.
C THE MODE IS SPECIFIED BY THE INDICES L(J), M(J), AND N(J).
C L(J) IS THE AXIAL MODE NUMBER AND MUST NOT EXCEED 10.
C M(J) IS THE AZIMUTHAL MODE NUMBER AND MUST NOT EXCEED 8.
C N(J) IS THE RADIAL MODE NUMBER AND MUST NOT EXCEED 5.
C THE INTEGER NS(J) IS ASSIGNED AS FOLLOWS:
C NS = 1 A-FUNCTION SIN(M*THETA) * COSH(I*B*Z)
C NS = 2 B-FUNCTION COS(M*THETA) * COSH(I*B*Z)
C NAME(J) IS A FOUR-CHARACTER NAME.*
DIMENSION L(10), NX(10), NAME(10), S(10), SJ(10), TITLE(10)
RHO0(10), RVAL(10,5), CI(10,20), CC(3,20,20)

D(20,20,20), AMPL(10), PHASE(10), AZI(2)

BES1(9,9,9), BES2(9,9,9), BES3(9,9,9)

V(x), JC(20), TS(x,20), TS(20,20), KMAX(4)

COMPLEX

CRSLT, CI, ZEJ, ZEF, ZEF2, ZE2, CAZ, CRAD

G1, DCOEF, CGAM, CA, BC(10), BC(10), YNOZ(10)

CNO(10), CSSO(10), TANINT(2), RAIINT(3)

AXINT(a,3), CC(a,10), CD(a,10,10)

CD(a,10,10,10), AX(a), T2, DI, D2, D3, D4

COMMON

B /BLK2/, M(10), NS(10)

C

DATA INPUT.

INPUT ROOTS AND VALUES OF BESSEL FUNCTIONS.

DATA CI, P0, I = 1,5, I = 1,9

1 3.63171, 7.01559, 10.17347, 13.32369, 16.47063

2 1.84118, 5.33144, 8.53632, 11.70600, 14.86359

3 3.05424, 6.70613, 9.96947, 13.17037, 16.34752

4 5.31755, 9.28240, 12.68191, 15.90411, 19.16033

5 6.41562, 10.51966, 13.98719, 17.31284, 20.57551

6 7.50127, 11.73494, 15.26818, 18.63744, 21.93172

7 8.57784, 12.93239, 16.52937, 19.94185, 23.26805

8 9.64742, 14.11552, 17.77401, 21.22906, 24.58720

DATA CI, P0, I = 1,5, I = 1,9

1 -0.40276, 0.30012, 0.24970, 0.21836, -0.19647

2 -0.58187, -0.34613, -0.27330, -0.23330, -0.20701

3 -0.68650, -0.31353, -0.25474, -0.22086, -0.19794

4 -0.43439, -0.29116, -0.24074, -0.20970, -0.19042

5 -0.39965, -0.27438, -0.22959, -0.20276, -0.18403

6 -0.28709, -0.22105, -0.19039, -0.16580, -0.14789

7 -0.25414, -0.20517, -0.17661, -0.15588, -0.13688

8 -0.37993, -0.30096, -0.24096, -0.18486, -0.16989

9 -0.32438, -0.28303, -0.22998, -0.19799, -0.16539

C

INPUT PARAMETERS.

READ (5,5000, END = 600) TITLE(I), I = 1,72

READ (5,5001) GAMMA, UEF, R, ZCOMB, NDROPS, NOZZLE

IF (GAMMA) 600, 600, 8

READ (5,5004) NMAX, NONLIN, NEGL, NOUT

IF (NEGL *.EQ. 1) READ (5,5005) SM1, SM2.

72
IF (NOZZLE .EQ. 1) GO TO 5
C COMPUTE ADMITTANCE FOR QUASI-STEADY NOZZLE.
Y = (GAMMA - 1.0) * UE/(2.0 * GAMMA)
DO 3 J = 1, NJMAX
AMPL(J) = Y
PHASE(J) = 0.0
3 CONTINUE
GO TO 7
C DO 6 I = 1, NJMAX
READ (5,5003) J, AMPL(J), PHASE(J)
6 CONTINUE
C DO 10 I = 1, NJMAX
READ (5,5002) J, L(J), M(J), N(J), NS(J), NAME(J)
10 CONTINUE
C DO 12 J = 1, NJMAX
THETA = PHASE(J) * PI/180.0
YR = AMPL(J) * COS(THETA)
YI = AMPL(J) * SIN(THETA)
YNOZ(J) = ONFLXCYR,YI)
12 CONTINUE
C ZE = 2.0 * RLD
CZE = CMPLX(ZE,0.0)
CGAM = CMPLX(GAMMA,0.0)
CAX = CGAM
IF (NDROPS .EQ. 1) CAX = CMPLX(1.0,0.0)
****************************************************************
C ASSIGN ARRAYS FOR ROOTS OF BESSEL FUNCTIONS.
DO 20 J = 1, NJMAX
IF ((M(J) .EQ. 0) .AND. (N(J) .EQ. 0)) GO TO 15
MM = M(J) + 1
NN = N(J)
S(J) = RJROOT(MM,NN)
SJ(J) = RJVAL(MM,NN)
GO TO 25
15 S(J) = 0.0
SJ(J) = 1.0
25 CSSQ(J) = CMPLX(SSQ,0.0)
20 CONTINUE
****************************************************************
C CALCULATE AXIAL ACOUSTIC EIGENVALUES.
C FIND MAXIMUM VALUES OF L(J), M(J), AND N(J).
KN = 0
LMAX = 0
MMAX = 0
NMAX = 0
DO 30 J = 1, NJMAX
  IF (L(J) .GT. LMAX) LMAX = L(J)
  IF (M(J) .GT. MMAX) MMAX = M(J)
  IF (N(J) .GT. NMAX) NMAX = N(J)
  IF (N(J) .NE. N(1)) KN = 1
30 CONTINUE
LMAX = LMAX + 1
MMAX = MMAX + 1

C COMPUTE EIGENVALUES.
   DO 40 J = 1, NJMAX
     LL = L(J)
     SMN = S(J)
     YAMPL = AMPL(J)
     YPHASE = PHASE(J)
     CALL EIGVAL(LL, SMN, GAMMA*Z, YAMPL, YPHASE, CRSLT)
     BC(J) = CRSLT
     BC(J) = CONJG(CRSLT)
40 CONTINUE

C ************************************************************
C CALCULATE LINEAR COEFFICIENTS.
C   DO 100 NJ = 1, NJMAX
   DO 100 NP = 1, NJMAX
C ZERO COEFFICIENT ARRAYS.
   DO 105 KC = 1, 4
     CC(KC,NJ,NP) = (0.0,0.0)
   105 CONTINUE
C ORTHOGONALITY PROPERTY OF TANGENTIAL EIGENFUNCTIONS.
   IF (NS(NP) .NE. NS(NJ)) GO TO 100
   IF (M(NP) .NE. M(NJ)) GO TO 100
   IF (H(NJ) .EQ. 0) GO TO 112
   AZ = PI
   GO TO 120
112 IF (NS(NJ) .EQ. 1) GO TO 100
   AZ = 2.0 * PI
C ORTHOGONALITY PROPERTY OF RADIAL EIGENFUNCTIONS.
   120 IF (NP(NP) .NE. N(NJ)) GO TO 100
   IF (S(NP)) 125, 122, 125
125 SGM = M(NJ) * M(NJ)
   SSQ = S(NP) * S(NP)
   SJSQ = SJ(NJ) * SJ(NJ)
RAD = (SSQ - SQM) * SJSQ/(2.0 * SSQ)
GO TO 127

C

122 RAD = 0.5

C

CALCULATE AXIAL INTEGRALS.

127 DO 130 NOPT = 1, 4
CALL AXIAL1(NOPT,NP,NJ,UE,ZE,ZCOMB,CRSLT)
AX(NOPT) = CRSLT

130 CONTINUE

C

EVALUATE FUNCTIONS AT NOZZLE END.
ZEJ = CCOSH(CI*BC(NJ)*CZE)
ZEP1 = CCOSH(CI*B(NP)*CZE)
ZEP2 = CI * B(NP) * CSINH(CI*B(NP)*CZE)

C

CAZ = CMPLX(AZ,0.0)
CRAD = CMPLX(RAD,0.0)

C

COEFFICIENT OF THE SECOND DERIVATIVE OF A(P).
CC(1,NJ,NP) = AX(1) * CAZ * CRAD

C

COEFFICIENT OF A(P).
CC(2,NJ,NP) = (CSSQ(NP)*AX(1) - AX(2) + ZEP2*ZEJ) * CAZ * CRAD

C

COEFFICIENT OF THE FIRST DERIVATIVE OF A(P).
CC(3,NJ,NP) = (CAK*AX(3) + (2.0,0.0)*AX(4)
1 + CGAM*YNO2(NP)*ZEP1*ZEJ) * CAZ * CRAD

C

COEFFICIENT OF THE RETARDED DERIVATIVE OF A(P).
CC(4,NJ,NP) = CGAM * AX(3) * CAZ * CRAD

100 CONTINUE

C

NORMALIZE LINEAR COEFFICIENTS.
DO 140 NJ = 1, NJMAX
CNORM(NJ) = CC(NJ,NJ,NJ)
DO 140 NP = 1, NJMAX
DO 140 KC = 1, 4
CC(KC,NJ,NP) = CC(KC,NJ,NP)/CNORM(NJ)

140 CONTINUE

C

****************************************************************

C

COMPUTE NONLINEAR COEFFICIENTS.
C

IF (NONLIN EQ 0) GO TO 402
G1 = (CGAM - (1.0,0.0)) * (0.5,0.0)

C

COMPUTATIONS OF BESSEL INTEGRALS WHEN ALL SERIES TERMS HAVE THE
SAME RADIAL MODE NUMBER N(J).
IF (KN .EQ. 1) GO TO 170
DO 150 MP = 1, MMAX
DO 150 MQ = 1, MMAX
DO 150 MJ = 1, MMAX
BES1(MP, MQ, MJ) = 0.0
BES2(MP, MQ, MJ) = 0.0
BES3(MP, MQ, MJ) = 0.0
L1 = MP - 1
L2 = MQ - 1
L3 = MJ - 1
LM = L1 + L2
LN = L1 + L3
MN = L2 + L3
IF ((LM .EQ. LN) .OR. (L2 .EQ. LN) .OR. (L1 .EQ. MN)) GO TO 160
GO TO 150
150 CONTINUE
C
170 DO 200 NJ = 1, NJMAX
DO 200 NP = 1, NJMAX
DO 200 NQ = 1, NJMAX
CD1(NJ, NP, NQ) = (0.0, 0.0)
CD2(NJ, NP, NQ) = (0.0, 0.0)
C
DO 210 J = 1, 2
CALL AZIMITL(J, NP, NQ, NJ, RESULT)
AZI(J) = RESULT
TANINTC(J) = CMPLX(RESULT, 0.0)
210 CONTINUE
C
220 IF (AZI(1)) 220, 225, 220
225 IF (AZI(2)) 220, 200, 220
C
220 IF (KN .EQ. 0) GO TO 222
L1 = M(NP)
L2 = M(NQ)
L3 = M(NJ)
A1 = S(NP)
A2 = S(NQ)
A3 = S(NJ)
GO TO 244

C
222 MP = M(NP) + 1
MQ = M(NQ) + 1
MJ = M(NJ) + 1
RADINT(1) = CMPLX(BES1(MP, MQ, MJ), 0.0)
RADINT(2) = CMPLX(BES2(MP, MQ, MJ), 0.0)
RADINT(3) = CMPLX(BES3(MP, MQ, MJ), 0.0)

C
244 DO 240 J = 1, 3
IF (KN .EQ. 0) GO TO 242
CALL RADIAL (J, L1, L2, L3, A1, A2, A3, RESULT)
RADINT(J) = CMPLX(RESULT, 0.0)

242 DO 240 NC = 1, 4
CALL AXIAL2(J, NC, NP, NJ, ZE, CRSLT)
AXINT(NC, J) = CRSLT

240 CONTINUE

C
C
C
DO 250 J = 1, 4
T1 = G1 * CSSQ(NP) * AXINT(J, 1)
T2 = G1 * AXINT(J, 3)
D1 = AXINT(J, 1) * TANINT(1) * RADINT(3)
D2 = AXINT(J, 1) * TANINT(2) * RADINT(2)
D3 = AXINT(J, 2) * TANINT(1) * RADINT(1)
D4 = (T2 - T1) * TANINT(1) * RADINT(1)
DCOEF = (0.5 * 0.0) * (D1 + D2 + D3 + D4) / CNORM(NJ)
IF (J .EQ. 1) CD1(NJ, NP, NJ) = (1.0, -1.0) * DCOEF
IF (J .EQ. 3) CD3(NJ, NP, NJ) = (1.0, -1.0) * DCOEF
IF (J .EQ. 4) CD4(NJ, NP, NJ) = (1.0, -1.0) * DCOEF

250 CONTINUE

200 CONTINUE

C
C
C
****************************************************************
C
CALCULATE COEFFICIENTS FOR EQUIVALENT REAL SYSTEM.
C
402 DO 350 NJ = 1, NJMAX
NEWJ = (2 * NJ) - 1
NEWJ1 = NEWJ + 1
DO 350 NP = 1, NJMAX
NEWP = (2 * NP) - 1
NEWP1 = NEWP + 1
C
C
COEFFICIENTS OF LINEAR TERMS.

C
CCR = REAL(CC(1,NJ,NP))
CCI = AIMAG(CC(1,NJ,NP))
C1(NEWJ,NEWP) = CCR
C1(NEWJ,NEWP1) = -CCI
C1(NEWJ1,NEWP) = CCI
C1(NEWJ1,NEWP1) = CCR
DO 360 KC = 1,3
CCR = REAL(CC(KC+1,NJ,NP))
CCI = AIMAG(CC(KC+1,NJ,NP))
C(KC,NEWJ,NEWP) = CCR
C(KC,NEWJ,NEWP1) = -CCI
C(KC,NEWJ1,NEWP) = CCI
C(KC,NEWJ1,NEWP1) = CCR
360 CONTINUE
C
C COEFFICIENTS OF NONLINEAR TERMS.
IF (NONLIN .EQ. 0)' 
DO 370 NO = 1, NJMAX
NEWQ = (2 * NO) - 1
NEWQ1 = NEWQ + 1
CD1R = REAL(CD1(NJ,NEWP,NO))
CD1I = AIMAG(CD1(NJ,NEWP,NO))
CD2R = REAL(CD2(NJ,NEWP,NO))
CD2I = AIMAG(CD2(NJ,NEWP,NO))
CD3R = REAL(CD3(NJ,NEWP,NO))
CD3I = AIMAG(CD3(NJ,NEWP,NO))
CD4R = REAL(CD4(NJ,NEWP,NO))
CD4I = AIMAG(CD4(NJ,NEWP,NO))
D(NEWJ,NEWP,NEWQ) = CD1R + CD2R + CD3R + CD4R
D(NEWJ,NEWP,NEWQ1) = -CD1I + CD2I - CD3I + CD4I
D(NEWJ,NEWP1,NEWQ) = -CD1I - CD2I + CD3I + CD4I
D(NEWJ,NEWP1,NEWQ1) = -CD1I + CD2I - CD3I + CD4I
D(NEWJ1,NEWP,NEWQ) = CD1I + CD2I + CD3I + CD4I
D(NEWJ1,NEWP1,NEWQ) = CD1R + CD2R + CD3R + CD4R
D(NEWJ1,NEWP1,NEWQ1) = -CD1I + CD2I + CD3I + CD4I
370 CONTINUE
350 CONTINUE
C
C ***************************************************************
C
C COMPUTE COEFFICIENTS FOR THE EQUATIONS WHICH ARE DECOUPLED
C IN THE SECOND DERIVATIVES.
C
DO 405 KC = 1, 4
KMAX(KC) = 0
405 CONTINUE
C
C CALCULATE INVERSE OF THE MATRIX C1(I,J).
JMAX = NJMAX
NJMAX = 2 * NJMAX

V(1) = 1
CALL GJR(C1,20,20,NJMAX,0,$500,JC,VC)

USE INVERSE TO CALCULATE DECOUPLED COEFFICIENTS.

DO 410 NP = 1, NJMAX

LINEAR COEFFICIENTS.
DO 420 NJ = 1, NJMAX
DO 420 KC = 1, 3
TS(KC,NJ) = 0.0
DO 420 K = 1, NJMAX
TS(KC,NJ) = TS(KC,NJ) + C1(NJ,K) * C(KC,K)*NP)
420 CONTINUE
DO 430 NJ = 1, NJMAX
DO 430 KC = 1, -3
C(KC,NJ,NP) = TS(KC,NJ)
ABSVAL = ABS(C(KC,NJ,NP))
IF (ABSVAL .GE. SM1) KMAX(KC) = KMAX(KC) + 1
430 CONTINUE

NONLINEAR COEFFICIENTS.
IF ( CNONLIN .EQ. 0) GO TO 410
DO 415 NO = 1, NJMAX
DO 440 NJ = 1, NJMAX
TSIUNJ) = 0.0
DO 440 K = 1, NJMAX
TSA(NJ) = TSA(NJ) + C1(NJ,K) * D(11,NP,NA)
440 CONTINUE
DO 445 NJ = 1, NJMAX
D(NJDNP,N0) = TSO(NJ)
ABSVAL = ABS(D(NJ,NP,NQ))
IF (ABSVAL .GE. SM2) KMAX(4) = KMAX(4) + 1
445 CONTINUE

410 CONTINUE

****************************************************************
OUTPUT.
IF (NOUT .GE. 2) GO TO 455

PRINTED OUTPUT.
WRITE (6,6001) (TITLE(I), I = 1, 72)
WRITE (6,6002) GAMMA, UE, RLD, ZCOMB
IF (NDROPS .EQ. 0) WRITE (6,6020)
IF (NDROPS .EQ. 1) WRITE (6, 6021)
IF (NOZZLE .EQ. 0) WRITE (6, 6012)
WRITE (6, 6004)
DO 310 J = 1, JMAX
WRITE (6, 6003) NAME(J), J, L(J), M(J), N(J), NS(J),
1 S(J), SJ(J), B(J), YNOZ(J)
310 CONTINUE
IF (NONLIN .EQ. 0) WRITE (6, 6013)
C
C OUTPUT OF LINEAR COEFFICIENTS.
DO 320 KC = 1, 3
IF (KC .EQ. 1) WRITE (6, 6005)
IF (KC .EQ. 2) WRITE (6, 6006)
IF (KC .EQ. 3) WRITE (6, 6007)
WRITE (6, 6008) (J, J = 1, NMAX)
WRITE (6, 6014)
DO 320 NJ = 1, NMAX
WRITE (6, 6009) NJ, (C(KC*NJNP), NP = 1, NMAX)
320 CONTINUE
C
C OUTPUT OF NONLINEAR COEFFICIENTS.
IF (NONLIN .EQ. 0) GO TO 452
DO 400 NJ = 1, NMAX
WRITE (6, 6010) NJ
WRITE (6, 6011) (J, J = 1, NMAX)
WRITE (6, 6015)
DO 400 NP = 1, NMAX
WRITE (6, 6009) NP, (D(NJNP, NP), NP = 1, NMAX)
400 CONTINUE
452 IF (NOUT .EQ. 0) GO TO 4
C
455 IF (NOUT .EQ. 3) GO TO 480
C
WRITE COEFFICIENTS ON FASTRAND FILE.
C
WRITE (9, 7001) GAMMA, UE,ZE, ZCOMB, NDROPS, NMAX
C
DO 450 J = 1, JMAX
WRITE (9, 7002) J, L(J), M(J), N(J), NS(J), S(J), SJ(J),
1 NAME(J)
450 CONTINUE
C
DO 457 J = 1, JMAX
WRITE (9, 7006) J, YNOZ(J), B(J)
457 CONTINUE
C
DO 460 KC = 1, 3
WRITE (9, 7003) KMAX(KC)
DO 460 NJ = 1, NMAX
DO 460 NP = 1, NMAX

ABSVAL = ABS(C(KC,NJ,NP))

IF (ABSVAL .GE. SM1) WRITE (9,7004) NJ, NP, C(KC,NJ,NP)

460 CONTINUE

C
WRITE (9,7003) KMAX(4)
IF (NONLIN .EQ. 0) GO TO 4
DO 470 NJ = 1, NJMAX
DO 470 NP = 1, NJMAX
DO 470 NQ = 1, NJMAX
ABSVAL = ABS(D(NJ,NP,NQ))
IF (ABSVAL .GE. SM2) WRITE (9,7005) NJ, NP, NQ, D(NJ,NP,NQ)

470 CONTINUE
GO TO 4
C
PUNCHED CARD OUTPUT
C
480 PUNCH 7001 GAMMA, UE, ZE, ZCOMB, NDROPS, NJMAX
C
DO 482 J = 1, JMAX
PUNCH 7002 J, L(J), M(J), N(J), NS(J), S(J), SJ(J), NAME(J)

482 CONTINUE
C
DO 484 J = 1, JMAX
PUNCH 7006 J, YNOZ(J), B(J)

484 CONTINUE
C
DO 486 KC = 1, 3
PUNCH 7003 KMAX(KC)
DO 486 NJ = 1, NJMAX
DO 486 NP = 1, NJMAX
ABSVAL = ABS(C(KC,NJ,NP))
IF (ABSVAL .GE. SM1) PUNCH 7004 NJ, NP, C(KC,NJ,NP)

486 CONTINUE
C
PUNCH 7003 KMAX(4)
IF (NONLIN .EQ. 0) GO TO 4
DO 488 NJ = 1, NJMAX
DO 488 NP = 1, NJMAX
DO 488 NQ = 1, NJMAX
ABSVAL = ABS(D(NJ,NP,NQ))
IF (ABSVAL .GE. SM2) PUNCH 7005 NJ, NP, NQ, D(NJ,NP,NQ)

488 CONTINUE
GO TO 4
C
ERROR EXIT
500 IF (JC(1)) 510, 510, 520
510 JC(1) = ABS(JC(1))
WRITE (6,6017) JC(1)
GO TO 4
520 WRITE (6,6018) JC(1)
GO TO 4
600 CONTINUE
C
C ***************************************************************
C FORMATT SPECIFICATIONS.
C 5000 FORMAT (72A1)
5001 FORMAT (4F10.0,215)
5002 FORMAT (5I5*1X,4A4)
5003 FORMAT (5I5*2F10.0)
5004 FORMAT (4I5)
5005 FORMAT (2F10.0)
6001 FORMAT (1H1,1X,72A1//)
6002 FORMAT (2X,8HGAMMA = ,F5.2,5X,5HUE = ,F5.2,5X,6HL/D = ,F8.5,
1 5X,6HZCOMB = ,F5.2/)       72
6003 FORMAT (2X,A4*5I5,6F10.5/)     72
6004 FORMAT (2X///2X,29HNAME J . \ M 
7HJM,7X,3HEPS,7X,3HETAS,8X,2HYR,6X,2HYI///)
6005 FORMAT (1H1,45H DECOUPLED COEFFICIENT OF B(P): C(1,J,P)///
6006 FORMAT (1H1,44H DECOUPLED COEFFICIENT OF THE DERIVATIVE OF,
1 6H B(P)*,5X,8HC(2,J,P)///)
6007 FORMAT (1H1,39H DECOUPLED COEFFICIENT OF THE RETARDED,
1 20H IN EQUATION FOR B(P) = 5X,8HC(3,J,P)///)
6008 FORMAT (7X,1HF18,9I12)
6009 FORMAT (2X,13,3X,10F12.6)
6010 FORMAT (2X///2X,24HLINEAR COEFFICIENTS ONLY)
6011 FORMAT (1H1,42H DECOUPLED COEFFICIENT OF B(P) * DB(Q)/DT,
1 19H IN EQUATION FOR B(P*1H)///)
6012 FORMAT (7X,1HF18,9I12)
6013 FORMAT (2X,19HQUASI-STEADY NOZZLE/)
6014 FORMAT (4X,1HJ)
6015 FORMAT (4X,1HF)
6017 FORMAT (1H1,31H OVERFLOW DETECTED, LAST ROW = ,I5)
6018 FORMAT (1H1,34H SINGULARITY DETECTED, LAST ROW = ,I5)
6020 FORMAT (2X,'DROPLET MOMENTUM SOURCE NEGLECTED'/)
6021 FORMAT (2X,'DROPLET MOMENTUM SOURCE INCLUDED'/)
7001 FORMAT (4F10.5,2I5)
7002 FORMAT (5I5*2F10.5,1X,A4)
7003 FORMAT (4I5)
7004 FORMAT (2I5*F15.6)
7005 FORMAT (3I5*F15.6)
7006 FORMAT (4I5*4F10.5)
END
SUBROUTINE EIGVAL(L, SMN, GAMMA, ZE, YAMPL, YPHASE, RESULT)

C
C COMPLEX RESULT
C COMMON /BLK1/ GSQ, ABSQ, ALBET, SMNSQ

C
C **************************************************************
C C THIS SUBROUTINE COMPUTES THE COMPLEX AXIAL ACOUSTIC EIGENVALUES
C FOR A CYLINDRICAL CHAMBER WITH A NOZZLE AND STORES THEM IN
C RESULT.
C THE EIGENVALUES ARE COMPUTED BY MEANS OF NEWTON'S METHOD.
C
C THE INPUT PARAMETERS ARE AS FOLLOWS:
C L IS THE AXIAL MODE NUMBER.
C SMN IS THE DIMENSIONLESS ACOUSTIC FREQUENCY.
C GAMMA IS THE SPECIFIC HEAT RATIO.
C ZE IS THE LENGTH-TO-RADIUS RATIO.
C YAMPL IS THE NOZZLE AMPLITUDE FACTOR.
C YPHASE IS THE NOZZLE PHASE SHIFT IN DEGREES.
C
C PI = 3.1415927
ERR = 0.0000001

C
C IF (YAMPL) 5, 60, 5
C CALCULATE CONSTANTS.
5 PHASE = YPHASE * PI/180.0
ALPHA = YAMPL * COS(PHASE)
BETA = YAMPL * SIN(PHASE)
GSQ = GAMMA * GAMMA
ABSQ = (ALPHA * ALPHA) - (BETA * BETA)
ALBET = ALPHA * BETA
SMNSQ = SMN * SMN

C
C ASSIGN INITIAL GUESS FOR EIGENVALUE.
IF (L .EQ. 0) GO TO 45
RL = L
PHI = PI/2.0 + PHASE
XM = RL * PI/ZE
A = YAMPL/ZE
XO = XM + A*COS(PHI)
YO = A*SIN(PHI)
GO TO 47
45 PHI = PI/4.0 + 0.5*PHASE
A = YAMPL * 10.0/ZE
XO = A * COS(PHI)
YO = A * SIN(PHI)

C
C ITERATION USING NEWTON'S METHOD FOR A SYSTEM OF TWO EQUATIONS
IN TWO UNKNOWNS.

47 L1 = 0
X = X0
Y = Y0

40 CALL FCNS(X,Y,ZE,F,G,FX,FY,GX,GY)
IF (L1.EQ. 40) GO TO 50
RJFG = (FX * FY) - (GX * FY)
IF (RJFG) 20, 30, 20

20 DELTAX = (-F * GY + G * FY)/RJFG
DELTAY = (-G * FX + F * GX)/RJFG
L1 = L1 + 1
X = X + DELTAX
Y = Y + DELTAY

C
TEST FOR CONVERGENCE.
IF (ABS(DELTAX) .GE. ERR .OR. ABS(DELTAY) .GE. ERR) GO TO 40
GO TO 10

C
WARNING MESSAGES
30 WRITE (6,6005)
GO TO 10
50 WRITE (6,6006)
GO TO 10

C
CASE OF HARD WALL (YAMPL = 0).

60 RL = L
X = RL * PI/ZE
Y = 0.0

10 RESULT = CMPLX(X,Y)

C
FORMAT SPECIFICATIONS.
6005 FORMAT (2X//2X,I6HJACOBIAN IS ZERO//)
6006 FORMAT (2X//2X$35HFAILED TO CONVERGE IN 40 ITERATIONS//)
RETURN
END
SUBROUTINE FCNS(X,Y,Z,E,F,G,FX,FY,GX,GY)

C
C THIS SUBROUTINE COMPUTES THE FUNCTIONS F(X,Y) AND G(X,Y)
AND THEIR PARTIAL DERIVATIVES WITH RESPECT TO X AND Y.
C
COMMON /BLKL1/ GQSQ, ABSQ, ALBET, SMNSQ

C COMPUTE THE TRIGONOMETRIC FUNCTIONS, THE HYPERBOLIC FUNCTIONS
AND THEIR SQUARES.
C
I = 1
ARGX = ZE * X
ARGY = ZE * Y

10 SX = SINCARGX)
CX = COS(ARGX)
SHY = SINHCARGY)
CHY = COSHCARGY)
IF (I .EQ. 2) GO TO 20
SXSQ = SX * SX
CXSQ = CX * CX
SHYSQ = SHY * SHY
CHYSQ = CHY * CHY
ARGX = 2.0 * ARGX
ARGY = 2.0 * ARGY
I = 2
GO TO 10

C COMPUTE TRANSCENDENTAL FUNCTIONS AND THEIR DERIVATIVES
C

20 FF = (SXSO * CHYSQ) - (CXSQ * SHYSQ)
GG = (CXSQ * CHYSQ) - (SXSO * SHYSQ)
HH = 0.25 * SX * SHY
FFX = ZE * SX * CHY
GGY = ZE * CX * SHY
FFY = -GGY
GGX = -FFX
HHX = 0.5 * GGY
HHY = 0.5 * FFX

C COMPUTE FACTORS
XYSQ = (X * X) - (Y * Y)
XY = X * Y
SMNXY = SMNSQ + XYSQ

F1 = (ABSQ * SMNXY) - (4.0 * ALBET * XY)
F2 = (ALBET * SMNXY) + (ABSQ * XY)
G1 = (ABSQ * SMNXY) - (4.0 * ALBET * XY)
FX1 = (2.0 * X * ABSQ) - (4.0 * ALBET * Y)
FX2 = (2.0 * X * ALBET) + (ABSQ * Y)
FY1 = (-2.0 * Y * ABSQ) - (4.0 * ALBET * X)
FY2 = (-2.0 * Y * ALBET) + (ABSQ * X)
GX1 = (2.0 * X * ABM + (4.0 * ALBET * Y)
GY1 = (-2.0 * Y * ABSQ + (4.0 * ALBET * X)

COMPUTE F(X,Y) AND G(X,Y)

F = (XYSQ * FF) - (4.0 * XY * HH)
G = (XYSQ * HH) + (XY * FF)

COMPUTE THE PARTIAL DERIVATIVES OF F AND G

FX = (2.0 * X * FF) + (XYSQ * FFX)
FY = (-2.0 * Y * FF) + (XYSQ * FFY)
GX = (2.0 * X * HH) + (XYSQ * HHX)
GY = (-2.0 * Y * HH) + (XYSQ * HHY)

RETURN
END
SUBROUTINE AXIAL1(NOPT,NP,NJ,UE,E,ZC,RESULT)

C
C THIS SUBROUTINE CALCULATES THE INTEGRAL OVER THE INTERVAL
C (0,ZE) OF THE FOLLOWING FUNCTIONS ACCORDING TO THE VALUE
C OF NOPT:
C
NOPT = 1    Z(NP) * ZC(NJ)
NOPT = 2    ZPP(NP) * ZC(NJ)
NOPT = 3    U * Z(NP) * ZC(NJ)
NOPT = 4    U * ZP(NP) * ZC(NJ)

C IN THE ABOVE EQUATIONS:
C Z(NP) IS THE AXIAL ACOUSTIC EIGENFUNCTION OF INDEX NP.
C Z(NJ) IS THE AXIAL ACOUSTIC EIGENFUNCTION OF INDEX NJ.
C ZC IS THE COMPLEX CONJUGATE OF THE AXIAL EIGENFUNCTION.
C ZP AND ZPP ARE THE FIRST AND SECOND DERIVATIVES OF THE
C AXIAL EIGENFUNCTIONS RESPECTIVELY.
C U IS THE STEADY STATE VELOCITY DISTRIBUTION AND UP IS ITS
C AXIAL DERIVATIVE.
C THE VELOCITY DISTRIBUTION IS COMPUTED BY THE SUBROUTINE UBAR.
C
REAL MAG
COMPLEX CI, CZE, BP, BJ, T1, T2, CH, F1, F2, F3, CZ, ARG, 1
     S1, S2, S3, RESULT, FUNCT500, B(10)
COMMON B

CI = (0.0,1.0)
CZE = COMPLEX(ZE,0.0)
BP = B(NP)
BJ = CONJG(B(NJ))

IF (NOPT .GT. 2) GO TO 50
CALCULATE INTEGRALS BY MEANS OF ANALYTICAL EXPRESSIONS FOR
NOPT = 1 AND NOPT = 2.
ARG = (BP + BJ) * CI
MAG = CABSC(ARG)
IF (MAG) 20, 25, 20
20 T1 = CSINH(ARG*CZE)/ARG
GO TO 30
25 T1 = CZE
30 ARG = (BP - BJ) * CI
MAG = CABSC(ARG)
IF (MAG) 35, 40, 35
35 T2 = CSINH(ARG*CZE)/ARG
GO TO 45
40 T2 = CZE
45 RESULT = (T1 + T2) * (0.5,0.0)
IF (NOPT .EQ. 2) RESULT = -B(NP) * B(NP) * RESULT
GO TO 100

C
C NUMERICAL EVALUATION OF INTEGRALS FOR NOPT = 3 AND NOPT = 4.
C
C COMPUTE STEP SIZE FOR SIMPSON INTEGRATION.
50 N = 50
RN = N
RESULT = (0.0, 0.0)
IC = ZCOMB
IC = 2 - IC
C
DO 90 J = 1, IC
IF (J .EQ. 1) H = ZCOMB * ZE/RN
IF (J .EQ. 2) H = (1.0 - ZCOMB) * ZE/RN
IF (J .EQ. 1) ZO = 0.0
IF (J .EQ. 2) ZO = ZCOMB * ZE
NP1 = N + 1
CH = CMPLX(H, 0.0)
C
COMPUTE INTEGRANDS.
DO 60 I = 1, NP1
STEP = I - 1
Z = (STEP * H) + ZO
IF ((I .EQ. 1) .AND. (J .EQ. 2)) Z = Z + H/100.0
IF (NOPT .EQ. 3) CALL UBAR(2 * UE * ZE, ZCOMB * ZE, F)
IF (NOPT .EQ. 4) CALL UBAR(1, UE * ZE, ZCOMB, Z, F)
F1 = CMPLX(F0, 0.0)
CZ = CMPLX(Z, 0.0)
ARG = CI * BP
IF (NOPT .EQ. 3) F2 = CCOSH(ARG * CZ)
IF (NOPT .EQ. 4) F2 = ARG * CSINH(ARG * CZ)
ARG = CI * BJ
F3 = CCOSH(ARG * CZ)
FUNCT(I) = F1 * F2 * F3
60 CONTINUE
C
PERFORM SIMPSON INTEGRATION.
NM1 = N - 1
S1 = FUNCT(1) + FUNCT(NP1)
S2 = (0.0, 0.0)
S3 = (0.0, 0.0)
DO 70 I = 2, N, 2
S2 = S2 + FUNCT(I)
70 CONTINUE
DO 80 I = 3, NM1, 2
S3 = S3 + FUNCT(I)
80 CONTINUE
RESULT = RESULT +
1 CH * (S1 + (4.0, 0.0) * S2 + (2.0, 0.0) * S3) / (3.0, 0.0)
90 CONTINUE
C
100 CONTINUE
RETURN
END
SUBROUTINE UBAR(NOPT, UE, ZE, ZCOMB, Z, RESULT)

C
C THIS SUBROUTINE CALCULATES THE STEADY STATE VELOCITY
C DISTRIBUTION FOR UNIFORMLY DISTRIBUTED COMBUSTION COMPLETED AT
C Z = ZCOMB * ZE WHERE:
C UE IS THE EXIT MACH NUMBER.
C ZE IS THE DIMENSIONLESS LENGTH.
C Z IS THE AXIAL COORDINATE.
C
C IF NOPT = 1 THE DISTRIBUTION IS CALCULATED.
C IF NOPT = 2 THE DERIVATIVE IS CALCULATED.
C IF NOPT = 3 THE SECOND DERIVATIVE IS CALCULATED.
C
ECZ = ZCOMB * ZE
GO TO (10, 20, 30), NOPT
10 IF (Z .LE. ECZ) RESULT = UE * Z/ECZ
   IF (Z .GT. ECZ) RESULT = UE
   GO TO 40
20 IF (Z .LE. ECZ) RESULT = UE/ECZ
   IF (Z .GT. ECZ) RESULT = 0.0
   GO TO 40
30 RESULT = 0.0
40 CONTINUE
RETURN
END
SUBROUTINE AZIMTLC(NOPT, NP, NQ, NJ, RESULT)

DIMENSION NFCN(3), SG(2)
COMMON /BLK2/ M(10), NS(10)

**************************************************************1

THIS SUBROUTINE CALCULATES THE INTEGRAL OVER THE INTERVAL
(0, 2*PI) OF THE FOLLOWING FUNCTIONS ACCORDING TO THE VALUE
OF NOPT:

NOPT = 1  TH(NP) * TH(NQ) * TH(NJ)
NOPT = 2  THP(NP) * THP(NQ) * TH(NJ)

IN THE ABOVE EQUATIONS:
TH(NP), TH(NQ), AND TH(NJ) ARE THE TANGENTIAL EIGENFUNCTIONS
AND NP, NQ, AND NJ ARE THEIR INDICES.
THP IS THE DERIVATIVE OF THE TANGENTIAL EIGENFUNCTIONS.

IF NS = 1  TH = SIN(M*THETA)
IF NS = 2  TH = COS(M*THETA)

RESULT = 0.0
FACTOR = 1.0
PI = 3.1415927

DISTINGUISH BETWEEN SINES AND COSINES.
DO 10 K1 = 1, 3
   NFCN(K1) = 1
10 CONTINUE
IF (NS(NJ) .EQ. 2) NFCN(3) = 2
IF (NOPT .EQ. 2) GO TO 20
IF (NS(NP) .EQ. 2) NFCN(1) = 2
IF (NS(NQ) .EQ. 2) NFCN(2) = 2
GO TO 30
20 IF (NS(NP) .EQ. 1) NFCN(1) = 2
   IF (NS(NQ) .EQ. 1) NFCN(2) = 2
   DO 40 K1 = 1, 2
      SG(K1) = 1.0
   40 CONTINUE
   IF (NFCN(K1) .EQ. 1) SG(K1) = -1.0
30 NSUM = 0
   DO 50 K1 = 1, 3
      NSUM = NSUM + NFCN(K1)
50 CONTINUE

FACTOR = SG(1) * SG(2) * M(NP) * M(NQ)
RESULT = NSUM * FACTOR

**************************************************************1
IF (NSUM .EQ. 3) .OR. (NSUM .EQ. 5)) GO TO 60
IF (NSUM .EQ. 4) GO TO 70
IF (NSUM .EQ. 6) GO TO 80
C
70 KOPT = 2
IF (NFCN(1) .EQ. 2) GO TO 72
GO TO 74
72 LL = M(NP)
MM = M(NQ)
NN = M(NJ)
GO TO 90
74 IF (NFCN(2) .EQ. 2) GO TO 76
GO TO 78
76 LL = M(NQ)
MM = M(NP)
NN = M(NJ)
GO TO 90
78 LL = M(NJ)
MM = M(NP)
NN = M(NQ)
GO TO 90
C
80 KOPT = 1
LL = M(NP)
MM = M(NQ)
NN = M(NJ)
C
90 IF ((LL.NE.0) *AND* (MM.NE.0) *AND* (NN.NE.0)) GO TO 101
GO TO 103
101 LM = LL + MM
LN = LL + NN
MN = MM + NN
IF ((NN.EQ.LM) .OR. (MM.EQ.LN)) RESULT = PI/2.0
IF (LL .EQ. MN) GO TO 102
GO TO 104
102 IF (KOPT .EQ. 1) RESULT = PI/2.0
IF (KOPT .EQ. 2) RESULT = -PI/2.0
GO TO 104
103 IF ((LL .EQ. 0) *AND* (MM .EQ. 0) *AND* (NN .EQ. 0)) GO TO 105
IF ((KOPT .EQ. 1) *AND* (NN .EQ. 0) *AND* (LL .EQ. MM)) RESULT = PI
IF ((KOPT .EQ. 1) *AND* (MM .EQ. 0) *AND* (LL .EQ. NN)) RESULT = PI
IF ((LL .EQ. 0) *AND* (MM .EQ. NN)) RESULT = PI
GO TO 104
105 IF (KOPT .EQ. 1) RESULT = 2.0 * PI
104 CONTINUE
RESULT = FACTOR * RESULT
60 CONTINUE
RETURN
END
SUBROUTINE RADIALCNOPT,L,M,N,A,B,C,RESULT)

C THIS SUBROUTINE CALCULATES THE INTEGRAL OVER THE INTERVAL
C (0,1) OF THE FOLLOWING PRODUCTS OF THREE BESSEL FUNCTIONS:
C
C NOPT = 1  JL(A*R) * JM(B*R) * JN(C*R) * R
C
C NOPT = 2  JL(A*R) * JM(B*R) * JN(C*R)/R
C
C NOPT = 3  JPL(A*R) * JPM(B*R) * JN(C*R) * R
C
JL IS THE BESSEL FUNCTION OF FIRST KIND OF ORDER L
JPL IS THE DERIVATIVE OF JL WITH RESPECT TO R
L, M, N ARE NON-NEGATIVE INTEGERS
A, B, C ARE REAL NUMBERS

DIMENSION FUNCT(200)
DOUBLE PRECISION DN, DH, DSTEP, DR, ARG1, ARG2, ARG3,
1  BES1, BES2, BES3, BESH, BESL, PROD,
2  FUNCT, BESLIM, S1, S2, S3

NN = 100
DN = NN
DH = 1.0/DN
NP1 = NN + 1

DO 10 I = 1, NP1
   DSTEP = I - 1
   DR = DH * DSTEP
   ARG1 = A * DR
   ARG2 = B * DR
   ARG3 = C * DR

   CALL JBES(N,ARG3,BES3,$500)
   IF (NOPT .EQ. 3) GO TO 101
   CALL JBESCL,ARG1,BES1,$500)
   CALL JBES(M,ARG2,BES2,$500)
   GO TO 102
101 IF (L .EQ. 0) GO TO 103
   CALL JBES(L+1,ARG1,BESH,$500)
   CALL JBES(L-1,ARG1,BESL,$500)
   BES1 = A * (BESL - BESH)/2.0
   GO TO 104
103 CALL JBES(L,ARG1,BES1,$500)
   BES1 = -BES1 * A
104 IF (M .EQ. 0) GO TO 105
   CALL JBES(M+1,ARG2,BESH,$500)
   CALL JBES(M-1,ARG2,BESL,$500)
   BES2 = B * (BESL - BESH)/2.0
   GO TO 102
105 CALL JBES(1, ARG2, BES2, $500)
BES2 = -BES2 * B
102 PROD = BES1 * BES2 * BES3

C

IF (NOPT .EQ. 2) GO TO 110
FUNCT(I) = PROD * DR
GO TO 10
110 IF (I .EQ. 1) GO TO 111
FUNCT(I) = PROD/DR
GO TO 10
111 BESLIM = 0.0
IF ((L.EQ.1) .AND. (M.EQ.0) .AND. (N.EQ.0)) BESLIM = A/2.0
IF ((L.EQ.0) .AND. (M.EQ.1) .AND. (N.EQ.0)) BESLIM = B/2.0
IF ((L.EQ.0) .AND. (M.EQ.0) .AND. (N.EQ.1)) BESLIM = C/2.0
FUNCT(I) = BESLIM
10 CONTINUE

C

NMI = NN - 1
S1 = FUNCT(1) + FUNCT(NP1)
S2 = 0.0
S3 = 0.0
DO 20 I = 2, NMI, 2
S2 = S2 + FUNCT(I)
20 CONTINUE
DO 30 I = 3, NMI, 2
S3 = S3 + FUNCT(I)
30 CONTINUE
RESULT = DH * CS1 + 4.0*S2 + 2.0*S3)/3.0
GO TO 501
500 WRITE (6, 6000)
6000 FORMAT (1H1e1OHERROR JBES)
501 CONTINUE
RETURN
END
SUBROUTINE AXIAL2(NOPT, NCONJ, NP, NQ, NJ, ZE, RESULT)

THIS SUBROUTINE CALCULATES THE INTEGRAL OVER THE INTERVAL
(0, ZE) OF THE FOLLOWING FUNCTIONS ACCORDING TO THE VALUES
OF NOPT AND NCONJ:

FOR NCONJ = 1 AND:
NOPT = 1   Z(NP) * Z(NQ) * ZC(NJ)
NOPT = 2   ZP(NP) * ZP(NQ) * ZC(NJ)
NOPT = 3   ZPP(NP) * Z(NQ) * ZC(NJ)

FOR NCONJ = 2 AND:
NOPT = 1   Z(NP) * ZC(NQ) * ZC(NJ)
NOPT = 2   ZP(NP) * ZPC(NQ) * ZC(NJ)
NOPT = 3   ZPP(NP) * ZC(NQ) * ZC(NJ)

FOR NCONJ = 3 AND:
NOPT = 1   ZC(NP) * Z(NQ) * ZC(NJ)
NOPT = 2   ZPC(NP) * ZP(NQ) * ZC(NJ)
NOPT = 3   ZPPC(NP) * ZC(NQ) * ZC(NJ)

FOR NCONJ = 4 AND:
NOPT = 1   ZC(NP) * ZC(NQ) * ZC(NJ)
NOPT = 2   ZPC(NP) * ZPC(NQ) * ZC(NJ)
NOPT = 3   ZPPC(NP) * ZC(NQ) * ZC(NJ)

IN THE ABOVE EQUATIONS:
Z(NP), Z(NQ), AND Z(NJ) ARE THE AXIAL ACOUSTIC EIGENFUNCTIONS
AND NP, NQ, AND NJ ARE THEIR INDICES.
ZP IS THE FIRST DERIVATIVE OF THE AXIAL EIGENFUNCTIONS.
ZPP IS THE SECOND DERIVATIVE OF THE AXIAL EIGENFUNCTIONS.
ZC AND ZPC ARE COMPLEX CONJUGATES OF Z AND ZP RESPECTIVELY.

REAL CI, MAG
COMPLEX CI, CF, CZE, BP, BQ, BJ, SUM, RESULT
COMMON B

CALCULATE INTEGRALS BY MEANS OF ANALYTICAL EXPRESSIONS.
CI = (0.0, 1.0)
CF = (0.25, 0.0)
CZE = CMPLX(ZE, 0.0)
BP = B(NP)
BQ = B(NQ)
BJ = CONJG(B(NJ))
IF ((NCONJ .EQ. 2) .OR. (NCONJ .EQ. 4)) BQ = CONJG(BQ)
IF (NCONJ .GT. 2) BP = CONJG(BP)
ARG(1) = (BP + BQ + BJ) * CI
ARG(2) = (BP + BQ - BJ) * CI
ARG(3) = (BP - BQ + BJ) * CI
ARG(4) = (BP - BQ - BJ) * CI
DO 10 J = 1, 4
MAG = CABS(ARG(J))
IF (MAG) 12, 15, 12
12 FUNCT(J) = CSINH(ARG(J) * CZE) / ARG(J)
GO TO 10
15 FUNCT(J) = CZE
10 CONTINUE
IF (NOPT .EQ. 2) GO TO 30
SUM = FUNCT(1) + FUNCT(2) + FUNCT(3) + FUNCT(4)
RESULT = CF * SUM
IF (NOPT .EQ. 3) RESULT = -BP * BP * RESULT
GO TO 50
30 SUM = FUNCT(1) + FUNCT(2) - FUNCT(3) - FUNCT(4)
RESULT = -CF * BP * BQ * SUM
50 CONTINUE
RETURN
END
APPENDIX D
PROGRAM LCYC3D: A USER'S MANUAL

General Description

Using the three-dimensional second-order theory described in this rep, Program LCYC3D calculates the nonlinear stability characteristics of a cylindrical combustion chamber with distributed combustion and a conventional nozzle. The response of the burning rate to pressure oscillations is described by Crocco's time-lag model. For given values of the operating parameters (i.e., n, $\bar{\tau}$, $\gamma$, $\bar{\nu}_e$, and L/D), a given series expansion, and a given initial disturbance Program LCYC3D integrates Eqs. (C-38) to obtain the time behavior of the unknown mode-amplitude functions (i.e., $B_j(t)$). From this information a time history of the pressure oscillation is determined. The program determines the final amplitude of the pressure oscillation attained in a linearly unstable engine (i.e., limit-cycle amplitude). Since the second-order analysis does not predict "triggering", however, the threshold amplitude above which a finite amplitude disturbance can trigger instability in linearly stable engine (i.e., triggering limit) is not calculated by Program LCYC3D. For either transient or limit-cycle conditions, the program prints out time histories of both pressure and axial velocity perturbations from which the amplitude, frequency, and wave shapes can be determined. The option to produce plotted output using a CALCOMP plotter is also provided.

Program Structure

A flow chart for Program LCYC3D is given in Fig. (D-1). This program performs the following operations: (1) reads the input data, (2) calculates the initial conditions, (3) numerically integrates the differential equations (4) tests for limit cycles (optional), and (5) prints and plots the results solutions.

The inputs to the program include the data generated by Program COEFS: the combustion parameters n and $\bar{\tau}$, various control numbers, and a description of the initial disturbance. The data from COEFS3D is read first and then printed out. Next the space dependent coefficients appearing in the series
Figure D-1. Flow Chart for Program LCYC3D.
expansions for $\xi, \phi$, and $\psi$ are computed and printed out. These coefficients are calculated by Subroutine PHICFS for use in the computation of pressure and axial velocity perturbations. The remaining input data is read, and following program execution, control is returned to this point (Fig. D-1) so that several cases (i.e., different values of $a$ and $\bar{a}$) may be run for a given set of coefficients generated by COEFFS3D.

After input of the initial amplitudes of the real parts (i.e., $B_{2j-1}(t)$ of the complex amplitude functions, the initial amplitudes of the imaginary parts (i.e., $B_{2j}(t)$) are calculated such that the nozzle admittance condition is satisfied for $-\bar{a} \leq t \leq 0$. These amplitudes are then printed out. Next the integration step-size, $\Delta t$, is calculated such that the interval $-\bar{a} \leq t \leq 0$ is divided into NDIV equal increments. Assuming a sinusoidal initial disturbance, the initial amplitudes of $B_{2j-1}(t)$ and $B_{2j}(t)$ are used to calculate these functions and their derivatives at each of the NDIV + 1 discrete points in $-\bar{a} \leq t \leq 0$. These values are needed in order to start the numerical solution of the differential equations (i.e., Eqs. (C-38)). The initial values of the amplitude functions are stored in the array $U(I,J)$ where the index $I$ varies from 1 ($t = -\bar{a}$) to NDIV + 1 ($t = 0$) and the index $J$ identifies the function. The corresponding initial values of the pressure and velocity perturbations are then printed out. This section also calculates the coefficients $C_2(j,p) - nC_3(j,p)$ and $nC_3(j,p)$ which are the coefficients of $dB^2/\partial t$ and $dB^2/(\bar{a} - t)/\partial t$ in Eqs. (C-38).

After the starting values are calculated, Eqs. (C-38) are solved using a modified form of the fourth order Runge Kutta method. Starting at $t = 0$ ($I = NDIV+1$), the amplitude functions at $t + \Delta t$ are calculated, using the Subroutine RHS to evaluate the functions $f_j(B_1, B_2, ..., B_{2N})$ on the right hand sides of Eqs. (C-38). The amplitude functions and the coefficients from PHICFS are then used to compute the pressure and axial velocity perturbations by Subroutine PRSVEL. The values of the amplitude functions at $t + \Delta t$ are stored in $U(I + 1,J)$, while the pressure and axial velocity perturbations are stored in the arrays PRESS(NPRES) and AXVEL(NPRES) where NPRES specifies the locations in the chamber where the data is calculated. Pressure data at one location (specified by NLOC) is also stored in the array PRS(I + 1). After checking for maximum and minimum values of $U(I,J)$ and PRS(I), the data may
be printed out (if $NTEST = 0$ and $TSTART \leq t \leq TQUIT$) or stored in plot arrays as desired. The time is then increased by $\Delta t$ (i.e., $I$ is increased by 1) and the calculations are repeated. This process continues until 250 integration steps have been computed ($t = 250\Delta t$), after which transfer is made to the limit-cycle section.

In the limit-cycle section a test for a limit-cycle is made if $NTEST = 1$. If the test is satisfied, $NTEST$ is set to zero so that no further tests will be made and the results can be printed or plotted. In either case the final values (for $250 - NDIV \leq I \leq 250$) replace the initial values (for $1 \leq I \leq NDIV + 1$) in the arrays $U(I,J)$ and $PRS(I)$, $I$ is again assigned the value $NDIV + 1$, and another 250 integration steps are calculated. This process continues until one of the following conditions is satisfied: (1) $NTEST = 0$ and $t > TQUIT$, (2) a limit-cycle is reached and $t > TQUIT$, and (3) more than 250 cycles of the pressure oscillation have been computed ($MAXNO > 500$). At this point the numerical calculations are terminated and the time history of the pressure amplitude (maxima and minima) are printed out and/or plotted as desired.

As can be seen from Fig. D-1 the output is not confined to a single section of the program but is produced in several different sections. Thus data is printed out or plotted shortly after it is calculated, which greatly reduces the amount of core storage required. All plots are generated by Subroutine GRAPHS which uses standard Univac 1108 plot routines. FORTRAN listings of Program LCYC3D and Subroutines PHICFS, PRSVEL, RHS, and GRAPHS are provided at the end of this appendix.

**Input Data**

A precise definition of the input data required to run the computer program is given below. This input data consists of three parts: (1) the control number $NOUTCF$, (2) the parameters and coefficients generated by Program COEFFS3D and (3) the data describing the cases to be run (see Fig. D-1). For each input case the following information must be provided: (1) the combustion parameters $n$ and $\bar{T}$; (2) a series of control numbers; and (3) information describing the initial disturbance.

The control number $NOUTCF$ determines whether the coefficients from COEFFS3D will be printed, and it appears on the first card of input. This
card is followed by the coefficient deck generated by COEFFS3D and the data
describing the cases to be run. Since the coefficient data has already be
-described in Appendix C, it will be omitted from the following detailed des-
cription of the input. As in Appendix C the location number refers to the
columns of the card. Again three formats are used for input: "A" indicates
alphanumeric characters, "I" indicates integers, and "F" indicates real num-
ers with a decimal point. For the "I" formats the values are placed in
fields of five locations, while a field of ten locations is used with the
"F" formats. In either case the numbers must be placed in the rightmost
locations of the allocated field.

<table>
<thead>
<tr>
<th>No. of Cards</th>
<th>Location</th>
<th>Type</th>
<th>Input Item</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1-5</td>
<td>I</td>
<td>NOUTCF</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>If 0: coefficients are not printed out.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>If 1: linear coefficients only are printed out.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>If 2: all coefficients are printed out.</td>
</tr>
<tr>
<td></td>
<td>1-72</td>
<td>A</td>
<td>TITLE</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Title used to label plots.</td>
</tr>
<tr>
<td>1</td>
<td>1-10</td>
<td>F</td>
<td>EN</td>
<td>Interaction index, n.</td>
</tr>
<tr>
<td></td>
<td>11-20</td>
<td>F</td>
<td>TAU</td>
<td>Time-lag, $\tau$.</td>
</tr>
<tr>
<td></td>
<td>21-30</td>
<td>F</td>
<td>H</td>
<td>Time-increment for numerical integration, $\Delta t$. *</td>
</tr>
<tr>
<td></td>
<td>31-40</td>
<td>F</td>
<td>TSTART</td>
<td>Time at which output of solutions begins.</td>
</tr>
<tr>
<td></td>
<td>41-50</td>
<td>F</td>
<td>TQUIT</td>
<td>Time at which output of solutions ends.</td>
</tr>
<tr>
<td>1</td>
<td>1-5</td>
<td>I</td>
<td>NTTEST</td>
<td>If 0: compute transient behavior.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>If 1: compute limit-cycle behavior.</td>
</tr>
<tr>
<td></td>
<td>6-10</td>
<td>I</td>
<td>JMODE</td>
<td>Identifies the amplitude function used to test for limit-cycles.</td>
</tr>
</tbody>
</table>

* This value is adjusted slightly by the program to divide the interval
$-\tau \leq t \leq 0$ into NDIV equal parts.
<table>
<thead>
<tr>
<th>No. of Cards</th>
<th>Location</th>
<th>Type</th>
<th>Input Item</th>
<th>Comments</th>
</tr>
</thead>
</table>
| 11-15        | I        | NLOC | Determines location for wall pressure maxima and minima. If 1: \( z = 0, \theta = 0^\circ \)  
If 2: \( z = 0, \theta = 45^\circ \)  
If 3: \( z = 0, \theta = 90^\circ \) |
| 16-20        | I        | NTERMS | Number of amplitude functions given initial values. |
| 21-25        | I        | NPZ | Determines how secondary instability zones are handled. If 0: all instability zones retained. If 1: secondary zones eliminated. |
| 26-30        | I        | NOUT | Determines output. If 0: printed output only. If \( 1 \leq \text{NOUT} \leq 6 \): both printed and plotted output, \( \text{NOUT} \) gives number of last plot produced. |

If \( 1 \leq \text{NOUT} \leq 6 \) the following two cards are read:

| 1  | 1-10     | F   | YHI(1) | Maximum ordinate for pressure plots. |
|    | 11-20    | F   | YHI(5) | Maximum ordinate for velocity plots. |
|    | 21-30    | F   | YLAB(1) | Interval for ordinate labeling of pressure plots. |
|    | 31-40    | F   | YLAB(5) | Interval for ordinate labeling of velocity plots. |
| 1  | 1-5      | I   | ITICY(1) | Number of ordinate tic marks for pressure plots. |
|    | 6-10     | I   | ITICY(5) | Number of ordinate tic marks for velocity plots. |
|    | 11-15    | I   | NFIRST | Gives the number of the first plot produced. |
|    | 16-20    | I   | NOMIT  | If 0: amplitude plot produced. If 1: amplitude plot omitted. |
The input data describing the cases to be run is given on a series of three or more cards. These cards are preceded by a title card which gives a title (TITLE) to be used to identify any plots produced by the run. This title appears before the first plot generated and does not appear on the printed output. The title card is included only for the first case of the run, on all subsequent cases it is omitted.

The first card of the series gives the interaction index, n, and the time-lag, \( \tau \), for the motor under consideration (EN and TAU); the time-increment, \( \Delta t \), used in the numerical integrations (H); and the times (TSTAR and TQUIT) at which output begins and ends. For all cases considered in the report a time-increment (dimensionless) of \( H = 0.050 \) was used, which gives about 70 steps per cycle for the LT mode. For \( \tau = 1.7 \) this input value was adjusted by the program to obtain \( H = 0.04857 \) which divides \( -\tau \leq t \leq 0 \) into 35 equal parts. For transient cases (NTEST = 0) printed output is given for TSTART \( \leq t \leq TQUIT \). When the limit-cycle behavior is calculated (NTEST = 1) TSTART and TQUIT are measured from the time at which the limit-cycle is reached, \( t_{LC} \). Thus the limit-cycle solutions are printed out for \( (t_{LC} + TSTART) \leq t \leq (t_{LC} + TQUIT) \). Two or three cycles of limit-cycle data for the LT mode are obtained with TSTART = 0 and TQUIT = 10. For plotted output, the time axis is always 10 units long, therefore \( (TQUIT - TSTART) > 10 \) to obtain plots.

The second card of the series gives the control numbers, NTEST, JMODE, NLOC, NTERMS, NPZ, and NOUT. The task to be performed by Program LCYC3D is specified by NTEST. If NTEST = 0 the transient behavior (growth or decay) of the pressure oscillation is determined, while for NTEST = 1 the program
searches for a limit-cycle amplitude. JMODE identifies the "principal" series term, the amplitude function used in the limit-cycle test. This is usually the lowest frequency mode (i.e., 1T or 1L) in the approximating series expansion. NLOC gives the location at which the amplitude-time history (maxima and minima) of the wall pressure perturbation is calculated. The number of complex series terms $A_j(t)$ receiving initial values is specified by NTERMS, while all other series terms are initially zero. The parameter NPZ determines how the secondary instability zones (phantom zones) are handled by Program LCYC3D. For NPZ = 1 the phantom zones are eliminated by dropping the combustion terms for a given mode when $\tau > \bar{\tau}_{\text{cut}}$ where:

$$\bar{\tau}_{\text{cut}} = \frac{2\pi}{\omega} = 2\pi \left[ \frac{s}{\omega_{mn}} + \frac{\omega}{z_e^2} \right]^{\frac{1}{2}} \quad (D-1)$$

A similar procedure was used in the axial instability studies by Lores and Zinn. The transverse instability data presented herein was obtained with NPZ = 0, while NPZ = 1 was used in the axial instability studies to facilitate comparison with the results of Ref. (3). The last control number NOUT determines which plots, if any, are produced. For NOUT = 0 no plots are produced. For $1 \leq \text{NOUT} \leq 6$, NOUT gives the number of the last plot produced, where the plots are numbered as given in Table D-1 below:

<table>
<thead>
<tr>
<th>No. of Plot (NPLOT)</th>
<th>Quantity Plotted</th>
<th>Axial Location</th>
<th>Azimuthal Coordinate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Pressure</td>
<td>Injector</td>
<td>$0^\circ$</td>
</tr>
<tr>
<td>2</td>
<td>&quot;</td>
<td>&quot;</td>
<td>$45^\circ$</td>
</tr>
<tr>
<td>3</td>
<td>&quot;</td>
<td>&quot;</td>
<td>$90^\circ$</td>
</tr>
<tr>
<td>4</td>
<td>&quot;</td>
<td>Nozzle</td>
<td>$0^\circ$</td>
</tr>
<tr>
<td>5</td>
<td>Axial Velocity</td>
<td>&quot;</td>
<td>$0^\circ$</td>
</tr>
<tr>
<td>6</td>
<td>Nozzle Boundary</td>
<td>&quot;</td>
<td>$0^\circ$</td>
</tr>
</tbody>
</table>
The nozzle boundary term given on the last plot is discussed later in this appendix.

If plots are produced, two additional cards are needed to give the minimum and minimum values of the variables to be plotted, YHI(NPLOT) and YLO(NPLOT); the intervals for ordinate labeling (YLAB(NPLOT)); and the number of ordinate tic marks, ITICY(NPLOT). All of the plots are symmetric about the time-axis so that YLO(NPLOT) = -YHI(NPLOT), and ITICY(NPLOT) may be negative to obtain the centerline. Since the ordinate scales and labels are the same for all pressure plots (NPLOT = 1, 2, 3, 4) this data is read for NPLOT = 1 only; likewise the data for the last two plots is read for NPLOT 5 only. In addition NFIRST gives the number of the first plot produced, giving additional control over the number of plots produced. NOMIT determines whether a plot of pressure amplitude versus time (location specified by NLOC) is produced.

The remaining cards give the initial amplitudes of the complex series terms, \( A_j(t) \), needed to start the numerical integration. Only the amplitude of the real parts, \( B_{2j-1}(t) \), are given on these cards, while the amplitudes of the imaginary parts, \( B_{2j}(t) \), are determined from the nozzle admittance condition. For each value of \( J \) the amplitudes \( A_{ST} \) and \( A_{CT} \) are assigned to the arrays \( AS(NP) \) and \( AC(NP) \) where \( NP = 2J - 1 \). The computation of the amplitudes of the imaginary parts, \( AS(NP + 1) \) and \( AC(NP + 1) \), is discussed later. The initial values of the series terms are then calculated from the formula:

\[
B_{p}(t) = AS(NP)\sin(\omega_p t) + AC(NP)\cos(\omega_p t) \quad (-\pi \leq t \leq 0) \tag{D-1}
\]

where \( \omega_p \) is the acoustic frequency. The derivatives, \( dB_p/dt \), are also required for starting the numerical integration; they are obtained simply by differentiating Eq. (D-2).

The proper input for pure standing and pure spinning single-mode initial disturbances is given as follows. For a standing mode, only the \( \cos(\omega_p t) \) term are retained in the series and NTERMS = 1. A single card is read giving the amplitude of the initial disturbance. For a spinning mode, both \( \sin(\omega_p t) \) and
cos(mφ) terms are included in the series expansion. It is convenient to pair these terms such that the index J corresponds to a sin(mφ) term and J + 1 corresponds to a cos(mφ) term. For an initial disturbance of amplitude A spinning in the counterclockwise direction (φ increasing), NTERMS = 2 and two cards are read giving the following data:

\[ J : \text{AST} = A \text{ and } \text{ACT} = 0 \quad \text{(D-3)} \]

\[ J + 1 : \text{AST} = 0 \text{ and } \text{ACT} = A \]

In both cases above initial amplitudes are required only for the mode initially present, and the initial amplitudes of all other modes included in the series expansion are zero.

The proper input for Program LCYC3D will be illustrated with the following example. Assuming that the velocity potential \( \phi \) is expressed in terms of the 1R, 1T, and 2T modes*, it is desired to determine the limit-cycle behavior of a linearly unstable engine \( (n = 0.57486, \quad \tau = 1.7, \quad \varphi_e = 0.2, \quad L/D = 0.5) \) with a nozzle admittance of \( A = 0.02 \) and \( \varphi = 45^\circ \). Sample input is given for the case of a spinning 1T mode disturbance of amplitude 0.3. The principal series term is the cos(mφ) term for the 1T mode \( (i.e., B_{011}(t)), \) thus JMODE = 2. Plots are desired for the pressure, axial velocity, and nozzle boundary condition at the nozzle entrance, thus NOUT = 6 and NFIRST = 4.

To run the case described above the data deck must be assembled as follows. The card specifying NOUTCF is followed by the coefficient deck produced by Program COEFFS3D; in this example it contains the information given in the sample output for COEFFS3D shown in Appendix C. The coefficient deck is followed by the data for the case to be run as shown in the sample input below:

* This is the same case used to illustrate Program COEFFS3D.
As seen from Eq. (13) the real parts of the time and space derivatives of the velocity potential (i.e., $\Phi_t$, $\Phi_r$, $\Phi_\theta$, $\Phi_z$) are needed in order to compute the pressure perturbation. Differentiating the complex series expansion given by Eq. (9) and evaluating at the chamber wall ($r = 1$) gives the following expansions:

$$
\Phi_t = \sum_{p=1}^{N} \frac{dA_p}{dt} Z_p(z) \Theta_p(\theta) R_p(l) = \sum_{p=1}^{N} C_{t}(p,z,\theta) \frac{dA_p}{dt} \tag{D-1}
$$

$$
\Phi_\theta = \sum_{p=1}^{N} A_p(t) Z_p(z) \Theta_p(\theta) R_p(l) = \sum_{p=1}^{N} C_{\theta}(p,z,\theta) A_p(t) \tag{D-5}
$$
\[
\phi_z = \sum_{p=1}^{N} A_p(t) Z_p(z) \Theta_p(\theta) R_p(\theta) = \sum_{p=1}^{N} C_{z,p} z_p, \theta) A_p(t) \quad (D-6)
\]

where the complex coefficients \(C_t, C_\theta, \text{ and } C_z\) are functions of \(z\) and \(\theta\). The quantity \(\phi_r\) is not needed since \(\phi_r = 0\) at the chamber wall. The complex coefficients \(C_t, C_\theta, \text{ and } C_z\) are calculated by Subroutine PHICFS and are assigned to the variables, \(C1, C2, \text{ and } C3\) respectively. The coefficients in the series expansions for the corresponding real parts (i.e., \(\varphi_t, \varphi_\theta, \varphi_z\)) are related to the complex coefficients by:

\[
\begin{align*}
C_t'(2p-1, z, \theta) &= \text{Re} [C_t(p, z, \theta)] \\
C_t'(2p, z, \theta) &= -\text{Im} [C_t(p, z, \theta)]
\end{align*}
\quad (D-7)
\]

where similar relations hold for \(C_\theta'\) and \(C_z'\). The real coefficients are stored in the arrays \(CFT(NPRES, NP), CFTH(NPRES, NP), \text{ and } CFZ(NPRES, NP)\) where \(NPRES\) determines the location in the chamber as given in Table D-3 below:

<table>
<thead>
<tr>
<th>NPRES</th>
<th>Axial Location (z)</th>
<th>Azimuthal Location ((\theta))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0°</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>45°</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>90°</td>
</tr>
<tr>
<td>4</td>
<td>(z_e)</td>
<td>0°</td>
</tr>
<tr>
<td>5</td>
<td>(z_e)</td>
<td>45°</td>
</tr>
<tr>
<td>6</td>
<td>(z_e)</td>
<td>90°</td>
</tr>
</tbody>
</table>

**Table D-3. Chamber Locations for Pressure Calculations.**

**Initial Amplitudes**

The initial amplitudes of the real parts of the complex series terms (i.e., \(B_{2j-1}(t)\)) are specified in the input to the program. The initial
amplitudes of the imaginary parts (i.e., $B_{2j}(t)$), however, are calculated such that the nozzle admittance condition is satisfied for $-\pi \leq t \leq 0$. This is done by introducing the linear expressions for $u'$ and $p'$ into the nozzle admittance relation and assuming periodic solutions. This yields a set of linear algebraic equations relating the amplitudes of the real and imaginary parts of the complex series terms. For given values of the amplitudes of the real parts, $AS(NP)$ and $AC(NP)$, these equations are solved to obtain the amplitudes of the imaginary parts, $AS(NP + 1)$ and $AC(NP + 1)$. The following formulas are used in this calculation.

\[
AS(NJ + 1) = -(r_2a_1 - r_1a_2) / (a_1^2 + a_2^2)
\]

\[
AC(NJ + 1) = (r_1a_1 + r_2a_2) / (a_1^2 + a_2^2)
\]

where

\[
r_1 = a_3 [AC(NJ)] - a_4 [AS(NJ)]
\]

\[
r_2 = -a_4 [AC(NJ)] - a_3 [AS(NJ)]
\]

and

\[
a_1 = (1 + \gamma Y \tilde{u}_e)CFZ(NPRES, NJ+1) - \gamma Y \tilde{u}_jCFT(NPRES, NJ+1)
\]

\[
a_2 = \gamma Y \tilde{u}_jCFT(NPRES, NJ+1) + \gamma Y \tilde{u}_eCFZ(NPRES, NJ+1)
\]

\[
a_3 = -(1 + \gamma Y \tilde{u}_e)CFZ(NPRES, NJ) + \gamma Y \tilde{u}_jCFT(NPRES, NJ)
\]

\[
a_4 = \gamma Y \tilde{u}_jCFT(NPRES, NJ) + \gamma Y \tilde{u}_eCFZ(NPRES, NJ)
\]

In Eqs. (D-8) through (D-10) $\omega_j$ is the acoustic frequency and CFT and CFZ are
the coefficients in the series for \( \phi_t \) and \( \phi_z \) computed previously. The above conditions are applied at a pressure anti-node for each series term, therefore \( \text{NPRES} = 4 \) \((z = z_e, \theta = 0^\circ)\) for a \( \cos(m\phi) \) term and \( \text{NPRES} = 6 \) \((z = z_e, 0 = 90^\circ)\) for a \( \sin(m\phi) \) term.

For nozzles with phase shifts of \( \phi = 90^\circ \) and \( \phi = 270^\circ \) the quantity \( a_1^2 + a_2^2 \) vanishes and Eqs. (D-8) become indeterminate. In these cases the amplitudes of the imaginary parts are given by:

\[
\begin{align*}
\text{AS}(NJ + 1) &= \text{AC}(NJ) \\
\text{AC}(NJ + 1) &= \text{AS}(NJ)
\end{align*}
\]  

which provides a good approximation to the nozzle admittance condition.

**Integration of the Differential Equations**

For purposes of numerical integration Eqs. (C-38) are written as an equivalent system of first order differential equations as follows:

\[
\frac{dB_j}{dt} = B_j' \tag{D-12}
\]

\[
\frac{dB_j'}{dt} = f_j(B_p, B_p') \tag{D-13}
\]

where the dependent variables are now \( B_j \) and \( B_j' \). These equations are solved numerically using the fourth order Runge-Kutta method. Due to the presence of retarded variables in Eqs. (D-12) and (D-13) the formulas (see Ref. 21) used in the Runge-Kutta method must be slightly modified.

The appropriate formulas for applying the Runge-Kutta method to problems involving a time-delay are readily obtained by considering a single equation of the following form:

\[
\frac{dx}{dt} = f(x, t) + g[x(t - \tau)] \tag{D-14}
\]
Noting that at any step of the integration the value of \( x(t - \tau) \) has already been determined from previous steps, the function \( g \) can be considered to be a known function of time \( g(t) \).

Since \( x(t) \) is computed only at discrete points \( x_n(t_n) \) it is desired that the retarded variable \( x(t_n - \tau) \) will coincide with such previously computed points. This can be accomplished by choosing the step-size \( \Delta t \) such that it divides the time-lag \( \tau \) into \( k \) equal increments. Thus \( \tau = k \Delta t \) and the Runge-Kutta formulas which apply to Eq. (D-14) can now be written as:

\[
x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\]

\[
k_1 = \left\{ f(x_n, t_n) + g(x_{n-\Delta t}) \right\} \Delta t
\]

\[
k_2 = \left\{ f(x_n + k_1/2, t_n + \Delta t/2) + g(x_{n-\Delta t/2}) \right\} \Delta t \tag{D-15}
\]

\[
k_3 = \left\{ f(x_n + k_2/2, t_n + \Delta t/2) + g(x_{n-\Delta t/2}) \right\} \Delta t
\]

\[
k_4 = \left\{ f(x_n + k_3, t_n + \Delta t) + g(x_{n-\Delta t}) \right\} \Delta t
\]

Equations (D-15) are readily extended to handle the system of equations given by Eqs. (D-12) and (D-13). It is seen from Eqs. (D-15) that \( k \) values of the dependent variables prior to the initial values are needed to start the integration.

Although the initial wave shape can be an arbitrary function of time, it is assumed that initially the mode-amplitudes are sinusoidal functions of time oscillating with the natural frequency \( \omega_j \). Thus each mode-amplitude function is expressed in the following form:

\[
B_j(t) = A_S(J) \sin(\omega_j t) + A_C(J) \cos(\omega_j t)
\]

(D-16)
\[ B_j'(t) = w_j \left[ AS(J) \cos(w_j t) - AC(J) \sin(w_j t) \right] \]

where \( -\tau \leq t \leq 0 \).

In Program LCYC3D both the functions \( B_j(t) \) and the derivatives \( B_j'(t) \) are stored in the same array \( U(I,J) \). The \( B_j(t) \) (N functions) are stored in the first half of the array (\( 1 \leq J \leq N \)), while the remaining space (\( N + 1 \leq J \leq 2N \)) is used to store the values of \( B_j'(t) \). Thus for a given value of \( j \) (\( 1 \leq j \leq N \)), \( B_j(t) \) is stored in \( U(I,J) \) and \( B_j'(t) \) is stored in \( U(I,J + N) \). In addition the retarded variables \( B_j'(t - \tau) \) are stored in the array \( RV(J,K) \) as follows:

\[
RV(J,1) = B_j'(t - \tau) \\
RV(J,2) = RV(J,3) = B_j'(t - \tau + \Delta t/2) \quad (D-17) \\
RV(J,4) = B_j'(t - \tau + \Delta t)
\]

The values of \( B_j'(t - \tau + \Delta t/2) \) are computed from \( B_j'(t - \tau) \), \( B_j'(t - \tau + \Delta t) \), and \( B_j'(t - \tau + 2\Delta t) \) using a three-point interpolation.

Pressure and Axial Velocity Perturbations

From the calculated time dependence of the series terms Program LCYC3D computes the dimensionless pressure perturbation, \( p' \), with the aid of Eqs. (D-4) through (D-6) and either Eq. (13) for \( NDROPS = 0 \) or Eq. (A-6) for \( NDROPS = 1 \). The pressure is calculated at the injector face (\( z = 0 \)) and the nozzle entrance plane (\( z = z_e \)) for three angular positions along the periphery of the chamber (i.e., \( r = 1; \theta = 0°, 45°, 90° \)). The results are stored in the array \( PRESS(NPRES) \) where \( NPRES \) gives the location according to Table D-3. The axial velocity perturbation at the nozzle entrance, \( u'_e \), is calculated for \( \theta = 0°, 45°, 90° \) using the relation \( u' = \varphi_z \) and Eq. (D-6), and the results are stored in \( AXVEL(K) \), where \( K = NPRES-3 \). In addition the quantity, \( Re\left[-y\varphi_t\right] \), is calculated at the nozzle entrance for \( \theta = 0° \) and assigned to the variable \( YPHI \). From Eq. (2) it is seen that \( YPHI \) is the axial velocity.
at the nozzle entrance (i.e., $u^_'_e$) if the nozzle admittance condition is exactly satisfied. Since the solutions generated by Program LCYC3D are approximate, the difference between $u^_'_e$ and $\Phi$ is a measure of the accuracy of this approximation at the nozzle boundary.

**Maximum and Minimum Values**

In order to determine the transient behavior and limit-cycle amplitudes it is necessary to follow the growth or decay of the amplitudes of the series terms and the pressure perturbation. The maxima and minima of the principal series term (specified by JMODE) are assigned to the array $\text{UMAX(MAXNO)}$ where MAXNO is a counter variable. For the pressure perturbation, maximum and minimum values at the location specified by NLOC are stored in $\text{PMAX(MAXP)}$, and the corresponding times of maximum and minimum are stored in $\text{TIMAX(MAXP)}$. Since the solutions are calculated only at discrete points, the maximum and minimum values are computed using a three-point interpolation scheme.

**Calculation of Limit-Cycle Amplitude**

A limit-cycle amplitude is calculated by specifying an initial disturbance and continuing the step-by-step integration of Eqs.(D-12) and (D-13) until a periodic solution is obtained; that is, the amplitude of the oscillation remains essentially constant. The test for convergence to a limit cycle is performed upon a single series term, usually the most important term in the series, in the following manner. After the first 500 integration steps, usually about 10 cycles for the 1T mode, the amplitude of the principal series term $A_1$ is compared with its amplitude after 250 integration steps $A_0$. If the change in amplitude $|A_1 - A_0|$ is greater than the maximum permissible change $\epsilon$, the calculations are continued and the change in amplitude during the next 250 integration steps is calculated. The process is repeated until $|A_k - A_{k-1}| < \epsilon$ at which point the computation is terminated. The amplitude used in the above calculations are determined by averaging the absolute value of $\text{UMAX(MAXNO)}$ over the last two complete cycles for each 250 integration steps. A value of $\epsilon = 0.001$ is used in Program LCYC3D which gives sufficient accuracy for most cases.
Output

Printed Output. The printed output produced by Program LCYC3D consists of the five sections discussed below.

Section 1 is a restatement of the input from Program COEFFS3D. It includes the following information: (a) the ratio of specific heats (GAMMA), the mean flow Mach number at the nozzle entrance (UE), the dimensionless chamber length (ZE), the length of the combustion zone as a fraction of the chamber length (ZCOMB), and the number of series terms (real) NJMAX; (b) a statement regarding the presence or absence of the droplet momentum source; (c) the parameters which describe and identify each term in the series expansion; (d) the nozzle admittance (YR and YI) and the axial acoustic eigenvalue (EPS and ETA) for each series term; (e) the nonzero linear coefficients, C(KC, NJ, NP); and (f) the nonzero nonlinear coefficients, D(NJ, NP, NQ). The nonlinear coefficients are omitted from the output for NOUTCF = 1, and no coefficients are printed out for NOUTCF = 0.

Section 2 gives the coefficients needed for computation of the wall pressure waveforms; that is, the coefficients in the series for \( \varphi_t \), \( \varphi_\theta \), and \( \varphi_z \). These are given for each of the NJMAX series terms at each of the six locations specified by NPRES (see Table D-3).

Section 3 gives the initial amplitudes (AS(J) and AC(J)) of all series terms included in the assumed initial disturbance. This section also states whether the limit-cycle behavior is calculated and whether plots are produced.

Section 4 gives the time-dependent solutions for the following quantities: (a) the injector pressure perturbation at \( \theta = 0^\circ, 45^\circ, 90^\circ \); (b) the nozzle pressure perturbation at \( \theta = 0^\circ, 45^\circ, 90^\circ \); (c) the nozzle axial velocity perturbation at \( \theta = 0^\circ, 45^\circ, 90^\circ \); and (d) the nozzle boundary term, \( \text{Re}\left[-\gamma Y_{t}\right] \), at \( \theta = 0^\circ \). This output is given in two parts: (1) the initial values for \(-\pi \leq t \leq 0\) and (2) the solutions for \( t_1 \leq t \leq t_f \), where \( t_1 \) and \( t_f \) are determined by the input parameters TSTART and TQUIT (see discussion on Input). On the first page of each part a heading gives the interaction index, \( n \), and the time-lag, \( \tau \), and the chamber parameters, \( \gamma \), \( \bar{U}_e \), and L/D.

Section 5 gives the time history of the pressure amplitude (maximum and minimum values) for the chamber location specified by NLOC. This information
is printed as an array of number pairs giving the value of the pressure maximum or minimum (upper number) and the corresponding time of maximum or minimum (lower number). This information is useful in determining the growth (or decay) rate of the transient solutions, and it provides a check on the convergence of the solution to a limit-cycle.

**Plotted Output.** According to the values of NOUT and NFIRST the pressure and axial velocity waveforms given in Section 4 of the printed output may be plotted using a Calcomp plotter. The data over the dimensionless time interval for printed output, \( t_i \leq t \leq t_f \), is plotted in sections of 10 units in length, beginning at \( t = t_i \). Thus for each quantity plotted, \( N \) plots are produced where \( N \) is the largest multiple of 10 contained in the interval \( t_i \leq t \leq t_f \). The data left over (i.e., for \( t_i + 10N \leq t \leq t_f \)) is not plotted. All quantities to be plotted for a given time interval are plotted before proceeding to the next time interval.

The data given in Section 5 of the printed output (pressure maxima only) is also plotted if NOUT > 0 and NOMET = 0. The abscissa and ordinate range for this plot are not specified in the input, but are calculated such that all of the data falls within these ranges. This plot is always the last plot produced.

All of the above plots are scaled to fit on standard \( 8\frac{1}{2}'' \times 11'' \) paper and scissor-lines are plotted for trimming plots to this size. The data is plotted as individual points using a small circle symbol, and all of the values computed during the given time interval are plotted. Before the first plot produced the identifying title (see Input) is printed.

**Sample Output.** The following sample output illustrates the printed and plotted output produced by Program LCYC3D for the sample input given in Table D-2.
Table D-4. Sample Output, Section 1.

\[ \Gamma = 1.200 \quad U = .200 \quad Z = 1,000,000 \quad Z_{\text{comb}} = 1.00 \quad N_{\text{max}} = 10 \]

**DROPLET MOMENTUM SOURCE IS NEGLECTED**

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**NUMBER OF COEFFICIENTS C(1,NJ,NP) IS 10**

| C(1, 1, 1) | = | 3.39060 |
| C(1, 2, 2) | = | 3.39060 |
| C(1, 3, 3) | = | 3.39060 |
| C(1, 4, 4) | = | 3.39060 |
| C(1, 5, 5) | = | 9.33021 |
| C(1, 6, 6) | = | 9.33021 |
| C(1, 7, 7) | = | 9.33021 |
| C(1, 8, 8) | = | 9.33021 |
| C(1, 9, 9) | = | 14.68491 |
| C(1, 10, 10) | = | 14.68491 |

**NUMBER OF COEFFICIENTS C(2,NJ,NP) IS 10**

| C(2, 1, 1) | = | .26153 |
| C(2, 2, 2) | = | .26153 |
| C(2, 3, 3) | = | .26153 |
| C(2, 4, 4) | = | .26153 |
| C(2, 5, 5) | = | .26457 |
| C(2, 6, 6) | = | .26457 |
| C(2, 7, 7) | = | .26457 |
| C(2, 8, 8) | = | .26457 |
| C(2, 9, 9) | = | .26654 |
| C(2, 10, 10) | = | .26654 |

**NUMBER OF COEFFICIENTS C(3,NJ,NP) IS 10**

| C(3, 1, 1) | = | .24000 |
| C(3, 2, 2) | = | .24000 |
| C(3, 3, 3) | = | .24000 |
| C(3, 4, 4) | = | .24000 |
| C(3, 5, 5) | = | .24000 |
| C(3, 6, 6) | = | .24000 |
| C(3, 7, 7) | = | .24000 |
Table D-4. (Continued)

\[
\begin{align*}
C(3, 8, 8) &= 0.24000 \\
C(3, 9, 9) &= 0.24000 \\
C(3, 10, 10) &= 0.24000
\end{align*}
\]

NUMBER OF COEFFICIENTS \( D(NJ, NP, NG) \) IS 50

\[
\begin{align*}
D(1, 1, 7) &= -1.73504 \\
D(1, 1, 9) &= -2.33866 \\
D(1, 3, 5) &= 1.73504 \\
D(1, 5, 3) &= 1.49783 \\
D(1, 7, 1) &= -1.49783 \\
D(1, 9, 1) &= -1.96281 \\
D(2, 2, 8) &= -1.73505 \\
D(2, 2, 10) &= -2.33867 \\
D(2, 4, 6) &= 1.73505 \\
D(2, 6, 4) &= 1.49784 \\
D(2, 8, 2) &= -1.49784 \\
D(2, 10, 2) &= -1.96282 \\
D(3, 1, 5) &= 1.73504 \\
D(3, 3, 7) &= 1.73504 \\
D(3, 3, 9) &= -2.33866 \\
D(3, 5, 11) &= 1.49783 \\
D(3, 7, 3) &= 1.49783 \\
D(3, 9, 3) &= -1.96281 \\
D(4, 2, 6) &= 1.73505 \\
D(4, 4, 6) &= 1.73505 \\
D(4, 4, 10) &= -2.33867 \\
D(4, 6, 2) &= 1.49784 \\
D(4, 8, 4) &= 1.49784 \\
D(4, 10, 2) &= -1.96282 \\
D(5, 1, 3) &= 1.13133 \\
D(5, 1, 5) &= -1.13133 \\
D(5, 5, 9) &= -3.07465 \\
D(5, 9, 5) &= -2.81865 \\
D(6, 2, 4) &= -1.13132 \\
D(6, 4, 2) &= -1.13132 \\
D(6, 6, 10) &= -3.07469 \\
D(6, 10, 6) &= -2.81868 \\
D(7, 1, 1) &= 1.13133 \\
D(7, 3, 3) &= -1.13133 \\
D(7, 7, 9) &= -3.07465 \\
D(7, 9, 7) &= -2.81865 \\
D(8, 2, 2) &= 1.13132 \\
D(8, 4, 4) &= -1.13132 \\
D(8, 8, 10) &= -3.07469 \\
D(8, 10, 8) &= -2.81868 \\
D(9, 1, 1) &= 1.04087 \\
D(9, 3, 3) &= 1.04087 \\
D(9, 5, 5) &= -2.1090 \\
D(9, 7, 7) &= -2.1090 \\
D(9, 9, 9) &= 4.18784 \\
D(10, 2, 2) &= 1.04087 \\
D(10, 4, 4) &= 1.04087 \\
D(10, 6, 6) &= -2.1091 \\
D(10, 8, 8) &= -2.1091 \\
D(10, 10, 10) &= 4.18793
\end{align*}
\]
Table D-5. Sample Output, Section 2.

COEFFICIENTS FOR COMPUTATION OF WALL PRESSURE WAVEFORMS

COEFFICIENTS IN SERIES FOR:

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<th>TIME DERIVATIVE</th>
<th>THETA DERIVATIVE</th>
<th>AXIAL DERIVATIVE</th>
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### Table D-6. Sample Output, Section 3.

INITIAL CONDITIONS ARE OF THE FORM:

\[ U(I,J) = AC(J) \cos(FREQ \times T) + AS(J) \sin(FREQ \times T) \times \exp(DAMP \times T) \]

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</table>

THE LIMIT-CYCLE BEHAVIOR IS CALCULATED.

THIS RUN PRODUCES PLOTTED OUTPUT.
Table D -7

Sample Output; Section'

COM8UsTION PARANFTERS1 INTERACTION INDEX = .57486
MOTuR PARAMETERS;
GAMMA = 1.20000

4.

TI-ME-LAG = 1.70000
EXIT MACH NUMBER = .20000

LENGTH/DIAMETER = .50000

INITIAL CONDITIONS

STEP

TIME

INJECTOR PRESSURE
O. DEG. - 45. DEG. 90. DEG.

.
- 1.70000
-.00446 .29088
.42934
..71.65143, -.03845
.26129
.42696
- 1.60286-.07144 '
.22989.
.42083
-3 e - - 1.55429
-.10320
.19702
.41103
- - 31
- 1.50574. , -.13352
.16303
.39768
30 .. - 1.45714 - -.16221
.12826
.380 9 3
-29
- 1.40857
-.18912
.09306 - .36100.
-28 .
- 1,360.00-.21410
.05775
.33812
-27
- 1.31143. -.23704
.02265.
.31256
-26
- 1.26286: -.25785
-.01193-,., .28461
-25 . -1.21429
7.27645
-.04572 :
.25460
- 1.16571
- 29 -,
-.29280
-.07847
.22285_
-23,` -1,11714- -.30685- -.11)993 . .18970
-22
- 1. 05857
- .31856. -.11991 .15552
*21,
-1.02000
-.32792
-.16823 . .12062
-20 ..
7.97143_ -.33491
-.19472
.08537
- 19.- --.92286
-.33953_ -.21926
.05008
-18.
-.87429
-.34176
-.24175
.01507
-17
.82571. -.34160
-.26209. -.01936
- 16 -.77714' -.33906
-.28021
-.05294 -15
-.72857
-.33414
-.29606
-.0858.3
-14
-.68000
-.32684
-.30960
-.11659.
-13 --.63143
-.31718
-.32080
-.14622
-12
-.58286
-.30516
-.32964
-.17415
-11
-.53429
-.29081
-.33612
-.20023
-AO,
- . 45 571
-.27417
-.34021
-.22434
'-9
-.43714 =.25527 . -.34193
7.24636
-8
-.38857
-.23418 - -.34125
-.26622
-7
-.34001)
-.21097
-.33819
-.28385
-6
-.29143
-.18574
-.33275
-.29920
-5
-.24286
-.15859
-.32494
-.31223
-4
-.19429
-.12968
-.31476
-.32293
-3
-.14571
-.09916
-.30223
-.33126
-2
-.09714
-.06723
-.28739
-.33721
-1 -.04857
-.03410
-.27025
-.34079
0
.00000
.00000
-.25087
-.34198
-35.

-34
-33,
. ,.

-

NOZZLE PRESSURE
O. DEG, 05. DEG.: 90. DEG.

NOZZLE AXIAL VELOCITY

O. DEG. 45. DEG.„ 90. bEG.,

.00294..
.30.085 :
.43542
-.00548
.00006
.00556
- .03161
.27126..43380
- .00595 . -.001364:
.00 5 05
- - .06520.
.23977.42837, -.006481
-.00133
.00450
- .09758.
.20673:
.41910
-.00676
-.00201,
.00391
- .12854
.17250•.
.40636
-.00708
-.00268 .00329.
- .15788.
.13741
.39006
- .00744
-.00332 •
.00265
- .18 5 43
.10182
.3 7 0 4 8-:-.00755 - -.00394
.00198
- .21107
.06806:
- .34786-.00770.
-.00453.. ,
.00130
- .23466,..,.
.03047,
.32246
-.00778
-.00508 .
.00060
- .25614
-.00465-.29459
-.00781 -.00559 ' -.00009
- .27536-01390. . .26456 , -.00777. -.00605 . -.00079
- .29234.
- .07236- .23270
-.00766
-.00647
-.00148,
- .30700
-.10445.
.19937
-.00750,-. -.00683 . -.00.216
- .31932 . -.13507
.16492, - .00728. -.00714
-.00282
- .32927.
- .16403 .12969
-.00700 • -.00739
-.00346- .33684
-.19i18
.09404, -.00666
-.00759
- .00407
-.34202- - .21648,-„ .05829
-.00627
-.00772.
-.00465
-.34480: -.23951_ .02278. -.00583
-.00779
-.00519
-.34518.
- .26049, -.01220
-.0U534
-.00780. -.00569
- .34317
- .27925 - -.04636
-.00481
-.00775
-.00614
- .33875
- .29572
-.07946
- .00425
-.00763
-.00055
- .33194.,
-.30908.
-.11124
-.00364
-.00746
-.00690
- .32273
-7,32169
.7,14152
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- .00744
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- .33817
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-.00658
-.00762
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-..00100.
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- .34509
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- .34495
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- .21860
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- .16653
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.00489
-.00085
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- .25803
-.34527
.00541
- .00015
- .00562

TPMI

-.00557
-.00605
- .00648
-.00685
- .00718.
-.00744 .
-.00765
-.00779
-.00788. - .00790-.00785
-.00775
- .00758
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- .00301
- .00235 -.
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..g=
.00251
.00317
.00380

.(000:97
.00550


### Table D-7. (Continued)

**Combustion Parameters:** Interaction Index = 0.57486

**Time-Lag** = 1.70000

**Motor Parameters:**
- Gamma = 1.20000
- Exit Mach Number = 0.20000
- Length/Diameter = 0.50000

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**Injector Pressure**

**Nozzle Pressure**

**Nozzle Axial Velocity**

**YPHI**
Table D-8. Sample Output, Section 5.

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<td>THETA = .0</td>
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Figure D-2. Sample Pressure Plot.
Figure D-3. Sample Amplitude Plot.
*************** PROGRAM LCYC3D ***********************

THIS PROGRAM CALCULATES THE NONLINEAR BEHAVIOR OF
TRANSVERSE, AXIAL, OR COMBINED LONGITUDINAL-TRANSVERSE
INSTABILITIES IN A CYLINDRICAL COMBUSTION CHAMBER WITH
UNIFORM PROPELLANT INJECTION, DISTRIBUTED COMBUSTION
PROCESS, AND A CONVENTIONAL NOZZLE. THE COMBUSTION PROCESS
IS DESCRIBED BY CRÖCCO'S TIME-LAG MODEL. BOTH TRANSIENT
AND LIMIT-CYCLE SOLUTIONS ARE CALCULATED.

THE FOLLOWING INPUTS ARE REQUIRED:

(1) THE CONTROL NUMBER, NOUTCF.
(2) THE COEFFICIENTS FROM PROGRAM COEFFS3D.
(3) THE DATA DECK.

NOUTCF DETERMINES PRINTOUT OF COEFFICIENTS.
   IF NOUTCF = 0 COEFFICIENTS ARE NOT PRINTED OUT.
   IF NOUTCF = 1 LINEAR COEFFICIENTS ONLY ARE PRINTED OUT.
   IF NOUTCF = 2 ALL COEFFICIENTS ARE PRINTED OUT.

THE DATA DECK CONSISTS OF THE FOLLOWING CARDS:

FIRST CARD:

EN IS THE INTERACTION INDEX.
TAU IS THE TIME LAG.
H IS THE INTEGRATION STEP SIZE.
TSTART IS THE TIME AT WHICH OUTPUT STARTS.
TQUIT IS THE TIME AT WHICH COMPUTATIONS ARE TERMINATED.

SECOND CARD:

NTEST IS TASK CONTROL NUMBER:
   IF NTEST = 0 COMPUTE TRANSIENT BEHAVIOR.
   IF NTEST = 1 COMPUTE THE LIMIT-CYCLE BEHAVIOR.
JMODE IS THE MODE-AMPLITUDE USED TO TEST FOR LIMIT-CYCLES.
NLOC DETERMINES THE LOCATION OF THE WALL PRESSURE MAXIMA
AND MINIMA:
   IF NLOC = 1 LOCATION IS Z = 0, THETA = 0 DEGREES.
   IF NLOC = 2 LOCATION IS Z = 0, THETA = 45 DEGREES.
   IF NLOC = 3 LOCATION IS Z = 0, THETA = 90 DEGREES.
NTERMS IS THE NUMBER OF TERMS GIVEN INITIAL VALUES.
NPZ DETERMINES HOW SECONDARY STABILITY ZONES (PHANTOM
ZONES) ARE HANDLED.
   IF NPZ = 0 PHANTOM ZONES ARE RETAINED.
   IF NPZ = 1 PHANTOM ZONES ARE ELIMINATED.
NOUT IS THE OUTPUT CONTROL NUMBER:
   IF NOUT = 0 PRINTED OUTPUT ONLY.
   IF NOUT > 0 BOTH PRINTED AND PLOTTED OUTPUT, NOUT
DETERMINES THE NUMBER OF THE LAST PLOT PRODUCED.

DATA FOR SETTING UP PLOTS (THIRD AND FOURTH CARDS):

YHI(1) IS THE MAXIMUM ORDI NATE FOR PRESSURE PLOTS.
YHI(5) IS THE MAXIMUM ORDI NATE FOR VELOCITY PLOTS.
NOTE: THE ORDI NATE SCALES FOR PRESSURE AND VELOCITY PLOTS
ARE SYMMETRIC ABOUT ZERO.
YLAB IS THE INTERVAL FOR ORDI NATE LABELING FOR ABOVE PLOTS.
ITYC Y IS THE NUMBER OF ORDI NATE TIC MARKS FOR ABOVE PLOTS.
NOTE: ITICY SHOULD BE NEGATIVE FOR PRESSURE AND VELOCITY PLOTS
TO OBTAIN CENTERLINE.
NFIRST IS THE NUMBER OF THE FIRST PLOT PRODUCED.
NOMIT DETERMINES WHETHER AMPLITUDE PLOT IS PRODUCED:
  IF NOMIT = 0 AMPLITUDE PLOT IS PRODUCED.
  IF NOMIT = 1 AMPLITUDE PLOT IS OMITTED.

INITIAL AMPLITUDES OF F-FUNCTIONS (REMAINING CARDS):

AS(J) IS THE AMPLITUDE OF THE SINE TERM.
AC(J) IS THE AMPLITUDE OF THE COSINE TERM.

COMPLEX YNOZ(10), B(10), C1, C2, C3, CPHIT(10), CSUM, A
DIMENSION L(10), M(10), S(10), NAME(10), AS(20), AC(20),
1 U(20*40), AA(4), Y(40), FZ(4*40), YP(40), UZ(40),
2 CP(3*20*20), FRQ(20), DMP1(20), UMAX(500), UAVG(100),
3 Z(6), ANGLE(6), THETA(6), CFT(6*20), YL(20),
4 CFTH(6*20), CFZ(6*20), PRESS(6), AXVEL(3), YR(20),
5 TPLOT(500), YPLOT(6*500), DUMMYT(500), DUMMYY(500),
6 IBUF(3000), ITT(4), ITY1(7), ITY2(7), ITY3(7),
7 ITY4(7), ITY5(6), TAUCUT(20), ITY6(8),
8 ITF(3), TITLE(12), PRS(500), TI(500), PMAX(500),
9 TIMAX(500), YLO(6), YHI(6), YLAB(6), ITICY(6)

COMMON RBU(20*4), C(3*20*20), D(20*400),
1 KFMAX(3*20), IC(3*20*20), KPMAX(20),
2 IDP(20*400), IDQ(20*400)
COMMON /BLK2/ M(10), NS(10), SX(10), B
COMMON /BLK3/ NUJ, NLMAX, GAMMA, COEF(3*20)

DATA ITT/"DIMENSIONLESS TIME, T'/,
1 ITY1/"INJECTOR PRESSURE PERTURBATION, THETA = 0'/,
2 ITY2/"INJECTOR PRESSURE PERTURBATION, THETA = 45'/,
3 ITY3/"INJECTOR PRESSURE PERTURBATION, THETA = 90'/,
4 ITY4/"NOZZLE PRESSURE PERTURBATION, THETA = 0'/,
5 ITY5/"NOZZLE AXIAL VELOCITY, THETA = 0'/,
6 ITY6/"NOZZLE B.C. (REC - GAMMA*Y*PHIT)) AT THETA = 0'/
ITP/'PRESSURE PEAKS'/

LAST = 250
ERR = 0.001
TDEL = 10.0
NPT = 0
AA(1) = 0.0
AA(2) = 0.5
AA(3) = 0.5
AA(4) = 1.0
PI = 3.1415927
READ (5,5003) NOUTCF

*************** COEFFICIENT INPUT SECTION ***********************

THIS VERSION OF LCYC3D READS THE COEFFICIENT DATA FROM A FASTRAND FILE GENERATED BY PROGRAM COEFFS3D. TO READ THIS DATA FROM CARDS, USE READ (5,XXXX) INSTEAD OF READ (9,XXXX) IN THIS SECTION.

INPUT OF MOTOR PARAMETERS AND NUMBER OF TERMS.
READ (9,5001) GAMMA, UE, ZE, ZCOMB, NDROPS, NJMAX
WRITE (6,6001) . GAMMA, UE, ZE, ZCOMB, NJMAX
IF (NDROPS .EQ. 0) WRITE (6,6030)
IF (NDROPS .EQ. 1) WRITE (6,6031)
NU = 2 * NJMAX
JMX = NJMAX/2
RLD = 0.5 * ZE

WRITE (6,6002)

INPUT OF DESCRIPTION OF SERIES EXPANSION.
DO 10 K = 1, JMX
READ (9,5002) NJ, L(NJ), M(NJ), N(NJ), NS(NJ), S(NJ), SJ(NJ), NAME(NJ)
WRITE (6,6003) NAME(NJ), NJ, L(NJ), M(NJ), N(NJ), NS(NJ), S(NJ), SJ(NJ)
10 CONTINUE

WRITE (6,6010)
DO 15 K = 1, JMX
READ (9,5010) J, YNOZ(J), B(J)
WRITE (6,6015) J, YNOZ(J), B(J)
NJ = (2 * J) - 1
YR(NJ) = REAL(YNOZ(J))
YI(NJ) = AIMAG(YNOZ(J))
YR(NJ+1) = YR(NJ)
YI(NJ+1) = YI(NJ)
15 CONTINUE
ZERO LINEAR COEFFICIENT ARRAYS.
DO 20 KC = 1, 3
DO 20 NJ = 1, 20
DO 20 NP = 1, 20
C(KC,NJ,NP) = 0.0
CP(KC,NJ,NP) = 0.0
20 CONTINUE

ZERO NONLINEAR COEFFICIENT ARRAY.
DO 30 NJ = 1, 20
DO 30 NPQ = 1, 400
D(NJ,NPQ) = 0.0
30 CONTINUE

INPUT OF LINEAR COEFFICIENTS.
DO 40 KC = 1, 3
READ (9,5003) KMAX
IF (NOUTCF.GT.0) WRITE (6,6004) KC, KMAX
IF (KMAX.EQ.0) GO TO 40
DO 45 K = 1, KMAX
READ (9,5004) NJ, NP, CP(KC,NJ,NP)
IF (NOUTCF.GT.0) WRITE (6,6005) KC, NJ, NP, CP(KC,NJ,NP)
45 CONTINUE
40 CONTINUE

INPUT OF NONLINEAR COEFFICIENTS.
READ (9,5003) NLMAX
IF (NOUTCF.EQ.2) WRITE (6,6006) NLMAX
IF (NLMAX.EQ.0) GO TO 50
DO 52 NJ = 1, 20
KPQMAX(NJ) = 0
52 CONTINUE
DO 55 K = 1, NLMAX
READ (9,5005) NJ, NP, NO, DT
IF (NOUTCF.EQ.2) WRITE (6,6007) NJ, NP, NO, DT
KPQMAX(NJ) = KPQMAX(NJ) + 1
KPQ = KPQMAX(NJ)
IDP(NJ,KPQ) = NP
IDQ(NJ,KPQ) = NQ
D(NJ,KPQ) = DT
55 CONTINUE
50 CONTINUE

************* PRESSURE COEFFICIENT SECTION **********************

CALCULATE SPATIAL COORDINATES FOR PRESSURE COMPUTATION.
DO 51 NFRES = 1, 3
Z(NFRES) = 0.0
RTHETA = NFRES - 1
ANGLE(NPRES) = RTHETA * 45.0
THETA(NPRES) = RTHETA * PI/4.0
Z(NPRES + 3) = ZE
ANGLE(NPRES + 3) = ANGLE(NPRES)
THETA(NPRES + 3) = THETA(NPRES)

51 CONTINUE

CALCULATE COEFFICIENTS FOR PRESSURE TIME HISTORIES.
DO 53 NPRES = 1, 6
DO 53 J = 1, JMX
NP = (2 * J) - 1
Z1 = Z(NPRES)
ANG = THETA(NPRES)
CALL PHICFS(J,Z1,ANG,C1,C2,C3)
IF (NPRES .EQ. 4) CPHIT(J) = C1
CFT(NPRES,NP) = REAL(C1)
CFT(NPRES,NP+1) = -AIMAG(C1)
CFTH(NPRES,NP) = REAL(C2)
CFTH(NPRES,NP+1) = -AIMAG(C2)
CFZ(NPRES,NP) = REAL(C3)
CFZ(NPRES,NP+1) = -AIMAG(C3)
53 CONTINUE

OUTPUT OF COEFFICIENTS FOR PRESSURE TIME HISTORIES.
WRITE (6,6020)
DO 56 NPRES = 1, 6
WRITE (6,6014)
DO 56 J = 1, NJMAX
WRITE (6,6021) J, Z(NPRES), ANGLE(NPRES), CFT(NPRES,J), CFTH(NPRES,J), CFZ(NPRES,J)
56 CONTINUE

************* DATA INPUT SECTION ******************:
READ (5,5000) TITLE

ZERO INITIAL VALUE AND FREQUENCY ARRAYS.
5 DO 57 K = 1, NJMAX
AS(K) = 0.0
AC(K) = 0.0
FRQ1(K) = 0.0
57 CONTINUE

READ COMBUSTION AND CONTROL PARAMETERS.
READ (5,5006, END = 300) EN, TAU, H, TSTART, TQUIT

READ CONTROL NUMBERS.
READ (5,5008) NTEST, JMODE, NLOC, NTERMS, NPZ, NOUT
JMODE = (2 * JMODE) - 1
JPMODE = JMODE + NJMAX
IF (NOUT .GT. 0)  NPT = 1

IF (NOUT .EQ. 0) GO TO 9
READ DATA FOR SETTING UP PLOTS.
READ (5,5009) YHI(1), YHI(5), YLAB(1), YLAB(5)
READ (5,5008) ITICY(1), ITICY(5), NFIRST, NOMIT

********** INITIAL AMPLITUDES SECTION ***********************

9 DO 58 K = 1, NTERMS

INPUT INITIAL AMPLITUDES FOR F-FUNCTIONS.
READ (5,5007) J, AST, ACT
NJ = (2 * J) - 1
AS(NJ) = AST
AC(NJ) = ACT

CALCULATE FREQUENCY AND DAMPING.
RL = L(J)
AX = RL * PI/ZE
AXSQ = AX * AX
SSQ = S(J) * S(J)
FRQ1(NJ) = SQRT(SSQ + AXSQ)
DMP1(NJ) = 0.0
FRQ1(NJ+1) = FRQ1(NJ)
DMP1(NJ+1) = DMP1(NJ)

CALCULATE INITIAL AMPLITUDES FOR G-FUNCTIONS.

IF (FRQ1(NJ)) 58s 58s 581
581 GYRU = GAMMA*YR(NJ)*UE
GYIF = GAMMA*YI(NJ)*FRQ1(NJ)
GYRF = GAMMA*YR(NJ)*FRQ1(NJ)
GYIU = GAMMA*YI(NJ)*UE

NPRES = 4
IF (NS(J) .EQ. 1) NPRES = 6

A1 = (1.0 + GYRU)*CFZ(NPRES,NJ+1)
1  - GYIF*CFT(NPRES,NJ+1)
A2 = GYRF*CFT(NPRES,NJ+1) + GYIU*CFZ(NPRES,NJ+1)
A3 = -(1.0 + GYRU)*CFZ(NPRES,NJ) + GYIF*CFT(NPRES,NJ)
A4 = GYRF*CFT(NPRES,NJ) + GYIU*CFZ(NPRES,NJ)

DET = A1*A1 + A2**2
IF (DET .LT. 0.0000)01) GO TO 583
R1 = A3*AC(NJ) - A4*AS(NJ)
R2 = -A4*AC(NJ) - A3*A5(NJ)
AC(NJ+1) = (R1*A1 + R2*A2)/DET
AS(NJ+1) = -(R2*A1 - R1*A2)/DET
GO TO 58

583 AC(NJ+1) = -AS(NJ)
AS(NJ+1) = AC(NJ)

58 CONTINUE

C

C OUTPUT OF INITIAL AMPLITUDES.
WRITE (6,6016)
DO 590 J = 1, NJMAX
IF (AS(J)) 591, 592, 591
592 IF (AC(J)) 591 , 590, 591
591 WRITE (6,6017) J, DMP1(J), FRG1(J), AC(J), AS(J)
590 CONTINUE
IF (NTEST .EQ. 0) WRITE (6,6025)
IF (NTEST .EQ. 1) WRITE (6,6026)
IF (NPZ .EQ. 1) WRITE (6,6028)
IF (NOUT .GE. 1) WRITE (6,6027)

C

************* LINEAR COEFFICIENTS SECTION **************

C

DO 59 KC = 1, 3
DO 59 s14.1 = 1, 10
KPMAX(KC,NJ) = 0
59 CONTINUE

C

IF (NPZ .EQ. 0) GO TO 605
DO 602 J = 1, JMX
NJ = (J * J) - 1
RL = L(J)
AX = RL * PI/ZE
AXSQ = AX * AX
SSQ = S(J) * S(J)
OMEGA = SQRT(SSQ + AXSQ)
TAUCUT(NJ) = 2.0 * PI/OMEGA
TAUCUT(NJ+1) = TAUCUT(NJ)
602 CONTINUE

C

DO 604 NJ = 1, NJMAX
DO 604 NP = 1, NJMAX
IF (TAU .GT. TAUCUT(NP)) CP(3,NJ,NP) = 0.0
604 CONTINUE

C

COMPUTE LINEAR COEFFICIENTS FOR GIVEN VALUES OF EN AND TAU.

605 DO 60 NJ = 1, NJMAX
DO 60 NP = 1, NJMAX
CT = CP(1,NJ,NP)
60 CONTINUE
IF (CT) 61, 62, 61
61 KPMAX(1,NJ) = KPMAX(1,NJ) + 1
KP = KPMAX(1,NJ)
IC(1,NJ,KP) = NP
C(1,NJ,KP) = CT
62 CT = CP(2,NJ,NP) - EN*CP(3,NJ,NP)
IF (CT) 63, 64, 63
63 KPMAX(2,NJ) = KPMAX(2,NJ) + 1
KP = KPMAX(2,NJ)
IC(2,NJ,KP) = NP
C(2,NJ,KP) = CT
64 CT = EN * CP(3,NJ,NP)
IF (CT) 65, 60, 65
65 KPMAX(3,NJ) = KPMAX(3,NJ) + 1
KP = KPMAX(3,NJ)
IC(3,NJ,KP) = NP
C(3,NJ,KP) = CT
60 CONTINUE
C
C ************* STEP-SIZE COMPUTATION *****************************
C
NDIV = 1.0 + TAU/H
RN = NDIV
H = TAU/RN
H6 = H/6.0
C
C ************* INITIAL VALUES SECTION *****************************
C
WRITE (6,6008) EN, TAU, GAMMA, UE, RLD
WRITE (6,6009)
WRITE (6,6022) (ANGLE(J), J = 1,6), (ANGLE(J), J = 1,3)
WRITE (6,6012)
NP1 = NDIV + 1
DO 70 I = 1, NP1
NSTEP = I - NP1
RSTEP = NSTEP
TIME = RSTEP * H
TI(I) = TIME
70 CONTINUE
751 IF (AC(J)) 751, 753, 751
752 U(I,J) = 0.0
U(I,JP) = 0.0
G0 TO 75
751 ARG = FRG1(J) * TIME
FSIN = SIN(ARG)
FCOS = COS(ARG)
FEXP = EXP(DM1P1(J)*TIME)
U(I,J) = (AS(J)*FSIN + AC(J)*FCOS) * FEXP
U(I,JP) = ((AS(J) * FCOS) - (AC(J) * FSIN)) * FRQ1(J) * FEXP
1 + DMP1(J) * U(I,J)

75 CONTINUE

C CALCULATE INITIAL VALUES OF PRESSURE AND VELOCITY.
DO 704 NPRES = 1, 6
DO 702 J = 1, NJMAX
COEF(1,J) = CFT(NPRES, J)
COEF(2,J) = CFTH(NPRES, J)
COEF(3,J) = CFZ(NPRES, J)
702 CONTINUE
DO 703 J = 1, NU
YC(J) = U(I, J)
703 CONTINUE
UBAR = 0.0
IF (NPRES .GT. 3) UBAR = UE
UMS = 0.0
IF (NPRES .LT. 4) UMS = UE/(Z*ZCOMB)
CALL PRSVEL(UBAR, UMS, Y, P, VTH, VZ)
PRES(NPRES) = P
704 CONTINUE
PRESS(I) = PRES(NLOC)

C CALCULATE INITIAL VALUES OF NOZZLE B.C.
CSUM = (0.0, 0.0, 0.0)
DO 710 J = 1, JMX
JP = J, NJMAX + (2 * J) - 1
FT = Y(JP)
GT = Y(JP+1)
A = CMPLX(FT, GT)
CSUM = CSUM + YNOZ(J) * CFHIT(J) * A
710 CONTINUE
SUM = REAL(CSUM)
YPHI = -GAMMA * SUM
WRITE (6, 6011) NSTEP, TIME, PRESS(J), J = 1, 6,
1 (AXVEL(J), J = 1, 3), YPHI
70 CONTINUE

C WRITE (6, 6008) EN, TAU, GAMMA, UE, RLD
WRITE (6, 6022) (ANGLE(J), J = 1, 6), (ANGLE(J), J = 1, 3)

C ************** INITIALIZE CONTROL NUMBERS **************

C LINE = 8
K = 0
MAXNO = 0
MAXP = 0
IF (NOUT .EQ. 0) GO TO 100
JPLT = 0
TMIN = TSTART
TMAX = TSTART + TDEL
YLO(1) = -YHI(1)
DO 90 J = 2,4
  YHI(J) = YHI(1)
  YLO(J) = YLO(1)
  YLAB(J) = YLAB(1)
  ITICY(J) = ITICY(1)
90 CONTINUE
YLO(5) = -YHI(5)
YHI(6) = YHI(5)
YLO(6) = YLO(5)
YLAB(6) = YLAB(5)
ITICY(6) = ITICY(5)

C
*************** NUMERICAL CALCULATIONS SECTION ******************
C
100 I = NP1
C
RUNGE-KUTTA INTEGRATION SCHEME.
105 NSTEP = (I - NP1 + (LAST - NP1) * K)
RSTEP = NSTEP
TIME = RSTEP * H
TI(I) = TIME
DO 110 J = 1, NJMAX
  JP = J + NJMAX
  RV(J,1) = U(I-NDIV+JP)
  RV(J,4) = U(I-NDIV+1,JP)
  RV(J,2) = 0.375*RV(J,1) + 0.75*RV(J,4)
  RV(J,3) = RV(J,2)
110 CONTINUE
DO 120 J = 1, NU
  Y(J) = U(I,J)
120 CONTINUE
CALL RHS(NU,1,Y,YP)
DO 130 J = 1, NU
  FZ(1,J) = YP(J)
130 CONTINUE
DO 140 II = 2,4
DO 144 J = 1, NU
  UZ(J) = Y(J) + AA(II) * H * FZ(II,J)
144 CONTINUE
CALL RHS(NU,II,UZ,YP)
DO 148 J = 1, NU
  FZ(II,J) = YP(J)
148 CONTINUE
DO 150 J = 1, NU
  U(I+1,J) = Y(J) + CFZ(1,J) + 2.0*(FZ(2,J)+FZ(3,J)) + FZ(4,J) * H6
150 CONTINUE
C
CALCULATE PRESSURE TIME HISTORIES.
DO 154 NPRES = 1, 6
   DO 152 J = 1, NJMAX
      COEF(1, J) = CFT(NPRES, J)
      COEF(2, J) = CFTH(NPRES, J)
      COEF(3, J) = CFZ(NPRES, J)
   152 CONTINUE
   UBAR = 0.0
   IF (NPRES .GT. 3) UBAR = UE
   UMS = 0.0
   IF ((NDROPS .LE. 1) .AND. (NPRES .LT 4)) UMS = UE / (ZESZCOMB)
   CALL PRSVEL(UBAR, UMS, Y, F, VH, VZ)
   PRESS(NPRES) = P
   IF (NPRES .GT. 3) AXVEL(NPRES - 3) = VZ
154 CONTINUE
PRS(I) = PRESS(NLOC)
C
C CALCULATE VALUES OF NOZZLE B.C.
CSUM = (0.0, 0.0, 0.0)
DO 650 J = 1, JMX
   JP = NJMAX + (2 * J) - 1
   FT = Y(JP)
   GT = Y(JP + 1)
   A = CMFLX(FT, GT)
   CSUM = CSUM + YNOZ(J) * CPHIT(J) * A
650 CONTINUE
SUM = REAL(CSUM)
YPHI = -GAMMA * SUM
C
C DETERMINE MAXIMA AND MINIMA OF PRINCIPAL MODE-AMPLITUDE
FUNCTION FOR USE IN DETERMINING LIMIT-CYCLE BEHAVIOR.
IF (U(I, JPMODE) * U(I + 1, JPMODE)) 170, 170, 160
170 FDEN = U(I, JPMODE) - U(I + 1, JPMODE)
   IF (FDEN) 171, 170, 171
171 PP = U(I, JPMODE)/FDEN
   PA = (PP - 1.0) * PP * 0.5
   PB = 1.0 - (PP * PP)
   PC = (PP + 1.0) * PP * 0.5
   MAXNO = MAXNO + 1
   MAX(MAXNO) = PA*U(I - 1, JMODE) + PB*U(I, JMODE) + FC*WI + 1, JMODE)
   IF (MAXNO .GE. 500) GO TO 250
160 CONTINUE

C DETERMINE MAXIMUM AND MINIMUM PRESSURE AT LOCATION SPECIFIED
BY NLOC.
DPL = PRS(I) - PRS(I - 1)
DPS = PRS(I - 1) - PRS(I - 2)
IF (DPL*DPS) 173, 173, 175
173 FNUM = PRS(I - 2) - PRS(I)
C PLOT TIME HISTORIES.
C DO 1020 NPLOT = OFIRST, NOUT, JPLOT = 0
C
PDEN = 2.0 * (FRS(I-2) + PRS(I) - 2.0*FRS(I-1))
174 IF (PDEN) 174, 175, 174
174 FP = FNUM/PDEN
FA = (PF - 1.0) * PP * 0.5
PB = 1.0 - (FP * PF)
PC = (FP + 1.0) * PP * 0.5
MAXF = MAXF + 1
PMAX(MAXF) = PA*FRS(I-2) + PB*FRS(I-1) + PC*FRS(I)
TIMAX(MAXF) = TI(I-1) + FF+H
IF (MAXP .GE. 500) GO TO 250
175 CONTINUE
C
C ************ TIME HISTORY PLOTTING SECTION **********************
C
IF (NTEST .EQ. 1) GO TO 155
IF (TIME .LT. TSTART) GO TO 155
IF ((NOUT .EQ. 0) .OR. (NOUT .GT. 6)) GO TO 156
C
C FILL TIME ARRAY FOR PLOTTING.
TPLOT(JPLOT) = TIME
C
C FILL INJECTOR PRESSURE ARRAYS FOR PLOTTING (THETA = 0, 45, 90)
DO 1001 J = 1,3
YPLOT(J,JFLOT) = PRESS(J)
1001 CONTINUE
C
C FILL NOZZLE PRESSURE ARRAY FOR PLOTTING (THETA = 0)
YPLOT(4,JFLOT) = PRESS(4)
C
C FILL NOZZLE AXIAL VELOCITY ARRAY FOR PLOTTING (THETA = 0)
YPLOT(5,JPLOT) = AXVEL(1)
C
C FILL NOZZLE B.C. ARRAY FOR PLOTTING (THETA = 0).
YPLOT(6,JPLOT) = YPHI
C
GO TO 156
C 1000 NUM = JPLOT
C
C PLOT TIME HISTORIES.
C DO 1020 NPLOT = NFIRST, NOUT
C
JPLOT = 0
C ASSIGN PLOTTING PARAMETERS.
  YMIN = YLO(NFLOT)
  YMAX = YHI(NFLOT)
  NTICY = ITICY(NFLOT)
  DELY = YLAB(NFLOT)

C ELIMINATE POINTS THAT ARE OUT OF THE ORDINATE RANGE.
DO 1010 J = 1, NUM
  IF ((YFLOT(NFLOT,J) .LT. YMIN) .OR. (YFLOT(NFLOT,J) .GT. YMAX))
    GO TO 1010
  JFLOT = JFLOT + 1
  DUMMY(JFLOT) = TFLT(J)
  DUMMYY(JFLOT) = YFLOT(NFLOT,J)
1010 CONTINUE

C IF (JFLOT .EQ. 0) GO TO 1020
   GO TO (1011, 1012, 1013, 1014, 1015, 1016, NFLOT)

C PLOT INJECTOR PRESSURE AT THETA = 0 DEGREES.
1011 CALL GRAPHS(1BUF, 3000, 4, JFLOT, 11, NTICY, TMAX, YMAX, TMIN, YMIN,
  ITTY, ITY1, 21, 41, DUMMY, DUMMYY, 2.0, DELY, TITLE)
   GO TO 1020

C PLOT INJECTOR PRESSURE AT THETA = 45 DEGREES.
1012 IF (M(JMODE) .EQ. 0) GO TO 1020
   CALL GRAPHS(1BUF, 3000, 4, JFLOT, 11, NTICY, TMAX, YMAX, TMIN, YMIN,
  ITTY, ITY2, 21, 42, DUMMY, DUMMYY, 2.0, DELY, TITLE)
   GO TO 1020

C PLOT INJECTOR PRESSURE AT THETA = 90 DEGREES.
1013 IF (M(JMODE) .EQ. 0) GO TO 1020
   CALL GRAPHS(1BUF, 3000, 4, JFLOT, 11, NTICY, TMAX, YMAX, TMIN, YMIN,
  ITTY, ITY3, 21, 43, DUMMY, DUMMYY, 2.0, DELY, TITLE)
   GO TO 1020

C PLOT NOZZLE PRESSURE AT THETA = 0 DEGREES.
1014 CALL GRAPHS(1BUF, 3000, 4, JFLOT, 11, NTICY, TMAX, YMAX, TMIN, YMIN,
  ITTY, ITY4, 21, 39, DUMMY, DUMMYY, 2.0, DELY, TITLE)
   GO TO 1020

C PLOT NOZZLE AXIAL VELOCITY AT THETA = 0 DEGREES.
1015 CALL GRAPHS(1BUF, 3000, 4, JFLOT, 11, NTICY, TMAX, YMAX, TMIN, YMIN,
  ITTY, ITY5, 21, 32, DUMMY, DUMMYY, 2.0, DELY, TITLE)
   GO TO 1020

C PLOT NOZZLE B.C. AT THETA = 0 DEGREES.
1016 CALL GRAPHS(1BUF, 3000, 4, JFLOT, 11, NTICY, TMAX, YMAX, TMIN, YMIN,
  ITTY, ITY6, 21, 44, DUMMY, DUMMYY, 2.0, DELY, TITLE)

C 1020 CONTINUE
REASSIGN PLOTTING PARAMETERS FOR NEXT SET OF PLOTS.
JPLT = 0
TMIN = TMAX
TMAX = TMAX + TDEL

************* TIME HISTORY PRINTED OUTPUT SECTION **************

156 WRITE (6,6011) NSTEP, TIME, (FRESS(J), J = 1,6),
1 (AXVEL(J), J = 1,3), YPHI
LINE = LINE + 1
157 IF (TIME .GT. TQUIT) GO TO 250
IF (LINE .LT. 52) GO TO 155
WRITE (6,6013)
WRITE (6,6022) (ANGLE(J), J = 1,6), (ANGLE(J), J = 1,3)
LINE = 4
155 I = I + 1
IF (I .LT. LAST) GO TO 105

************* LIMIT-CYCLE SECTION *******************************

TEST FOR LIMIT CYCLE.
K = K + 1
IF ((NTEST .EQ. 0) .OR. (MAXNO .LT. 80)) GO TO 190
UTOT = 0.0
DO 180 J = 0, 3
JMAX = MAXNO - J
UTOT = UTOT + ABS(UMAX(JMAX))
180 CONTINUE
UAVG(K) = UTOT/4.0
IF (K .EQ. 1) GO TO 190
CHANGE = UAVG(K) - UAVG(K-1)
ABSCHG = ABS(CHANGE/UAVG(K))
IF (ABSCHG .GT. ERR) GO TO 190
TM = TIME/2.0
ITM = TM
ITM = 2*ITM + 2
TM = ITM
TSTART = TM + TSTART
TQUIT = TM + TQUIT
TMIN = TSTART
TMAX = TSTART + TDEL
NTEST = 0

RE-ASSIGN ARRAYS.
190 DO 200 I = 1, NP1
ILAST = LAST - NP1 + I
FRS(I) = FHS(ILAST)
TI(I) = TI(ILAST)
DO 200 J = 1, NU
UI(J) = U(ILAST,J)
200 CONTINUE
GO TO 100

C
C
C *************** PRESSURE MAXIMA AND MINIMA PRINTOUT ***************
C
250 WRITE (6,6023) Z(NLOC), ANGLE(NLOC), MAXP
LINE = 4
DO 255 JST = 1, MAXP, 8
JSTART = JST
JSTOP = JST + 7
IF (JSTOP .GT. MAXP) JSTOF = MAXP
WRITE (6,6024) (FMAXCJ), J = JSTART, JSTOF)
WRITE (6,6024) (TIMAX(J), J = JSTART, JSTOF)
WRITE (6,6014)
LINE = LINE + 3
IF (LINE .LT. 52) GO TO 255
LINE = 0
WRITE (6,6013)
255 CONTINUE
IF ((NOUT .LT. 0) .OR. (NOMIT .LT. 1)) GO TO 5

C
C *************** PRESSURE MAXIMA PLOTTING SECTION *******************
C
C DETERMINE LARGEST VALUE OF PMAX.
AMPMAX = 0.0
DO 260 J = 1, MAXP
IF (PMAX(J) .LT. AMPMAX) GO TO 260
AMPMAX = PMAX(J)
260 CONTINUE

C RANGE OF PLOT AND COORDINATE LABELING.
ITM = AMPMAX + 1.0
AMPMAX = ITM
ITM = 1.0 + TIMAX(MAXP)/50.0
TMAX = ITM * 50
DELX = TMAX/10.0
DELY = AMPMAX/10.0

C ELIMINATE NEGATIVE VALUES.
JFLOT = 0
DO 262 J = 1, MAXP
IF (PMAX(J)) 262, 264, 264
264 JFLOT = JFLOT + 1
DUMMY(JFLOT) = TIMAX(J)
DUMMYY(JFLOT) = PMAX(J)
262 CONTINUE

C
PLT VALUES.
CALL GRAPH (IBUF, 3000, JFLOT, 101, TMAX, TMAX, 0.0, 0.0, 101, 101, TMAX, TMAX, "DUMMY", "DUMMY", "DELT", "DELY", "TITLE")

GO TO 5

TURN OFF PLOTTING ROUTINE.
300 IF (NPT EQ. 1) CALL SHFARG

************* READ FORMAT SPECIFICATIONS *************

5000 FORMAT (12A6)
5001 FORMAT (4F10.0, 2I5)
5002 FORMAT (5I5, 2F10.5, 1X, A4)
5003 FORMAT (15)
5004 FORMAT (2I5, F15.6)
5005 FORMAT (3I5, F15.6)
5006 FORMAT (5F10.0)
5007 FORMAT (15, 2F10.0)
5008 FORMAT (7I5)
5009 FORMAT (7F10.0)
5010 FORMAT (15, 4F10.5)

************* WRITE FORMAT SPECIFICATIONS *************

6001 FORMAT (1H1, 9H GAMMA = , F5.3, 5X, SHOE = F5.3,
1 5X, 5HZE = F8.5, 5X, 8HZ COMB = I5.2,
2 5X, 8HNJMAX = I2/)
6002 FORMAT (2X, A4, 515, 2F10.5)
6003 FORMAT (2X, 2HC(I1, I2, I3), =, F10.5)
6004 FORMAT (1H1, 45H COMBUSTION PARAMETERS: INTERACTION INDEX = F7.5,
1 12X, 11H TIME-LAG =, F7.5/2X, 17H MOTOR PARAMETERS:
2 8H GAMMA =, F7.5, 23H EXIT MACH NUMBER = P7.5
3 22H LENGTH/DIAMETER =, F7.5/)
6005 FORMAT (2X, A4, 5I5, 2F10.5)
6006 FORMAT (1H0, 18H INITIAL CONDITIONS/)
6007 FORMAT (1H0, 5X, 1HJ, 8X, 7H YR, 8X, 7H I7, 3H EPS, 7X, 3H ETA/)
6008 FORMAT (2X, I5, F12.5, 10F10.5)
6009 FORMAT (1I0)
6010 FORMAT (1I0)
6011 FORMAT (1I1)
6012 FORMAT (1I0)
6013 FORMAT (1H1)
6014 FORMAT (1H1)
6015 FORMAT (2X, I5, 4F10.5)
6016 FORMAT (1H1, 36H INITIAL CONDITIONS ARE OF THE FORM://
1 2X, 49HUC(I,J) = ACC(J)*COS(FREQ*T) + AS(J)*SIN(FREQ*T),
2 14H * EXP(DAMP*T) //6X, 1HJ, 8X, 7H DAMPING,
3 6X, 9H FREQUENCY, 10X, 5H AC(J), 10X, 5H AS(J)/

139
6017 FORMAT (2X, I5, 4F15.8)
6020 FORMAT (1H1, 46H COEFFICIENTS FOR COMPUTATION OF WALL PRESSURE,
1 10H WAVEFORMS /// 43X, 27H COEFFICIENTS IN SERIES FOR ///
2 22X, 5HTHETA, 10X, 4HTIME, 10X, 5HTHETA, 10X, 5HAXIAL //
3 6X, 1HJ, 9X, 1HZ, 3X, 9H(DEGREES), 5X, 10H DERIVATIVE, 5X, 10H DERIVATIVE ///)
6021 FORMAT (2X, I5, F10.3, F12.1, 3F15.7)
6022 FORMAT (26X, 17H INJECTOR PRESSURE, 14X, 15H NOZZLE PRESSURE,
1 12X, 21H NOZZLE AXIAL VELOCITY, 12X, 4H STEP, 8X, 4HTIME //
2 / F5.0, 5H DEG., F5.0, 5H DEG., F5.0, 5H DEG., 
3 F5.0, 5H DEG., F5.0, 5H DEG., F5.0, 5H DEG., 
4 F5.0, 5H DEG., F5.0, 5H DEG., F5.0, 5H DEG., 
6X, 4H YPHI ///)
6023 FORMAT (1H1, 38H PRESSURE MAXIMA AND MINIMA AT: 
1 2X, 11H THETA = , F4.2, 11H VALUES COMPUTED: , 13 //)
6024 FORMAT (1H, 7X, 8F13.6)
6025 FORMAT (2X, 37H THE TRANSIENT BEHAVIOR IS CALCULATED.)
6026 FORMAT (2X, 39H THE LIMIT-CYCLE BEHAVIOR IS CALCULATED.)
6027 FORMAT (2X, 33H THIS RUN PRODUCES PLOTTED OUTPUT.)
6028 FORMAT (2X, 'THE PHANTOM ZONES ARE ELIMINATED.')
6030 FORMAT (2X, 'DROPLET MOMENTUM SOURCE IS NEGLECTED')
6031 FORMAT (2X, 'DROPLET MOMENTUM SOURCE IS INCLUDED')
END
SUBROUTINE PHICFS(NP, Z, THETA, CT, CTH, CZ)

C
C
C THIS SUBROUTINE COMPUTES THE COEFFICIENTS NEEDED TO
C CALCULATE THE WALL PRESSURE PERTURBATION.
C
C NP IS THE INDEX OF THE COMPLEX SERIES TERM.
C Z IS THE AXIAL LOCATION.
C THETA IS THE AZIMUTHAL LOCATION.
C CT IS THE COEFFICIENT IN THE SERIES FOR THE TIME DERIVATIVE OF
C THE VELOCITY POTENTIAL.
C CTH IS THE COEFFICIENT IN THE SERIES FOR THE THETA DERIVATIVE
C OF THE VELOCITY POTENTIAL.
C CZ IS THE COEFFICIENT IN THE SERIES FOR THE AXIAL DERIVATIVE
C OF THE VELOCITY POTENTIAL.
C
C COMPLEX CI, CZ, CAXI, CAXIZ, CRAD, CAZI, CAZITH,
1 B(10), CT, CTH, CZ

COMMON /BLK2/ MC10, NS(10), SJ(10), B

CI = (0.0, 1.0)
CZ = CMPLX(Z, 0.0)
CAXI = CCOSH(CI * B(NP) * CZ)
CAXIZ = CI * B(NP) * CSINH(CI * B(NP) * CZ)
CRAD = CMPLX(SJ(NP), 0.0)
EM = M(NP)
ARG = EM * THETA
FSIN = SINCARG
FCOS = COSCARG
AZI = FCOS
IF (NS(NP) .EQ. 1) AZI = FSIN
AZITH = EM * FCOS
IF (NS(NP) .EQ. 2) AZITH = -EM * FSIN
CAZI = CMPLX(AZI, 0.0)
CAZITH = CMPLX(AZITH, 0.0)

CT = CAZI * CAXI * CRAD
CTH = CAZITH * CAXI * CRAD
CZ = CAZI * CAXIZ * CRAD

RETURN
END
SUBROUTINE PRSVELCUBAR, UMS, Y, P, VTH, VZ

C
C THIS SUBROUTINE COMPUTES THE WALL PRESSURE AND VELOCITY.
C
C UBAR IS THE LOCAL AXIAL STEADY STATE MACH NUMBER.
C UMS IS THE DERIVATIVE OF THE MACH NUMBER FOR THE CASE
C WHEN DROPLET MOMENTUM SOURCES ARE INCLUDED.
C Y IS THE ARRAY CONTAINING VALUES OF THE MODE-AMPLITUDE
C FUNCTIONS AND THEIR DERIVATIVES.
C P IS THE VALUE OF THE WALL PRESSURE PERTURBATION.
C VTH IS THE TANGENTIAL COMPONENT OF VELOCITY AT THE WALL.
C VZ IS THE AXIAL COMPONENT OF VELOCITY AT THE WALL.
C
DIMENSION Y(40), SUM(4), SUMSQ(3)
COMMON /BLK3/ NJMAX, NLMAX, GAMMA, COEF(3,20)

DO 10 I = 1, 4
  SUM(I) = 0.0
10 CONTINUE

DO 20 I = 1, 4
  DO 20 J = 1, NJMAX
    JY = J
    IF (I .EQ. 1) JY = J + NJMAX
    II = I
    IF (I .EQ. 4) II = 1
    SUM(I) = SUM(I) + Y(JY) * COEF(II,J)
20 CONTINUE

PLIN = SUM(1) + UBAR*SUM(3) + UMS*SUM(4)
PNL = 0.0
IF (NLMAX .EQ. 0) GO TO 40
DO 30 I = 1, 3
  SUMSQ(I) = SUM(I) * SUM(I)
30 CONTINUE
PNL = 0.5 * (SUMSQ(2) + SUMSQ(3) - SUMSQ(1))

P = -GAMMA * (PLIN + PNL)
VTH = SUM(2)
VZ = SUM(3)

RETURN
END
SUBROUTINE RHS(Nu, II, U, UP)

C
DIMENSION U(Nu), UF(Nu)
COMMON HV(20,4), C(3,20,20), D(20,400)
1 KMAX(3,20), IC(3,20,20), KPMAX(20),
2 IDF(20,400), IDG(20,400)
COMMON /BLK3/ NMAX, NLMAX, GAMMA, COEF(3,20)
C
DO 10 NJ = 1, NMAX
NJP = NJ + NMAX
UP(NJ) = UF(NJP)
SL1 = 0.0
SL2 = 0.0
SL3 = 0.0
SNL = 0.0
MAX = KPMAX(1,NJ)
IF (MAX .EQ. 0) GO TO 25
DO 20 KP = 1, MAX
NP = IC(1,NJ,KP)
SL1 = SL1 + (C(1,NJ,KP) * UF(NP))
20 CONTINUE
25 MAX = KPMAX(2,NJ)
IF (MAX .EQ. 0) GO TO 35
DO 30 KP = 1, MAX
NFP = IC(2,NJ,KP) + NLMAX
SL2 = SL2 + (C(2,NJ,KP) * UF(NFP))
30 CONTINUE
35 MAX = KPMAX(3,NJ)
IF (MAX .EQ. 0) GO TO 45
DO 40 KP = 1, MAX
NP = IC(3,NJ,KP)
SL3 = SL3 + (C(3,NJ,KP) * HV(NP,II))
40 CONTINUE
45 IF (NLMAX .EQ. 0) GO TO 55
MAX = KPMAX(NJ)
IF (MAX .EQ. 0) GO TO 55
DO 50 KP = 1, MAX
NP = IDG(NJ,KPG)
NQP = IDG(NJ,KPG) + NLMAX
SNL = SNL + (D(NJ,KPG) * UF(NP) * UF(NQP))
50 CONTINUE
55 UP(NJP) = -(SL1 + SL2 + SL3 + SNL)
10 CONTINUE
RETURN
END
SUBROUTINE GRAPHS(IBUF,NLOC,LDEV,NTOT,NTICX,NTICY,
1 XMAX,YMAX,XMIN,YMIN,ITITLX,ITITLY,LTITLX,LTITLY,LTITLE,XARRAY,
2 YARRAY,DELX,DELY,TITLE)

C IDENTIFIER MEANING TYPE
C
C IBUF: ADDRESS OF BUFFER AREA FOR PLOT OUTPUT INTEGER
C NLCC: NUMBER OF LOCATIONS IN BUFFER AREA (>=2000) INTEGER
C LDEV: LOGICAL DEVICE NUMBER FOR PLOT INTEGER
C NTOT: NUMBER OF POINTS TO BE PLOTTED INTEGER
C NTICX: NUMBER OF TIC MARKS ON ABSCISSA (>=2) INTEGER
C NTICY: NUMBER OF TIC MARKS ON ORDINATE (>=2) INTEGER
C XMAX: UPPER LIMIT OF ABSISSA DOMAIN REAL
C YMAX: UPPER LIMIT OF ORDINATE RANGE REAL
C XMIN: LOWER LIMIT OF ABSISSA DOMAIN REAL
C YMIN: LOWER LIMIT OF ORDINATE RANGE REAL
C ITITLX: ABSISSA LABEL FIELD DATA AREA
C ITITLY: ORDINATE LABEL FIELD DATA AREA
C LTITLX: NUMBER OF CHARACTERS IN ITITLX INTEGER
C LTITLY: NUMBER OF CHARACTERS IN ITITLY INTEGER
C XARRAY: ABSISSA POINTS IN TERMS OF XMIN-XMAX COORD'S REAL ARRAY
C YARRAY: ORDINATE POINTS IN TERMS OF YMIN-YMAX COORD'S REAL ARRAY
C DELX: INTERVALS OF ABSISSA TIC MARK LABELING REAL
C DELY: INTERVALS OF ORDINATE TIC MARK LABELING REAL
C IN TERMS OF XMIN-XMAX COORDINATES REAL
C TITLE: LABEL FOR THE WHOLE RUN FIELD DATA AREA
C
DIMENSION IBUF(NLCC),XARRAY(NTOT),YARRAY(NTOT),ITITLX(1),
1 ITITLY(1),YDIT(100)
DIMENSION TITLE(1)
C
C FIXED BASIC PARAMETERS
C
C LOGICAL ZERO
DEFINEZERO=NDEC.LT.0.AND.ABS(FPN).LT.5
1 .OR.NDEC.GT.0.AND.ABS(FPN).LT.10.*10.**(NDEC-1)
DEFINE DNDEC=NDEC-FLD(0,36,ZERO)*NDEC-FLD(0,36,ZERO)
DEFINE IFIX(FARG)=INT(FARG*.5)
DATA J/1/
DATA HEIGHT/105/
DATA INTEG/1/
DATA ABSCIS/8/
DATA ORDINA/6/
DATA ICODE/-1/
DATA TOPMAR/1.0/
DATA BOTMAR/1.5/
REAL LEFMAR
DATA LEFMAR/1.9/
DATA RYTMAR/1.1/
DATA FACT/1.0/
DATA MAXIS/1.0/
DATA MLINE/1.0/
DATA HLAB/-.105/
C
C 19 INITIAL COMPUTATION OF DERIVED PARAMETERS
C AND INITIAL PLOTS CALL
C 20 SKIPS PRELIMINARIES FOR 2ND AND SUBSEQUENT CALLS
C
GO TO (19,20),J
19 YDIT(1) = 3./19*
TICKLE = HEIGHT/2.0
ROTFAC = -3./14.*HEIGHT - 4./7.*HEIGHT
STARTL = 6.*HEIGHT + ROTFAC + TICKLE
SEPLAB = STARTL + 1.5*HEIGHT
SYMBLH = 0.070
REAL LABSEP
LABSEP = 4.*HEIGHT
ASTART = 2.*HEIGHT
DO 1 I = 2,100
1 YDIT(I) = YDIT(I - 1) + (2 * MOD(I,2) + 1)/19*
YDIT(100) = YDIT(100) + .5
CALL PLOTS(IBUF,NLOC,LDEV)
CALL FACTOR(1.0)
J = 2
CALL SYMBOL (HEIGHT,36*HEIGHT + 5.5*HEIGHT,TITLE,270.0,72)
CALL PLOT(1.,-5.,-3.)
3 DO 2 I = 1,100
2 CALL PLOT(0.,YDIT(I),3 - MOD(I,2))
DO 33 I = 1,100
33 YDIT(I) = YDIT(I) - ABSCIS - RYTMAR
C
C RESET ORIGIN
C
XPAGE = BOTMAR + ORDINA
GO TO 2019
20 XPAGE = BOTMAR + ORDINA + TOPMAR
2019 CALL WHERE(RXPAGE,RYPAGE,FACT)
YPAGE = RYPAGE - LEFMAR
CALL PLOT(XPAGE,YPAGE, - 3)
CALL FACTOR(FACT)
DRAW AXES AND LABELING MAXIS TIMES

DO 100 I = 1,MAXIS
100 CALL MYAXIS

DRAW POINTS, OPTIONAL CENTERLINE, AND PAGE SCISSORLINE

DO 200 I = 1,MLINE
200 CALL MYLINE
RETURN

ENTRY POINT SHPARG
TERMINATE PLOTTING SEQUENCE

ENTRY SHPARG
CALL WHERE(RXPAGE,RYPAGE,I)
CALL PLOT(RXPAGE,RYPAGE,999)
RETURN

SUBROUTINE MYAXIS (INTERNAL)

SUBROUTINE MYAXIS
STARTL = 6 * HEIGHT + ROTFAC + TICKLE
IMAX = IFIX((YMAX - YMIN)/DELY)
TICSEP = ORDINA/CABS(NTICY)
CALL DENDEC(YMAX,DELY,NTICY)
K = 1
N = (ABS(NTICY)/IMAX) - 1 + MOD(ABS(NTICY),2)
DO 9 I = 0,IMAX
GO TO (11,12),K
11 IF(2 * I.LT.IMAX)GO TO 12
CALL AXLAB(0,ITITLY,LTITLE,HITLE)
K = 2
12 FPN = YMAX - I * DELY
IF(ZERO)FFN = 0.
TMID = 1.
XPAGE = - I * ORDINA/IMAX - .5 * HEIGHT
IF(FFN)113,122,118
113 IF(NDEC = 2)115,114,112
114YPAGE = STARTL
GO TO 112

115 IF(NDEC = 1) 117, 116, 112
116 YPAGE = STARTL - HEIGHT
GO TO 112

117 IF(ABS(FPN) = 100) 119, 116, 116
119 IF(ABS(FPN) = 10) 120, 121, 121
120 YPAGE = STARTL - 3 * HEIGHT
GO TO 112

121 YPAGE = STARTL - 2 * HEIGHT
GO TO 112

122 YPAGE = STARTL - 4 * HEIGHT
GO TO 112

118 IF(NDEC = 2) 123, 116, 112
123 IF(NDEC = 1) 125, 124, 112
124 IF(FPN = 10) 126, 122, 126
125 IF(FPN = 100) 120, 121, 127
126 IF(FPN = 1000) 121, 116, 128
127 IF(FPN = 10000) 116, 114, 114
128 IF(FPN = 100000) 115, 114, 115
112 NNDEC = NNDEC + 1

CALL NUMBER(XPAGE, YPAGE, HEIGHT, FPN, 270, *NNDEC)
XPAGE = - I * (ORDINA/IMAX)
DO 10 JJ = 1, N
YPAGE = TICKLE * MID
CALL PLOT(XPAGE, YPAGE, 3)
YPAGE = YPAGE * (1 - I/IMAX)
CALL PLOT(XPAGE, YPAGE, 2)

110 IF(I/IMAX = 0) 110
CALL PLOT(XPAGE, YPAGE, 3)
XPAGE = XPAGE - TICSEP
CALL PLOT(XPAGE, YPAGE, 2)
T MID = .5
10 CONTINUE
9 CONTINUE
K = 1
IMAX = IFIX((XMAX - XMIN)/DELX)
TICSEP = ABS(ABS(INTIC - 1))
XPAGE = - ASTART - ORDINA
CALL DENDEC(XMAX, DELX, NDEC)
DO 28 I = 0, IMAX
STARTL = - I * TICSEP

24 IF(2 * I = IMAX) GO TO 25
CALL AXLAB(270, ITITLE, TITLE, HTAYLE)
K = 2
XPAGE = - ASTART - ORDINA
25 FPN = XMIN + I * DELX
IF(ZERO) FPN = 0
IF(FPN) 813, 822, 818
813 IF(NDEC = 2) 815, 817, 23
814 YPAGE = STARTL + 10 * 7 * HEIGHT
GO TO 23
815 IF(NDEC = 1) 817, 816, 23
YPAGE = STARTL + 25./14.* HEIGHT
GO TO 23
IF(ABS(FPN) <= 100.) YPAGE = STARTL + 11./14.* HEIGHT
GO TO 23
IF(ABS(FPN) <= 10.) YPAGE = STARTL + 9./7.* HEIGHT
GO TO 23
YPAGE = STARTL + 2./7.* HEIGHT
GO TO 23
YPAGE = STARTL + 2./7.* HEIGHT
GO TO 23
IF(NDEC = 2) YPAGE = STARTL + 11./14.* HEIGHT
GO TO 23
IF(NDEC = 1) YPAGE = STARTL + 9./7.* HEIGHT
GO TO 23
IF(FPN = 10.) YPAGE = STARTL + 2./7.* HEIGHT
GO TO 23
IF(FPN = 100.) YPAGE = STARTL + 2./7.* HEIGHT
GO TO 23
IF(FPN = 1000.) YPAGE = STARTL + 2./7.* HEIGHT
GO TO 23
IF(FPN = 10000.) YPAGE = STARTL + 2./7.* HEIGHT
GO TO 23
NDEC = NNDEC
CALL NUMBER(XPAGE,YPAGE,HEIGHT,FPN,270.,NNDEC)
N = (NTICX/IMAX) - 1 + MOD(NTICX,2)
DO 26 I = IMAX,0,-1
TMID = 1.*
YPAGE = - I * ABSCIS/IMAX
DO 27 JJ = 1,N
XPAGE = - ORDINA - TICKLE * TMID
CALL PLOT(XPAGE,YPAGE,3)
XPAGE = XPAGE + (TICKLE + FLD(0,36,I,N.E.0) * TICKLE) * TMID
CALL PLOT(XPAGE,YPAGE,2)
IF(I) 111,26,111
111 XPAGE = ORDINA
CALL PLOT(XPAGE,YPAGE,3)
YPAGE = YPAGE + TICSEP
CALL PLOT(XPAGE,YPAGE,2)
TMID = .5
27 CONTINUE
26 CONTINUE
RETURN
SUBROUTINE MYLINE (INTERNAL)
ITOP = IFIX((ABSCIS + RYTMAR + .5)/11.* 99.)
IBOT = IFIX(RYTMAR/11.* 99.)
DO 17 I = 1,NTOT
XPAGE = (YARRAY(I) - YMAX)/(YMAX - YMIN) * ORDINA
YPAGE = (XMIN - XARRAY(I))/(XMAX - XMLN) * ABSCIS
17 CALL SYMBOL(XPAGE,YPAGE,SYMBLH,INTEA,270.,ICODE)
IF(NTICY.GE.0) GO TO 22
XPAGE = ORDINA/2.
YPAGE = ABSCIS
CALL PLOT(XPAGE,YPAGE,3)
DO 18 I = IBOT,ITOP

148
CALL PLOT(XPAGE, YDIT(I), 3 - MOD(I, 2))
XPAGE = TOPMAR
YPAGE = - ABSCIS - RYTMAR - 5
CALL PLOT(XPAGE, YPAGE, 3)
DO 21 I = 1, 100
21 CALL PLOT(XPAGE, YDIT(I), 3 - MOD(I, 2))
RETURN

SUBROUTINE AXLAB (INTERNAL)

SUBROUTINE AXLAB(ANGLE, IBCD, NCHARX, HEIGHT)
DIMENSION IBCD(7)
LOGICAL S
INTEGER 4:60/ S/ 1
K = 2
NCHAR = NCHARX
S = .FALSE.
IF(ABS(ANGLE) GT .1) GO TO 30
XPAGE = - ORDINA/2. - NCHAR * HEIGHT/2
YPAGE = SEPLAB
GO TO 31
30 XPAGE = - ORDINA - LABSEP
YPAGE = - ABSCIS/2. + NCHAR * HEIGHT/2
31 LSTART = 6 * MODCNCHAR, 6) - 12
IF(LSTART EQ. - 12) LSTART = 24
LOOK = NCHAR/6 + 1
S = .FALSE.
CALL SYMBOL(XPAGE, YPAGE, HEIGHT, IBCD, ANGLE, NCHAR)
RETURN

SUBROUTINE DENDEC (INTERNAL)

SUBROUTINE DENDEC(QMAX, DELG, NDEC)
IF(INT(ABS(QMAX)) GE 10) GO TO 5
IF(AMOD(ABS(QMAX - DELG), 1) GE .01) GO TO 7
NDEC = 1
RETURN
5 NDEC = - 1
RETURN
7 NDEC = 2
RETURN
END
General Description

Two auxiliary programs, LINSOL and LSTB3D, calculate the linear stability characteristics of a cylindrical combustion chamber with distributed combustion and a conventional nozzle. For given values of the operating parameter (i.e., n, 7, y, 5_e, and L/D) and a given nozzle admittance (i.e., A and 0), Program LINSOL calculates the growth rate, A, and the frequency, w, of a given acoustic mode. For given values of 0, Program LSTB3D calculates the corresponding values of n and w for neutral stability (A = 0). These programs are based on an analytical solution of the linearized version of Eqs. (12). After a discussion of the linear analysis, Programs LINSOL and LSTB3D will be described.

Linear Analysis

For a single acoustic mode, dropping the nonlinear terms in Eqs. (12) yields the following linear equation:

\[
\frac{d^2 A}{dt^2} + C_1 A + (C_2 - n C_3) \frac{dA}{dt} + n C_3 \frac{d[A(t - \tilde{t})]}{dt} = 0
\]  

(E-1)

where A(t) is the unknown complex amplitude function for the mode under consideration and the coefficients are obtained from Eqs. (C-1) through (C-4) by dividing by -C_0. Thus the coefficients are complex numbers given by:

\[
C_1 = S^2 \frac{S'(z_e)Z^*(z_e) - \int_{z_e}^{z} Z''Z^* dz}{\int_{z_e}^{z} ZZ^* dz} 
\]

(E-2)
\[ c_2 = \frac{2 \int_0^{\infty} \bar{u}(z)Z^*Z'Z^* \, dz + \int_0^{\infty} \frac{d\bar{u}}{dz}ZZ^* \, dz + \gamma \bar{y}Z'(z_e)Z^*(z_e)}{\int_0^{\infty} ZZ^* \, dz} \]  
\[ (E-3) \]

where the droplet momentum source has been neglected. When the droplet momentum source is included, the \( \gamma \) in the second term of Eq. (E-3) is replaced by \( \gamma + 1 \) (see Appendix A).

The linear solutions are determined by substituting a solution of the form:

\[ A(t) = ae^{(\Lambda + i\omega)t} \]  
\[ (E-5) \]

into Eq. (E-1) and separating real and imaginary parts to obtain:

\[ \omega^2 = c_{1r} + \Lambda^2 + (c_{2r} - nC_3)\Lambda - c_{2i}^2 + c_3 ne^{-\Lambda \bar{T}}(\Lambda \cos \omega \bar{T} + \omega \sin \omega \bar{T}) \]  
\[ (E-6) \]

\[ \Lambda = -\left\{ \frac{c_{1r} + (c_{2r} - nC_3)\omega + nC_3 e^{-\Lambda \bar{T}} \omega \cos \omega \bar{T}}{2\omega + c_{2i} e^{-\Lambda \bar{T}} \sin \omega \bar{T}} \right\} \]  
\[ (E-7) \]

where \( C_1 = c_{1r} + iC_1i, \, C_2 = c_{2r} + iC_2i, \) and \( C_3 \) is always real. The above equations are solved numerically by Program LINSOL to obtain the growth rate, \( \Lambda \), and the frequency, \( \omega \), for given values of \( n \) and \( \bar{T} \).

The equations describing the neutral stability limits are obtained by substituting \( \Lambda = 0 \) into Eqs. (E-6) and (E-7). Solving the resulting equations...
for \( n \) and \( \omega^2 \) gives:

\[
n = \frac{C_2r + C_{11}/\omega}{C_3(1 - \cos \omega \tau)}
\]

(E-8)

\[
\omega^2 = C_{1r} + \omega(nC_3 \sin \omega \tau - C_{21})
\]

(E-9)

which are solved numerically by Program LSTB3D.

**Program LINSOL**

**Program Structure.** A flow chart for Program LINSOL is given in Fig. (E-1). This program consists of the following major sections: (1) input, (2) calculation of the coefficients \( C_1, C_2, \) and \( C_3 \), (3) iterative solution for \( A \) and \( \omega \), and (4) output.

**Input.** The input data required by Program LINSOL includes: (1) a title for the run, (2) the chamber parameters \( \gamma, u_e, L/D, \) and \( z_c/z_e \), (3) several control numbers, (4) the nozzle admittance, (5) the mode under consideration, and (6) the values of \( n \) and \( \tau \) for the cases to be run. This data is described in the following table where the location number refers to the columns of the card and the following three formats are used: alphanumeric characters (A), integers (I), and numbers with a decimal point (F). For the "I" formats the values are placed in fields of five locations, while a field of ten locations is used with the "F" formats. In either case the numbers must be placed in the rightmost locations of the allocated field.

<table>
<thead>
<tr>
<th>No. of Cards</th>
<th>Location</th>
<th>Type</th>
<th>Input Item</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1-72</td>
<td>A</td>
<td>TITLE</td>
<td>Title of run.</td>
</tr>
<tr>
<td>1</td>
<td>1-10</td>
<td>F</td>
<td>GAMMA</td>
<td>Specific heat ratio, ( \gamma ).</td>
</tr>
<tr>
<td></td>
<td>11-20</td>
<td>F</td>
<td>UE</td>
<td>Steady state Mach number at nozzle entrance, ( u_e ).</td>
</tr>
<tr>
<td></td>
<td>21-30</td>
<td>F</td>
<td>RLD</td>
<td>Length-to-diameter ratio, ( L/D = z_e/2 ).</td>
</tr>
</tbody>
</table>
Acoustic Frequencies

Input of Chamber Parameters

Input of $\tau_j$ and $n_j$

Subroutine EIGVAL

Subroutine AXIAL1

Calculate Coefficients $C_1, C_2, C_3$

Output of Chamber Parameters

K=1

Initial Guesses $\Lambda_1, \omega_1$

Compute $\Lambda_{k+1}, \omega_{k+1}$

Print "FAILED TO CONVERGE"

Yes

Is $K=0$?

K=1

Increase $J$ by 1

Yes

Is $J<NCASES$?

STOP

No

Output of $\tau_j, n_j, \Lambda_k, \omega_k$

Increase $J$ by 1

No

Is $\Delta \Lambda < \varepsilon$ and $\Delta \omega < \varepsilon$?

Compute $\Delta \Lambda$ & $\Delta \omega$

Increase $K$ by 1

No

Print "FAILED TO CONVERGE"

Yes

Figure E-1. Flow Chart for Program LINSOL.
<table>
<thead>
<tr>
<th>No. of Cards</th>
<th>Location</th>
<th>Type</th>
<th>Input Item</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>31-40</td>
<td>F</td>
<td>ZCOMB</td>
<td>Length of combustion zone, $z_c/z_e$.</td>
<td></td>
</tr>
<tr>
<td>41-45</td>
<td>I</td>
<td>NDROPS</td>
<td>If 0: droplet momentum source neglected.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>If 1: droplet momentum source included.</td>
</tr>
<tr>
<td>46-50</td>
<td>I</td>
<td>NOZZLE</td>
<td>If 0: quasi-steady nozzle.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>If 1: conventional nozzle.</td>
</tr>
<tr>
<td>51-55</td>
<td>I</td>
<td>NOPT</td>
<td>If 1: all coefficients included.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>If 2: imaginary parts neglected.</td>
</tr>
</tbody>
</table>

If NOZZLE = 1:

1 1-10  F  YAMPL Amplitude factor of nozzle admittance, A.
11-20  F  YPHASE Phase of nozzle admittance, $\phi$.

End of input for NOZZLE = 1.

1 1-5  I  L Axial mode number, $L$ ($0 \leq L \leq 10$).
6-10  I  M Tangential mode number, $M$ ($0 \leq M \leq 6$).
11-15  I  N Radial mode number, $N$ ($0 \leq N \leq 5$).
16-20  I  NCASES Number of cases to be run (NCASES $\leq 100$).

NCASES 1-10  F  TAU Time-lag, $\tau$.
11-20  F  EN Interaction Index, $n$.

The title on the first card should identify the mode under consideration.
On the second card of input all quantities are the same as those given in the input to COEPPS3D (see Appendix C) except NOPT. NOPT gives the option to neglect the imaginary parts of the coefficients $C_1$ and $C_2$ which are an order of magnitude smaller than the corresponding real parts. Neglecting these
imaginary parts (NOPT = 2) yields linear solutions consistent with the non-linear solutions obtained when the small coefficients are neglected (NEGL = 1 in input to COEFFS3D). The values of \( n \) and \( \tau \) for the cases to be run are given on a series of NCASES cards. These cards are all read and the values of \( \tau \) and \( n \) are stored in the arrays TAU(J) and EN(J) before any computations are made.

In addition to the above card input, the acoustic frequencies \( S_{mn} \) are also needed for these calculations. As in Program COEFFS3D these values are given in a DATA statement, which is an integral part of the program.

Calculation of \( C_1, C_2, \) and \( C_3 \). In this section the coefficients \( C_1, C_2, \) and \( C_3 \) appearing in Eqs. (E-6) and (E-7) are calculated using Eqs. (E-2) through (E-4). As in Program COEFFS3D the axial acoustic eigenvalues necessary for these computations are calculated by Subroutines EIGVAL and FCNS, and the integrals of the products of two axial eigenfunctions appearing in Eqs. (E-2) through (E-4) are computed by Subroutines AXIAL1 and UBAR. Listings of these subroutines are given in Appendix C.

Iterative Solution for \( \Lambda \) and \( \omega \). Equations (E-6) and (E-7) are of the form:

\[
\omega^2 = C_{1r} + f(\Lambda, \omega) \]

\[
\Lambda = g(\Lambda, \omega) \tag{E-10}
\]

where the quantity \( f(\Lambda, \omega) \) is small compared to \( C_{1r} \) and \( \Lambda \) is small in most cases. Starting with an initial guess of

\[
\omega_1 = \sqrt{S_{mn}^2 + \frac{2 \pi}{z_e}} \tag{E-11}
\]

\[
\Lambda_1 = 0
\]
Eqs. (E-10) are solved iteratively using the following recursion formulas:

\[ \omega_{k+1}^2 = c_{1r} + f(\lambda_k, \omega_k) \]

\[ \lambda_{k+1} = \varphi(\lambda_k, \omega_k) \]  

At each step of the iteration the quantities \( \Delta \lambda \) and \( \Delta \omega \) are calculated, where

\[ \Delta \lambda = | \lambda_{k+1} - \lambda_k | \]

\[ \Delta \omega = | \omega_{k+1} - \omega_k | \]  

and the computations are terminated when \( k = 40 \) or when \( \Delta \lambda \) and \( \Delta \omega \) are less than \( \epsilon = 10^{-6} \). The process usually converges in less than 15 iterations.

Output. The output generated by Program LINSOL consists of a restatement of the input data followed by the calculated results in tabular form. For each case the tabulated results give the values of \( \vec{T} \) and \( n \) (TAU and RN), the corresponding values of the growth rate \( \Lambda \) and the frequency \( \omega \) (LAMBDAB and OMEGA), and the number of iterations (ITER). When ITER is 40 the last values of \( \Lambda \) and \( \omega \) are given followed by the warning message "FAILED TO CONVERG.

Sample Input and Output. A sample input for the IT mode is given in Table E-1 followed by the resulting output in Table E-2.

Program LSTB3D

Program Structure. A flow chart for Program LSTB3D is given in Figure (E-2). This program consists of the following major sections: (1) input, (2) calculation of the coefficients \( C_1, C_2, \) and \( C_3 \), (3) iterative solution for \( n \) and \( \omega \) for neutral stability, and (4) output.
Table E-1. Sample Input for LINSOL.

<table>
<thead>
<tr>
<th>IT MODE</th>
<th>1.2</th>
<th>0.2</th>
<th>0.5</th>
<th>1 0 0 1 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>45 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.4</td>
<td>0.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.4</td>
<td>0.58596</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>0.57562</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>0.6</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table E-2. Sample Output for LINSOL.

**IT MODE.**

DROPLET MOMENTUM SOURCE NEGLECTED

**GAMMA = 1.20**  **UE = 0.20**  **L/D = 0.50000**  **ZCOMB = 1.00**

**AMPL = 0.02000**  **PHASE = 45.0**

<table>
<thead>
<tr>
<th>TAU</th>
<th>EN</th>
<th>LAMBDA</th>
<th>OMEGA</th>
<th>ITER</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.400</td>
<td>50000</td>
<td>0.01769</td>
<td>1.86593</td>
<td>7</td>
</tr>
<tr>
<td>1.400</td>
<td>58396</td>
<td>0.00000</td>
<td>1.87005</td>
<td>7</td>
</tr>
<tr>
<td>1.400</td>
<td>60000</td>
<td>0.00339</td>
<td>1.87078</td>
<td>7</td>
</tr>
<tr>
<td>1.700</td>
<td>50000</td>
<td>0.00975</td>
<td>1.83602</td>
<td>7</td>
</tr>
<tr>
<td>1.700</td>
<td>54490</td>
<td>0.00000</td>
<td>1.83612</td>
<td>6</td>
</tr>
<tr>
<td>1.700</td>
<td>60000</td>
<td>0.01176</td>
<td>1.83618</td>
<td>7</td>
</tr>
<tr>
<td>2.000</td>
<td>50000</td>
<td>0.01537</td>
<td>1.80691</td>
<td>8</td>
</tr>
<tr>
<td>2.000</td>
<td>57562</td>
<td>0.00000</td>
<td>1.80410</td>
<td>8</td>
</tr>
<tr>
<td>2.000</td>
<td>60000</td>
<td>0.00487</td>
<td>1.80322</td>
<td>8</td>
</tr>
</tbody>
</table>
Figure E-2. Flow Chart for Program LSTB3D.
The input data required by Program LSTB3D is basically the same as required by Program LINSOL. The first two cards, which give the title of the case, the chamber parameters, and the control numbers, are identical in content and format to those required by LINSOL. The third card gives the mode numbers $l$, $m$, and $n$ and is followed by a card giving the nozzle admittance if a conventional nozzle is specified. The last card gives the values of $\tau$ for the cases to be run. A detailed description of this input is given below.

<table>
<thead>
<tr>
<th>No. of Cards</th>
<th>Location</th>
<th>Type</th>
<th>Input Item</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1-72</td>
<td>A</td>
<td>TITLE</td>
<td>See input for LINSOL.</td>
</tr>
<tr>
<td>1</td>
<td>1-40</td>
<td>F</td>
<td>GAMMA, UE, RLD, ZCOMB</td>
<td>See input for LINSOL.</td>
</tr>
<tr>
<td>41-55</td>
<td>I</td>
<td>NDROPS, NOZZLE, NOPT</td>
<td>See input for LINSOL.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1-15</td>
<td>I</td>
<td>L, M, N</td>
<td>See input for LINSOL</td>
</tr>
</tbody>
</table>

If NOZZLE = 1:

<table>
<thead>
<tr>
<th>No. of Cards</th>
<th>Location</th>
<th>Type</th>
<th>Input Item</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1-20</td>
<td>F</td>
<td>YAMPL, YPHASE</td>
<td>See input for LINSOL.</td>
</tr>
</tbody>
</table>

End of input for NOZZLE = 1.

<table>
<thead>
<tr>
<th>No. of Cards</th>
<th>Location</th>
<th>Type</th>
<th>Input Item</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1-10</td>
<td>F</td>
<td>TAUMIN</td>
<td>Smallest value of $\tau$.</td>
</tr>
<tr>
<td>11-20</td>
<td>F</td>
<td>TAUMAX</td>
<td>Largest value of $\tau$.</td>
<td></td>
</tr>
<tr>
<td>21-30</td>
<td>F</td>
<td>DELTAU</td>
<td>Increment in $\tau$.</td>
<td></td>
</tr>
</tbody>
</table>

The last card gives the values of $\tau$ which are used in the computation of the neutral stability limit. Thus computations are begun for $\tau = TAUMIN$, $\tau$ is increased by increments of DELTAU, and computations are terminated when $\tau \geq TAUMAX$.

After completion of the computations program control returns to the read statement for the nozzle admittance, thus neutral stability curves can be calculated for several different nozzles for the same set of chamber and mode parameters.

Calculation of $C_1$, $C_2$, and $C_3$. The calculation of the coefficients $C_1$, $C_2$, and $C_3$ appearing in Eqs. (E-8) and (E-9) is performed in the same manner as
Iterative Solution for \( n \) and \( \omega \). The values of \( n \) and \( \omega \) for neutral stability are calculated for each value of \( \bar{\tau} \) by solving Eqs. (E-8) and (E-9) using the following iteration scheme:

\[
\begin{align*}
n_k &= \frac{C_{2r} + C_{1i}/\omega_k}{C_3(1 - \cos \omega_k \bar{\tau})} \\
\omega_{k+1}^2 &= C_{1r} + \omega_k(n_k C_{3i} \sin \omega_k \bar{\tau} - C_{2i})
\end{align*}
\]

(E-14)

The iteration is started by using \( \omega_1 = \sqrt{C_{1r}} \) and is stopped when \( k = 40 \) or \( \Delta n \) and \( \Delta \omega \) are less than \( \varepsilon = 10^{-6} \). Convergence is usually obtained in less than 20 iterations.

Output. The output generated by Program LSTB3D consists of a restatement of the input data followed by the calculated results in tabular form. For each value of \( \bar{\tau} \) in the range \( \text{TAMIN} \leq \bar{\tau} \leq \text{TAMAX} \), the tabulated results give the value of \( \bar{\tau} \) (TAU), the corresponding values of \( n \) and \( \omega \) for neutral stability (EN and OMEGA), and the number of iterations (ITER). If ITER is 40 the last values of \( n \) and \( \omega \) computed are given followed by the warning message "FAILED TO CONVERGE."

Sample Input and Output. A sample input for the 1T mode is given in Table E-3 and is followed by the resulting output in Table E-4.
Table E-4. Sample Output for LSTB3D.

IT MODE.

DROPLET MOMENTUM SOURCE NEGLECTED

<table>
<thead>
<tr>
<th>GAMMA = 1.20</th>
<th>UE = 0.20</th>
<th>RLD = 0.50000</th>
<th>ZCOMB = 1.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>AMPL = 0.02000</td>
<td>PHASE = 45.00</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TAU</th>
<th>EN</th>
<th>OMEGA</th>
<th>ITER</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.60000</td>
<td>1.66353</td>
<td>2.03102</td>
<td>6</td>
</tr>
<tr>
<td>0.70000</td>
<td>1.31671</td>
<td>1.99646</td>
<td>6</td>
</tr>
<tr>
<td>0.80000</td>
<td>1.08482</td>
<td>1.96911</td>
<td>6</td>
</tr>
<tr>
<td>0.90000</td>
<td>0.92333</td>
<td>1.94663</td>
<td>6</td>
</tr>
<tr>
<td>1.00000</td>
<td>0.80765</td>
<td>1.92753</td>
<td>6</td>
</tr>
<tr>
<td>1.10000</td>
<td>0.72330</td>
<td>1.91089</td>
<td>6</td>
</tr>
<tr>
<td>1.20000</td>
<td>0.61337</td>
<td>1.89605</td>
<td>6</td>
</tr>
<tr>
<td>1.30000</td>
<td>0.51616</td>
<td>1.88255</td>
<td>6</td>
</tr>
<tr>
<td>1.40000</td>
<td>0.58396</td>
<td>1.87005</td>
<td>6</td>
</tr>
<tr>
<td>1.50000</td>
<td>0.56230</td>
<td>1.85827</td>
<td>6</td>
</tr>
<tr>
<td>1.60000</td>
<td>0.56961</td>
<td>1.84702</td>
<td>6</td>
</tr>
<tr>
<td>1.70000</td>
<td>0.54490</td>
<td>1.83612</td>
<td>6</td>
</tr>
<tr>
<td>1.80000</td>
<td>0.54769</td>
<td>1.82542</td>
<td>6</td>
</tr>
<tr>
<td>1.90000</td>
<td>0.55785</td>
<td>1.81479</td>
<td>6</td>
</tr>
<tr>
<td>2.00000</td>
<td>0.57562</td>
<td>1.80410</td>
<td>6</td>
</tr>
<tr>
<td>2.10000</td>
<td>0.60157</td>
<td>1.79325</td>
<td>6</td>
</tr>
<tr>
<td>2.20000</td>
<td>0.63666</td>
<td>1.78210</td>
<td>6</td>
</tr>
<tr>
<td>2.30000</td>
<td>0.68221</td>
<td>1.77055</td>
<td>6</td>
</tr>
<tr>
<td>2.40000</td>
<td>0.74006</td>
<td>1.75847</td>
<td>11</td>
</tr>
<tr>
<td>2.50000</td>
<td>0.81258</td>
<td>1.74575</td>
<td>13</td>
</tr>
<tr>
<td>2.60000</td>
<td>0.90278</td>
<td>1.73224</td>
<td>14</td>
</tr>
<tr>
<td>2.70000</td>
<td>1.01446</td>
<td>1.71783</td>
<td>17</td>
</tr>
<tr>
<td>2.80000</td>
<td>1.15226</td>
<td>1.70240</td>
<td>21</td>
</tr>
</tbody>
</table>
*** PROGRAM LINSOL ***

THIS PROGRAM COMPUTES THE DAMPING (LAMBDA) AND FREQUENCY (OMEGA) FOR GIVEN VALUES OF THE INTERACTION INDEX (EN) AND THE TIME-LAG (TAU). THIS PROGRAM IS BASED ON AN ANALYTICAL SOLUTION OF THE COMPLEX DIFFERENTIAL EQUATION.

THE FOLLOWING INPUTS ARE REQUIRED:

FIRST CARD:
THE TITLE OF THE CASE.

SECOND CARD:
GAMMA IS THE SPECIFIC HEAT RATIO.
UE IS THE STEADY STATE MACH NUMBER AT THE NOZZLE ENTRANCE.
RLD IS THE LENGTH-TO-DIAMETER RATIO.
ZCOMB IS THE LENGTH OF THE COMBUSTION ZONE, EXPRESSED AS A FRACTION OF THE CHAMBER LENGTH.
NDROPS DETERMINES THE PRESENCE OF DROPLET MOMENTUM SOURCES:
NDROPS = 0 DROPLET MOMENTUM SOURCE NEGLECTED.
NDROPS = 1 DROPLET MOMENTUM SOURCE INCLUDED.
NOZZLE SPECIFIES THE TYPE OF NOZZLE USED:
NOZZLE = 0 QUASI-STeadY
NOZZLE = 1 CONVENTIONAL NOZZLE
NOPT SPECIFIES THE SOLUTIONS DESIRED:
NOPT = 1 COUPLING COEFFICIENTS INCLUDED.
NOPT = 2 COUPLING COEFFICIENTS NEGLECTED.

THIRD CARD (FOR CONVENTIONAL NOZZLE ONLY):
YAMPL IS THE AMPLITUDE OF THE NOZZLE ADMITTANCE.
YPHASE IS THE PHASE OF THE NOZZLE ADMITTANCE.

FOURTH CARD:
The mode is specified by the indices L, M, AND N.
L IS THE AXIAL MODE NUMBER AND MUST NOT EXCEED 10.
M IS THE AZIMUTHAL MODE NUMBER AND MUST NOT EXCEED 8.
N IS THE RADIAL MODE NUMBER AND MUST NOT EXCEED 5.
NCASES IS THE NUMBER OF CASES TO BE RUN.

REMAINING CARDS:
TAU IS THE TIME LAG.
EN IS THE INTERACTION INDEX.

*****************************************************************************
************** DATA INPUT SECTION **************

ERR = 0.000001
PI = 3.1415927
CI = (0.0,1.0)

INPUT ROOTS AND VALUES OF BESSEL FUNCTIONS.
DATA ((RJROOT(I,J)), J = 1,5), I = 1,9)
1 3.83171, 7.01559, 10.17437, 13.32836, 16.47063
2 1.84118, 5.33144, 8.53632, 11.70600, 14.68359
3 3.05424, 6.70613, 9.96447, 13.17037, 16.34752
4 4.20119, 8.01524, 11.80559, 14.58585, 17.78857
5 5.31755, 9.28240, 12.68191, 15.96411, 19.19603
6 6.41562, 10.51986, 13.98719, 17.34752, 20.57551
7 7.50127, 11.73494, 15.26818, 18.63744, 21.93172
8 8.57784, 12.93239, 16.52937, 19.94185, 23.26805

INPUT PARAMETERS.
READ (5*5000) (TITLE(I), I = 1, 72)
READ (5,5001) GAMMA, UE, RLD, ZCOMB, NDROP$, NOZZLE, NOPT
IF (NOZZLE .EQ. 1) GO TO 5

COMPUTE ADMITTANCE FOR QUASI-STEADY NOZZLE.
YAMPL = (GAMMA + 1.0) * UE/(2.0 + GAMMA)
YPHASE = 0.0
GO TO 7
5 READ (5,5002) YAMPL, YPHASE.
7 READ (5,5003) L, M, N, NCASES

THETA = YPHASE * PI/180.0
YR = YAMPL * COS(THETA)
YI = YAMPL * SIN(THETA)
YNOZ = OMPLX(YR,YI)

ZE = 2.0 * RLD
CZE = OMPLX(ZE,0.0)
CGAM = OMPLX(GAMMA,0.0)
CAX = CGAM
IF (NDROPS .EQ. 1) CAX = CGAM + (1.0,0.0)
C
DO 10 J = 1, NCASES
READ (5,5002) TAU(J), EN(J)
10 CONTINUE

************* PRELIMINARY CALCULATIONS *************

ASSIGN ARRAYS FOR ROOTS OF BESSEL FUNCTIONS.
IF (M .EQ. 0) .AND. (N .EQ. 0) GO TO 15
M = M + 1
N = N
SMN = RJROOT(MM,NN)
GO TO 20
15 SMN = 0.0
C
20 SSG = SMN * SMN
CSSQ = CMPLX(SSQ,0.0)
C
CALCULATE AXIAL ACOUSTIC EIGENVALUES.
CALL EIGVAL(L,SMN,GAMMA,ZE,YAMPL,YPHASE,RESULT)
B(1) = RESULT
BC = CONJG(RESULT)
C
************* CALCULATE AXIAL INTEGRALS ************************
C
DO 100 NT = 1, 4
CALL AXIAL1(NT,1,UE,ZE,ZCOMB,RESULT)
AX(NT) = RESULT
100 CONTINUE
C
************* CALCULATE VALUES AT NOZZLE ENTRANCE ****************
C
ZEJ = CCOSH(CI*BC*CZE)
ZEP1 = CCOSH(CI*B(1)*CZE)
ZEP2 = CI * B(1) * CSINH(CI*B(1)*CZE)
C
************* CALCULATE COEFFICIENTS ******************************
C
CC = (CSSQ*AX(1) - AX(2) + ZEP2*ZEJ)/AX(1)
CD = (CAX*AX(3) + (2.0,0.01*AX(4) + CGAM*YNOZ*ZEP1*ZEJ)/AX(1)
CE = CGAM*AX(3)/AX(1)
C
D(1) = REAL(CC)
D(3) = REAL(CD)
D(5) = REAL(CE)
IF (NOPT .EQ. 2) GO TO 50
D(2) = AIMAG(CC)
D(4) = AIMAG(CD)
GO TO 55
50 D(2) = 0.0
D(4) = 0.0
C
***** CALCULATION OF DAMPING AND FREQUENCY ***********************
C
55 WRITE (6,6001) (TITLE(I), I = 1, 72)
IF (NDROPS .EQ. 0) WRITE (6,6020)
IF (NDROPS .EQ. 1) WRITE (6,6021)
IF (NOPT = 0) WRITE (6,6015)
WRITE (6,6002) GAMMA, UE, RLDF, ZCOMB
IF (NOZZLE = 0) WRITE (6,6012)
WRITE (6,6005) YAMFL, YPHASE
WRITE (6,6011)
LINE = 14
CALCULATE INITIAL GUESSES FOR FREQUENCY.

\[ RL = L \]
\[ AXI = RL \times \frac{\pi}{Z}\text{E} \]
\[ AXS\text{Q} = AXI \times AXI \]
\[ SSQ = SMN \times SMN \]
\[ FRQ = \sqrt{SSQ + AXS\text{Q}} \]

DO 200 J = 1, NCASES

\[ C2R = D(3) - EN(J) \times D(5) \]
\[ C3 = EN(J) \times D(5) \]

\[ \text{LAMBDA}(1) = 0.0 \]
\[ \text{OMEGA}(1) = FRQ \]

\[ K = 1 \]

210 \[ X = \text{LAMBDA}(K) \]
\[ Y = \text{OMEGA}(K) \]
\[ XT = X \times TAU(J) \]
\[ YT = Y \times TAU(J) \]
\[ EX = \text{EXP}(-XT) \]
\[ SN = \text{SIN}(YT) \]
\[ CS = \text{COS}(YT) \]
\[ XSQ = X \times X \]
\[ VSQ = D(1) + XSQ + C2R \times X - D(4) \times Y \]
\[ 1 + C3 \times \text{EXP}(X \times CS + Y \times SN) \]
\[ A = D(2) + C2R \times Y + C3 \times \text{EXP} \times Y \times CS \]
\[ BB = 2 \times 0 \times Y + D(4) - C3 \times \text{EXP} \times SN \]

\[ \text{OMEGA}(K+1) = \sqrt{VSQ} \]
\[ \text{LAMBDA}(K+1) = \frac{A}{BB} \]

IF (K .EQ. 40) GO TO 216

DX = ABS(\text{LAMBDA}(K+1) - \text{LAMBDA}(K))
DY = ABS(\text{OMEGA}(K+1) - \text{OMEGA}(K))

K = K + 1

IF ((DX .LT. ERR) .AND. (DY .LT. ERR)) GO TO 217

GO TO 210

216 WRITE (6,6009) TAU(J), EN(J), LAMBDA(K), OMEGA(K), K
GO TO 220

217 WRITE (6,6008) TAU(J), EN(J), LAMBDA(K), OMEGA(K), K

220 LINE = LINE + 2
IF (LINE .LT. 54) GO TO 200
WRITE (6,6007)
WRITE (6,6011)
LINE = 4

200 CONTINUE
C
C ************ FORMAT SPECIFICATIONS ************
C
C READ FORMATS
5000 FORMAT (72A1)
5001 FORMAT (4F10.0,315)
5002 FORMAT (2F10.0)
5003 FORMAT (415)
C
C WRITE FORMATS
6001 FORMAT (1H1,1X,72A1/)
6002 FORMAT (2X,8HGAMMA = ,F5+2,5X,SHUE = ,F5+2,5X,6HL/D = ,F6+5,
1 5X,6HCOMB = ,F5+2/)
6005 FORMAT (2X,7HFL = ,F8+5,5X,8PHASE = ,F6+1/)
6007 FORMAT (1H )
6008 FORMAT (2X,F5+2,F6+5,2F10+5,16/)
6009 FORMAT (2X,F5+3,F6+5,2F10+5,I6,5X,16HFAILED TO CONVERGE/)
6011 FORMAT (2X//4X,3HTAU,6X,2HEN,4X,6HLAMBD2,5X,5HOMEGA,
1 2X,4HITER/)
6012 FORMAT (2X,19HQUASI-STEADY NOZZLE/
6015 FORMAT (2X,24HCOUPLING TERMS NEGLECTED/
6020 FORMAT (2X,'DROPLET MOMENTUM SOURCE NEGLECTED/
6021 FORMAT (2X,'DROPLET MOMENTUM SOURCE INCLUDED/
END
*************** PROGRAM LSTB3D ***************

THIS PROGRAM COMPUTES THE LINEAR STABILITY LIMITS CONSISTENT WITH THE THREE-DIMENSIONAL SECOND-ORDER THEORY.

THE FOLLOWING INPUTS ARE REQUIRED:

FIRST CARD:
THE TITLE OF THE CASE.

SECOND CARD:
GAMMA IS THE SPECIFIC HEAT RATIO.
UE IS THE STEADY-STATE MACH NUMBER AT THE NOZZLE ENTRANCE.
RLD IS THE LENGTH-TO-DIAMETER RATIO.
ZCOMB IS THE LENGTH OF THE COMBUSTION ZONE, EXPRESSED AS A FRACTION OF THE CHAMBER LENGTH.
NDROPS DETERMINES THE PRESENCE OF DROPLET MOMENTUM SOURCES:
   NDROPS = 0 DROPLET MOMENTUM SOURCE NEGLECTED.
   NDROPS = 1 DROPLET MOMENTUM SOURCE INCLUDED.
NOZZLE SPECIFIES THE TYPE OF NOZZLE USED:
   NOZZLE = 0 QUASI-STEADY
   NOZZLE = 1 CONVENTIONAL NOZZLE
NOPT SPECIFIES WHICH SOLUTION WILL BE COMPUTED:
   NOPT = 1 COUPLING COEFFICIENTS INCLUDED.
   NOPT = 2 COUPLING COEFFICIENTS NEGLECTED.

THIRD CARD:
THE MODE IS SPECIFIED BY THE INDICES L, M, AND N.
L IS THE AXIAL MODE NUMBER AND MUST NOT EXCEED 10.
M IS THE AZIMUTHAL MODE NUMBER AND MUST NOT EXCEED 8.
N IS THE RADIAL MODE NUMBER AND MUST NOT EXCEED 5.

FOURTH CARD (IF CONVENTIONAL NOZZLE):
YAMPL IS THE AMPLITUDE OF THE NOZZLE ADMITTANCE.
YPHASE IS THE PHASE OF THE NOZZLE ADMITTANCE.

REMAINING CARDS:
TAUMIN IS THE MINIMUM VALUE OF THE TIME-LAG.
TAUMAX IS THE MAXIMUM VALUE OF THE TIME-LAG.
DELTAY IS THE INCREMENT IN TIME-LAG.

*****************************************************************

YNOZ, RESULT, BC10), BC, AXC4), CI, CZE,
CGAM, ZEJ, ZEP1, ZEP2, CC, CD, CE, CSSQ, CAX
DIMENSION TITLE(72),
            RJROOT(10,5),
            OMEGA(100), EN(100)
COMMON B
*************** DATA INPUT SECTION ****************************************

ERR = 0.000001
PI = 3.1415927
CI = (0.0, 1.0)

INPUT ROOTS AND VALUES OF BESSEL FUNCTIONS:
DATA (CRJROOT(I..J), J = 1, 5), I = 1, 9/
 1 3.63171, 7.01559, 10.17347, 13.32369, 16.47063,
 2 1.84112, 5.33144, 8.53632, 11.70600, 14.86399,
 3 3.05424, 6.70613, 9.96947, 13.17347, 16.34752,
 4 4.20119, 8.01524, 11.34928, 14.58635, 17.88775,
 5 5.31755, 9.28240, 12.68191, 15.96411, 19.26030,
 6 6.41562, 10.51926, 13.98719, 17.31284, 20.57059,
 7 7.50127, 11.73494, 15.17594, 18.63744, 21.93172,
 8 8.57764, 12.92399, 16.39372, 19.84856, 23.26805,

INPUT PARAMETERS.
READ (5,5000) (TITLE(I), I = 1, 72)
READ (5,5001) GAMMA, UE, RLD, ZCOMB, NDROPS, NOZZLE, NOPT
READ (5,5002) L, M, N
8 IF (NOZZLE .EQ. 1) GO TO 5

COMPUTE ADMITTANCE FOR QUASI-STEADY NOZZLE:
YAMPL = (GAMMA - 1.0) * UE / (2.0 * GAMMA)
YPHASE = 0.0
GO TO 7
5 READ (5,5003, END = 300) YAMPL, YPHASE
7 READ (5,5003, END = 300) TAUMIN, TAUMAX, DELTAU

THETA = YPHASE * PI / 180.0
YR = YAMPL * COS(THETA)
YI = YAMPL * SIN(THETA)
YNOZ = CMPLX(YR, YI)

ZE = 2.0 * RLD
CZE = CMPLX(ZE, 0.0)
CGAM = CMPLX(GAMMA, 0.0)
CAX = CGAM
IF (NDROPS .EQ. 1) CAX = CGAM + (1.0, 0.0)

*************** PRELIMINARY CALCULATIONS ***********************************

ASSIGN ARRAYS FOR ROOTS OF BESSEL FUNCTIONS.
IF (M .EQ. 0) .AND. (N .EQ. 0) GO TO 15
MM = M + 1
NN = N
SMN = RJROOT(MM, NN)
GO TO 20
15 SMN = 0.0
20 SSQ = SMN * SMN
CSSQ = CMPLX(SSQ, 0.0)
C CALCULATE AXIAL ACOUSTIC EIGENVALUES.
CALL EIGVAL(L,SMN,GAMMA,ZE,YAMPL,YPHASE,RESULT)
B(1) = RESULT
BC = CONJG(RESULT)
C
************* CALCULATE AXIAL INTEGRALS ***********************
C DO 100 NT = 1, 4
CALL AXIAL1(NT, 1, UE, ZE, ZCOMB, RESULT)
AX(NT) = RESULT
100 CONTINUE
C
************* CALCULATE VALUES AT NOZZLE ENTRANCE **************
C
ZEJ = CCOSH(CI*BC*CZE)
ZEPI = CCOSH(CI*B(1)*CZE)
ZEP2 = CI * B(1) * CSINH(CI*B(1)*CZE)
C
************* CALCULATE COEFFICIENTS ******************************
C
CC = (CSSO*AX(1) - AX(2) + ZEP2*ZEJ)/AX(1)
CD = (CAX*AX(3) + (2.0,0.0)*AX(4)
1 + CGAM*YN0Z*ZEP1*ZEJ)/AX(1)
CE = CGAM*AX(3)/AX(1)
C
C1 = REAL(CC)
D1 = REAL(CD)
E = REAL(CE)
IF (NOPT .EQ. 2) GO TO 50
C2 = AIMAG(CC)
D2 = AIMAG(CD)
GO TO 55
50 C2 = 0.0
D2 = 0.0
C
************* CALCULATION OF LINEAR STABILITY LIMIT **********
C
55 OMEGA(1) = SQRT(C1)
C
WRITE (6,6001) (TITLE(J), J = 1,72)
IF (NDROPS .EQ. 0) WRITE (6,6025)
IF (NDROPS .EQ. 1) WRITE (6,6026)
IF (NOPT .EQ. 2) WRITE (6,6022)
WRITE (6,6002) GAMMA, UE, RLD, ZCOMB
IF (NOZZLE .EQ. 0) WRITE (6,6012)
WRITE (6,6005) YAMPL, YPHASE
WRITE (6,6010)
LINE = 12
C
TAU = TAUMIN
370 IF (TAU .GT. TAUMAX) GO TO 8
C
K = 1
310 WT = OMEGA(K) * TAU
BB = (D1 + C2/OMEGA(K))/E
EN(K) = BB/(1.0 - COS(WT))
G = (E*EN(K)*SIN(WT) - D2) * OMEGA(K)
OMEGA(K+1) = SQRT(C1 + G)
IF (K .EQ. 40) GO TO 316
IF (K .EQ. 1) GO TO 311
DN = ABS(EN(K) - EN(K-1))
DW = ABS(OMEGA(K+1) - OMEGA(K))
IF ((DN .LT. ERR) .AND. (DW .LT. ERR)). GO TO 317
311 K = K + 1
GO TO 310
C
316 WRITE (6,6013) TAU, EN(K), OMEGA(K), K
GO TO 318
317 WRITE (6,6014) TAU, EN(K), OMEGA(K), K
C
318 LINE = LINE + 2
TAU = TAU + DELTAU
IF ((LINE .LT. 60) .OR. (TAU .GT. TAOMAX)) GO TO 370
WRITE (6,6015)
WRITE (6,6010)
LINE = 6
GO TO 370
C
300 CONTINUE
C
************* FORMAT SPECIFICATIONS *****************************
C
READ FORMATS
5000 FORMAT (72A1)
5001 FORMAT (4F10.0,315)
5002 FORMAT (315)
5003 FORMAT (3F10.0)
C
WRITE FORMATS
6001 FORMAT (1H1,1X,72A1/)
6002 FORMAT (2X,8HGAMMA = sF5.2, 5X,SHUE = sF5.2, 5X,6HRLD = sF8.5,
1 5X,8HZCOMB = sF5.2/)  
6003 FORMAT (2X,A4,515,4F10.5/)
6005 FORMAT (2X,7HAKFL = sF8.5, 5X,8PHASE = sF7.2/)  
6007 FORMAT (1H )
6008 FORMAT (1H0)
6010 FORMAT (2X,5X,3HTAU,6X,2HEN, 5X,SHOMEGA,6X,4HITER/)
6012 FORMAT (2X,19HQUASI-STEADY NOZZLE/)
6013 FORMAT (2X,3F10.5,110,5X,19H FAILED TO CONVERGE/)
6014 FORMAT (2X,3F10.5,110/)
6015 FORMAT (1H1)
6022 FORMAT (2X,24HCOUPLING TERMS NEGLECTED/)  
6025 FORMAT (2X,'DROPLET MOMENTUM SOURCE NEGLECTED'/)
6026 FORMAT (2X,'DROPLET MOMENTUM SOURCE INCLUDED'/)
END
REFERENCES


Univac 1108 Math-Pack Programmers Reference, UP-7542, Sperry Rand Corporation (Univac Division).

University of Illinois
Aeronautics/Astronautic Engineering Department
Attn: R. A. Strehlow
Transportation Building, Room 101
Urbana, Illinois 61801

NASA
Manned Spacecraft Center
Attn: J. C. Thibadaux
Houston, Texas 77058

Massachusetts Institute of Technology
Department of Mechanical Engineering
Attn: T. Y. Toong
77 Massachusetts Avenue
Cambridge, Massachusetts 02139

Illinois Institute of Technology
Attn: T. P. Torda
Room 200 M. H.
3300 S. Federal Street
Chicago, Illinois 60616

U. S. Army Missile Command
AMSMI-RKL, Attn: W. W. Wharton
Redstone Arsenal, Alabama 35808

University of California
Aerospace Engineering Department
Attn: F. A. Williams
Post Office Box 109
LaJolla, California 92037

Georgia Institute of Technology
School of Aerospace Engineering
Attn: B. T. Zinn
Atlanta, Georgia 30332

Marshall Industries
Dynamic Science Division
2400 Michelson Drive
Irvine, California 92664

Mr. Donald H. Dahlene
U. S. Army Missile Command
Research, Development, Engineering and Missile Systems Laboratory
Attn: AMSMI-RK
Redstone Arsenal, Alabama 35809

TISIA
Defense Documentation Center
Cameron Station
Building 5
5010 Duke Street
Alexandria, Virginia 22314

Office of Assistant Director
(Chemical Technician)
Office of the Director of Defense Research and Engineering
Washington, D. C. 20301

D. E. Mock
Advanced Research Projects Agency
Washington, D. C. 20525

Dr. H. K. Doetsch
Arnold Engineering Development Center
Air Force Systems Command
Tullahoma, Tennessee 37389

Library
Air Force Rocket Propulsion Laboratory (RPR)
Edwards, California 93523

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Wright-Patterson AFB, Ohio 45433

Technical Information Department
Aeronutronic Division of Philco Ford Corporation
Ford Road
Newport Beach, California 92663